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ON THE DIGITAL REPRESENTATION OF INTEGERS WITH BOUNDED PRIME FACTORS

To Noriko Hirata-Kohno on her sixtieth birthday

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Abstract

Let $b \geq 2$ be an integer. Not much is known on the representation in base b of prime numbers or of numbers whose prime factors belong to a given, finite set. Among other results, we establish that any sufficiently large integer which is not a multiple of b and has only small (in a suitable sense) prime factors has at least four nonzero digits in its representation in base b.

1. Introduction and results

We still do not know whether there are infinitely many prime numbers of the form $2^n + 1$ (that is, with only two nonzero binary digits) or of the form $11\dots11$ (that is, with only the digit 1 in their decimal representation). Both questions are notorious, very difficult open problems, which at present seem to be completely out of reach. However, there have been recently several spectacular advances on the digital representation of prime numbers. In 2010, Mauduit and Rivat [18] established that the sum of digits of primes is well-distributed. Subsequently, Bourgain [8] showed the existence of prime numbers in the sparse set defined by prescribing a positive proportion of the binary digits. This year, Maynard [19] proved that, if d is any digit in $\{0, 1, \dots, 9\}$, then there exist infinitely many prime numbers which do not have the digit d in their decimal representation. The proofs of all these results depend largely on Fourier analysis techniques. On a neighboring topic, Shparlinski [23, 24] and Bourgain [7] obtained lower bounds on the number of prime divisors of integers whose representation in a given integer base contains a fixed number of nonzero digits; see also Elsholtz [14]. In these four papers the proofs use bounds for exponential sums.

Lastly, we mention a result of Stewart [25], who established that, if a and b are multiplicatively independent positive integers, then for every sufficiently large integer n, the representation in base b of the integer a^n has more than $(\log n)/(2 \log\log n)$ nonzero digits. The proof rests on a subtle application of Baker's theory of linear forms in the logarithms of algebraic numbers.

In the present note we study a related problem, namely the digital representation of integers all of whose prime factors belong to a given, finite set of prime numbers. We apply techniques from Diophantine approximation to discuss the following general (and left intentionally vague) question:

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Do there exist arbitrarily large integers which have only small prime factors and, at the same time, few nonzero digits in their representation in some integer base?

The expected answer is *no* and our results are a modest step in this direction.

Let n be a positive integer n and $P[n]$ denote its greatest prime factor, with the convention that $P[1] = 1$. Let $S = \{q_1, \dots, q_s\}$ be a finite, non-empty set of distinct prime numbers. Write $n = q_1^{r_1} \dots q_s^{r_s} M$, where $r_1, \dots, r_s$ are non-negative integers and M is an integer relatively prime to $q_1 \dots q_s$. We define the S-part $[n]_S$ of n by

$$[n]_S := q_1^{r_1} \dots q_s^{r_s}.$$

The S-parts of linear recurrence sequences and of integer polynomials and decomposable forms evaluated at integer points have been studied in [15, 9, 10].

In the sequel, for a given integer $k \geq 2$, we denote by $(u_j^{(k)})_{j\geq 1}$ the sequence, arranged in increasing order, of all positive integers which are not divisible by b and have at most k nonzero digits in their b-ary representation. Said differently, $(u_j^{(k)})_{j\geq 1}$ is the ordered sequence composed of the integers $1, 2, \dots, b-1$ and those of the form

$$d_k b^{n_k} + \dots + d_2 b^{n_2} + d_1, \quad n_k > \dots > n_2 > 0, \quad d_1, \dots, d_k \in \{0, 1, \dots, b-1\}, \quad d_1 d_k \neq 0.$$

We stress that, for the questions investigated in the present note, it is natural to restrict our attention to integers not divisible by b. Obviously, the sequence $(u_j^{(k)})_{j\geq 1}$ depends on b, but, for shortening the notation, we have decided not to mention this dependence.

Our first result shows that, for any base b, there are only finitely many integers not divisible by b which have a given number of nonzero b-ary digits and whose prime divisors belong to a given finite set.

Theorem 1.1. *Let $b \geq 2, k \geq 2$ be integers and ε a positive real number. Let S be a finite, non-empty set of prime numbers. Then, we have*

$$[u_j^{(k)}]_S < (u_j^{(k)})^{\varepsilon},$$

for every sufficiently large integer j. In particular, the greatest prime factor of $u_j^{(k)}$ tends to infinity as j tends to infinity.

