UNSTABILIZED WEAKLY REDUCIBLE HEEGARDA SPLITTINGS

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Abstract

In this paper, we give a sufficient condition for a (weakly reducible) Heegaard splitting to be unstabilized and uncritical. We also give a sufficient condition for a Heegaard splitting to be critical.

1. Introduction

All 3-manifolds in this paper are assumed to be compact and orientable. All surfaces in 3-manifolds are assumed to be properly embedded and orientable.

Let $M$ be a 3-manifold. If there is a closed surface $S$ which cuts $M$ into two compression bodies $V$ and $W$ with $S = \partial V = \partial W$, then we say $M$ has a Heegaard splitting, denoted by $M = V \cup_S W$; and $S$ is called a Heegaard surface of $M$. Moreover, if the genus $g(S)$ of $S$ is minimal among all Heegaard surfaces of $M$, then $g(S)$ is called the genus of $M$, denoted by $g(M)$. If there are essential disks $B \subset V$ and $D \subset W$ such that $\partial B = \partial D$ (resp. $\partial B \cap \partial D = \emptyset$), then $V \cup_S W$ is said to be reducible (resp. weakly reducible). Otherwise, it is said to be irreducible (resp. strongly irreducible). If there are essential disks $B \subset V$ and $D \subset W$, such that $|B \cap D| = 1$, then $M = V \cup_S W$ is said to be stabilized; otherwise, $M = V \cup_S W$ is said to be unstabilized. If a surface $F$ in a 3-manifold $M$ is incompressible and not parallel to $\partial M$, then $F$ is said to be essential. If a separating surface $F$ in $M$ is compressible on both sides of $F$, then $F$ is said to be bicompressible. If every compressing disk in one side of $F$ intersects every compressing disk in the other side, then $F$ is said to be strongly irreducible. If $F$ is incompressible except for $\partial F$, then $F$ is said to be almost incompressible; if $F$ is bicompressible except for $\partial F$, then $F$ is said to be almost bicompressible; if $F$ is strongly irreducible except for $\partial F$, then $F$ is said to be almost strongly irreducible, where $[\partial F]$ is the isotopy class of $\partial F$.

Let $M$ be a 3-manifold, and $S$ be a closed separating compressible surface in $M$. $S$ is said to be critical (see [1]), if the compressing disks for $S$ can be partitioned into two sets $C_0$ and $C_1$, and there is at least one pair of disks $V_i, W_i \in C_i$ (i = 0, 1) on opposite sides of $S$, such that $V_i \cap W_i = \emptyset$, and if $V \in C_i$ and $W \in C_j$, lie on opposite sides of $S$, then $V \cap W \neq \emptyset$. If $S$ is not critical, then $S$ is said to be uncritical. There are some examples, see [2]–[4], [8]–[10].

Let $S$ be a closed surface with $g(S) \geq 2$. The curve complex of $S$ (see [5]) is the complex whose vertices are the isotopy classes of essential simple closed curves on $S$, and $k + 1$
vertices determine a \( k \)-simplex if they are represented by pairwise disjoint curves. If \( S \) is a torus, the curve complex of \( S \) (see [11], [12]) is the complex whose vertices are the isotopy classes of essential simple closed curves on \( S \), and \( k + 1 \) vertices determine a \( k \)-simplex if they can be represented by a collection of curves, any two of which intersect in only one point. We denote the curve complex of \( S \) by \( C(S) \). For any two vertices in \( C(S) \), one can define the distance \( d_{C(S)}(x, y) \) to be the minimal number of 1-simplices in a simplicial path joining \( x \) to \( y \) over all such possible paths.

If \( S \) is a surface with \( \partial S \neq \emptyset \), then we can define the curve complex \( C(S) \) of \( S \) and \( d_{C(S)}(x, y) \) for any two vertices \( x \) and \( y \) in \( C(S) \) by the same way, where the vertex of \( C(S) \) is the isotopy class of non-\( \partial \)-parallel essential simple closed curves on \( S \). The distance of the Heegaard splitting \( M = V \cup_S W \) with \( g(S) \geq 2 \) (see [6]) is \( d(S) = \min\{d_{C(S)}(\alpha, \beta) \ | \ \alpha \ \text{bounds a disk in} \ V \ \text{and} \ \beta \ \text{bounds a disk in} \ W \} \). If \( S' \) is an almost bicompressible subsurface of \( S \), then \( d(S') = \min\{d_{C(S')}(\alpha, \beta) \ | \ \alpha \ \text{bounds a disk in} \ V \ \text{and} \ \beta \ \text{bounds a disk in} \ W \} \) is said to be local Heegaard distance of \( S' \) respect to \( d(S) \) (see [7], [13]).

In this paper, we give a sufficient condition for a (weakly reducible) Heegaard splitting to be unstabilized and uncritical. We also give a sufficient condition for a Heegaard splitting to be critical as follows:

**Theorem 1.** Let \( M \) be a 3-manifold, \( M = V \cup_S W \) be a Heegaard splitting of \( M \), \( D \) be an essential disk in \( V \) such that \( \partial D \) cuts \( S \) into an almost incompressible surface \( F \) and an almost strongly irreducible surface \( S' \). If \( d(S') \geq 5 \), then \( M = V \cup_S W \) is unstabilized and uncritical.

