Nakaguchi, E. and Osaki, K.
Osaka J. Math.
55 (2018), 51–70

GLOBAL EXISTENCE OF SOLUTIONS TO AN n-DIMENSIONAL PARABOLIC-PARABOLIC SYSTEM FOR CHEMOTAXIS WITH LOGISTIC-TYPE GROWTH AND SUPERLINEAR PRODUCTION

Dedicated to Professor Masayasu Mimura on the occasion of his 75th birthday.

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(Received October 24, 2016)

Abstract

We study the global existence of solutions to an n-dimensional parabolic-parabolic system for chemotaxis with logistic-type growth. We introduce superlinear production of a chemoattractant. We then show the global existence of solutions in L_p space ($p > n$) under certain relations between the degradation and production orders.

1. Introduction

In the present paper we study a chemotaxis system with logistic growth:

$$\text{(E)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u) & \text{in } \Omega\times(0,\infty),\\ \tau\dfrac{\partial v}{\partial t} = \Delta v - v + g(u) & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x) & \text{in } \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and the space dimension $n \in \mathbb{N}$ is an arbitrary positive integer. The unknown functions $u(x,t)$ and $v(x,t)$ are the population density of bacteria and the concentration of a chemical substance at the position x and time t, respectively. The term $-\chi\nabla\cdot(u\nabla v)$ expresses the advection of bacteria due to chemotaxis. The coefficient χ is a positive constant, which shows chemotactic intensity. The function $f(u)$ is the proliferation and the reduction in numbers due to death of bacteria (we refer to the combined effects of proliferation and reduction in numbers simply as growth). Typical $f(u)$'s are quadratic $u(1-u)$ and cubic $u(1-u)(u-\gamma)$, $0 < \gamma < 1/2$, logistic growth functions [12]. The coefficient τ is a positive constant, which shows the time scale of reaction and diffusion of v. The function $g(u)$ is the secretion of chemical substance v by bacteria. A typical $g(u)$ is a linear function; and some nonlinear forms of $g(u)$ have been proposed, such as the saturating function $u/(1+\gamma u)$, as used in the nonlinear signal kinetics

2010 Mathematics Subject Classification. Primary 35K51; Secondary 35A01, 35B45, 35K57, 35Q92, 92C17.

model. For these topics, see the book by Murray [15], and the review articles by Hillen and Painter [4] and by Tindall, Maini, Porter and Armitage [23].

We consider the global existence of solutions to (E). In the context of global existence, the degradation of the growth $f(u)$ can be considered as an inhibitory effect on the increase of u. Indeed, if there is no growth ($f(u) \equiv 0$) and the production $g(u)$ is linear, then the system (E) reduces to the classical parabolic-parabolic Keller-Segel system [10]. In the Keller-Segel system, it is known that when $n = 2$, a finite-time blow-up with a δ-function singularity of u occurs if $\chi \|u_0\|_{L_1}$ is sufficiently large [3, 7]. In contrast, when $n \geq 3$, no restriction on χ and $\|u_0\|_{L_1}$ is necessary for the occurrence of blow-up [26]. For other topics on the Keller-Segel system, see Horstmann's review papers [5, 6] and the references therein. On the contrary, if $f(u)$ is quadratic and $g(u)$ is linear, then blow-up does not occur and global existence of solutions is assured even if $\|u_0\|_{L^1}$ and χ are large. This has been shown for $n = 2$ by one of the authors et al. [19] and for $n \geq 1$ with convex Ω and large μ by Winkler [27]. See also the recent related works [1, 11, 14].

