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# UNIFORM WELL-POSEDNESS FOR A TIME-DEPENDENT GINZBURG-LANDAU MODEL IN SUPERCONDUCTIVITY

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**Abstract**

We study the initial boundary value problem for a time-dependent Ginzburg-Landau model in superconductivity. First, we prove the uniform boundedness of strong solutions with respect to diffusion coefficient $0 < \epsilon < 1$ in the case of Coulomb gauge. Our second result is the global existence and uniqueness of the weak solutions to the limit problem when $\epsilon = 0$.

## 1. Introduction

This paper is concerned with the following Ginzburg-Landau model in superconductivity:

$$\eta\partial_t\psi + i\eta k\phi\psi + \left(i\frac{\epsilon}{k}\nabla + A\right)^2\psi + (|\psi|^2 - 1)\psi = 0, \tag{1.1}$$

$$\partial_t A + \nabla\phi + \operatorname{curl}^2 A + \operatorname{Re}\left\{\left(i\frac{\epsilon}{k}\nabla\psi + \psi A\right)\overline{\psi}\right\} = 0 \tag{1.2}$$

in $Q_T := (0,T)\times\Omega$, with boundary and initial conditions

$$\epsilon\nabla\psi\cdot\nu = 0,\ \ A\cdot\nu = 0,\ \ \operatorname{curl} A\times\nu = 0\ \text{ on }\ (0,T)\times\partial\Omega, \tag{1.3}$$

$$(\psi, A)(x,0) = (\psi_0, A_0)(x)\ \text{ in }\ \Omega. \tag{1.4}$$

Here $\Omega\subset\mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$, $\nu$ is the outward normal to $\partial\Omega$, and $T$ is any given positive constant. The unknowns $\psi, A$, and $\phi$ are $\mathbb{C}$-valued, $\mathbb{R}^d$-valued, and $\mathbb{R}$-valued functions, respectively, and they stand for the order parameter, the magnetic potential, and the electric potential, respectively. $\eta$ and $k$ are Ginzburg-Landau positive constants. $\overline{\psi}$ denotes the complex conjugate of $\psi$, $\operatorname{Re}\psi := (\psi+\overline{\psi})/2$, $|\psi|^2 := \psi\overline{\psi}$ is the density of superconducting carriers, and $i := \sqrt{-1}$. $\epsilon$ is a positive constant.

It is well known that the Ginzburg-Landau equations are gauge invariant, namely if $(\psi, A, \phi)$ is a solution of (1.1)-(1.4), then for any real-valued smooth function $\chi$, $(\psi e^{ik\chi}, A + \nabla\chi, \phi - \partial_t\chi)$ is also a solution of (1.1)-(1.4). So, in order to obtain the well-posedness of the problem, we need to impose suitable gauge condition. From the physical point of view, one usually has four types of the gauge conditions:

- Coulomb gauge: $\operatorname{div} A = 0$ in $\Omega$ and $\int_\Omega \phi dx = 0$.
- Lorentz gauge: $\phi = -\operatorname{div} A$ in $\Omega$.
- Lorenz gauge: $\partial_t\phi = -\operatorname{div} A$ in $\Omega$.
- Temporal gauge(Weyl gauge): $\phi = 0$ in $\Omega$.

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For the initial data $\psi_0 \in H^1(\Omega), |\psi_0| \leq 1, A_0 \in H^1(\Omega)$, Chen, Elliott and Tang [1], Chen, Hoffmann and Liang [2], Du [3] and Tang [4] proved the existence and uniqueness of global strong solutions to (1.1)-(1.4) in the case of the Coulomb and Lorentz as well as temporal gauges. For the initial data $\psi_0 \in H^1(\Omega), A_0 \in H^1(\Omega)$, Tang and Wang [5] obtained the existence and uniqueness of global strong solutions, while Fan and Jiang [6] showed the existence of global weak solutions when $\psi_0, A_0 \in L^2$. Fan and Ozawa [7] (2-D) and Fan, Gao and Guo [8, 9] (3-D) prove the uniqueness of weak solutions for $\psi_0, A_0 \in L^d$ with $d = 2, 3$, which is critical. This comes from a scaling argument for (1.1) and (1.2). Move precisely, if $(\psi(t,x), A(t,x), \phi(t,x))$ is a solution of (1.1) and (1.2) associated with the initial data $(\psi_0(x), A_0(x))$ without linear lower order term $\psi$, then