**Corollary 2.** Let \( M \) be a 3-manifold, \( M = V \cup_S W \) be a Heegaard splitting of \( M \), \( \psi \) be an essential simple closed curve on \( S \) which cuts \( S \) into an almost incompressible surface \( F \) and an almost strongly irreducible surface \( S' \). If \( d(S') \geq 9 \), then \( M = V \cup_S W \) is unstabilized.

**Theorem 3.** Let \( M \) be an irreducible 3-manifold, \( M = V \cup_S W \) be a Heegaard splitting of \( M \), \( D \) be an essential disk in \( V \) such that \( \partial D \) cuts \( S \) into an almost incompressible surface \( F \) and an almost strongly irreducible surface \( S' \).

1. If \( S \) is critical, then \( d(S') \leq 4 \).
2. If there are two essential disks \( D_V \subset V \) and \( D_W \subset W \), such that \( D_V \) is not isotopic to \( D_D \), \( D_W \cap D \neq \emptyset \) and \( D_W \cap D_V = \emptyset \), then \( S \) is critical.

### 2. The proof of Theorem 1

Firstly, we show that \( M = V \cup_S W \) is unstabilized. Assume on the contrary that \( M = V \cup_S W \) is stabilized. Then, there are two essential disks \( D_V \subset V \) and \( D_W \subset W \), such that \( |D_V \cap D_W| = 1 \). So, there is an essential simple closed curve \( \gamma \) on \( S \) which bounds an essential disk \( D_v \) in \( V \) and an essential disk \( D_w \) in \( W \) such that the 2-sphere \( S' = D_v \cup D_w \) bounds a once-punctured standard genus one Heegaard splitting of the 3-sphere (i.e. a 3-ball).

**Proposition 4.** \( \gamma \cap \partial D \neq \emptyset \).

Proof. Assume on the contrary that \( \gamma \cap \partial D = \emptyset \). If \( \gamma \) is parallel to \( \partial D \), then \( F \) and \( S' \) lie in opposite sides of \( S' \). Since \( F \) is almost incompressible, \( S' \) lies in the 3-ball bounded by \( S' \). Then, \( S' \) is a once-punctured torus. Hence, \( d(S') \leq 1 \), a contradiction. So, \( \gamma \) is a non-\( \partial \)-parallel essential simple closed curve on \( F \) or \( S' \). Since \( F \) is almost incompressible,
By Proposition 4, we may assume that $\gamma \cap \partial D \neq \emptyset$ and $|\gamma \cap \partial D|$ is minimal. So, each component of $\gamma \cap S'$ (resp. $\gamma \cap F$) is an essential arc on $S'$ (resp. $F$). Recall that $\gamma$ bounds an essential disk $D^\gamma_t$ in $V$ and an essential disk $D^\gamma_w$ in $W$. If $|\gamma \cap S'| = |\gamma \cap F| = n$, then $D^\gamma_t$ (resp. $D^\gamma_w$) is said to be an n-disk in $V$ (resp. $W$).

Since $D^\gamma_t \cap D \neq \emptyset$, we may assume that each component of $D^\gamma_t \cap D$ is an arc on both $D^\gamma_t$ and $D$. Let $\alpha$ be a component of $D^\gamma_t \cap D$. Then, $\alpha$ cuts a disk $D_\alpha$ from $D^\gamma_t$. If $\partial D_\alpha \cap D = \emptyset$, then $D_\alpha$ is said to be an outermost disk of $D^\gamma_t$, and $\alpha$ is said to be an outermost arc of $D^\gamma_t \cap D$ on $D^\gamma_t$. Since $F$ is almost incompressible, all outermost disks of $D^\gamma_t$ lie in the component of $cl(V - D)$ which contains $S'$. Let $D_0$ be an outermost disk of $D^\gamma_t$. Then, $|\partial D_0 \cap S'| = |\partial D_0 \cap D| = 1$, and $\partial D_0 \cap S'$ is an essential arc on $S'$. Let $l_1 = \partial D_0 \cap S'$ and $l'_1 = \partial D_0 \cap D$. We push $l'_1$ into $\partial D$ and denote it by $l'_1$. Let $l^1 = l_1 \cup l'_1$. After isotopy, we may assume that $l^1$ lies in $S'$. Since $l_1$ is essential on $S'$, $l_1^1$ is non-$\partial$-parallel essential on $S'$ and bounds an essential disk $D_1$ in $V$. So, $d_{c(S')}(l^1, \partial D_0) = 0$.

