Matsumura, A. and Yoshida, N.
Osaka J. Math.
57 (2020), 187–205

GLOBAL ASYMPTOTICS TOWARD THE RAREFACTION WAVES FOR SOLUTIONS TO THE CAUCHY PROBLEM OF THE SCALAR CONSERVATION LAW WITH NONLINEAR VISCOSITY

Akitaka MATSUMURA and Natsumi YOSHIDA

(Received October 15, 2018, revised November 12, 2018)

Abstract

In this paper, we investigate the asymptotic behavior of solutions to the Cauchy problem for the scalar viscous conservation law where the far field states are prescribed. Especially, we deal with the case when the viscosity is of non-Newtonian type, including a pseudo-plastic case. When the corresponding Riemann problem for the hyperbolic part admits a Riemann solution which consists of single rarefaction wave, under a condition on nonlinearity of the viscosity, it is proved that the solution of the Cauchy problem tends toward the rarefaction wave as time goes to infinity, without any smallness conditions.

1. Introduction and main theorems

In this paper, we consider the asymptotic behavior of solutions to the Cauchy problem for a one-dimensional scalar conservation law with nonlinear viscosity

$$(1.1)\qquad \begin{cases} \partial_t u + \partial_x(f(u) - \sigma(\partial_x u)) = 0 & (t > 0,\ x \in \mathbb{R}), \\ u(0,x) = u_0(x) & (x \in \mathbb{R}), \\ \displaystyle\lim_{x\to\pm\infty} u(t,x) = u_\pm & (t \geq 0). \end{cases}$$

Here, $u = u(t, x)$ is the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity, the functions f and $-\sigma$ stand for the convective flux and viscous/diffusive one, respectively, u_0 is the initial data, and $u_\pm \in \mathbb{R}$ are the prescribed far field states. We suppose that f is a smooth function, and σ is a smooth function satisfying

$$(1.2)\qquad \sigma(0) = 0, \quad \sigma'(v) > 0 \quad (v \in \mathbb{R}),$$

and for some $p > 0$

$$(1.3)\qquad |\,\sigma(v)\,| \sim |\,v\,|^p, \quad |\,\sigma'(v)\,| \sim |\,v\,|^{p-1} \quad (\,|\,v\,| \to \infty\,).$$

A typical example of σ in the field of viscous fluid, where u corresponds to the fluid velocity, is

$$(1.4)\qquad \sigma(\partial_x u) = \mu \left(1 + |\,\partial_x u\,|^2 \right)^{\frac{p-1}{2}} \partial_x u$$

2010 Mathematics Subject Classification. 35K55, 35B40, 35L65.

where $\mu > 0$ is a positive constant, which describes a nonlinear relation between the internal stress σ and the deformation velocity $\partial_x u$, and it is noted that the cases $p > 1$, $p = 1$ and $p < 1$ physically correspond to where the fluid is of dilatant type, Newtonian and pseudo-plastic type, respectively (see [3], [4], [5], [14], [23], [26], [27], [38] and so on). We are interested in the global asymptotics for the solution of (1.1), in particular, the pseudo-plastic case $p < 1$, since there seems no results ever on this case. First, when $u_- = u_+ (=: \tilde{u})$, we expect the solution globally tends toward the constant state $\tilde{u}$ as time goes to infinity. In fact, we can show the following

Theorem 1.1. *Assume the far field states satisfy* $u_- = u_+ (=: \tilde{u})$*, the viscous flux* σ*,* (1.2)*,* (1.3)*, and* $p > 3/7$*. Further assume the initial data satisfy* $u_0 - \tilde{u} \in H^2$*. Then the Cauchy problem* (1.1) *has a unique global in time solution u satisfying*

$$\begin{cases} u - \tilde{u} \in C^0 \cap L^\infty(\,[\,0,\,\infty\,)\,; H^2\,), \\ \partial_x u \in L^2(\,0,\,\infty\,; H^2\,), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\,u(t,x) - \tilde{u}\,| = 0, \qquad \lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\,\partial_x u(t,x)\,| = 0.$$

Remark 1.1. Under the same assumptions in Theorem 1.1, if $\|\,\partial_x u_0\,\|_{H^1}$ is suitably small, the stability results in Theorems 1.1 follow for any $p > 0$.

Next, we consider the case where the convective flux function f is fully convex, that is,

$$f''(u) > 0 \quad (u \in \mathbb{R}), \tag{1.5}$$

and $u_- < u_+$. Then, since the corresponding Riemann problem (cf. [22], [37])

$$\begin{cases} \partial_t u + \partial_x(\,f(u)\,) = 0 \quad (\,t > 0,\ x \in \mathbb{R}\,), \\ u(0,x) = u_0^{\mathrm{R}}(x) := \begin{cases} u_- & (x < 0), \\ u_+ & (x > 0), \end{cases} \end{cases} \tag{1.6}$$

turns out to admit a single rarefaction wave solution, we expect that the solution of the Cauchy problem (1.1) globally tends toward the rarefaction wave as time goes to infinity. Here, the rarefaction wave connecting u_- to u_+ is given by

$$u^r\left(\frac{x}{t}\,;\,u_-,\,u_+\right) = \begin{cases} u_- & (\,x \le f'(u_-)\,t\,), \\ (f')^{-1}\left(\dfrac{x}{t}\right) & (\,f'(u_-)\,t \le x \le f'(u_+)\,t\,), \\ u_+ & (\,x \ge f'(u_+)\,t\,). \end{cases} \tag{1.7}$$

Then we can show the following

Theorem 1.2. *Assume the far field states satisfy* $u_- < u_+$*, the convective flux* f*,* (1.5)*, the viscous flux* σ*,* (1.2)*,* (1.3)*, and* $p > 3/7$*. Further assume the initial data satisfy* $u_0 - u_0^{\mathrm{R}} \in L^2$*, and* $\partial_x u_0 \in H^1$*. Then the Cauchy problem* (1.1) *has a unique global in time solution u satisfying*

$$\begin{cases} u - u_0^{\mathrm{R}} \in C^0 \cap L^\infty(\,[\,0,\ \infty\,)\,;L^2\,), \\ \partial_x u \in C^0 \cap L^\infty(\,[\,0,\ \infty\,)\,;H^1\,) \cap L^2_{\mathrm{loc}}(\,0,\ \infty\,;H^2\,), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} \left| u(t,x) - u^r\left(\frac{x}{t}\ ;\ u_-,\ u_+\right)\right| = 0,$$

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}} \left| \partial_x u(t,x) - \partial_x U\,(\,t,\ x\,;\ u_-,\ u_+\,)\,\right| = 0,$$

where U is a smooth approximation for u^r which is defined by (2.6).

Remark 1.2. Under the same assumptions in Theorem 1.2, if $|\,u_- - u_+\,|$ and

$$\|\,\partial_x u_0 - \partial_x U\,(\,0,\ \cdot\,;\ u_-,\ u_+\,)\,\|_{H^1}$$

are suitably small, the stability results in Theorems 1.2 follow for any $p > 0$.

It should be emphasized again that as far as the global asymptotic stability for either constant states or rarefaction waves, there have been no results for the case $p < 1$ (pseudo-plastic type viscosity). For the case $p = 1$ (Newtonian type viscosity), global nonlinear stability of both rarefaction wave and viscous shock wave were first obtained by Il'in-Oleĭnik [13]. For the case $p > 1$ (dilatant type viscosity), when the convective flux satisfies (1.5) and viscous flux is even the Ostwald-de Waele type (p-Laplacian type, see [6], [35]), that is,

$$\sigma(v) = \mu\,|\,v\,|^{p-1}\,v. \tag{1.8}$$

Matsumura-Nishihara [30] proved that if the far field states satisfy $u_- = u_+ (=:\ \tilde{u})$, then the solution globally tends toward the constant state $\tilde{u}$, and if $u_- < u_+$, then toward the rarefaction wave. Yoshida [43] also obtained the precise time-decay estimates of the solution toward the constant state and the single rarefaction wave. For $p \geq 1$, it is further considered a case where the flux function f is smooth and convex on the whole $\mathbb{R}$ except a finite interval $I := (a,\ b) \subset \mathbb{R}$, and linearly degenerate on I, that is,

$$\begin{cases} f''(u) > 0 \quad (\,u \in (-\infty,\ a\,] \cup [\,b,\ \infty\,)\,), \\ f''(u) = 0 \quad (\,u \in (a,\ b)\,). \end{cases} \tag{1.9}$$