We henceforth assume that the function $f(u)$ is a real, smooth function of $u \in [0, \infty)$ such that $f(0) \geq 0$ and

$$f(u) = u - \mu u^{\alpha} \quad \text{for sufficiently large } u \geq 0;$$

and the function $g(u)$ is given by

$$g(u) = u(1 + u)^{\beta - 1} \quad \text{for } u \geq 0,$$

where the exponents α and β satisfy the relations

$$\alpha > 1 \quad \text{and} \quad 0 < \beta \leq 2, \tag{1}$$

and μ is a positive constant. From the results quoted above, we find that in the n-dimensional domain ($n \geq 2$), a blow-up can occur when $\alpha = 1$ and $\beta = 1$ with a special choice of $\mu = 1$, and the blow-up of solutions is prevented and the global existence of solutions is assured when $\alpha = 2$ and $\beta = 1$. We can then conjecture that the critical degradation order α_{cr} is in the interval $1 \leq \alpha_{\mathrm{cr}} \leq 2$ under linear production $\beta = 1$; however, it has not been determined for the parabolic-parabolic chemotaxis-growth system (E). Recently, Xiang [29] showed global existence of solutions under $\beta = 1$ when $\alpha > 19/9$ if $n = 3$ and when $\alpha > n - 1$ if $n > 3$.

In the two- and three-dimensional cases, the authors [16, 17] introduced sublinear production order $\beta < 1$, and showed a sufficient condition $2(n + 4)/(n + 6) < \alpha \leq 2$ and $0 < \beta < (n + 6)(\alpha - 1)/[2(n + 2)]$ for the existence of global and bounded solutions to (E) in a Hilbert space $H_2^{(n/2)-1}(\Omega) \times H_2^{(n/2)+\varepsilon}(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$ (their results would include Xiang's results [29] when $n = 3$ if the existence of local solutions were assured for $\alpha > 2$). The authors have also shown in the previous paper [18] the global existence of solutions in L_p-space of arbitrary space dimension n with $p > n$, where (α, β) is merely allowed for $0 < \beta < (\alpha - 1)/2$.

In this paper, we revise the results obtained in [18] considerably by combining the semigroup method and the energy estimates and by applying the technique of trace operator [9, 13] (see Step 2 of Proof of Lemma 9). The main theorem of this paper is as follows:

Theorem 1. *Assume that the exponents α and β satisfy the relations* (1) *and*

$$\beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1). \tag{2}$$

Let p be an arbitrarily fixed exponent with

$$\max\{2, n, (\alpha-2)n\} < p < \infty. \tag{3}$$

Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega) \times H_p^1(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$, the system (E) *admits a unique global solution (u, v) in the function space*

$$\begin{cases} 0 \le u \in C([0,\infty); L_p(\Omega)) \cap C((0,\infty); H^2_{p,N}(\Omega)) \cap C^1((0,\infty); L_p(\Omega)), \\ 0 \le v \in C([0,\infty); H^1_p(\Omega)) \cap C((0,\infty); H^3_{p,N}(\Omega)) \cap C^1((0,\infty); H^1_p(\Omega)). \end{cases} \tag{4}$$

Moreover the solution satisfies the estimate

$$\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p} \le \psi\left(\|u_0\|_{L_p} + \|v_0\|_{H^1_p}\right), \quad t \ge 0 \tag{5}$$

with some increasing function $\psi(\cdot)$.

The definition and notation of function spaces will be given below and in Section 2. Theorem 1 above does not yet cover the case $(\alpha, \beta) = (2, 1)$ for $n \ge 2$ shown by Winkler [27], but the theorem requires no assumption on the largeness of μ nor the convexity of Ω considered in [27]. Our new results also contain the uniform boundedness of solutions with respect to the size of initial data.

We conclude this introduction by referring the results on the parabolic-elliptic chemotaxis systems. The parabolic-elliptic simplifications correspond to the situation where the chemical substance diffuses very quickly, which implies that the time scale τ tends to 0 in (E). For the n-dimensional parabolic-elliptic system with α-th order growth and linear secretion, that is, in the case of $\tau = 0$ and $\beta = 1$ in (E), the problem on the global existence and blow-up of solutions has largely been solved by Winkler [25, 28]: global existence and boundedness are assured when $\alpha > \max\{n/2, 2 - (1/n)\}$ [25]; also, there exists a blow-up solution when $1 < \alpha < 3/2 + 1/(2n-2)$ with $n \ge 5$ [28].

This paper is organized as follows. We provide preliminary results that we utilize in subsequent sections. In Section 3 we show the local existence of solutions by using a semigroup method (Theorem 5). In the final section we construct several a priori energy estimates by combining semigroup and energy methods. After obtaining the a priori estimates, we give the proof of the main theorem.