If there is an essential disk $D_0$ in $W$ with $\partial D_0 \subset S'$, such that $\partial D_0$ is non-$\partial$-parallel on $S'$ and disjoint from a component $h$ of $\gamma \cap S'$, then $h$ cuts $\partial D$ into two arcs $h_1$ and $h'_1$. Let $h^1 = h \cup h_1$. After isotopy, we may assume that $h^1$ lies in $S'$ and $h^1 \cap \partial D_0 = \emptyset$. Since $h$ is essential on $S'$, $h^1$ is non-$\partial$-parallel on $S'$. So, $d_{c(S')}(l^1, \partial D_0) \leq 1$. Since $h \cap l_1 = \emptyset$, $d_{c(S')}(h^1, l^1) \leq 2$. So, $d(S') \leq d_{c(S')}(\partial D_0, \partial D_0) \leq d_{c(S')}(\partial D, l^1) + d_{c(S')}(l^1, h^1) + d_{c(S')}(h^1, \partial D_0) \leq 3$, a contradiction.

By the argument as above, we may assume that for any essential disk $D^W$ in $W$ with $\partial D^W \subset S'$ and any component $\eta$ of $\gamma \cap S'$, if $\partial D^W$ is non-$\partial$-parallel on $S'$, then $\partial D^W \cap \eta \neq \emptyset$.

If $D^W_\eta$ (which is bounded by $\gamma$) is a 1-disk in $W$, then $|\gamma \cap S'| = 1$. Then, $|\partial D^W_\eta \cap D| = 1$.

Hence, there are two outermost disks of $D^W_\eta$ which lie in different components of $cl(V - D)$, a contradiction. So, we may assume that $D^W_\eta$ is an n-disk with $n \geq 2$.

**Proposition 5** ([2]). There are an essential disk $D_k$ in $W$ with $\partial D_k \subset S'$ and a component $l_2$ of $\gamma \cap S'$, such that $\partial D_k$ is non-$\partial$-parallel on $S'$ and $d_{c(S')}(l^2, \partial D_k) \leq 3$, where $l^2$ is obtained from $l_2$ by attaching a component of $cl(\partial D - \partial l_2)$, after isotopy, $l^2$ is non-$\partial$-parallel essential on $S'$.

Proof. Recall that for any essential disk $D^W$ in $W$ with $\partial D^W \subset S'$ and any component $\alpha$ of $\partial D^W \cap S'$, if $\partial D^W$ is non-$\partial$-parallel on $S'$, then $\partial D^W \cap \alpha \neq \emptyset$. We may assume that $|\partial D^W \cap D^W_\eta|$ is minimal among all essential disks in $W$, whose boundaries lie in $S'$ and are non-$\partial$-parallel. So, each component of $D^W \cap D^W_\eta$ is an arc on both $D^W$ and $D^W_\eta$. Since $|\partial D^W \cap D^W_\eta|$ is minimal, and for each component $\alpha$ of $\partial D^W_\eta \cap S'$, $\alpha \cap \partial D^W \neq \emptyset$, both endpoints of each arc of $\partial D^W_\eta \cap D^W$ on $D^W_\eta$ lie in different components of $\partial D^W_\eta \cap S'$. For each subdisk $D^W_\alpha$ of $D^W_\eta$ which is cut by $D^W$, if $\partial D^W_\alpha$ contains $m$ components or subcomponents of $\partial D^W_\eta \cap S'$, then $D^W_\alpha$ is said to be a pseudo $m$-disk. For each component $\alpha$ of $\partial D^W_\eta \cap S'$, there are two components $\alpha_1$ and $\alpha_2$ of $\partial D^W_\eta \cap F$, which are adjacent to $\alpha$. Let $L_\alpha = \{l \mid l$ is an arc of $D^W_\eta \cap D^W_\alpha$ on $D^W_\alpha$, such that $l \cap \alpha \neq \emptyset\}$.

Suppose $\alpha \in \partial D^W_\eta \cap S'$ and $l_\alpha$ is a component of $L_\alpha$. Then, $l_\alpha$ cuts $D^W_\eta$ into two disks $D'$ and $D''$. We may assume that $D'$ is a pseudo $m_1$-disk, and $D''$ is a pseudo $m_2$-disk. Then, $m_2 = n - m_1 + 2$, see Figure 1. If $D'$ (resp. $D''$) is a pseudo 2-disk, then $l_\alpha$ is said to be $\partial$-parallel to $\partial D^W_\eta \cap F$ in $D^W_\eta$. If all components of $L_\alpha$ are $\partial$-parallel to $\partial D^W_\eta \cap F$ in $D^W_\eta$, then
is only one component \( \alpha \) and \( \partial D \) is not \( \partial \) of \( D \). Hence, \( \alpha \) is a component of \( D \), which is disjoint from \( \partial D \). Thus, we may assume that there is a component \( \partial D \) of \( \partial D \) from \( (\partial D \cup F) \cap \gamma \) in \( D \). So, we may assume that \( \partial D \cap \gamma \) is an \( n \)-disk with \( n \) cuts a disk \( \gamma \) in \( D \), such that \( D \cap \gamma \) is disjoint from \( \partial D \cap \gamma \). Then, \( k_1 \) cuts \( D \) into two disks \( D_1 \) and \( D_2 \). Suppose \( D_1 \) is a pseudo \( n_1 \)-disk and \( D_2 \) is a pseudo \( n_2 \)-disk. Since \( k_1 \) is not parallel to \( \partial D \cap F \) in \( D \), \( 3 \leq n_1, n_2 < n \).