Under the conditions $p \geq 1$, $u_- < u_+$, (1.8), and (1.9), it is proved that the unique global in time solution to the Cauchy problem (1.1) globally tends toward the multiwave pattern of the combination of the viscous contact wave and the rarefaction waves as time goes to infinity, where the viscous contact wave is constructed by the linear heat kernel for $p = 1$ by Matsumura-Yoshida ([32]) (see also [47]), and also by the Barenblatt-Kompaneec-Zel'dovič solution (see also [1], [2], [11], [15], [39], [40], [48]) of the porous medium equation for $p > 1$ by Yoshida ([43], [44]). Yoshida ([41], [42], [45]) also obtained the precise time-decay estimates for these stability results. On the other hand, under the Rankine-Hugoniot condition

$$-s\,(\,u_+ - u_-) + f(u_+) - f(u_-) = 0, \tag{1.10}$$

and Oleĭnik's shock condition

$$-s\,(\,u-u_{\pm}\,)+f(u)-f(u_{\pm})\begin{cases} <0 & (\,u\in(u_{+},\,u_{-})\,),\\ >0 & (\,u\in(u_{-},\,u_{+})\,),\end{cases} \tag{1.11}$$

the local asymptotic stability of viscous shock waves is proved for $p = 1$ by Matsumura-Nishihara ([31]), and very recently for any $p > 0$, more generally, for the case where smooth σ satisfies

$$\sigma(0)=0, \qquad \sigma'(v)>0 \quad (v\in\mathbb{R}), \qquad \lim_{v\to\pm\infty}\sigma(v)=\pm\infty, \tag{1.12}$$

by Yoshida ([46]), though the global asymptotic stability is still open.

The proofs of Theorem 1.1 and Theorem 1.2 are given by a technical energy method, and a Sobolev type inequality motivated by an idea in Kanel' ([16]). Because the proof of Theorem 1.1 and the proof of Theorem 1.2 with $p \geq 1$ are much easier than that of Theorem 1.2 with $0 < p < 1$, we only show Theorem 1.2 under the assumption $0 < p < 1$ in the present paper.

This paper is organized as follows. In Section 2, we prepare the basic properties of the rarefaction wave. In Section 3, we reformulate the problem in terms of the deviation from the asymptotic state. Also, in order to show the global solution in time and its asymptotic behavior for the reformulated problem, we show the strategy how the local existence and the *a priori* estimates are combined. In the remaining Section 4, Section 5, and Section 6, we give the proof of the *a priori* estimates step by step by using a technical energy method.

Some Notations. We denote by C generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha,\beta,\cdots}$ when we emphasize the dependency on $\alpha,\ \beta,\ \cdots$. For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and k-th order Sobolev space on the whole space $\mathbb{R}$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^k}$, respectively.

2. Preliminaries

In this section, we prepare a couple of lemmas concerning with the basic properties of the rarefaction wave. Since the rarefaction wave u^r is not smooth enough, we need some smooth approximated one. We start with the rarefaction wave solution w^r to the Riemann problem for the non-viscous Burgers equation:

$$\begin{cases} \partial_t w+\partial_x\Big(\dfrac{1}{2}\,w^2\Big)=0 & (\,t>0,\ x\in\mathbb{R}\,),\\[2mm] w(0,x)=w_0^{\mathrm{R}}(\,x\,;\,w_-,\ w_+\,):=\begin{cases} w_+ & (x>0),\\ w_- & (x<0),\end{cases} \end{cases} \tag{2.1}$$

where $w_{\pm}\in\mathbb{R}$ are the prescribed far field states satisfying $w_- < w_+$. The unique global weak solution $w = w^r\,(\,x/t\,;\ w_-,\ w_+\,)$ of (2.1) is explicitly given by

$$w^r\Big(\,\frac{x}{t}\,;\ w_-,\ w_+\Big)=\begin{cases} w_- & (\,x\le w_-\,t\,),\\ \dfrac{x}{t} & (\,w_-\,t\le x\le w_+\,t\,),\\ w_+ & (\,x\ge w_+\,t\,).\end{cases} \tag{2.2}$$

Next, under the condition $f''(u) > 0$ $(u\in\mathbb{R})$ and $u_- < u_+$, the rarefaction wave solution $u = u^r\,(x/t\,;\ u_-,\ u_+\,)$ of the Riemann problem (1.6) for hyperbolic conservation law is exactly given by

$$u^r\Big(\frac{x}{t}\,;\,u_-,\,u_+\Big)=(\lambda)^{-1}\Big(w^r\Big(\frac{x}{t}\,;\,\lambda_-,\,\lambda_+\Big)\Big) \tag{2.3}$$

which is nothing but (1.7), where $\lambda(u) := f'(u)$ and $\lambda_\pm := \lambda(u_\pm) = f'(u_\pm)$. We define a smooth approximation of $w^r(x/t\,;\,w_-,\,w_+)$ by the unique classical solution

$$w = w(t,\,x\,;\,w_-,\,w_+) \in C^\infty([0,\,\infty)\times\mathbb{R})$$

to the Cauchy problem for the following non-viscous Burgers equation

$$\begin{cases} \partial_t w + \partial_x\Big(\dfrac{1}{2}w^2\Big) = 0 & (t>0,\ x\in\mathbb{R}),\\ w(0,x) = w_0(x) := \dfrac{w_-+w_+}{2} + \dfrac{w_+-w_-}{2}K_q\displaystyle\int_0^x \frac{\mathrm{d}y}{(1+y^2)^q} & (x\in\mathbb{R}), \end{cases} \tag{2.4}$$

where K_q is a positive constant such that

$$K_q\int_0^\infty \frac{\mathrm{d}y}{(1+y^2)^q} = 1 \quad \Big(q>\frac{1}{2}\Big).$$

By applying the method of characteristics, we get the following formula

$$\begin{cases} w(t,x) = w_0(x_0(t,x)) = \dfrac{\lambda_-+\lambda_+}{2} + \dfrac{\lambda_+-\lambda_-}{2}K_q\displaystyle\int_0^{x_0(t,x)} \frac{\mathrm{d}y}{(1+y^2)^q},\\ x = x_0(t,x) + w_0(x_0(t,x))\,t. \end{cases} \tag{2.5}$$

By making use of (2.5) similarly as in [29], we obtain the properties of the smooth approximation $w(t,\,x\,;\,w_-,\,w_+)$ in the next lemma.

Lemma 2.1. *Assume $w_- < w_+$. Then the classical solution $w = w(t,\,x\,;\,w_-,\,w_+)$ given by* (2.5) *satisfies the following properties:*

(1) $w_- < w(t,x) < w_+$ *and* $\partial_x w(t,x) > 0 \quad (t>0,\ x\in\mathbb{R})$.

(2) *For any $q > 1/2$ and $r \in [1,\,\infty]$, there exists a positive constant $C_{q,r}$ such that*

$$\|\partial_x w(t)\|_{L^r} \le C_{q,r}(1+t)^{-1+\frac{1}{r}} \quad (t\ge 0),$$
$$\|\partial_x^2 w(t)\|_{L^r} \le C_{q,r}(1+t)^{-1-\frac{1}{2q}(1-\frac{1}{r})} \quad (t\ge 0),$$
$$\|\partial_x^3 w(t)\|_{L^r} \le C_{q,r}(1+t)^{-1-\frac{1}{2q}(2-\frac{1}{r})} \quad (t\ge 0).$$

(3) $\displaystyle\lim_{t\to\infty}\sup_{x\in\mathbb{R}}\left|w(t,x) - w^r\Big(\frac{x}{t}\Big)\right| = 0.$

We now define the approximation for the rarefaction wave $u^r(x/t\,;\,u_-,\,u_+)$ by

$$U(t,\,x\,;\,u_-,\,u_+) = (\lambda)^{-1}(w(t,\,x\,;\,\lambda_-,\,\lambda_+)). \tag{2.6}$$

Noting the assumption of the smooth flux function f, we have the next lemma.

Lemma 2.2. *Assume $u_- < u_+$ and $f''(u) > 0$ $(u\in\mathbb{R})$. Then we have the following:*

(1) *$U(t,x)$ defined by* (2.6) *is the unique smooth global solution to the Cauchy problem*

$$\begin{cases} \partial_t U + \partial_x(f(U)) = 0 \quad (t > 0,\ x \in \mathbb{R}), \\ U(0, x) = (\lambda)^{-1} \left(\dfrac{\lambda_- + \lambda_+}{2} + \dfrac{\lambda_+ - \lambda_-}{2} K_q \displaystyle\int_0^x \dfrac{\mathrm{d}y}{(1 + y^2)^q} \right) \quad (x \in \mathbb{R}), \\ \displaystyle\lim_{x \to \pm\infty} U(t, x) = u_\pm \quad (t \geq 0). \end{cases}$$

(2) $u_- < U(t, x) < u_+$ *and* $\partial_x U(t, x) > 0 \quad (t > 0,\ x \in \mathbb{R})$.