The proof of Theorem 1.1 rests on the Schmidt Subspace Theorem and does not allow us to estimate the speed with which $P[u_j^{(k)}]$ tends to infinity with j. It turns out that, by means of the theory of linear forms in logarithms, we are able to derive such an estimate, but (apparently) only for $k \leq 3$.

The case $k = 2$ has already been considered. It reduces to the study of a finite union of binary linear recurrence sequences of the form

$$(d_2 b^n + d_1 1^n)_{n\geq 1}, \quad \text{where } d_1, d_2 \text{ are digits in } \{1, \dots, b-1\}.$$

We gather in the next theorem a recent result of Bugeaud and Evertse [9] and an immediate consequence of a lower bound for the greatest prime factor of terms of binary linear recurrence sequences, established by Stewart [27].

Theorem BES. *Let $b \geq 2$ be an integer. Let S be a finite, non-empty set of prime numbers. Then, there exist an effectively computable positive number c_1, depending only on b, and an effectively computable positive number c_2, depending only on b and S, such that*

$$[u_j^{(2)}]_S \le (u_j^{(2)})^{1-c_1}, \quad \textit{for every } j \ge c_2.$$

Furthermore, there exists an effectively computable positive number c_3, depending only on b and S, such that

$$P[u_j^{(2)}] > (\log u_j^{(2)})^{1/2} \exp\Big(\frac{\log\log u_j^{(2)}}{105 \log\log\log u_j^{(2)}}\Big), \quad \textit{for } j > c_3.$$

We point out that the constant c_1 in Theorem BES does not depend on S.

The main new result of the present note is an estimate of the speed with which $P[u_j^{(3)}]$ tends to infinity with j.

Theorem 1.2. *Let $b \ge 2$ be an integer. Let S be a finite, non-empty set of prime numbers. Then, there exist effectively computable positive numbers c_4 and c_5, depending only on b and S, such that*

$$[u_j^{(3)}]_S \le (u_j^{(3)})^{1-c_4}, \quad \textit{for every } j \ge c_5.$$

Furthermore, for every positive real number ε, there exists an effectively computable positive number c_6, depending only on b and ε, such that

$$P[u_j^{(3)}] > (1-\varepsilon) \log\log u_j^{(3)} \frac{\log\log\log u_j^{(3)}}{\log\log\log\log u_j^{(3)}}, \quad \textit{for } j > c_6. \tag{1.1}$$

The proof of Theorem 1.2 yields a very small admissible value for c_4.

We point out the following reformulation of the second assertion of Theorem 1.2. Recall that a positive integer is called B-smooth if all its prime factors are less than or equal to B.

Corollary 1.3. *Let $b \ge 2$ be an integer. Let ε be a positive integer. There exists an effectively computable positive integer n_0, depending only on b and ε, such that any integer $n > n_0$ which is not divisible by b and is*

$$(1-\varepsilon)(\log\log n)\frac{\log\log\log n}{\log\log\log\log n}\textit{-smooth}$$

has at least four nonzero digits in its b-ary representation.

It is very likely that any large integer cannot be 'very' smooth and, simultaneously, have only few nonzero digits in its b-ary representation. Corollary 1.3 provides a first result in this direction.

The proofs of our theorems are obtained by direct applications of classical deep tools from Diophantine approximation, namely the Schmidt Subspace Theorem and the theory of linear forms in the logarithms of algebraic numbers. The latter theory has already been applied to get lower bounds for the greatest prime factor of linear recurrence sequences (under some assumptions, see [26]) and for the greatest prime factor of integer polynomials and decomposable forms evaluated at integer points (see e.g. [16]). The bounds obtained in [26, 16] have exactly the same order of magnitude as our bound in Theorem 1.2, that is, they involve a double logarithm times a triple logarithm divided by a quadruple logarithm. A brief explanation is given at the end of Section 2.

An interesting feature of the proof of Theorem 1.2 is that it combines estimates for Archimedean and non-Archimedean linear forms in logarithms. Similar arguments appeared when searching for perfect powers with few digits; see [3, 4].

2. Auxiliary results from Diophantine approximation

The Schmidt Subspace Theorem [20, 21, 22] is a powerful multidimensional extension of the Roth Theorem. We quote below a version of it which is suitable for our purpose, but the reader should keep in mind that there are more general formulations.