$$(\lambda\psi(\lambda^2 t, \lambda x), \lambda A(\lambda^2 t, \lambda x), \lambda^2\phi(\lambda^2 t, \lambda x)) =: (\psi_\lambda, A_\lambda, \phi_\lambda) \tag{1.5}$$

is also a solution for any $\lambda > 0$. A Banach space $\mathbf{B}$ of distributions on $\mathbb{R} \times \mathbb{R}^d$ is a critical space if its norm verifies for any $\lambda$ and any $u \in \mathbf{B}$,

$$\|u\|_{\mathbf{B}} = \|\lambda u(\lambda^2 \cdot, \lambda \cdot)\|_{\mathbf{B}}.$$

If we choose $\mathbf{B}$ as $L^r(0,\infty; L^p(\mathbb{R}^d))$, then $(r, p)$ should satisfy

$$\frac{2}{r} + \frac{d}{p} = 1.$$

In this paper, we will choose the Coulomb gauge.

First, we will prove

**Theorem 1.1.** *Let $d = 3$ and $0 < \epsilon < 1$. Let $\psi_0 \in H^1, |\psi_0| \leq 1$ and $A_0 \in H^1$. Then the solution $(\psi, A, \phi)$ satisfies*

$$\begin{aligned} &|\psi| \leq 1, \|\psi\|_{L^\infty(0,T;H^1)} \leq C, \|\partial_t\psi\|_{L^2(0,T;L^2)} \leq C, \\ &\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} + \|\partial_t A\|_{L^2(0,T;L^2)} \leq C, \\ &\|\phi\|_{L^2(0,T;H^1)} \leq C \end{aligned} \tag{1.6}$$

*for any $0 < T < \infty$. Here and later $C$ will denote a constant independent of $\epsilon$.*

When $\epsilon = 0$, we will prove

**Theorem 1.2.** *Let $d = 3, \epsilon = 0$, and $\psi_0, A_0 \in L^2$. If $\psi, A \in L^2(0,T;H^1) \cap W$ with $W := \{(\psi, A); \psi \in L^\infty(0,T;L^3) \cap L^2(0,T;L^\infty), A \in L^\infty(0,T;L^3) \cap L^{\frac{2p}{p-3}}(0,T;L^p)$ with some $3 < p \leq \infty\}$, then the problem (1.1)-(1.4) has at most a unique weak solution.*

REMARK 1.1. *The space $W$ is scaling invariant due to (1.5).*

**Theorem 1.3.** *Let $d = 3, \epsilon = 0, \psi_0 \in H^1, |\psi_0| \leq 1$ and $A_0 \in L^4$. Then the problem (1.1)-(1.4) has a unique weak solution.*

REMARK 1.2. *Our results also hold true with the choice of Lorentz gauge.*

In our proofs, we will use the following lemmas.

**Lemma 1.1** ([10, 11]). *Let $\Omega$ be a smooth and bounded open set in $\mathbb{R}^3$. Then there exists $C > 0$ such that*

$$\|f\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\Omega)}^{1-\frac{1}{p}}\|f\|_{W^{1,p}(\Omega)}^{\frac{1}{p}} \tag{1.7}$$

*for any* $1 < p < \infty$ *and* $f : \Omega \to \mathbb{R}^3$ *be in* $W^{1,p}(\Omega)$.

**Lemma 1.2** ([12]). *Let* $\Omega$ *be a regular bounded domain in* $\mathbb{R}^3$, *let* $f : \Omega \to \mathbb{R}^3$ *be a smooth enough vector field, and let* $1 < p < \infty$. *Then, the following identity holds true:*

$$\begin{aligned} & -\int_\Omega \Delta f \cdot f|f|^{p-2}\mathrm{d}x \\ = & \int_\Omega |f|^{p-2}|\nabla f|^2\mathrm{d}x + \frac{4(p-2)}{p^2}\int_\Omega |\nabla |f|^{\frac{p}{2}}|^2\mathrm{d}x - \int_{\partial\Omega}|f|^{p-2}(\nu\cdot\nabla)f\cdot f\mathrm{d}S. \end{aligned} \tag{1.8}$$