(3) *For any* $q > 1/2$ *and* $r \in [\,1,\ \infty\,]$, *there exists a positive constant* $C_{q,r}$ *such that*

$$\begin{aligned} &\| \partial_x U(t) \|_{L^r} \leq C_{q,r} (1 + t)^{-1+\frac{1}{r}} \quad (t \geq 0), \\ &\| \partial_x^2 U(t) \|_{L^r} \leq C_{q,r} (1 + t)^{-1-\frac{1}{2q}(1-\frac{1}{r})} \quad (t \geq 0), \\ &\| \partial_x^3 U(t) \|_{L^r} \leq C_{q,r} (1 + t)^{-1-\frac{1}{2q}(2-\frac{1}{r})} \quad (t \geq 0). \end{aligned}$$

(4) $\displaystyle\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| U(t, x) - u^r \left(\frac{x}{t} \right) \right| = 0.$

Because the proofs of them are well-known, we omit the proofs here (see [9], [10], [25], [29], [32], [41], and so on).

3. Reformulation of the problem

In this section, we reformulate our problem (1.1) in terms of the deviation from the asymptotic state. Now letting

$$u(t, x) = U(t, x) + \phi(t, x), \tag{3.1}$$

we reformulate the problem (1.1) in terms of the deviation ϕ from U as

$$\begin{cases} \partial_t \phi + \partial_x(f(U + \phi) - f(U)) - \partial_x(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U)) \\ \quad = \partial_x(\sigma(\partial_x U)) \quad (t > 0,\ x \in \mathbb{R}), \\ \phi(0, x) = \phi_0(x) := u_0(x) - U(0, x) \quad (x \in \mathbb{R}), \\ \displaystyle\lim_{x \to \pm\infty} \phi(t, x) = 0 \quad (t \geq 0). \end{cases} \tag{3.2}$$

Then we look for the unique global in time solution ϕ which has the asymptotic behavior

$$\sup_{x \in \mathbb{R}} (\, |\phi(t, x)| + |\partial_x \phi(t, x)| \,) \to 0 \qquad (t \to \infty). \tag{3.3}$$

Here we note that $\phi_0 \in H^2$ by the assumptions on u_0, and Lemma 2.2. Then the corresponding theorem for ϕ to Theorem 1.2 we should prove is as follows.

Theorem 3.1. *Assume the far field states satisfy* $u_- < u_+$, *the convective flux* f, (1.5)*, the viscous flux* σ, (1.2), (1.3), *and* $p > 3/7$. *Further assume the initial data satisfy* $\phi_0 \in H^2$. *Then the Cauchy problem* (3.2) *has a unique global in time solution u satisfying*

$$\begin{cases} \phi \in C^0 \cap L^\infty([\,0,\ \infty\,) ; H^2), \\ \partial_x \phi \in L^2(0,\ \infty ; H^2), \end{cases}$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \big(\, |\phi(t,x)| + |\partial_x \phi(t,x)| \, \big) = 0.$$

Theorem 3.1 is shown by combining the local existence of the solution together with the *a priori* estimates as in the previous papers. To state the local existence precisely, the Cauchy problem at general initial time $\tau \geq 0$ with the given initial data $\phi_\tau \in H^2$ is formulated:

$$(3.4)\qquad \begin{cases} \partial_t \phi + \partial_x \big(f(U+\phi) - f(U) \big) - \partial_x \big(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U) \big) \\ \quad = \partial_x \big(\sigma(\partial_x U) \big) \quad (t > \tau, \ x \in \mathbb{R}), \\ \phi(\tau, x) = \phi_\tau(x) := u_\tau(x) - U(\tau, x) \quad (x \in \mathbb{R}), \\ \lim_{x\to\pm\infty} \phi(t, x) = 0 \quad (t \geq \tau). \end{cases}$$

Theorem 3.2 (local existence)**.** *For any $M > 0$, there exists a positive constant $t_0 = t_0(M)$ not depending on τ such that if $\phi_\tau \in H^2$ and $\|\phi_\tau\|_{H^2} \leq M$, then the Cauchy problem* (3.4) *has a unique solution ϕ on the time interval $[\,\tau, \tau + t_0(M)\,]$ satisfying*

$$\phi \in C^0([\,\tau, \tau + t_0\,]; H^2) \cap L^2(\tau, \tau + t_0; H^3).$$

The proof of Theorem 3.2 is given by standard iterative method with the aid of the semigroup theory by Kato [17], [18]. Because the proof is similar to the one in Yoshida [47], we omit the details here (cf. [21], [24], [43], [44]). The *a priori estimates* we establish in Section 4, Section 5 and Section 6 are the following.

Theorem 3.3 (*a priori* estimates)**.** *Under the same assumptions in Theorem 3.1, for any initial data $\phi_0 \in H^2$, there exists a positive constant C_{ϕ_0} such that if the Cauchy problem* (3.2) *has a solution ϕ on a time interval $[\,0, T\,]$ satisfying*

$$\phi \in C^0([\,0, T\,]; H^2) \cap L^2(0, T; H^3)$$

for some constant $T > 0$, then it holds that

$$(3.5)\qquad \|\phi(t)\|_{H^2}^2 + \int_0^t \big\| (\sqrt{\partial_x U}\, \phi)(\tau) \big\|_{L^2}^2 \, d\tau + \int_0^t \big(\|\partial_x \phi(\tau)\|_{H^2}^2 + \|\partial_t \partial_x \phi(\tau)\|_{L^2}^2 \big) \, d\tau \leq C_{\phi_0} \qquad (t \in [\,0, T\,]).$$

Once Theorem 3.3 is established, by combining the local existence Theorem 3.2 with $M = M_0 := \sqrt{C_{\phi_0}}$, $\tau = n\, t_0(M_0)$, and $\phi_\tau = \phi(n\, t_0(M_0))$ $(n = 0, 1, 2, \ldots)$ together with the *a priori* estimates with $T = (n+1)\, t_0(M_0)$ inductively, the unique solution of (3.3) $\phi \in C^0([\,0, n\, t_0(M_0)\,]; H^2) \cap L^2(0, n\, t_0(M_0); H^3)$ for any $n \in \mathbb{N}$ is easily constructed, that is, the global solution in time $\phi \in C^0([\,0, \infty); H^2) \cap L^2_{\mathrm{loc}}(0, \infty; H^3)$. Then, the *a priori* estimates again assert that

$$(3.6)\qquad \sup_{t\geq 0} \|\phi(t)\|_{H^2} < \infty, \qquad \int_0^\infty \big(\|\partial_x \phi(t)\|_{H^2}^2 + \|\partial_t \partial_x \phi(t)\|_{L^2}^2 \big) \, dt < \infty,$$

which easily gives

$$(3.7)\qquad \int_0^\infty \left| \frac{d}{dt} \|\partial_x \phi(t)\|_{L^2}^2 \right| dt < \infty.$$

Hence, it follows from (3.6) and (3.7) that

$$\| \partial_x \phi(t) \|_{L^2} \to 0 \quad (t \to \infty).$$

Due to the Sobolev inequality, the desired asymptotic behavior in Theorem 3.1 is obtained as

$$\sup_{x\in\mathbb{R}} | \phi(t, x) | \le \sqrt{2} \, \| \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} \to 0 \quad (t \to \infty),$$

$$\sup_{x\in\mathbb{R}} | \partial_x \phi(t, x) | \le \sqrt{2} \, \| \partial_x \phi(t) \|_{L^2}^{\frac{1}{2}} \| \partial_x^2 \phi(t) \|_{L^2}^{\frac{1}{2}} \to 0 \quad (t \to \infty).$$

Thus, Theorem 3.1 is shown by combining Theorem 3.2 together with Theorem 3.3. In the following sections, we give the proof of the *a priori* estimates, Theorem 3.3. To do that, in the whole remaining sections we assume $\phi \in C^0([\,0,\, T\,]\,; H^2) \cap L^2(\,0,\, T\,; H^3)$ is a solution of (3.2) for some $T > 0$, and for simplicity we use the notation C_0 to denote positive constants which may depend on the initial data $\phi_0 \in H^2$, and the shape of the equation but not depend on T.

4. A priori estimates I

In this section, we show the following basic L^2-energy estimate for ϕ.