Theorem 2.1. *Let $m \geq 2$ be an integer. Let S' be a finite set of prime numbers. Let $L_{1,\infty}, \dots, L_{m,\infty}$ be m linearly independent linear forms in m variables with integer coefficients. For any prime ℓ in S', let $L_{1,\ell}, \dots, L_{m,\ell}$ be m linearly independent linear forms in m variables with integer coefficients. Let ε be a positive real number. Then, there are an integer T and proper subspaces $S_1, \dots, S_T$ of $\mathbf{Q}^m$ such that all the solutions $\mathbf{x} = (x_1, \dots, x_m)$ in $\mathbf{Z}^m$ to the inequality*

$$\prod_{\ell \in S'} \prod_{i=1}^{m} |L_{i,\ell}(\mathbf{x})|_\ell \cdot \prod_{i=1}^{m} |L_{i,\infty}(\mathbf{x})| \leq (\max\{1, |x_1|, \dots, |x_m|\})^{-\varepsilon}$$

are contained in the union $S_1 \cup \dots \cup S_T$.

We quote an immediate corollary of a theorem of Matveev [17].

Theorem 2.2. *Let $n \geq 2$ be an integer. Let $x_1/y_1, \dots, x_n/y_n$ be positive rational numbers. Let $b_1, \dots, b_n$ be integers such that $(x_1/y_1)^{b_1} \dots (x_n/y_n)^{b_n} \neq 1$. Let $A_1, \dots, A_n$ be real numbers with*

$$A_i \geq \max\{|x_i|, |y_i|, \mathrm{e}\}, \quad 1 \leq i \leq n.$$

Set

$$B = \max\Big\{1, \max\Big\{|b_j| \, \frac{\log A_j}{\log A_n} : 1 \leq j \leq n\Big\}\Big\}.$$

Then, we have

$$\log\Big|\Big(\frac{x_1}{y_1}\Big)^{b_1} \dots \Big(\frac{x_n}{y_n}\Big)^{b_n} - 1\Big| > -8 \times 30^{n+3} \, n^{9/2} \, \log(\mathrm{e}B) \, \log A_1 \dots \log A_n.$$

The next statement was proved by Yu [29]. For a prime number p and a nonzero rational number z we denote by $v_p(z)$ the exponent of p in the decomposition of z in product of prime factors.

Theorem 2.3. *Let p be a prime number and $n \geq 2$ an integer. Let $x_1/y_1, \dots, x_n/y_n$ be nonzero rational numbers and $A_1, \dots, A_n$ real numbers with*

$$A_i \geq \max\{|x_i|, |y_i|, \mathrm{e}\}, \quad 1 \leq i \leq n.$$

Let $b_1, \dots, b_n$ be nonzero integers such that $(x_1/y_1)^{b_1} \dots (x_n/y_n)^{b_n} \neq 1$. Let B and B_n be real numbers such that

$$B \geq \max\{|b_1|, \dots, |b_n|, 3\} \quad \text{and} \quad B \geq B_n \geq |b_n|.$$

Assume that

$$v_p(b_n) \leq v_p(b_j), \quad j = 1, \dots, n.$$

Let δ be a real number with $0 < \delta \le 1/2$. Then, we have

$$v_p\Big(\Big(\frac{x_1}{y_1}\Big)^{b_1} \ldots \Big(\frac{x_n}{y_n}\Big)^{b_n} - 1\Big) < (16\mathrm{e})^{2(n+1)} n^{3/2} (\log(2n))^2 \, \frac{p}{(\log p)^2}$$
$$\max\Big\{(\log A_1) \cdots (\log A_n)(\log T), \frac{\delta B}{B_n}\Big\},$$

where

$$T = 2B_n \delta^{-1} \mathrm{e}^{(n+1)(6n+5)} p^{n+1} (\log A_1) \cdots (\log A_{n-1}).$$

There are two key ingredients in Theorems 2.2 and 2.3 which explain the quality of the estimates in Theorem 1.2. A first one is the dependence on n, which is only exponential: this allows us to get in (1.1) the extra factor triple logarithm over quadruple logarithm. The use of earlier estimates for linear forms in logarithms would give only the factor involving the double logarithm in (1.1). A second one is the factor $\log A_n$ occurring in the denominator in the definition of B in the statement of Theorem 2.2. The formulation of Theorem 2.3 is slightly different, but, in our special case, it yields a similar refinement. This allows us to save a (small) power of $u_j^{(3)}$ when estimating its S-part. Without this refinement, the saving would be much smaller, namely less than any power of $u_j^{(3)}$.