NOTATIONS. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. For $1 \le p \le \infty$, the space of complex-valued L_p functions in Ω is denoted by $L_p(\Omega)$ with the usual norm $\|\cdot\|_{L_p}$. The complex Sobolev space in Ω of order k, $k = 0, 1, 2, \ldots$, and exponent p, $1 \le p \le \infty$, is denoted by $H^k_p(\Omega)$ with norm $\|\cdot\|_{H^k_p}$. More generally, the Sobolev space of fractional order $s > 0$ and exponent $1 \le p \le \infty$ is denoted by $H^s_p(\Omega)$ with norm $\|\cdot\|_{H^s_p}$. The space of complex-valued continuous functions on $\overline{\Omega}$ is denoted by $C(\overline{\Omega})$ with norm $\|\cdot\|_C$. Let X be a Banach space and I an interval of $\mathbb{R}$. $C(I; X)$ and $C^1(I; X)$ denote the space of X-valued continuous functions and of X-valued continuously differentiable functions, respectively. $\mathcal{B}(I; X)$ denotes the space of X-valued bounded functions. For simplicity, we will use a universal notation C to denote various constants that are determined for each occurrence by Ω in a specific way. In a situation where C also depends on some parameter, say η, it will be denoted by C_η. In addition, by a universal notation $\psi(\cdot)$ we will denote continuous increasing functions, which may change depending on the context.

2. Preliminaries

In this section we shall list some well-known results in the theories of function spaces and linear operators [19, 22, 24, 30].

Interpolation of Sobolev spaces. For $0 \le s_0 < s < s_1 < \infty$ and $1 < p < \infty$, $H_p^s(\Omega)$ is the interpolation space $[H_p^{s_0}(\Omega), H_p^{s_1}(\Omega)]_\theta$ between $H_p^{s_0}(\Omega)$ and $H_p^{s_1}(\Omega)$, where $s = (1-\theta)s_0 + \theta s_1$, with the estimate

$$\|w\|_{H_p^s} \le C\|w\|_{H_p^{s_0}}^{1-\theta}\|w\|_{H_p^{s_1}}^{\theta} \quad \text{for } w \in H_p^{s_1}(\Omega). \tag{6}$$

See [30, Theorem 1.35].

Embedding theorem of Sobolev spaces. Let $1 < p < \infty$.

If $0 \le s < n/p$, then $H_p^s(\Omega) \subset L_r(\Omega)$ for any $p \le r \le pn/(n-ps) = [(1/p)-(s/n)]^{-1}$ with continuous embedding

$$\|w\|_{L_r} \le C_{s,p}\|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \tag{7}$$

If $s = n/p$, then $H_p^s(\Omega) \subset L_r(\Omega)$ for any finite $p \le r < \infty$ with continuous embedding

$$\|w\|_{L_r} \le C_{s,p}\|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \tag{8}$$

If $n/p < s < \infty$, then $H_p^s(\Omega) \subset \mathcal{C}(\overline{\Omega})$ with continuous embedding

$$\|w\|_{\mathcal{C}} \le C_{s,p}\|w\|_{H_p^s} \quad \text{for } w \in H_p^s(\Omega). \tag{9}$$

See [30, Theorem 1.36].

If $1 \le r \le p < \infty$, then $L_r(\Omega)$ is embedded in $(H_{p'}^s(\Omega))'$, the dual space of $H_{p'}^s(\Omega)$ with respect to L_2-inner product, for $(n/r)-(n/p) \le s < \infty$ and $p' = p/(p-1)$ with continuous embedding

$$\|w\|_{(H_{p'}^s)'} \le C_r\|w\|_{L_r} \quad \text{for } w \in L_r(\Omega). \tag{10}$$

Gagliardo-Nirenberg's inequality. Let $1 \le q \le p \le \infty$. Then the embedding $H_p^1(\Omega) \cap L_q(\Omega) \subset L_r(\Omega)$ holds for

$$\begin{cases} q \le r \le pn/(n-p) & \text{if } 1 \le p < n; \\ q \le r < \infty & \text{if } p = n; \\ q \le r \le \infty & \text{if } n < p \le \infty, \end{cases} \tag{11}$$

with the estimate

$$\|w\|_{L_r} \le C_{p,q,r}\|w\|_{H_p^1}^{a}\|w\|_{L_q}^{1-a} \quad \text{for } w \in H_p^1(\Omega), \tag{12}$$

where a is given by

$$\frac{1}{r} = a\left(\frac{1}{p} - \frac{1}{n}\right) + \frac{1-a}{q}. \tag{13}$$

See [30, Theorem 1.37].