Proposition 4.1. *For* $0 < p < 1$*, there exists a positive constant* C_0 *such that*

$$\| \phi(t) \|_{L^2}^2 + \int_0^t \int | \phi |^2 \, \partial_x U \, \mathrm{d}x\mathrm{d}\tau + \int_0^t \int < \partial_x \phi >^{p-1} | \partial_x \phi |^2 \, \mathrm{d}x\mathrm{d}\tau \le C_0 \qquad (\, t \in [\,0,\, T\,]\,),$$

where $< s > := (1 + s^2)^{1/2}$ $(s \in \mathbb{R})$.

Here and after the range $(\,-\infty,\, \infty\,)$ will be abbreviated without confusion, that is,

$$\int_{-\infty}^{\infty} \mathrm{d}x = \int \mathrm{d}x.$$

To obtain Proposition 4.1, we first show the uniform boundedness of $\|\phi\|_{L^\infty}$ by using the L^q $(q \ge 2)$ energy estimates as follows (cf. [12], [21], [24]).

Lemma 4.1. *There exists a positive constant* C_0 *such that*

$$\| \phi(t) \|_{L^\infty} \le C_0 \qquad (\, t \in [\,0,\, T\,]\,).$$

Proof of Lemma 4.1. For $r \ge 1$, multiplying the equation in (3.2) by $| \phi |^{r-1} \phi$, and integrating the resultant formula with respect to x, we have, after integration by parts,

$$\begin{aligned} (4.1) \qquad & \frac{1}{r+1} \frac{\mathrm{d}}{\mathrm{d}t} \| \phi(t) \|_{L^{r+1}}^{r+1} + r \int \int_0^{\phi} (\, f'(\eta + U) - f'(U)\,) \, | \eta |^{r-1} \, \partial_x U \, \mathrm{d}\eta \, \mathrm{d}x \\ & + r \int | \phi |^{r-1} \, \partial_x \phi \, (\, \sigma(\, \partial_x U + \partial_x \phi\,) - \sigma(\, \partial_x U\,)\,) \, \mathrm{d}x \\ & = \int | \phi |^{r-1} \, \phi \, \partial_x \, (\, \sigma(\, \partial_x U\,)\,) \, \mathrm{d}x. \end{aligned}$$

We estimate the right-hand side of (4.1) by the Hölder inequality as

$$(4.2) \qquad \left| \int | \phi |^{r-1} \, \phi \, \partial_x \, (\, \sigma(\, \partial_x U\,)\,) \, \mathrm{d}x \right| \le \| \phi \|_{L^{r+1}}^{r} \, \left\| \partial_x \, (\, \sigma(\, \partial_x U\,)\,) \right\|_{L^{r+1}}.$$

Note that by the assumptions (1.2), (1.5) on f and σ, the second and third terms on the left side of (4.1) are non-negative. Then, substituting (4.2) into (4.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\phi(t)\,\|_{L^{r+1}} \le \Big\|\,\partial_x\,(\,\sigma(\,\partial_x U\,)\,)\,(t)\,\Big\|_{L^{r+1}}. \tag{4.3}$$

Integrating (4.3) with respect to t, we have for any compact set $K \subset \mathbb{R}$,

$$\|\,\phi(t)\,\|_{L^{r+1}(K)} \le \|\,\phi(t)\,\|_{L^{r+1}} \le \|\,\phi_0\,\|_{L^{r+1}} + \int_0^t \Big\|\,\partial_x\,(\,\sigma(\,\partial_x U\,)\,)\,(\tau)\,\Big\|_{L^{r+1}}\,\mathrm{d}\tau. \tag{4.4}$$

Taking the limit $r \to \infty$ in (4.4), we immediately have

$$\max_{x\in K}|\,\phi(t,x)\,| \le \|\,\phi_0\,\|_{L^\infty} + \int_0^t \Big\|\Big(\,\sigma'(\,\partial_x U\,)\,\partial_x^2 U\Big)(\tau)\Big\|_{L^\infty}\,\mathrm{d}\tau. \tag{4.5}$$

Because the compact set $K \subset \mathbb{R}$ is arbitrary, we obtain

$$\sup_{x\in\mathbb{R}}|\,\phi(t,x)\,| \le \|\,\phi_0\,\|_{L^\infty} + C\int_0^t \Big\|\,\partial_x^2 U(\tau)\,\Big\|_{L^\infty}\,\mathrm{d}\tau. \tag{4.6}$$

Since $\|\partial_x^2 U(\cdot)\|_{L^\infty} \in L^1_t(0,\,\infty)$ by Lemma 2.2, the proof of Lemma 4.1 is completed. □

Proof of Proposition 4.1. Taking $r = 1$ in (4.1), and using the assumptions (1.2), (1.3), (1.5) together with Lemma 4.1, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\,\phi(t)\,\|_{L^2}^2 + C_0^{-1}\int \phi^2\,\partial_x U\,\mathrm{d}x + C_0^{-1}\int <\partial_x\phi>^{p-1}|\,\partial_x\phi\,|^2\,\mathrm{d}x \\ \le \Big|\int \phi\,\partial_x\,(\,\sigma(\,\partial_x U\,)\,)\,\mathrm{d}x\Big|.\end{aligned} \tag{4.7}$$

In order to estimate the right hand side of (4.7), we prepare the following □

Lemma 4.2. *For $g \in H^2$, it holds that*

$$\|\,g\,\|_{L^\infty} \le C\,Q_g^{\frac{1}{4}}\,\|\,g\,\|_{L^2}^{\frac{1}{2}} + C\,Q_g^{\frac{1}{3p+1}}\,\|\,g\,\|_{L^2}^{\frac{2p}{3p+1}}, \tag{4.8}$$

where

$$Q_g := \int <\partial_x g>^{p-1}\;|\,\partial_x g\,|^2\,\mathrm{d}x.$$

Proof of Lemma 4.2. We first note that

$$Q_g \sim \int_{|\partial_x g|\le 1}|\,\partial_x g\,|^2\,\mathrm{d}x + \int_{|\partial_x g|>1}|\,\partial_x g\,|^{p+1}\,\mathrm{d}x. \tag{4.9}$$

By simple calculation, we have

$$\|\,g\,\|_{L^\infty}^2 \le \int 2\,|\,g\,|\,|\,\partial_x g\,|\,\mathrm{d}x = \int_{|\,\partial_x g\,|\le 1} + \int_{|\,\partial_x g\,|>1} =: I_1 + I_2. \tag{4.10}$$

We estimate each $I_i\,(i = 1, 2)$ by using the Hölder and Young inequalities as follows:

$$I_1 \le 2\Big(\int_{|\,\partial_x g\,|\le 1}|\,\partial_x g\,|^2\,\mathrm{d}x\Big)^{\frac{1}{2}}\Big(\int_{|\,\partial_x g\,|\le 1}|\,g\,|^2\,\mathrm{d}x\Big)^{\frac{1}{2}} \le C\,Q_g^{\frac{1}{2}}\,\|\,g\,\|_{L^2}; \tag{4.11}$$

$$I_2 \le 2\Big(\int_{|\partial_x g|>1} |\partial_x g|^{p+1}\,\mathrm{d}x \Big)^{\frac{1}{p+1}} \Big(\int_{|\partial_x g|>1} |g|^{\frac{p+1}{p}}\,\mathrm{d}x \Big)^{\frac{p}{p+1}} \le C\, Q_g^{\frac{1}{p+1}} \|g\|_{L^2}^{\frac{2p}{p+1}} \|g\|_{L^\infty}^{\frac{1-p}{p+1}} \tag{4.12}$$

$$\le \epsilon \|g\|_{L^\infty}^2 + C_\epsilon\, Q_g^{\frac{2}{3p+1}} \|g\|_{L^2}^{\frac{4p}{3p+1}} \quad (\epsilon > 0).$$

Thus, substituting (4.11) and (4.12) into (4.10), and choosing ϵ suitably small, we complete the proof of Lemma 4.2. □

Let us turn to the estimate of the right hand side of (4.7) by using Lemma 2.2 and Lemma 4.2 as

$$\Big| \int \phi\, \partial_x(\sigma(\partial_x U))\,\mathrm{d}x \Big| \le C \|\phi\|_{L^\infty} \|\partial_x^2 U\|_{L^1} \tag{4.13}$$

$$\le \epsilon\, Q_\phi + C_\epsilon \|\phi\|_{L^2}^{\frac{2}{3}} \Big(\|\partial_x^2 U\|_{L^1}^{\frac{4}{3}} + \|\partial_x^2 U\|_{L^1}^{\frac{3p+1}{3p}} \Big)$$

$$\le \epsilon\, Q_\phi + C_\epsilon (1 + \|\phi\|_{L^2}^2)(1+t)^{-\frac{4}{3}} \qquad (\epsilon > 0).$$

Substituting (4.13) into (4.7), choosing ϵ suitably small, and using the Gronwall inequality, we obtain the desired *a priori* estimate for ϕ. Thus, the proof of Proposition 4.1 is completed.