First, we consider \( D_1 \). Note that \( D_1 \cap D \subseteq D \cap D \). If \( D_1 \) is a pseudo 3-disk, then there is only one component \( \alpha \) of \( \partial D \cap \gamma \) on \( \partial D \), such that \( \alpha \cap k_1 = \emptyset \). Hence, \( L_\alpha \) is parallel to \( \partial D \cap F \) in \( D \). So, we may assume that \( D_1 \) is a pseudo \( n_1 \)-disk with \( 4 \leq n_1 < n \). If all components of \( D_1 \cap D \) on \( D_1 \) are parallel to \( (\partial D \cap F) \cup k_1 \) in \( D \), then there is a component \( \alpha \) of \( \partial D \cap \gamma \), such that \( \alpha \cap k_1 = \emptyset \) and \( L_\alpha \) is parallel to \( \partial D \cap F \) in \( D \). So, we may assume that there is a component \( k_2 \) of \( D_1 \cap D \) on \( D_1 \), such that \( k_2 \) is not parallel to \( \partial D \cap F \) in \( D \). Then, \( k_2 \) cuts a disk \( D_2 \) from \( D_1 \), such that \( \partial D_2 \) does not contain \( k_1 \). Hence, \( D_2 \cap D \subseteq D_1 \cap D \subseteq D \cap D \). Since \( k_2 \) is not parallel to \( (\partial D \cap F) \cup k_1 \) in \( D_1 \), we may assume that \( D_2 \) is a pseudo \( n_2 \)-disk with \( 3 \leq n_2 < n_1 < n \). By the same argument as \( D_1 \), either there is a component \( \alpha \) of \( \partial D \cap \gamma \), which is disjoint from \( k_2 \), such that \( L_\alpha \) is parallel to \( \partial D \cap F \) in \( D \), or there is a component \( k_3 \) of \( D_3 \cap D \) on \( D_3 \), such that \( k_3 \) is not parallel to \( (\partial D \cap F) \cup k_2 \) in

\[ L_\alpha \text{ is said to be } \partial \text{-parallel to } \partial \cap \gamma \text{ in } D. \]

**Lemma 6.** There are at least two components \( \alpha \) and \( \beta \) of \( \partial \cap \gamma \), such that both \( L_\alpha \) and \( L_\beta \) are parallel to \( \partial \cap \gamma \) in \( D \).

Proof. If \( \partial \cap \gamma \) is an \( n \)-disk with \( n = 2, 3 \), then the Lemma holds, see Figure 2. So, we may assume that \( \partial \cap \gamma \) is an \( n \)-disk with \( n \geq 4 \). If all components of \( \partial \cap \gamma \) on \( \partial \cap \gamma \) are parallel to \( \partial \cap \gamma \), then the Lemma holds. So, we may assume that there is a component \( k_1 \) of \( \partial \cap \gamma \) on \( \partial \cap \gamma \), such that \( k_1 \) is not parallel to \( \partial \cap \gamma \) in \( D \). Then, \( k_1 \) cuts \( D \) into two disks \( D_1 \) and \( D_2 \). Suppose \( D_1 \) is a pseudo \( n_1 \)-disk and \( D_2 \) is a pseudo \( n_2 \)-disk. Since \( k_1 \) is not parallel to \( \partial \cap \gamma \) in \( D \), \( 3 \leq n_1, n_2 < n \).
$D^2_k$. Then, $k_3$ cuts a disk $D^1_k$ from $D^2_k$, such that $\partial D^1_k$ does not contain $k_2$. Then, $D^1_k \cap D^W \subseteq D^2_k \cap D^W \subseteq D^1_k \cap D^W \subseteq D^W \cap D^W$. Since $k_3$ is not $\partial$-parallel to $(\partial D^W \cap F) \cup k_2$ in $D^2_k$, we may assume that $D^1_k$ is a pseudo $n_3$-disk with $3 \leq n_3 < n_2 < n_1 < n$.

We continue this procedure as above, either there is a component $\alpha$ of $\partial D^m_k \cap S'$, such that $L_\alpha$ is $\partial$-parallel to $\partial D^m_k \cap F$ in $D^W_\gamma$, or there is a component $k_m$ of $D^m_k \cap D^W$ on $D^m$, such that $k_m$ is not $\partial$-parallel to $(\partial D^m_k \cap F) \cup k_{m-1}$ in $D^m$ $(m \geq 2)$. Then, $k_m$ cuts a disk $D^m_k$ from $D^m_{k-1}$, such that $\partial D^m_k$ does not contain $k_{m-1}$. Hence, $D^m_k \cap D^W \subseteq D^m_{k-1} \cap D^W \subseteq \cdots \subseteq D^1_k \cap D^W \subseteq D^W \cap D^W$. Since $k_m$ is not $\partial$-parallel to $(\partial D^m_k \cap F) \cup k_{m-1}$ in $D^m_{k-1}$, we may assume that $D^m_k$ is a pseudo $n_m$-disk with $3 \leq n_m < n_{m-1} < \cdots < n_2 < n_1 < n$. Since $n$ is finite, either there is a component $\alpha$ of $\partial D^m_\gamma \cap S'$, such that $L_\alpha$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$, or $n_m = 3$. If $D^m_k$ is a pseudo $n_m$-disk with $n_m = 3$, then there is only one component $\alpha$ of $\partial D^m_k \cap S'$, which is disjoint from $k_m$, such that $L_\alpha$ is $\partial$-parallel to $\partial D^m_k \cap F$ in $D^W_\gamma$. Finally, we obtain a component $\alpha$ of $\partial D^m_\gamma \cap S'$, such that $L_\alpha$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$.