## 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, we only need to show a priori estimates (1.6).

To begin with, it is easy to show that [1, 2, 3, 4]:

$$|\psi| \leq 1 \ \text{ in } \ \Omega\times(0,T). \tag{2.1}$$

Testing (1.1) by $\overline{\psi}$ and taking the real parts, we see that

$$\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\mathrm{d}x + \int\left|i\frac{\epsilon}{k}\nabla\psi + \psi A\right|^2\mathrm{d}x + \int|\psi|^4\mathrm{d}x = \int|\psi|^2\mathrm{d}x,$$

which gives

$$\int_0^T\int\left|i\frac{\epsilon}{k}\nabla\psi + \psi A\right|^2\mathrm{d}x\mathrm{d}t \leq C. \tag{2.2}$$

In [6], we have proved that

$$\nabla\phi\cdot\nu = 0 \ \text{ on } \ (0,T)\times\partial\Omega. \tag{2.3}$$

Testing (1.2) by $\partial_t A + \operatorname{curl}^2 A$, using (2.1), (2.2) and (2.3), we find that

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\int|\operatorname{curl} A|^2\mathrm{d}x + \int(|\partial_t A|^2 + |\operatorname{curl}^2 A|^2)\mathrm{d}x \\ \leq & \int\left|i\frac{\epsilon}{k}\nabla\psi + \psi A\right||\partial_t A + \operatorname{curl}^2 A|\mathrm{d}x \\ \leq & \frac{1}{2}\int(|\partial_t A|^2 + |\operatorname{curl}^2 A|^2)\mathrm{d}x + C\int\left|i\frac{\epsilon}{k}\nabla\psi + \psi A\right|^2\mathrm{d}x, \end{aligned}$$

which leads to

$$\|A\|_{L^\infty(0,T;H^1)} + \|A\|_{L^2(0,T;H^2)} + \|\partial_t A\|_{L^2(0,T;L^2)} \leq C, \tag{2.4}$$

whence

$$\|\phi\|_{L^2(0,T;H^1)} \leq C. \tag{2.5}$$

Multiplying (1.1) by $-\Delta\overline{\psi}$, integrating by parts and taking the real part, using (2.1), (2.4) and (2.5), we obtain

$$\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\psi|^2\mathrm{d}x+\frac{\epsilon^2}{k^2}\int|\Delta\psi|^2\mathrm{d}x\\
\leq\ &\left|\mathrm{Re}\int i\eta k\phi\psi\cdot\Delta\overline{\psi}\mathrm{d}x\right|+2\left|\mathrm{Re}\frac{\epsilon}{k}\int iA\nabla\psi\cdot\Delta\overline{\psi}\mathrm{d}x\right|\\
&+\mathrm{Re}\int A^2\psi\Delta\overline{\psi}\mathrm{d}x+\mathrm{Re}\int(|\psi|^2-1)\psi\cdot\Delta\overline{\psi}\mathrm{d}x\\
\leq\ &\frac{1}{2}\frac{\epsilon^2}{k^2}\int|\Delta\psi|^2\mathrm{d}x+C\int|\nabla\phi||\nabla\psi|\mathrm{d}x\\
&+C\|A\|_{L^\infty}^2\|\nabla\psi\|_{L^2}^2+C\|A\|_{L^\infty}\|\nabla A\|_{L^2}\|\nabla\psi\|_{L^2}+C\|\nabla\psi\|_{L^2}^2,
\end{aligned}$$

which yields

$$\|\psi\|_{L^\infty(0,T;H^1)}+\epsilon\|\psi\|_{L^2(0,T;H^2)}\leq C, \tag{2.6}$$

whence

$$\|\partial_t\psi\|_{L^2(0,T;L^2)}\leq C. \tag{2.7}$$

This completes the proof.

□

## 3. Proof of Theorem 1.2

In this section, we will prove the uniqueness. To this end, let $(\psi_i, A_i, \phi_i)$ $(i = 1, 2)$ be the two weak solutions and let

$$\psi := \psi_1-\psi_2, A := A_1-A_2, \phi := \phi_1-\phi_2.$$

Then it is easy to verify that

$$\eta\partial_t\psi+i\eta k\phi\psi_1+i\eta k\phi_2\psi+A_1^2\psi_1-A_2^2\psi_2+|\psi_1|^2\psi_1-|\psi_2|^2\psi_2-\psi=0, \tag{3.1}$$

$$\partial_tA+\nabla\phi+\mathrm{curl}\,^2A+|\psi_1|^2A_1-|\psi_2|^2A_2=0, \tag{3.2}$$