5. A priori estimates II

In this section, we proceed to the *a priori* estimate for the derivative $\partial_x \phi$.

Proposition 5.1. *For $0 < p < 1$, there exists a positive constant C_0 such that*

$$\int <\partial_x \phi>^{p-1} |\partial_x \phi|^2\,\mathrm{d}x + \int_0^t \int <\partial_x \phi>^{2(p-1)} |\partial_x^2 \phi|^2\,\mathrm{d}x\mathrm{d}\tau$$

$$\le C_0 \ll \partial_x \phi \gg_\infty^{1-p} \qquad (t \in [0, T]),$$

where

$$\ll v \gg_\infty := \operatorname*{ess\,sup}_{t\in[0,T],\ x\in\mathbb{R}} < v(t,x) >.$$

Proof of Proposition 5.1. Multiplying the equation in (3.2) by

$$-\partial_x(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U)),$$

and integrating the resultant formula with respect to x, we have, after integration by parts,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \int_0^{\partial_x \phi} (\sigma(\partial_x U + \eta) - \sigma(\partial_x U))\,\mathrm{d}\eta\,\mathrm{d}x \tag{5.1}$$

$$- \int \partial_x(f(U+\phi) - f(U))\, \partial_x(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U))\,\mathrm{d}x$$

$$- \int (\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U) - \sigma'(\partial_x U)\partial_x \phi)\, \partial_t \partial_x U\,\mathrm{d}x$$

$$+ \int \big| \partial_x(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U)) \big|^2\,\mathrm{d}x$$

$$= - \int \partial_x(\sigma(\partial_x U + \partial_x \phi) - \sigma(\partial_x U))\, \partial_x(\sigma(\partial_x U))\,\mathrm{d}x.$$

By using the Young inequality, we estimate the second term on the left-hand side of (5.1) as

(5.2)
$$\Big|\int \partial_x(f(U+\phi)-f(U))\,\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\,\mathrm{d}x\Big|$$
$$\leq \epsilon\int\big|\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\big|^2\,\mathrm{d}x + C_\epsilon\int\big|\partial_x(f(U+\phi)-f(U))\big|^2\,\mathrm{d}x \quad (\epsilon>0).$$

Similarly, the right-hand side of (5.1) is estimated as

$$\Big|\int \partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\,\partial_x(\sigma(\partial_x U))\,\mathrm{d}x\Big| \tag{5.3}$$
$$\leq \epsilon\int\big|\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\big|^2\,\mathrm{d}x + C_\epsilon\int|\partial_x^2 U|^2\,\mathrm{d}x \qquad (\epsilon>0).$$

The third term on the left side of (5.1) is estimated by the Tayler formula, the uniform boundedness of σ' for $0<p<1$, and Lemma 2.1 as

$$\Big|\int(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U)-\sigma'(\partial_x U)\,\partial_x\phi)\,\partial_t\partial_x U\,\mathrm{d}x\Big| \tag{5.4}$$
$$\leq\int\big|(\sigma'(\partial_x U+\theta\,\partial_x\phi)-\sigma'(\partial_x U))\big|\,|\partial_x\phi|\,|\partial_t\partial_x U|\,\mathrm{d}x$$
$$\leq C\int|\partial_x\phi|^2\,\mathrm{d}x \qquad (\exists\theta=\theta(t,x)\in(0,\,1)).$$

Substituting (5.2), (5.3) and (5.4) into (5.1), and choosing ϵ suitably small, we have

(5.5)
$$\frac{\mathrm{d}}{\mathrm{d}t}\int\int_0^{\partial_x\phi}(\sigma(\partial_x U+\eta)-\sigma(\partial_x U))\,\mathrm{d}\eta\,\mathrm{d}x+\int\big|\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\big|^2\,\mathrm{d}x$$
$$\leq C\int\big|\partial_x(f(U+\phi)-f(U))\big|^2\,\mathrm{d}x + C\int(|\partial_x^2 U|^2+|\partial_x\phi|^2)\,\mathrm{d}x.$$

By Lemma 2.2 and Lemma 4.1, we estimate the first term on the right-hand side of (5.5) as

$$\int\big|\partial_x(f(U+\phi)-f(U))\big|^2\,\mathrm{d}x\leq C\int(\phi^2\,\partial_x U+|\partial_x\phi|^2)\,\mathrm{d}x. \tag{5.6}$$

Similarly, the second term on the left-hand side of (5.5) is estimated as

(5.7)
$$\int\big|\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\big|^2\,\mathrm{d}x\geq\int\big|\sigma'(\partial_x U+\partial_x\phi)\big|^2\,|\partial_x^2\phi|^2\,\mathrm{d}x$$
$$-\int\big|\sigma'(\partial_x U+\partial_x\phi)-\sigma'(\partial_x U)\big|^2\,|\partial_x^2 U|^2\,\mathrm{d}x$$
$$\geq\int\big|\sigma'(\partial_x U+\partial_x\phi)\big|^2\,|\partial_x^2\phi|^2\,\mathrm{d}x - C\int|\partial_x^2 U|^2\,\mathrm{d}x.$$

Furthermore, by the assumptions (1.2), (1.3), it holds

$$\int|\sigma'(\partial_x U+\partial_x\phi)|^2\,|\partial_x^2\phi|^2\,\mathrm{d}x\sim\int<\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x, \tag{5.8}$$

and

$$\int\int_0^{\partial_x\phi}(\sigma(\partial_x U+\eta)-\sigma(\partial_x U))\,\mathrm{d}\eta\mathrm{d}x\sim\int<\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x. \tag{5.9}$$

Hence, substituting (5.6) and (5.7) into (5.5), and integrating with respect to t, we have

$$\begin{aligned}&\int<\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x+\int_0^t\int<\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\mathrm{d}\tau\\&\le C_0\Big(1+\int_0^t\int(\phi^2\partial_x U+|\partial_x\phi|^2+|\partial_x^2 U|^2)\,\mathrm{d}x\mathrm{d}\tau\Big).\end{aligned} \tag{5.10}$$

Finally, by Lemma 2.1 and Proposition 4.1, it holds that

$$\int_0^t\int(\phi^2\partial_x U+|\partial_x^2 U|^2)\,\mathrm{d}x\mathrm{d}\tau\le C_0, \tag{5.11}$$

and

$$\begin{aligned}\int_0^t\int|\partial_x\phi|^2\,\mathrm{d}x\mathrm{d}\tau&\le\int_0^t\int\ll\partial_x\phi\gg_\infty^{1-p}<\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\mathrm{d}\tau\\&\le C_0\ll\partial_x\phi\gg_\infty^{1-p}.\end{aligned} \tag{5.12}$$

Substituting (5.11) and (5.12) into (5.10), we obtain the desired *a priori* estimate for $\partial_x\phi$. Thus, the proof of Proposition 5.1 is completed. □

6. A priori estimates III

In this section, we further show the *a priori* estimate for $\partial_x^2\phi$, establish the uniform boundedness of $\|\partial_x\phi\|_{L^\infty}$, and then accomplish the proof of Theorem 3.3.

Proposition 6.1. *For $0<p<1$, there exists a positive constant C_0 such that*

$$\begin{aligned}&\int<\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x+\int_0^t\int<\partial_x\phi>^{p-1}|\partial_t\partial_x\phi|^2\,\mathrm{d}x\mathrm{d}\tau\\&\qquad\le C_0\ll\partial_x\phi\gg_\infty^{3(1-p)}\quad(t\in[0,T]).\end{aligned}$$

Proof of Proposition 6.1. Differentiating the equation in (3.2) once with respect to x gives

$$\partial_t\partial_x\phi+\partial_x^2(f(U+\phi)-f(U))-\partial_x^2(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))=\partial_x^2(\sigma(\partial_x U)). \tag{6.1}$$

Multiplying (6.1) by

$$\partial_t(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U)),$$

and integrating the resultant formula with respect to x, we obtain, after integration by parts,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\big|\partial_x(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\big|^2\,\mathrm{d}x\\&+\int\partial_t\partial_x\phi\,\partial_t(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\,\mathrm{d}x\\&+\int\partial_x^2(f(U+\phi)-f(U))\,\partial_t(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\,\mathrm{d}x\\&=\int\partial_t(\sigma(\partial_x U+\partial_x\phi)-\sigma(\partial_x U))\,\partial_x^2(\sigma(\partial_x U))\,\mathrm{d}x.\end{aligned} \tag{6.2}$$