3. Proofs

Proof of Theorem 1.1. Let $k \ge 2$ be an integer and ε a positive real number. Let $\mathcal{N}_1$ be the set of k-tuples $(n_k, \ldots, n_2, n_1)$ such that $n_k > \ldots > n_2 > n_1 = 0$ and

$$[d_k b^{n_k} + \cdots + d_2 b^{n_2} + d_1]_S > (d_k b^{n_k} + \cdots + d_2 b^{n_2} + d_1)^{\varepsilon},$$

for some integers $d_1, \ldots, d_k$ in $\{0, \ldots, b-1\}$ such that $d_1 d_k \neq 0$.

Assume that $\mathcal{N}_1$ is infinite. Then, there exist an integer h with $2 \le h \le k$, positive integers $D_1, \ldots, D_h$, an infinite set $\mathcal{N}_2$ of h-tuples $(n_{h,i}, \ldots, n_{1,i})$ such that

$$n_{h,i} > \ldots > n_{2,i} > n_{1,i} = 0,$$

$$[D_h b^{n_{h,i}} + \cdots + D_2 b^{n_{2,i}} + D_1]_S > (D_h b^{n_{h,i}} + \cdots + D_2 b^{n_{2,i}} + D_1)^{\varepsilon}, \quad i \ge 1,$$

and

$$\lim_{i \to +\infty} (n_{\ell,i} - n_{\ell-1,i}) = +\infty, \quad \ell = 2, \ldots, h. \tag{3.1}$$

We are in position to apply Theorem 2.1.

Let S_1 denote the set of prime divisors of b. By (3.1), for any prime number p in S_1, we have

$$v_p(D_h b^{n_{h,i}} + \cdots + D_2 b^{n_{2,i}} + D_1) = v_p(D_1),$$

if i is sufficiently large. Consequently, we may assume that S and S_1 are disjoint. Consider the linear forms in $\mathbf{X} = (X_1, \ldots, X_h)$ given by

$$L_{j,\infty}(\mathbf{X}) := X_j, \quad j = 1, \ldots, h,$$

and, for every prime number p in S_1,

$$L_{j,p}(\mathbf{X}) := X_j, \quad j = 1, \dots, h,$$

and, for every prime number p in S,

$$L_{j,p}(\mathbf{X}) := X_j, \quad j = 1, \dots, h-1, \quad L_{h,p}(\mathbf{X}) := D_h X_h + \dots + D_1 X_1.$$

By Theorem 2.1 applied with $S' = S \cup S_1$, the set of tuples $\mathbf{b} = (b^{n_h}, \dots, b^{n_2}, b^{n_1})$ such that $n_h > \dots > n_1 \geq 0$ and

$$\prod_{j=1}^{h} |L_{j,\infty}(\mathbf{b})| \times \prod_{p \in S \cup S_1} \prod_{j=1}^{h} |L_{j,p}(\mathbf{b})|_p < b^{-\varepsilon n_h} \tag{3.2}$$

is contained in a finite union of proper subspaces of $\mathbf{Z}^h$. Since the left hand side of (3.2) is equal to $[D_h b^{n_h} + \cdots + D_1 b^{n_1}]_S^{-1}$, this shows that the set of tuples $(b^{n_h}, \dots, b^{n_1})$, where $(n_h, \dots, n_1)$ lies in $\mathcal{N}_2$, is contained in a finite union of proper subspaces of $\mathbf{Z}^h$.

Thus, there exist integers $t_1, \dots, t_h$, not all zero, and an infinite set $\mathcal{N}_3$, contained in $\mathcal{N}_2$, of integer tuples $(n_h, \dots, n_1)$ such that $n_h > \dots > n_1 \geq 0$ and

$$t_h b^{n_h} + \cdots + t_1 b^{n_1} = 0.$$

We then deduce from (3.1) that $t_1 = \dots = t_h = 0$, a contradiction. Consequently, the set $\mathcal{N}_1$ must be finite. This establishes the theorem. □

Proof of Theorem 1.2. Below, the constants $c_1, c_2, \dots$ are effectively computable and depend at most on b and the constants $C_1, C_2, \dots$ are absolute and effectively computable.