Norms of a product of two functions. For $1 < p < \infty$ and $s > n/p$, from (9),

$$\|uv\|_{L_p} \le C_p\|u\|_{L_p}\|v\|_{L_\infty} \le C_{p,s}\|u\|_{L_p}\|v\|_{H_p^s} \quad \text{for } u \in L_p(\Omega),\ v \in H_p^s(\Omega). \tag{14}$$

As a corollary,

$$\begin{aligned}\|\nabla\cdot(u\nabla v)\|_{L_p} &\le \|\nabla u\cdot\nabla v\|_{L_p} + \|u\Delta v\|_{L_p} \le \|\nabla u\|_{L_p}\|\nabla v\|_{L_\infty} + \|u\|_{L_\infty}\|\Delta v\|_{L_p}\\ &\le C_{p,s}(\|u\|_{H_p^1}\|v\|_{H_p^{1+s}} + \|u\|_{H_p^s}\|v\|_{H_p^2})\\ &\qquad \text{for } u \in H_p^1(\Omega)\cap H_p^s(\Omega),\ v \in H_p^2(\Omega)\cap H_p^{1+s}(\Omega).\end{aligned} \tag{15}$$

When $n < p < \infty$, since $H_p^1(\Omega) \subset L_\infty(\Omega)$ by (9), it holds that

$$\|\nabla\cdot(u\nabla v)\|_{L_p} \le C_p\|u\|_{H_p^1}\|v\|_{H_p^2} \quad \text{for } u \in H_p^1(\Omega),\ v \in H_p^2(\Omega). \tag{16}$$

Domains of fractional powers of Laplace operators in L_p-spaces. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and $A_0 = -\Delta + 1$, Δ being the Laplace operator with Neumann boundary condition. Then, for each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H_{p,N}^2(\Omega)$ (see [2, Theorem 2.4.1.3], [24, Theorem 5.3.4] or [30, Theorem 2.15]). Let us denote $A_p = A_0|_{L_p}$; then $\mathcal{D}(A_p) = H_{p,N}^2(\Omega)$. Moreover, by the shift property (see [2, Theorem 2.5.1.1] or [24, Theorems 5.3.4 and 5.4.1]) it holds that $\mathcal{D}(A_p|_{H_p^1}) = H_{p,N}^3(\Omega)$ with norm equivalence.

The domains of fractional powers of A_p are characterized by

$$\mathcal{D}(A_p^\theta) = \begin{cases} H_p^{2\theta}(\Omega) & \text{for } 0 \le \theta < \dfrac{1}{2} + \dfrac{1}{2p}\\[2mm] H_{p,N}^{2\theta}(\Omega) & \text{for } \dfrac{1}{2} + \dfrac{1}{2p} < \theta \le \dfrac{3}{2}\end{cases} \tag{17}$$

with norm equivalence. Here, $H_{p,N}^s(\Omega)$ for $s > 1 + (1/p)$ denotes a closed subspace of $H_p^s(\Omega)$ such that

$$H_{p,N}^s(\Omega) = \left\{w \in H_p^s(\Omega);\ \frac{\partial w}{\partial n} = 0 \text{ on } \partial\Omega\right\} \quad \text{for } s > 1 + \frac{1}{p}.$$

Indeed, we can see that A_p has a bounded H_∞ functional calculus (see Yagi [30, Sec.16.1.2]) in $L_p(\Omega)$ and $H_p^1(\Omega)$, and by Yagi [30, Theorem 16.5], that the interpolation $\mathcal{D}(A_p^\theta) = [L_p(\Omega), H_{p,N}^2(\Omega)]_\theta$ and $\mathcal{D}((A_p|_{H_p^1})^\theta) = [H_p^1(\Omega), H_{p,N}^3(\Omega)]_\theta$ hold for $0 < \theta < 1$ with norm equivalence. Then, carefully following the proof of [30, Theorem 16.11], we can verify the rest part of (17). For the detail see Appendix.