First, we estimate the second term on the left-hand side of (6.2)

$$
\begin{aligned}
(6.3)\qquad &\int \partial_t\partial_x\phi\, \partial_t(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U))\, \mathrm{d}x \\
&= \int \sigma'(\partial_x U + \partial_x\phi)\, |\partial_t\partial_x\phi|^2\, \mathrm{d}x \\
&\quad + \int \partial_t\partial_x\phi\, (\sigma'(\partial_x U + \partial_x\phi) - \sigma'(\partial_x U))\, \partial_t\partial_x U\, \mathrm{d}x,
\end{aligned}
$$

where note that

$$
(6.4)\qquad \int \sigma'(\partial_x U + \partial_x\phi)\, |\partial_t\partial_x\phi|^2\, \mathrm{d}x \sim \int <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi|^2\, \mathrm{d}x.
$$

The second term on the right-hand side of (6.3) is estimated by the Young inequality as

$$
\begin{aligned}
(6.5)\qquad &\left| \int \partial_t\partial_x\phi\, (\sigma'(\partial_x U + \partial_x\phi) - \sigma'(\partial_x U))\, \partial_t\partial_x U\, \mathrm{d}x \right| \\
&\leq C \int |\partial_x\phi|\,|\partial_t\partial_x\phi|\,|\partial_t\partial_x U|\, \mathrm{d}x \\
&\leq \epsilon \int <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi|^2\, \mathrm{d}x + C_\epsilon \int <\partial_x\phi>^{1-p} |\partial_x\phi|^2\, |\partial_t\partial_x U|^2\, \mathrm{d}x \\
&\leq \epsilon \int <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi|^2\, \mathrm{d}x + C_\epsilon \ll \partial_x\phi \gg_\infty^{2(1-p)} \int <\partial_x\phi>^{p-1} |\partial_x\phi|^2\, \mathrm{d}x.
\end{aligned}
$$

Next, we estimate the third term on the left-hand side of (6.2). Noting

$$
\begin{aligned}
(6.6)\qquad &\left| \partial_x^2(f(U+\phi) - f(U)) \right| \\
&\leq C \left(|\partial_x\phi|^2 + |\partial_x\phi|\,|\partial_x U| + |\partial_x^2\phi| + |\phi|\,|\partial_x U|^2 + |\phi|\,|\partial_x^2 U| \right),
\end{aligned}
$$

and

$$
(6.7)\qquad \left| \partial_t(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U)) \right| \leq C <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi| + C\,|\partial_x\phi|\,|\partial_t\partial_x U|,
$$

we have

$$
\begin{aligned}
(6.8)\qquad &\left| \int \partial_x^2(f(U+\phi) - f(U))\, \partial_t(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U))\, \mathrm{d}x \right| \\
&\leq C \int <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi| \\
&\qquad \times \left(|\partial_x\phi|^2 + |\partial_x\phi|\,|\partial_x U| + |\partial_x^2\phi| + |\phi|\,(|\partial_x U|^2 + |\partial_x^2 U|) \right) \mathrm{d}x \\
&\quad + C \int |\partial_x\phi|\,|\partial_t\partial_x U| \\
&\qquad \times \left(|\partial_x\phi|^2 + |\partial_x\phi|\,|\partial_x U| + |\partial_x^2\phi| + |\phi|\,(|\partial_x U|^2 + |\partial_x^2 U|) \right) \mathrm{d}x \\
&=: I_1 + I_2.
\end{aligned}
$$

Let us estimate each I_i $(i = 1, 2)$. By using the Young inequality, we have

$$
(6.9)\qquad I_1 \leq \epsilon \int <\partial_x\phi>^{p-1} |\partial_t\partial_x\phi|^2\, \mathrm{d}x
$$

$$+C_\epsilon \int <\partial_x\phi>^{p-1}\Big(|\partial_x\phi|^4+|\partial_x\phi|^2|\partial_x U|^2+|\partial_x^2\phi|^2+|\phi|^2(|\partial_x U|^4+|\partial_x^2 U|^2)\Big)\,\mathrm{d}x.$$

By using Lemma 2.2, Lemma 4.1, Proposition 4.1, and the Sobolev inequality, each term in the second term on the right-hand side of (6.9) is estimated as follows:

(6.10)
$$\begin{aligned}\int <\partial_x\phi>^{p-1}|\partial_x\phi|^4\,\mathrm{d}x &\le \|\partial_x\phi\|_{L^\infty}^2\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\\ &\le C_0\|\partial_x\phi\|_{L^2}\|\partial_x^2\phi\|_{L^2}\ll\partial_x\phi\gg_\infty^{1-p}\\ &\le C_0\ll\partial_x\phi\gg_\infty^{\frac{5}{2}(1-p)}\Big(\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\Big)^{\frac12}\Big(\int <\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\Big)^{\frac12};\end{aligned}$$

$$\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2|\partial_x U|^2\,\mathrm{d}x\le C\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\,; \tag{6.11}$$

$$\int <\partial_x\phi>^{p-1}|\partial_x^2\phi|^2\,\mathrm{d}x\le\ll\partial_x\phi\gg_\infty^{1-p}\int <\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\,; \tag{6.12}$$

(6.13)
$$\begin{aligned}\int <\partial_x\phi>^{p-1}|\phi|^2(|\partial_x U|^4+|\partial_x^2 U|^2)\,\mathrm{d}x &\le \int|\phi|^2(|\partial_x U|^4+|\partial_x^2 U|^2)\,\mathrm{d}x\\ &\le C_0(\|\partial_x U\|_{L^\infty}^4+\|\partial_x^2 U\|_{L^\infty}^2)\le C_0(1+t)^{-2}.\end{aligned}$$

Similarly, each term in I_2 is estimated as follows:

$$\begin{aligned}\int|\partial_x\phi|^3|\partial_t\partial_x U|\,\mathrm{d}x &\le 2\|\partial_x\phi\|_{L^2}^2\|\partial_x^2\phi\|_{L^2}\|\partial_t\partial_x U\|_{L^2}\\ &\le C\ll\partial_x\phi\gg_\infty^{2(1-p)}\Big(\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\Big)\Big(\int <\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\Big)^{\frac12}\\ &\le C\ll\partial_x\phi\gg_\infty^{\frac52(1-p)}\Big(\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\Big)^{\frac12}\Big(\int <\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\Big)^{\frac12};\end{aligned} \tag{6.14}$$

$$\int|\partial_x\phi|^2|\partial_t\partial_x U||\partial_x U|\,\mathrm{d}x\le C\ll\partial_x\phi\gg_\infty^{1-p}\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\,; \tag{6.15}$$

$$\begin{aligned}\int|\partial_x\phi||\partial_x^2\phi||\partial_t\partial_x U|\,\mathrm{d}x &\le C\Big(\|\partial_x\phi\|_{L^2}^2+\|\partial_x^2\phi\|_{L^2}^2\Big)\\ &\le C\ll\partial_x\phi\gg_\infty^{1-p}\int <\partial_x\phi>^{p-1}|\partial_x\phi|^2\,\mathrm{d}x\\ &\quad+C\ll\partial_x\phi\gg_\infty^{2(1-p)}\int <\partial_x\phi>^{2(p-1)}|\partial_x^2\phi|^2\,\mathrm{d}x\,;\end{aligned} \tag{6.16}$$

$$\begin{aligned}&\int|\phi||\partial_x\phi||\partial_t\partial_x U|(|\partial_x U|^2+|\partial_x^2 U|)\,\mathrm{d}x\\ &\le C\int\Big(|\partial_x\phi|^2+|\phi|^2|\partial_t\partial_x U|^2(|\partial_x U|^4+|\partial_x^2 U|^2)\Big)\,\mathrm{d}x\end{aligned} \tag{6.17}$$

$$\leq C \ll \partial_x\phi \gg_\infty^{1-p} \int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x + C_0 (1+t)^{-2}.$$

We finally estimate the right-hand side of (6.2) by (6.7) and the Young inequality as

$$\begin{aligned} &\left| \int \partial_t(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U)) \partial_x^2(\sigma(\partial_x U)) \,\mathrm{d}x \right| \\ &\quad \leq C \int < \partial_x\phi >^{p-1} |\partial_t\partial_x\phi| \left| \partial_x^2(\sigma(\partial_x U)) \right| \mathrm{d}x \\ &\quad + C \int |\partial_x\phi| |\partial_t\partial_x U| \left| \partial_x^2 (\sigma(\partial_x U)) \right| \mathrm{d}x. \end{aligned} \tag{6.18}$$