Second, we consider $D^1_k$. By the same argument as $D^1_k$, there is a component $\beta$ ($\neq \alpha$) of $\partial D^m_\gamma \cap S'$, such that $L_\beta$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$. So, the Lemma holds.

By Lemma 6, there is a component $l_2$ of $\partial D^m_k \cap S'$, such that $L_{l_2}$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$. Let $l'_2$ and $l''_2$ be two components of $\partial D^m_\gamma \cap F$, such that $l'_2$ and $l''_2$ are adjacent to $l_2$. Since $|\gamma \cap \partial D|$ is minimal, both $l'_2$ and $l''_2$ are essential on $F$.

**Lemma 7.** There is a 1-disk $D^1$ in $W$, such that $(\partial D^1 \cap S') \cap l_2 = \emptyset$, and $\partial D^1 \cap F$ is parallel to $l'_2$ or $l''_2$.

Proof. Let $k$ be a component of $L_{l_2}$. Since $L_{l_2}$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$, $k$ cuts a pseudo 2-disk $D^1$ from $D^W_\gamma$. If $\text{int} D^1 \cap L_{l_2} = \emptyset$, then $D^1$ is said to be an outermost disk of $D^W_\gamma$, and $k$ is said to be an outermost arc of $D^W \cap D^W_\gamma$ on $D^W_\gamma$. Let $k_1$ be a component of $L_{l_2}$, such that $k_1$ is an outermost arc of $D^W \cap D^W_\gamma$ on $D^W_\gamma$. Then, $k_1$ cuts an outermost disk $D^1_k$ from $D^W_\gamma$, such that $\text{int} D^1_k \cap L_{l_2} = \emptyset$. So, $D^1_k$ is a pseudo 2-disk. Since $L_{l_2}$ is $\partial$-parallel to $\partial D^m_\gamma \cap F$ in $D^W_\gamma$, we may assume that $k_1$ is parallel to $l'_2$, where $l'_2$ is adjacent to $l_2$ on $\partial D^W_\gamma$. Note that $k_1$ also cuts $D^W$ into two disks $D^W_1$ and $D^W_2$. Let $D_{l_1} = D^W_1 \cup D^W_2$ and $D_{l_2} = D^W_1 \cup D^W_2$. Since $k_1$ is parallel to $l'_2$ in $D^W_\gamma$, after isotopy, both $\partial D_{l_1} \cap F$ and $\partial D_{l_2} \cap F$ are parallel to $l_2$.

Since $l'_2$ is essential on $F$ and $F$ is almost incompressible, both $\partial D_{l_1} \cap S'$ and $\partial D_{l_2} \cap S'$ are essential on $S'$. Hence, $D_{l_1}$ and $D_{l_2}$ are 1-disks in $W$. After isotopy, $|D_{l_1} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$, $|D_{l_1} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$, $|D_{l_2} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$, and $|D_{l_2} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$. We may consider $D_{l_1}$, we only consider $D_{l_2}$. Let $L^1_{l_2} = \{k \mid k$ is a component of $D^W_1 \cap D_{l_2}$, such that $k \cap l_2 \neq \emptyset\}$. Then, $L^1_{l_2} \subseteq L_{l_2}$. Hence, $L^1_{l_2}$ is $\partial$-parallel to $\partial D^W_1 \cap F$ in $D^W_\gamma$. If $L^1_{l_2} = \emptyset$, let $D^1 = D_{l_2}$, then $l'_2 \cap (\partial D^1 \cap S') = \emptyset$ and $\partial D^1 \cap F$ is parallel to $l'_2$. Hence, the Lemma holds. If $L^1_{l_2} \neq \emptyset$, let $k_2$ be a component of $L^1_{l_2}$, such that $k_2$ is an outermost arc of $D_{l_2} \cap D^W_\gamma$ on $D^W_\gamma$. Then, $k_2$ cuts an outermost disk $D^2_k$ from $D^W_\gamma$, such that $\text{int} D^2_k \cap L^1_{l_2} = \emptyset$. So, $D^2_k$ is a pseudo 2-disk. Since $L^1_{l_2}$ is $\partial$-parallel to $\partial D^W_1 \cap F$ in $D^W_\gamma$, we may assume that $k_2$ is parallel to $l'_2$, where $l'_2$ is adjacent to $l_2$ in $D^W_\gamma$. Let $D^2_k$ be a subdisk of $D_{l_2}$, which is cut by $k_2$, such that $\partial D^2_k$ does not contain $\partial D_{l_2} \cap F$, and $D_{l_2} = D^W_1 \cup D^W_2$.