Let $q_1, \dots, q_s$ be distinct prime numbers written in increasing order. Let $j \geq b^4$ be an integer and write

$$u_j^{(3)} = d_3 b^m + d_2 b^n + d_1, \quad \text{where } d_1, d_2, d_3 \in \{0, 1, \dots, b-1\}, d_1 d_3 \neq 0, m > n > 0.$$

There exist non-negative integers $r_1, \dots, r_s$ and a positive integer M coprime with $q_1 \dots q_s$ such that

$$u_j^{(3)} = q_1^{r_1} \cdots q_s^{r_s} M.$$

Assume first that $m \geq 2n$. Since

$$\Lambda_a := |q_1^{r_1} \cdots q_s^{r_s} b^{-m} (M d_3^{-1}) - 1| \leq b^{1+n-m} \leq b^{-(m-2)/2},$$

we get the upper bound

$$\log \Lambda_a \leq -\Big(\frac{m}{2} - 1\Big) \log b. \tag{3.3}$$

For the lower bound, by setting

$$Q := (\log q_1) \cdots (\log q_s) \quad \text{and} \quad A := \max\{M, d_3, 2\},$$

and by using that $r_j \log q_j \leq (m+1) \log b$ for $j = 1, \dots, s$, Theorem 2.2 implies that

$$\log \Lambda_a \geq -c_1 C_1^s \, Q \, (\log A) \, \max\Big\{\log \frac{m}{\log A}, 1\Big\}. \tag{3.4}$$

If $m \geq 3 \log A$, we combine (3.3) and (3.4) to get

$$\frac{m}{\log A} \le c_2\, C_1^s\, Q\, \log \frac{m}{\log A}.$$

Since $X \le Y \log X$ implies $X \le 2Y \log Y$ for all real numbers $X, Y \ge 3$, we deduce that the estimate

$$m \le c_3\, C_2^s\, Q\,(\log Q)\,(\log A) \tag{3.5}$$

always holds, whether or not m exceeeds $3 \log A$.

Assume now that $m \le 2n$. Let p be the smallest prime divisor of b. Set

$$\Lambda_u := q_1^{r_1} \cdots q_s^{r_s} \frac{M}{d_1} - 1 = \frac{b^n}{d_1}(d_2 + d_3 b^{m-n})$$

and

$$A = \max\{M, d_1, 2\}, \quad B = \max\{r_1, \dots, r_s, 3\}.$$

Observe that

$$v_p(\Lambda_u) \ge n - \frac{\log b}{\log p} \ge \frac{m}{2} - \frac{\log b}{\log p}. \tag{3.6}$$

It follows from Theorem 2.3 applied with

$$\delta = \frac{Q(\log A)}{B}$$

that

$$B < 2Q \log A, \quad \text{if } \delta > 1/2, \tag{3.7}$$

and, otherwise,

$$v_p(\Lambda_u) < c_4 C_3^s Q(\log A) \max\Big\{\log\Big(\frac{B}{\log A}\Big), 1\Big\}. \tag{3.8}$$

Since $2^B \le u_j^{(3)} < b^{m+1}$, we get $B \le c_5 m$ and deduce from (3.6) and (3.8) that

$$m \le c_6\, C_4^s\, Q\,(\log Q)\,(\log A). \tag{3.9}$$

If (3.7) holds, then, setting

$$Q^* := (\log q_1) + \cdots + (\log q_s),$$

it follows from $u_j^{(3)} \ge b^m$ that

$$m \log b \le BQ^* + \log M \le c_7 Q Q^* (\log A). \tag{3.10}$$

Observe that $A \le \max\{M, b\}$. It then follows from (3.5), (3.9), and (3.10) that if

$$m > c_8\, C_5^s\, Q\, Q^*\, (\log b),$$

then $A = M$ and, using that $\log u_j^{(3)} < (m+1) \log b$, we conclude that

$$M \ge (u_j^{(3)})^{(c_9 C_5^s Q Q^*)^{-1}},$$

thus

$$[u_j^{(3)}]_S = \frac{u_j^{(3)}}{M} \le (u_j^{(3)})^{1-(c_9 C_5^s \, Q \, Q^*)^{-1}}.$$

In the particular case where $M = 1$ and $q_1, \dots, q_s$ are the first s prime numbers $p_1, \dots, p_s$, written in increasing order, the above proof shows that

$$m \le c_{10} \, C_6^s \, (\prod_{k=1}^{s} \log p_k) \, (\log p_s). \tag{3.11}$$

Let ε be a positive real number. We deduce from (3.11), the Prime Number Theorem, and the inequality $\log u_j^{(3)} < (m+1) \log b$ that

$$\log\log u_j^{(3)} \le c_{11} + C_7 s + \sum_{k=1}^{s} \log\log p_k \le (1+\varepsilon) p_s \frac{\log\log p_s}{\log p_s},$$

if j is sufficiently large in terms of ε. This establishes (1.1) and completes the proof of Theorem 1.2. □

4. Additional remarks

In this section, we present additional results, discuss related problems, and make some suggestions for further research.