Each term on the right-hand side of (6.18) is estimated as

$$\begin{aligned} &\int < \partial_x\phi >^{p-1} |\partial_t\partial_x\phi| \left| \partial_x^2 (\sigma(\partial_x U)) \right| \mathrm{d}x \\ &\quad \leq \epsilon \int < \partial_x\phi >^{p-1} |\partial_t\partial_x\phi|^2 \,\mathrm{d}x + C_\epsilon \left\| \partial_x^2 (\sigma(\partial_x U)) (t) \right\|_{L^2}^2 \\ &\quad \leq \epsilon \int < \partial_x\phi >^{p-1} |\partial_t\partial_x\phi|^2 \,\mathrm{d}x + C_\epsilon (1+t)^{-2}, \end{aligned} \tag{6.19}$$

and

$$\begin{aligned} &\int |\partial_x\phi| |\partial_t\partial_x U| \left| \partial_x^2(\sigma(\partial_x U)) \right| \mathrm{d}x \\ &\quad \leq C \ll \partial_x\phi \gg_\infty^{1-p} \int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x + C (1+t)^{-2}. \end{aligned} \tag{6.20}$$

Then, substituting all the estimates (6.5)~(6.20) into (6.2), choosing ϵ suitably small, and integrating the resultant formula with respect to t, we arrive at

$$\begin{aligned} &\int \left| \partial_x(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U)) \right|^2 \mathrm{d}x + \int_0^t \int < \partial_x\phi >^{p-1} |\partial_t\partial_x\phi|^2 \,\mathrm{d}x\mathrm{d}\tau \\ &\qquad \leq C_0 \ll \partial_x\phi \gg_\infty^{3(1-p)} \quad (t \in [0, T]), \end{aligned} \tag{6.21}$$

where we used the estimate

$$\begin{aligned} &\int_0^t \Big(\int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x\mathrm{d}\tau \Big)^{\frac12} \Big(\int < \partial_x\phi >^{2(p-1)} |\partial_x^2\phi|^2 \,\mathrm{d}x \Big)^{\frac12} \mathrm{d}\tau \\ &\quad \leq \Big(\int_0^t \int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x\mathrm{d}\tau \Big)^{\frac12} \Big(\int_0^t \int < \partial_x\phi >^{2(p-1)} |\partial_x^2\phi|^2 \,\mathrm{d}x\mathrm{d}\tau \Big)^{\frac12} \\ &\quad \leq C_0 \ll \partial_x\phi \gg_\infty^{\frac12 (1-p)} . \end{aligned} \tag{6.22}$$

Finally, if we note the estimates (5.7) and (5.8) imply

$$\int < \partial_x\phi >^{2(p-1)} |\partial_x^2\phi|^2 \,\mathrm{d}x \leq C \int \left| \partial_x(\sigma(\partial_x U + \partial_x\phi) - \sigma(\partial_x U)) \right|^2 \mathrm{d}x + C,$$

the estimate (6.21) immediately implies the desired *a priori* estimate for $\partial_x^2\phi$. Thus the proof of Proposition 6.1 is completed. □

Now, combining Proposition 4.1, Proposition 5.1, and Proposition 6.1, we show the the following uniform boundedness of $\|\partial_x\phi\|_{L^\infty}$ which plays the essential role to control the nonlinearity of σ. The proof is motivated by an idea in Kanel' [16].

Lemma 6.1. *For* $3/7 < p < 1$*, there exists a positive constant* C_0 *such that*

$$\| \partial_x \phi(t) \|_{L^\infty} \le C_0 \qquad (t \in [0, T]).$$

Proof of Lemma 6.1. By the Cauchy-Schwarz inequality, we have for $a > 0$

$$\begin{aligned}(6.23)\qquad < \partial_x\phi(t,x) >^a &= 1 + \int_{-\infty}^{x} \frac{\partial}{\partial y} < \partial_x\phi(t,y) >^a \,\mathrm{d}y \\ &\le 1 + a \int < \partial_x\phi >^{a-2} |\partial_x\phi| |\partial_x^2\phi| \,\mathrm{d}x \\ &\le 1 + a \Big(\int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int < \partial_x\phi >^{2a-3-p} |\partial_x^2\phi|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}}.\end{aligned}$$

If we choose $a = (3p+1)/2$, (6.23) gives

$$(6.24)\qquad \ll \partial_x\phi \gg_\infty^{\frac{3p+1}{2}} \le 1 + C \Big(\int < \partial_x\phi >^{p-1} |\partial_x\phi|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}} \Big(\int < \partial_x\phi >^{2(p-1)} |\partial_x^2\phi|^2 \,\mathrm{d}x \Big)^{\frac{1}{2}},$$

which deduces from Proposition 5.1 and Proposition 6.1 that

$$(6.25)\qquad \ll \partial_x\phi \gg_\infty^{\frac{3p+1}{2}} \le C_0 \ll \partial_x\phi \gg_\infty^{2(1-p)}.$$

Hence, if we assume

$$\frac{3p+1}{2} > 2(1-p) \qquad \Big(\Leftrightarrow p > \frac{3}{7} \Big),$$

we obtain for $3/7 < p < 1$

$$(6.26)\qquad \ll \partial_x\phi \gg_\infty \le C_0.$$

Thus, the proof of Lemma 6.1 is completed. □

By Proposition 4.1, Proposition 5.1, and Proposition 6.1 with the aid of Lemma 6.1, we obtain the energy estimate

$$\begin{aligned}(6.27)\qquad \|\phi(t)\|_{H^2}^2 &+ \int_0^t \big\| (\sqrt{\partial_x U}\,\phi)(\tau) \big\|_{L^2}^2 \,\mathrm{d}\tau \\ &+ \int_0^t \big(\|\partial_x\phi(\tau)\|_{H^1}^2 + \|\partial_t\partial_x\phi(\tau)\|_{L^2}^2 \big) \,\mathrm{d}\tau \le C_0 \qquad (t \in [0, T]).\end{aligned}$$

Therefore, in order to accomplish the proof of Theorem 3.3, it suffices to show the following *a priori* estimate:

$$(6.28)\qquad \int_0^t \|\partial_x^3\phi(\tau)\|_{L^2}^2 \,\mathrm{d}\tau \le C_0 \qquad (t \in [0, T]).$$

The estimate (6.28) is directly obtained by the equation (6.1) and the estimate (6.27) as follows. The equation (6.1) is rewritten as

$$\begin{aligned}\sigma'(\partial_x U + \partial_x\phi)\,\partial_x^3\phi &= \partial_t\partial_x\phi + \partial_x^2\big(f(U+\phi) - f(U)\big) \\ &\quad - \sigma''(\partial_x U + \partial_x\phi)\,|\partial_x U + \partial_x\phi|^2 - \sigma'(\partial_x U + \partial_x\phi)\,\partial_x^3 U.\end{aligned}$$

Then, by the estimate (6.27), Lemma 2.2, and the Sobolev inequality, it holds

$$\begin{aligned}\int_0^t \|\partial_x^3\phi\|_{L^2}^2\,\mathrm{d}\tau &\le C_0 + C_0\int_0^t\int|\partial_x^2\phi|^4\,\mathrm{d}x\mathrm{d}\tau\\ &\le C_0 + C_0\int_0^t \|\partial_x^2\phi\|_{L^2}^3\|\partial_x^3\phi\|_{L^2}\,\mathrm{d}\tau\\ &\le C_0 + \frac{1}{2}\int_0^t\|\partial_x^3\phi\|_{L^2}^2\,\mathrm{d}\tau + C_0\int_0^t\|\partial_x^2\phi\|_{L^2}^2\,\mathrm{d}\tau,\end{aligned}$$

which implies (6.28). Thus, the proof of Theorem 3.3 is completed.

Acknowledgements. The author would like to thank an anonymous referee for his/her helpful comments.

References

[1] G.I. Barenblatt: *On the motion of suspended particles in a turbulent flow taking up a half-space or a plane open channel of finite depth*, Prikl. Mat. Meh., **19** (1955), 61–88 (in Russian).

[2] J.A. Carrillo and G. Toscani: *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J, **49** (2000), 113–142.

[3] R.P. Chhabra: Bubbles, drops and particles in non-Newtonian Fluids, CRC, Boca Raton, FL, 2006.

[4] R.P. Chhabra: *Non-Newtonian Fluids: An Introduction*, available at http://www.physics.iitm.ac.in/˜compflu/Lect-notes/chhabra.pdf.

[5] R.P. Chhabra and J.F. Richardson: Non-Newtonian flow and applied rheology, 2nd edition, Butterworth-Heinemann, Oxford, 2008.