By the same argument as $D_{l_1}$, $D_{l_2}$ is a 1-disks in $W$ and $\partial D_{l_2} \cap F$ is parallel to $l'_2$. After isotopy, $|D_{l_1} \cap D^W_\gamma| < |D_{l_1} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$ and $|D_{l_2} \cap D^W_\gamma| < |D_{l_2} \cap D^W_\gamma| < |D^W \cap D^W_\gamma|$. Let $L^2_{l_2} = \{k \mid k$ is a component of $D^W_1 \cap D_{l_2}$ on $D^W_\gamma$, such that $k \cap l_2 \neq \emptyset\}$. Then, $L^2_{l_2} \subseteq L^1_{l_2} \subseteq L_{l_2}$. 

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Hence, $L_j^2$ is $\partial$-parallel to $\partial D_j^W \cap F$ in $D_j^W$. By the same proof as $D_k$, either $D^1 = D_k$ such that $l_2 \cap (D^1 \cap S') = \emptyset$ and $D^1 \cap F$ is parallel to $l_2$, or we obtain a 1-disk $D_k$, in $W$, such that $\partial D_k \cap F$ is parallel to $l_2$, where $l_2$ is adjacent to $l_2$ in $\partial D_j^W$. $D_k \cap D_j^W \subseteq D_k \cap D_j^W$ and $k \cap l_2 = \emptyset$ $\Rightarrow$ $l_2 \subseteq L_j^2 \subseteq L_j^1 \subseteq L_j^2$. Continue this procedure as above, since $|D_j^W \cap D_j^W|$, is finite, finally, we obtain a 1-disk $D_k$ ($m \geq 1$) in $W$, such that $\partial D_k \cap F$ is parallel to $l_2$, where $l_2$ is adjacent to $l_2$ in $\partial D_j^W$. $D_k \cap D_j^W \subseteq D_k \cap D_j^W \subseteq D_k \cap D_j^W$, and $k \cap l_2 = \emptyset$ $\Rightarrow$ $l_2 \subseteq L_j^2 \subseteq L_j^1 \subseteq L_j^2$. Let $D^1 = D_k$. Then, $l_2 \cap (D^1 \cap S') = \emptyset$ and $D^1 \cap F$ is parallel to $l_2$. Hence, the Lemma holds.

**Lemma 8.** If $D^1$ is a 1-disk in $W$, then there is an essential disk $D_k$ in $W$ with $\partial D_k \subset S'$, such that $D_k \cap D^1 = \emptyset$.