$$-\Delta\phi=\mathrm{div}\,(|\psi_1|^2A_1-|\psi_2|A_2). \tag{3.3}$$

Testing (3.1) by $\overline{\psi}$ and taking the real part, we get

$$\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\psi|^2\mathrm{d}x\\
\leq\ &\eta k\left|\int\phi\psi_1\psi\mathrm{d}x\right|+\left|\int(A_1^2-A_2^2)\psi_2\overline{\psi}\mathrm{d}x\right|+\int|\psi_2|^2|\psi|^2\mathrm{d}x+\int|\psi|^2\mathrm{d}x\\
\leq\ &C\|\phi\|_{L^2}\|\psi_1\|_{L^\infty}\|\psi\|_{L^2}+C\|A_1+A_2\|_{L^p}\|A\|_{L^{\frac{2p}{p-2}}}\|\psi_2\|_{L^\infty}\|\psi\|_{L^2}+\int|\psi_2|^2|\psi|^2\mathrm{d}x+\int|\psi|^2\mathrm{d}x\\
\leq\ &\delta\|\phi\|_{L^2}^2+C(\|\psi_1\|_{L^\infty}^2+\|\psi_2\|_{L^\infty}^2+1)\|\psi\|_{L^2}^2+C\|A_1+A_2\|_{L^p}^2\|A\|^2_{L^{\frac{2p}{p-2}}}
\end{aligned} \tag{3.4}$$

for any $0<\delta<1$.

On the other hand, we have

$$\begin{aligned}
\|\phi\|_{L^2}\ &\leq\ C\|\nabla\phi\|_{L^{\frac{6}{5}}}\leq C\||\psi_1|^2A_1-|\psi_2|^2A_2\|_{L^{\frac{6}{5}}}\\
&\leq\ C\||\psi_1|^2A\|_{L^{\frac{6}{5}}}+C\|(|\psi_1|-|\psi_2|)(|\psi_1|+|\psi_2|)A_2\|_{L^{\frac{6}{5}}}
\end{aligned} \tag{3.5}$$

$$
\begin{aligned}
&\leq C\|\psi_1\|_{L^3}^3\|A\|_{L^6} + C\||\psi_1| + |\psi_2|\|_{L^\infty}\|\psi\|_{L^2}\|A_2\|_{L^3} \\
&\leq C\|A\|_{L^6} + C(\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2} \\
&\leq C\|\operatorname{curl} A\|_{L^2} + C(\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2}.
\end{aligned}
$$

Using the Gagliardo-Nirenberg inequality

$$
\|A\|_{L^{\frac{2p}{p-2}}} \leq C\|A\|_{L^2}^{1-\frac{3}{p}}\|A\|_{H^1}^{\frac{3}{p}}, \tag{3.6}
$$

we have

$$
C\|A_1 + A_2\|_{L^p}^2\|A\|_{L^{\frac{2p}{p-2}}}^2 \leq \delta\|A\|_{H^1}^2 + C\|A_1 + A_2\|_{L^p}^{\frac{2p}{p-3}}\|A\|_{L^2}^2 \tag{3.7}
$$

for any $0 < \delta < 1$.

Inserting (3.5) and (3.7) into (3.4), we have

$$
\begin{aligned}
&\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\psi|^2 \mathrm{d}x \\
\leq\ & C\delta\|A\|_{H^1}^2 + C(1 + \|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C(\|A_1\|_{L^p}^{\frac{2p}{p-3}} + \|A_2\|_{L^p}^{\frac{2p}{p-3}})\|A\|_{L^2}^2
\end{aligned} \tag{3.8}
$$

for any $0 < \delta < 1$.

Testing (3.2) by $A$, we deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int A^2 \mathrm{d}x + \int |\operatorname{curl} A|^2 \mathrm{d}x + \int |\psi_1|^2 A \mathrm{d}x \\
=\ & -\int (|\psi_1|^2 - |\psi_2|^2)A_2 A \mathrm{d}x \\
\leq\ & (\|\psi_1\|_{L^\infty} + \|\psi_2\|_{L^\infty})\|\psi\|_{L^2}\|A_2\|_{L^p}\|A\|_{L^{\frac{2p}{p-2}}} \\
\leq\ & (\|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C\|A_2\|_{L^p}^2\|A\|_{L^{\frac{2p}{p-2}}}^2 \\
\leq\ & \delta\|A\|_{H^1}^2 + (\|\psi_1\|_{L^\infty}^2 + \|\psi_2\|_{L^\infty}^2)\|\psi\|_{L^2}^2 + C\|A_2\|_{L^p}^{\frac{2p}{p-3}}\|A\|_{L^2}^2
\end{aligned} \tag{3.9}
$$

for any $0 < \delta < 1$.