[6] A. de Waele: *Viscometry and plastometry*, J. Oil Colour Chem. Assoc., **6** (1923), 33–69.

[7] Q. Du and M.D. Gunzburger: *Analysis of a Ladyzhenskaya model for incompressible viscous flow*, J. Math. Anal. Appl, **155** (1991), 21–45.

[8] M.E. Gurtin and R.C. MacCamy: *On the diffusion of biological populations*, Math. Biosci. **33** (1979), 35–49.

[9] I. Hashimoto and A. Matsumura: *Large time behavior of solutions to an initial boundary value problem on the half space for scalar viscous conservation law*, Methods Appl. Anal. **14** (2007), 45–59.

[10] Y. Hattori and K. Nishihara: *A note on the stability of rarefaction wave of the Burgers equation*, Japan J. Indust. Appl. Math. **8** (1991), 85–96.

[11] F. Huang, R. Pan and Z. Wang: *L^1 Convergence to the Barenblatt solution for compressible Euler equations with damping*, Arch. Ration. Mech. Anal. **200** (2011), 665–689.

[12] A.M. Il'in, A.S. Kalašnikov and O.A. Oleĭnik, *Second-order linear equations of parabolic type*, Uspekhi Math. Nauk SSSR **17** (1962), 3–146 (in Russian), Russian Math. Surveys **17** (1962), 1–143 (in English).

[13] A.M. Il'in and O.A. Oleĭnik: *Asymptotic behavior of the solutions of the Cauchy problem for some quasi-linear equations for large values of the time*, Mat. Sb. **51** (1960), 191–216 (in Russian).

[14] P. Jahangiri, R. Streblow and D. Müller: *Simulation of Non-Newtonian Fluids using Modelica*: in Proceedings of the 9th International Modelica Conference September 3–5, Munich, Germany 57–62, 2012.

[15] S. Kamin: *Source-type solutions for equations of nonstationary filtration*, J. Math. Anal. Appl. **64** (1978), 263–276.

[16] Ya.I. Kanel': *A model system of equations for the one-dimensional motion of a gas*, Differencial'nya Uravnenija **4** (1968), 721–734 (in Russian).

[17] T. Kato: *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Univ. Tokyo Sect. I, **17** (1970), 241–258.

[18] T. Kato: *Linear evolution equations of "hyperbolic" type, II*, J. Math. Soc. Japan **19** (1973), 648–666.

[19] S. Kawashima and A. Matsumura: *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Comm. Pure Appl. Math. **47** (1994), 1547–1569.

[20] O.A. Ladyženskaja, *New equations for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them*; in Boundary Value Problems of Mathematical Physics V, AMS, Providence, RI, 1970.

[21] O.A. Ladyženskaja, V.A. Solonnikov and N.N. Ural'ceva: Linear and quasilinear equations of parabolic type (Russian), Translated from the Russian by S. Smith., Transl. Math. Monogr. **23**, AMS, Providence, RI, 1968.

[22] P.D. Lax: *Hyperbolic systems of conservation laws II*, Comm. Pure Appl. Math. **10** (1957), 537–566.

[23] H.W. Liepmann and A. Roshko: Elements of Gas Dynamics, John Wiley & Sons, Inc., New York, 1957.

[24] J.L. Lions: Quelques mèthodes de rèsolution des problèmes aux limites non linèaires, Dunod Gauthier-Villars, Paris, 1969 (in French).

[25] T.-P. Liu, A. Matsumura and K. Nishihara: *Behaviors of solutions for the Burgers equation with boundary corresponding to rarefaction waves*, SIAM J. Math. Anal. **29** (1998), 293–308.

[26] J. Málek: *Some frequently used models for non-Newtonian fluids*, available at http://www.karlin.mff.cuni.cz/~malek/new/images/Lecture4.pdf.

[27] J. Málek, D. Pražák and M. Steinhauer: *On the existence and regularity of solutions for degenerate power-law fluids*, Differential Integral Equations **19** (2006), 449–462.

[28] A. Matsumura and M. Mei: *Nonlinear stability of viscous shock profile for a non-convex system of viscoelasticity*, Osaka J. Math. **34** (1997), 589–603.

[29] A. Matsumura and K. Nishihara: *Asymptotic toward the rarefaction wave of solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. **3** (1986), 1–13.

[30] A. Matsumura and K. Nishihara: *Asymptotics toward the rarefaction wave of the solutions of Burgers' equation with nonlinear degenerate viscosity*, Nonlinear Anal. **23** (1994), 605–614.

[31] A. Matsumura and K. Nishihara: *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys. **165** (1994), 83–96.

[32] A. Matsumura and N. Yoshida: *Asymptotic behavior of solutions to the Cauchy problem for the scalar viscous conservation law with partially linearly degenerate flux*, SIAM J. Math. Anal. **44** (2012), 2526–2544.

[33] T. Nagai and M. Mimura: *Some nonlinear degenerate diffusion equations related to population dynamics*, J. Math. Soc. Japan **35** (1983), 539–562.

[34] T. Nagai and M. Mimura: *Asymptotic behavior for a nonlinear degenerate diffusion equation in population dynamics*, SIAM J. Appl. Math. **43** (1983), 449–464.

[35] W. Ostwald: *Über die Geschwindigkeitsfunktion der Viskositat disperser Systeme*, I. Colloid Polym. Sci. **36** (1925), 99–117 (in German).

[36] R.E. Pattle: *Diffusion from an instantaneous point source with a concentration-dependent coefficient*, Quart. J. Mech. Appl. Math. **12** (1959), 407–409.

[37] J. Smoller: Shock Waves and Reaction-diffusion Equations, Springer-Verlag, New York, 1983.

[38] T. Sochi: *Pore-Scale Modeling of Non-Newtonian Flow in Porous Media*, PhD thesis, Imperial College London, 2007.

[39] J.L. Vázquez: Smoothing and Decay Estimates for Nonlinear Diffusion Equations: Equations of Porous Medium Type, Oxford Math. and Appl. **33**, Oxford University Press, Oxford, 2006.

[40] J.L. Vázquez: The Porous Medium Equation: Mathematical Theory, Oxford Math. Monogr., Oxford: Clarendon Press, New York, 2007.

[41] N. Yoshida: *Decay properties of solutions toward a multiwave pattern for the scalar viscous conservation law with partially linearly degenerate flux*, Nonlinear Anal. **96** (2014), 189–210.

[42] N. Yoshida: *Decay properties of solutions to the Cauchy problem for the scalar conservation law with nonlinearly degenerate viscosity*, Nonlinear Anal. **128** (2015), 48–76.

[43] N. Yoshida: *Large time behavior of solutions toward a multiwave pattern for the Cauchy problem of the scalar conservation law with degenerate flux and viscosity*: in Sūrikaisekikenkyūsho Kōkyūroku "Mathematical Analysis in Fluid and Gas Dynamics", **1947**, 205–222, 2015.

[44] N. Yoshida: *Asymptotic behavior of solutions toward a multiwave pattern for the scalar conservation law with the Ostwald-de Waele-type viscosity*, SIAM J. Math. Anal. **49** (2017), 2009–2036.

[45] N. Yoshida: *Decay properties of solutions toward a multiwave pattern to the Cauchy problem for the scalar conservation law with degenerate flux and viscosity*, J. Differential Equations **263** (2017), 7513–7558.

[46] N. Yoshida: *Asymptotic behavior of solutions toward the viscous shock waves to the Cauchy problem for the scalar conservation law with nonlinear flux and viscosity*, SIAM J. Math. Anal. **50** (2018), 891–932.

[47] N. Yoshida: *Asymptotic behavior of solutions toward a multiwave pattern to the Cauchy problem for the dissipative wave equation with partially linearly degenerate flux*, appear in Funkcialaj Ekvacioj.
[48] Ya.B. Zel'dovič and A.S. Kompaneec: *On the theory of propagation of heat with the heat conductivity depending upon the temperature*: in Collection in honor of the seventieth birthday of academician A.F. Ioffe, Izdat. Akad. Nauk SSSR, 61–71, 1950 (in Russian).

Akitaka Matsumura
Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University, Suita, Osaka 565–0871
Japan
e-mail: akitaka@math.sci.osaka-u.ac.jp

Natsumi Yoshida
OIC Research Organization
Ritsumeikan University, Ibaraki, Osaka 567–8570
Japan
email: 14v00067@gst.ritsumei.ac.jp

[*Current Address*:]
Faculty of Culture and Information Science
Doshisha University, Kyotanabe, Kyoto 610–0394
Japan
e-mail: jt-bnk68@mail.doshisha.ac.jp