Proof. Assume on the contrary that for each essential disk $D_k$ in $W$ with $\partial D_k \subset S'$, $D_k \cap D^1 \neq \emptyset$. We may assume that $|D_k \cap D^1|$ is minimal among all essential disks in $W$ with $\partial D_k \subset S'$. If $\partial D_k \cap F$ is parallel to $\partial S'$, then $|D_k \cap D^1| = 1$. Let $\delta = D_k \cap D^1$. Then, there is a subdisk $D_\delta$ of $D^1$ which is cut by $\delta$, such that $D_\delta$ contains $\partial D^1 \cap F$. We can push $\delta$ into $F$. After isotopy, we denote $D_\delta$ by $D'_\delta$. So, $D'_\delta$ is an essential disk in $W$ with $\partial D'_\delta \subset F$ and $\partial D'_\delta$ is not parallel to $\partial F$. It is a contradiction to the fact that $F$ is almost incompressible. So, we may assume that $\partial D_\delta$ is not parallel to $\partial S'$. Since $|D_\delta \cap D^1|$ is minimal, each component of $D_\delta \cap D^1$ is an arc on both $D_\delta$ and $D^1$. Let $\lambda$ be an outermost arc of $D^1 \cap D_\delta$ on $D^1$, such that $\lambda$ cuts a subdisk $D_\lambda$ from $D^1$ with $\text{int} D_\lambda \cap D_\delta = \emptyset$, and $\partial D_\lambda$ does not contain $\partial D^1 \cap F$. Also, $\lambda$ cuts $D_\delta$ into $D_\delta^1$ and $D_\delta^2$. Let $D_\delta^1 = D_\lambda \cup D_\lambda^1$ and $D_\delta^2 = D_\lambda \cup D_\lambda^2$. Since $D_\lambda$ is essential in $W$ with $\partial D_\lambda \subset S'$ and $\partial D_\lambda$ is not parallel to $\partial S'$, at least one of $D_\lambda^1$ and $D_\lambda^2$ is essential in $W$ whose boundary lies in $S'$ and is not parallel to $\partial S'$. We may assume that $D_\lambda^1$ is essential in $W$ with $\partial D_\lambda^1 \subset S'$ and $\partial D_\lambda^1$ is not parallel to $\partial S'$. So, $|D_\lambda^1 \cap D^1| < |D_\lambda \cap D^1|$, a contradiction. By Lemma 7, we may assume that $D^1$ is a 1-disk in $W$, such that $l_2 \cap (\partial D^1 \cap S') = \emptyset$, and $\partial D^1 \cap F$ is parallel to $l_2$, where $l_2$ is adjacent to $l_2$ in $\partial D_j^W$. $l_2$ is essential on $F$. For convenience, let $\gamma_1 = \partial D^1 \cap S'$ and $\gamma_2 = \partial D^1 \cap F$. So, $l_2 \cap \gamma_1 = \emptyset$, and $\gamma_2$ is parallel to $l_2$. By Lemma 8, there is an essential disk $D_k$ in $W$ with $\partial D_k \subset S'$, such that $\partial D_k \cap \gamma_1 = \emptyset$. Let $l^2$ be a non-$\partial$-parallel essential simple closed curve on $S'$, which is obtained from $l_2$ by attaching a component of $\text{cl}(\partial D_\lambda \cap l_2)$, $\gamma_1^l$ be a non-$\partial$-parallel essential simple closed curve on $S'$, which is obtained from $\gamma_1$ by attaching a component of $\text{cl}(\partial D_\lambda \cap \gamma_1)$. Since $l_2 \cap \gamma_1 = \emptyset$, $|l^2| \leq 1$. So, $d_{\text{C}(S')}(|l^2|, \gamma_1^l) \leq 2$. Since $\partial D_k \cap \gamma_1 = \emptyset$, $|\partial D_k \cap \gamma_1 | = 0$. Then, $d_{\text{C}(S')}(|l^2|, \partial D_k) \leq 1$. Hence, $d_{\text{C}(S')}(|l^2|, \partial D_k) \leq d_{\text{C}(S')}(|l^2|, \gamma_1^l) + d_{\text{C}(S')}(|l^2|, \partial D_k) \leq 3$. So, the Proposition holds. By Proposition 5, there are an essential disk $D_k$ in $W$ with $\partial D_k \subset S'$ and a component $l_2$ of $\gamma \cap S'$, such that $\partial D_k$ is non-$\partial$-parallel on $S'$ and $d_{\text{C}(S')}(|l^2|, \partial D_k) \leq 3$, where $l^2$ is obtained from $l_2$ by attaching a component of $\text{cl}(\partial D_\lambda \cap l_2)$, after isotopy, $l^2$ is non-$\partial$-parallel essential on $S'$. Since both $l_1$ and $l_2$ are components of $\gamma \cap S'$, $l_1 \cap l_2 = \emptyset$. Then, $|l^1| \cap l^2 | \leq 1$. Since $l^1$ bounds an essential disk $D_1$ in $V$ with $\partial D_1 \subset S'$ and $\partial D_1$ is not $\partial$-parallel, there is an essential disk $D^1$ in $V$ with $\partial D^1 \subset S'$, such that $\partial D^1$ is non-$\partial$-parallel on $S'$ and $d_{\text{C}(S')}(|l^2|, \partial D^1) \leq 1$. So,
Secondly, we show that the Heegaard surface $S$ is uncritical. Assume on the contrary that $S$ is critical. Then, all compressing disks for $S$ can be partitioned into two sets $C_0$ and $C_1$, and there is at least one pair of disks $V_i, W_i \in C_i$ ($i = 0, 1$) on opposite sides of $S$, such that $V_i \cap W_i = \emptyset$, and if $V \in C_1$ and $W \in C_{1-i}$ lie on opposite sides of $S$, then $V \cap W = \emptyset$.

We may assume that $D$ lies in $C_0$, $D_V$ and $D_W$ lie in $C_1$ and $D_V \cap D_W = \emptyset$. By definition, $D \cap D_W \neq \emptyset$. Since $\partial D$ cuts $S$ into an incompressible surface $F$ and an almost strongly irreducible surface $S'$, by a similar argument as above, there are essential disks $D_V \in V$, $D_W \subset W$ and a component $l_2 \subset (\partial D_W \cap S')$, such that $\partial D_V$ is non-$\partial$-parallel on $S'$, $\partial D_W$ is non-$\partial$-parallel on $S'$, $d_{\partial S'}(\partial D_V, l_1) \leq 1$ and $d_{\partial S'}(\partial D_W, l_1) \leq 3$, where $l_1$ is obtained from $l_2$ by attaching a component of $c(l, \partial D - \partial l_2)$, after isotopy. $l_1$ is non-$\partial$-parallel essential on $S'$. So, $d(S') \leq d_{\partial S'}(\partial D_V, \partial D_W) \leq d_{\partial S'}(\partial D_V, l_1) + d_{\partial S'}(l_1, \partial D_W) \leq 4$, a contradiction.

\[\square\]

3. The proof of Corollary 2

Assume on the contrary that $M = V \cup W$ is stabilized. Then, there are two essential disks $D_V \subset V$ and $D_W \subset W$, such that $|D_V \cap D_W| = 1$. So, there is an essential simple closed curve $\gamma$ on $S$ which bounds an essential disk $D_V$ in $V$ and an essential disk $D_W$ in $W$ such that the 2-sphere $S = D_V \cup D_W$ bounds a once-punctured standard genus one Heegaard splitting of the 3-sphere (i.e. a 3-ball). By arguments similar to those for Proposition 4, we may assume that $\gamma \cap \psi \neq \emptyset$ and $|\gamma \cap \psi|$ is minimal. So, each component of $\gamma \cap S'$ (resp. $\gamma \cap F$) is an essential arc on $S'$ (resp. $F$).