Analytic semigroups generated by Laplace operators in L_p-spaces. For each $1 < p < \infty$, A_0 defined above generates in L_p-space an analytic semigroup e^{-tA_0} (it is independent of p in the sense that $e^{-tA_p}w = e^{-tA_2}w$ for $w \in L_p(\Omega)\cap L_2(\Omega)$). For $\gamma \ge 0$ it satisfies the estimate

$$\|A_0^\gamma e^{-tA_0}w\|_{L_p} \le Ct^{-\gamma}e^{-\delta_0 t}\|w\|_{L_p}, \qquad t > 0,\ w \in L_p(\Omega), \tag{18}$$

with some fixed constant $\delta_0 > 0$. See [8, Sec. 2] (see also [26, Lemma 1.3], [30, Theorems 2.19 and 2.27] and [22, Sec. 13.7]).

A differential geometric property of functions with Neumann boundary condition. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. If the function $w \in \mathcal{C}^2(\overline{\Omega})$ satisfies $\partial w/\partial\nu = 0$ on $\partial\Omega$, then it holds that

$$\frac{\partial|\nabla w|^2}{\partial\nu} \leq 2\kappa_\Omega |\nabla w|^2 \quad \text{on } \partial\Omega, \tag{19}$$

where κ_Ω is an upper bound for the curvatures of $\partial\Omega$; $\kappa_\Omega = 0$ when Ω is convex. See [13, Lemma 4.2]. See also [9].

Boundedness of trace operators. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Let $1 < p < \infty$ and $s > 1/p$. Then, the trace $T : f \mapsto f|_{\partial\Omega}$ is a bounded linear operator from $H_p^s(\Omega)$ to $L_p(\partial\Omega)$. Hence, we have

$$\|w\|_{L_p(\partial\Omega)} \leq C_{s,p}\|w\|_{H_p^s(\Omega)}, \qquad w \in H_p^s(\Omega). \tag{20}$$

See [30, Theorem 1.39] or [24, Theorem 4.7.1].

3. Local solutions

By similar argument to that in [17, 18, 19] or [30, Chap. 12], we can show the existence of local solutions to (E). We first review the existence theorem by Yagi [30, Chap. 4] (see also [20]) for local solutions to an abstract equation in a Banach space. Let X be a Banach space with norm $\|\cdot\|_X$. We consider the following Cauchy problem for a semilinear abstract evolution equation in X:

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & t > 0, \\ U(0) = U_0. \end{cases} \tag{21}$$

Here A is a sectorial operator of X satisfying that its spectral set is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbb{C};\ |\arg\lambda| \leq \phi\}$ with some $0 \leq \phi < \pi/2$, and $\|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \leq M/(|\lambda| + 1)$, $\lambda \notin \Sigma$ with constant M. The nonlinear operator F is a mapping from $\mathcal{D}(A^\eta)$ to X, where $0 < \eta < 1$, and it also satisfies a Lipschitz condition:

$$\begin{aligned} \|F(U) - F(\tilde{U})\|_X &\leq \varphi\left(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X\right) \\ &\times \left[\|A^\eta(U - \tilde{U})\|_X + \left(\|A^\eta U\|_X + \|A^\eta \tilde{U}\|_X\right)\|A^\gamma(U - \tilde{U})\|_X\right], \quad U, \tilde{U} \in \mathcal{D}(A^\eta), \end{aligned} \tag{22}$$

where γ is an exponent such that $0 < \gamma \leq \eta < 1$, and $\varphi(\cdot)$ is some increasing continuous function. The initial value U_0 is taken in $\mathcal{D}(A^\gamma)$. Then, from [30, Theorem 4.1] (or [20, Theorem 3.1]) we have the existence theorem of the local solutions to (21):