Using the well-known Poincaré inequality

$$
\|A\|_{H^1} \leq C\|\operatorname{curl} A\|_{L^2}, \tag{3.10}
$$

summing up (3.8) and (3.9), taking $\delta$ small enough, using the Gronwall inequality, we arrive at

$$
\psi = 0, A = 0
$$

and thus $\phi = 0$, whence $\psi_1 = \psi_2, A_1 = A_2$ and $\phi_1 = \phi_2$.

This completes the proof.

□

## 4. Proof of Theorem 1.3

This section is devoted to the proof of Theorem 1.3, we only need to show a priori estimates.

We still have (2.1).

Testing (1.2) by $A$, we see that

$$\|A\|_{L^2(0,T;H^1)} \le C. \tag{4.1}$$

Testing (1.2) by $|A|^2A$ and using (1.8), we have

$$\begin{aligned}(4.2)\qquad &\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |A|^4\mathrm{d}x + \int |A|^2|\nabla A|^2\mathrm{d}x + \frac{1}{2}\int |\nabla |A|^2|^2\mathrm{d}x + \int |\psi|^2|A|^4\mathrm{d}x \\ = &\int \nabla\phi\cdot|A|^2A\mathrm{d}x + \int_{\partial\Omega}|A|^2(\nu\cdot\nabla)A\cdot A\mathrm{d}S =: I_1 + I_2.\end{aligned}$$

Using the formula

$$\begin{aligned}(\nu\cdot\nabla)A\cdot A &= (A\cdot\nabla)A\cdot\nu + (\mathrm{curl}\,A\times\nu)\cdot A \\ &= (A\cdot\nabla)A\cdot\nu \\ &= -(A\cdot\nabla)\nu\cdot A,\end{aligned}$$

we observe that

$$\begin{aligned}I_2 &= -\int_{\partial\Omega}|A|^2(A\cdot\nabla)\nu\cdot A\mathrm{d}S \le C\int_{\partial\Omega}|A|^4\mathrm{d}S \\ &= C\int_{\partial\Omega} f^2\mathrm{d}S \le C\|f\|_{L^2(\Omega)}\|f\|_{H^1(\Omega)} (f := |A|^2) \\ &\le \frac{1}{8}\int |\nabla f|^2\mathrm{d}x + C\|f\|_{L^2}^2.\end{aligned}$$

Using (2.1), we bound $I_1$ as follows

$$\begin{aligned}I_1 &\le \|\nabla\phi\|_{L^4}\|A\|_{L^4}^3 \\ &\le C\||\psi|^2A\|_{L^4}\|A\|_{L^4}^3 \le C\|A\|_{L^4}^4.\end{aligned}$$

Inserting the above estimates into (4.2), we have

$$\|A\|_{L^\infty(0,T;L^4)} + \int_0^T\int |A|^2|\nabla A|^2\mathrm{d}x\mathrm{d}t \le C, \tag{4.3}$$

whence

$$\|A\|_{L^5(0,T;L^5)} \le C, \tag{4.4}$$

$$\|\nabla\phi\|_{L^\infty(0,T;L^4)} \le C. \tag{4.5}$$

Taking $\nabla$ to (1.1), testing by $\nabla\overline{\psi}$ and taking the real part, using (2.1), (4.3) and (4.5), we have

$$\begin{aligned}\frac{\eta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\nabla\psi|^2\mathrm{d}x &\le \eta k\int |\nabla\phi||\nabla\psi|\mathrm{d}x + \int |\nabla |A|^2||\nabla\psi|\mathrm{d}x + C\int |\nabla\psi|^2\mathrm{d}x \\ &\le C\|\nabla\phi\|_{L^2}\|\nabla\psi\|_{L^2} + C\|\nabla\psi\|_{L^2}^2 + C\int |A|^2|\nabla A|^2\mathrm{d}x,\end{aligned}$$

which implies

$$\|\psi\|_{L^\infty(0,T;H^1)} \le C.$$

This completes the proof. □

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