If $D_V$ (resp. $D_W$) is a 1-disk in $V$ (resp. $W$), then $|\gamma \cap S'| = 1$. Let $l = \gamma \cap S'$. By Lemma 10 in [2], there are essential disks $D_V' \subset V$ and $D_W' \subset W$, such that $\partial D_V'$ is non-$\partial$-parallel on $S'$, $\partial D_W'$ is non-$\partial$-parallel on $S'$, $d_{\partial S'}(\partial D_V', l_1) \leq 1$ and $d_{\partial S'}(\partial D_W', l_1) \leq 1$, where $l_1$ is obtained from $l$ by attaching a component of $c(l, \partial D - \partial l)$, after isotopy. $l_1$ is non-$\partial$-parallel essential on $S'$. So, $d(S') \leq d_{\partial S'}(\partial D_V', \partial D_W') \leq d_{\partial S'}(\partial D_V', l_1) + d_{\partial S'}(l_1, \partial D_W') \leq 2$, a contradiction.

So, we may assume that $D_V'$ (resp. $D_W'$) is an $n$-disk in $V$ (resp. $W$) with $n \geq 2$. By arguments in the proof of Theorem 1, there are essential disks $D_V \subset V$, $D_W \subset W$, and components $l_1$ and $l_2$ of $\gamma \cap S'$, such that $\partial D_V$ is non-$\partial$-parallel on $S'$, $\partial D_W$ is non-$\partial$-parallel on $S'$, $d_{\partial S'}(\partial D_V, l_1) \leq 3$ and $d_{\partial S'}(\partial D_W, l_1) \leq 3$, where $l_1$ is obtained from $l_1$ by attaching a component of $c(l, \partial D - \partial l_1)$, after isotopy. $l_1$ is non-$\partial$-parallel essential on $S'$. Since both $l_1$ and $l_2$ are components of $\gamma \cap S'$, $l_1 \cap l_2 = \emptyset$. Hence, $d_{\partial S'}(l_1, l_2) \leq 2$. So, $d(S') \leq d_{\partial S'}(\partial D_V, \partial D_W) \leq d_{\partial S'}(\partial D_V, l_1) + d_{\partial S'}(l_1, l_2) + d_{\partial S'}(l_2, \partial D_W) \leq 8$, a contradiction.

\[\square\]

4. The proof of Theorem 3

(1) By arguments in the proof of Theorem 1, if $S$ is critical, then $d(S') \leq 4$.

(2) For all compressing disks for $S$, we partition them into two sets $C_0$ and $C_1$. Let $V \cap C_0 = \{D\}$, $W \cap C_0 = \{D_W\}$. $D_W$ is an essential disk in $W$ and $D_W \cap D = \emptyset$, $V \cap C_1 = \{D_V\}$. $D_V$ is an essential disk in $V$ and $D_V$ is not isotopic to $D$, $W \cap C_1 = \{D_W\}$. $D_W$ is an essential disk in $W$ and $D_W \cap D \neq \emptyset$. Since $S'$ is almost strongly irreducible, $V \cap C_1 \neq \emptyset$ and
W \cap C_0 \neq \emptyset. Since there is an essential disk $D_W \subset W$ with $D_W \cap D \neq \emptyset$, $W \cap C_1 \neq \emptyset$.

In $C_0$, for any disk $D^0_W$ in $W \cap C_0$, $D^0_W \cap D = \emptyset$. In $C_1$, there are two essential disks $D^1_V \subset (V \cap C_1)$ and $D^1_W \subset (W \cap C_1)$, such that $D^1_W \cap D^1_V = \emptyset$. For any disk $D^0_W$ in $W \cap C_1$, $D^1_W \cap D \neq \emptyset$. For any disks $D^0_W \subset (W \cap C_0)$ and $D^1_V \subset (V \cap C_1)$, since $M$ is irreducible, $F$ is almost incompressible and $S'$ is almost strongly irreducible, $\partial D^0_W$ lies in $S'$ and $\partial D^0_W$ is non-\(\partial\)-parallel on $S'$. If $D^1_V \cap D = \emptyset$, since $S'$ is almost strongly irreducible, $D^0_W \cap D^1_V \neq \emptyset$. If $D^1_V \cap D \neq \emptyset$, we may assume that $|D^1_V \cap D|$ is minimal and each component of $D^1_V \cap D$ is an arc on both $D^1_V$ and $D$. Assume on the contrary that $D^0_W \cap D^1_V = \emptyset$. By arguments in the proof of Theorem 1, all outermost disks of $D^1_V$ lies in the component of $cl(V - D)$ which contains $S'$. Let $D_0$ be an outermost disk of $D^1_V$. We can push $\partial D_0$ into $S'$. After isotopy, we still denote it by $D_0$. Since $\partial D_0$ is non-\(\partial\)-parallel on $S'$ and $D^0_W \cap D_0 = \emptyset$, it is a contradiction to the fact that $S'$ is almost strongly irreducible. \(\square\)

References


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