Theorem 2 ([30, Theorem 4.1]). *Under the above assumptions, for any $U_0 \in \mathcal{D}(A^\gamma)$, (21) possesses a unique local solution U in the function space:*

$$\begin{cases} U \in \mathcal{C}((0, T_{U_0}]; \mathcal{D}(A)) \cap \mathcal{C}([0, T_{U_0}]; \mathcal{D}(A^\gamma)) \cap \mathcal{C}^1((0, T_{U_0}]; X), \\ t^{1-\gamma} U \in \mathcal{B}((0, T_{U_0}]; \mathcal{D}(A)) \end{cases}$$

with the estimate

$$t^{1-\gamma}\|AU(t)\|_X + \|A^\gamma U(t)\|_X \le C_{U_0}, \quad 0 < t \le T_{U_0},$$

where T_{U_0} and C_{U_0} are positive constants depending only on the norm $\|A^\gamma U_0\|_X$.

By applying Theorem 2, we can show the existence of the local solutions to (E). The following proposition has been proved in [18].

Proposition 3 ([18, Proposition 3]). *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an exponent satisfying*

$$\max\{n, (\alpha - 2)n\} < p < \infty. \tag{23}$$

Then, for each pair of initial functions $(u_0, v_0) \in L_p(\Omega)\times H^1_p(\Omega) \subset L_n(\Omega)\times C(\overline{\Omega})$, the problem (E) *admits a unique local solution (u, v) in the function space*

$$\begin{cases} u \in C((0,T]; H^1_p(\Omega)) \cap C([0,T]; L_p(\Omega)) \cap C^1((0,T]; (H^1_{p'}(\Omega))'), \\ v \in C((0,T]; H^2_{p,N}(\Omega)) \cap C([0,T]; H^1_p(\Omega)) \cap C^1((0,T]; L_p(\Omega)) \end{cases} \tag{24}$$

with the estimate

$$t^{\frac{1}{2}}\left\{\|u(t)\|_{H^1_p} + \|v(t)\|_{H^2_p}\right\} + \left\{\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p}\right\} \le C, \quad 0 < t \le T,$$

where $p' = p/(p-1)$, and T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$.

By a solution (u, v) to (E) in the function space (24) we mean that the pair of functions (u, v) contained in (24) satisfies

$$\begin{cases} \dfrac{d}{dt}\langle u, w\rangle_{L_2} = -\langle \nabla u, \nabla w\rangle_{L_2} + \chi\langle u\nabla v, \nabla w\rangle_{L_2} + \langle f(u), w\rangle_{L_2} \\ \qquad\qquad\qquad\qquad \text{for any } w \in H^1_{p'}(\Omega) \text{ and } 0 < t < \infty, \\ \tau\dfrac{\partial v}{\partial t} = \Delta v - v + g(u) \quad \text{in } \Omega \times (0, \infty). \end{cases}$$

Next, we will show the local existence of solutions in the second function space:

Proposition 4. *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an exponent satisfying $n < p < \infty$. Then, for each pair of initial functions $(u_0, v_0) \in H^1_p(\Omega) \times H^2_{p,N}(\Omega)$, the problem* (E) *admits a unique local solution (u, v) in the function space*

$$\begin{cases} u \in C((0,T]; H^2_{p,N}(\Omega)) \cap C([0,T]; H^1_p(\Omega)) \cap C^1((0,T]; L_p(\Omega)), \\ v \in C((0,T]; H^3_{p,N}(\Omega)) \cap C([0,T]; H^2_{p,N}(\Omega)) \cap C^1((0,T]; H^1_p(\Omega)) \end{cases}$$

with the estimate

$$t^{\frac{1}{2}}\left\{\|u(t)\|_{H^2_p} + \|v(t)\|_{H^3_p}\right\} + \left\{\|u(t)\|_{H^1_p} + \|v(t)\|_{H^2_p}\right\} \le C, \quad 0 < t \le T,$$

where T and C are positive constants depending only on the norm $\|u_0\|_{H^1_p} + \|v_0\|_{H^2_p}$.