Arguing as Stewart did in [26], we can apply the arithmetic-geometric mean inequality in the course of the proof of Theorem 1.2 to derive a lower bound for $Q[u_j^{(3)}]$, where $Q[n]$ denotes the greatest square-free divisor of a positive integer n.

Theorem 4.1. *Let $b \ge 2$ be an integer. There exist effectively computable positive numbers c_1, c_2, depending only on b, such that*

$$Q[u_j^{(3)}] > \exp\Big(c_1 \, \log\log u_j^{(3)} \, \frac{\log\log\log u_j^{(3)}}{\log\log\log\log u_j^{(3)}}\Big), \quad \textit{for } j > c_2.$$

Outline of the proof. We keep the notation of the proof of Theorem 1.2 and assume that m is large. We consider the case where $M = 1$ and $r_1, \dots, r_s$ are positive, thus $Q[u_j^{(3)}] = q_1 \cdots q_s$ and $Q^* = \log Q[u_j^{(3)}]$. We conclude from (3.5), (3.9), and (3.10) that there exist an absolute, effectively computable real number C_1 and an effectively computable real number c_3, depending at most on b, such that

$$m < c_3 Q (C_1^s \log Q + Q^*).$$

The arithmetic-geometric mean inequality gives us that $Q \le (Q^*/s)^s$. Then, as in [26], we distinguish the cases $s < (\log m)/(\log\log\log m)$ and $s \ge (\log m)/(\log\log\log m)$ in order to bound Q^* from below in terms of m. We omit the details. The theorem then follows since $b^m \le u_j^{(3)} < b^{m+1}$. □

It is not difficult to make Theorem 1.2 completely explicit. Even in the special case where the cardinality of the set S is small, the bounds obtained are rather large, since estimates for linear forms in three or more logarithms are needed. Thus, it is presumably not straightforward to solve completely an equation like

$$2^a + 2^b + 1 = 3^x 11^y, \quad \text{in non-negative integers } a, b, x, y \text{ with } a > b > 0.$$

Sometimes, however, congruences are very helpful: as observed by Mike Bennett, by arguing modulo 8, we deduce that if $2^a + 2^b + 1 = 3^x 5^y$ holds for non-negative integers a, b, x, y with $a > b > 0$, then x and y must be even and we can then apply Szalay's result [28] on squares with few binary digits to solve completely that equation.

Presumably, other techniques, based on the hypergeometric method, could yield effective improvements of Theorem 1.2 in some special cases.

Problem 4.2. *Take* $S = \{3, 5\}$*. Prove that there exists an effectively computable integer* m_0 *such that, for any integers* m, n *with* $m > m_0$ *and* $m > n > 0$*, we have*

$$[2^m + 2^n + 1]_S \le 2^{3m/4}. \tag{4.1}$$

No importance should be attached to the value $3/4$ in (4.1). Similar questions have been successfully addressed in [5, 6].

Perfect powers with few nonzero digits in some given integer base have been studied in [11, 13, 1, 3, 4, 2]; see also the references given therein. We briefly discuss a related problem. Let a, b be integers such that $a > b > 1$. Perfect powers in the bi-infinite sequence $(a^m + b^n + 1)_{m,n \ge 1}$ have been considered by Corvaja and Zannier [12] and also in [3, 4]. All the general results obtained so far have been established under the assumption that a and b are not coprime. To remove this coprimeness assumption seems to be a very difficult problem.

We note that the methods of the proof of Theorem 1.2 allow us to establish the following result.

Theorem 4.3. *Let* a, b *be distinct integers with* $\gcd(a, b) \ge 2$*. Let* $\mathbf{v} = (v_j)_{j \ge 1}$ *denote the increasing sequence composed of all the integers of the form* $a^m + b^n + 1$*, with* $m, n \ge 1$*. Then, for every positive* ε*, we have*

$$P[v_j] > (1 - \varepsilon) \log\log v_j \, \frac{\log\log\log v_j}{\log\log\log\log v_j},$$

when j *exceeds some effectively computable constant depending only on* a*,* b*, and* ε*.*

We point out the following problem, which is probably rather difficult.

Problem 4.4. *Give an effective lower bound for the greatest prime factor of* $2^m + 3^n + 1$ *in terms of* $\max\{m, n\}$*.*

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