Proof. The system (E) can be expressed as a semilinear parabolic equation

$$(25)\qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), \quad t > 0, \\ U(0) = U_0 = \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} \end{cases}$$

in a product Banach space $X = L_p(\Omega) \times H^1_p(\Omega)$. Here, we define the linear operator A by

$$A = \begin{bmatrix} -\Delta + 1 & 0 \\ 0 & \tau^{-1}(-\Delta + 1) \end{bmatrix}, \quad \mathcal{D}(A) = H^2_{p,N}(\Omega) \times H^3_{p,N}(\Omega).$$

The nonlinear operator F is defined by

$$F(U) = \begin{bmatrix} -\chi\nabla \cdot (u\nabla v) + \bar{f}(u) + u \\ \bar{g}(u) \end{bmatrix}, \quad U = \begin{bmatrix} u \\ v \end{bmatrix} \in \mathcal{D}(A^\eta) = H^1_p(\Omega) \times H^2_{p,N}(\Omega)$$

with $\eta = 1/2$. Here, $\bar{f}(u)$ and $\bar{g}(u)$ denote some smooth extensions of $f(u)$ and $g(u)$ for the variable $u \in \mathbb{C}$ satisfying $f(u) \geq 0$ for $u < 0$ and $g(u) = 0$ for $u < -1$, respectively. The initial value U_0 is taken in the function space $\mathcal{D}(A^\gamma) = \mathcal{D}(A^\eta)$, that is $\gamma = \eta$. Under this setting, we need to verify only the Lipschitz condition (22). For $U = \begin{bmatrix} u \\ v \end{bmatrix}, \tilde{U} = \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} \in \mathcal{D}(A^\eta)$,

$$\|F(U) - F(\tilde{U})\|_X \leq \chi\|\nabla \cdot (u\nabla v - \tilde{u}\nabla\tilde{v})\|_{L_p} + \|u - \tilde{u}\|_{L_p} + \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} + \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H^1_p}.$$

For the first term, applying (16), we see

$$\begin{aligned} \|\nabla \cdot (u\nabla v) - \nabla \cdot (\tilde{u}\nabla\tilde{v})\|_{L_p} &\leq \|\nabla \cdot ((u - \tilde{u})\nabla v)\|_{L_p} + \|\nabla \cdot (\tilde{u}\nabla(v - \tilde{v}))\|_{L_p} \\ &\leq C_{p,s}(\|u - \tilde{u}\|_{H^1_p}\|v\|_{H^2_p} + \|\tilde{u}\|_{H^1_p}\|v - \tilde{v}\|_{H^2_p}) \\ &\leq C(\|A^\gamma U\|_X + \|A^\gamma \tilde{U}\|_X)\|A^\gamma(U - \tilde{U})\|_X. \end{aligned}$$

For the third and forth terms, using (1) and $H^s_p(\Omega) \subset L_\infty(\Omega)$ by (9), we can easily see that

$$\begin{aligned} \|\bar{f}(u) - \bar{f}(\tilde{u})\|_{L_p} &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})^{\alpha-1}\|u - \tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H^1_p} + \|\tilde{u}\|_{H^1_p})^{\alpha-1}\|u - \tilde{u}\|_{L_p}, \end{aligned}$$

$$\begin{aligned} \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{H^1_p} &\leq \|\bar{g}(u) - \bar{g}(\tilde{u})\|_{L_p} + \|\bar{g}'(u)\nabla(u - \tilde{u})\|_{L_p} + \|\{\bar{g}'(u) - \bar{g}'(\tilde{u})\}\nabla\tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{L_\infty} + \|\tilde{u}\|_{L_\infty})(\|u - \tilde{u}\|_{L_p} + \|\nabla(u - \tilde{u})\|_{L_p}) + C\|u - \tilde{u}\|_{L_\infty}\|\nabla\tilde{u}\|_{L_p} \\ &\leq C(1 + \|u\|_{H^1_p} + \|\tilde{u}\|_{H^1_p})\|u - \tilde{u}\|_{H^1_p}. \end{aligned}$$

Thus $F(U)$ satisfies the Lipschitz condition (22). We complete the proof. □

Now we can state our main theorem of this section:

Theorem 5. *Let $n \in \mathbb{N}$, assume the relation* (1) *for α and β, and let p be an arbitrarily fixed exponent satisfying* (23)*. Then, for each pair of nonnegative initial functions $(u_0, v_0) \in L_p(\Omega) \times H^1_p(\Omega) \subset L_n(\Omega) \times C(\overline{\Omega})$, the problem* (E) *admits a unique local solution (u, v) in the function space*

$$(26) \qquad \begin{cases} 0 \le u \in \mathcal{C}([0,T]; L_p(\Omega)) \cap \mathcal{C}((0,T]; H^2_{p,N}(\Omega)) \cap \mathcal{C}^1((0,T]; L_p(\Omega)), \\ 0 \le v \in \mathcal{C}([0,T]; H^1_p(\Omega)) \cap \mathcal{C}((0,T]; H^3_{p,N}(\Omega)) \cap \mathcal{C}^1((0,T]; H^1_p(\Omega)) \end{cases}$$

with the estimate

$$(27) \qquad \|u(t)\|_{L_p} + \|v(t)\|_{H^1_p} \le C, \quad 0 < t \le T,$$

where T and C are positive constants depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$.

Proof. It is clear that the local solutions belong to the function space (26) from Propositions 3 and 4. The nonnegativity of solutions has been proved in [18, Theorem 4] with the aid of the truncation method [30, Section 12.1.3]. Hence we conclude the proof. □

4. A priori estimates and global solutions

In this section we will construct several a priori estimates. The a priori estimates hold with each of the inequalities of α and β in the lemmas. Throughout this section, except for in the global existence theorem, we assume that $0 \le u_0 \in H^2_{p,N}(\Omega) \subset H^1_\infty(\Omega)$ and $0 \le v_0 \in H^3_{p,N}(\Omega) \subset H^2_\infty(\Omega)$ with $n < p < \infty$. In this case, applying [30, Theorem 4.2], we can verify that $0 \le u \in \mathcal{C}([0,T]; H^2_{p,N}(\Omega))$ and $0 \le v \in \mathcal{C}([0,T]; H^3_{p,N}(\Omega))$ with the estimate $\|u(t)\|_{H^2_p} + \|v(t)\|_{H^3_p} \le C_{U_0}$ for $0 \le t \le T$, where C_{U_0} is some positive constant. For a local solution (u, v) to (E) and exponents $z > 0$ and $\omega > 0$, we define

$$I^z_\omega(t) = \int_0^t \omega e^{-\omega(t-s)} \int_\Omega u^z dx ds.$$

The following lemma will be used frequenty in this section.

Lemma 6 (Gronwall's inequality). *Assume that a smooth real function $h(t)$ satisfies the differential inequality*

$$h'(t) + ah(t) \le K(t), \quad t_0 \le t \le T,$$

with a positive constant a and an integrable real function $K(t)$. Then, $h(t)$ is estimated by

$$h(t) \le h(t_0)e^{-a(t-t_0)} + \int_{t_0}^t e^{-a(t-s)} K(s) ds, \quad t_0 \le t \le T.$$

Lemma 7. *Let (u, v) be a local solution to* (E)*, and assume that*

$$\alpha > 1.$$

Then, it holds that

$$(28) \qquad \|u\|_{L_1} = \int_\Omega u\, dx \le e^{-t}\|u_0\|_{L_1} + a_1|\Omega|$$

with a constant $a_1 = \max\{f(u) + u;\ u \ge 0\}$*. In addition, for an arbitrary constant $\omega > 0$,*

$$(29) \qquad I^\alpha_\omega(t) \le \frac{2}{\mu}\{(a + a_1\omega)|\Omega| + \omega\|u_0\|_{L_1}\} \equiv \bar{I}^\alpha_\omega$$

holds with a constant $a = \max\{f(u) + \mu u^\alpha/2;\ u \ge 0\}$.

Proof. (Just the same as [17, Lemmas 4.1 and 4.2] or the first half part of [18, Lemma 5].) Integrating the first equation of (E) over Ω, we have

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega f(u)\,dx \le \int_\Omega (a_1 - u)dx.$$

Then, by Lemma 6, we obtain (28). From these inequalities, we see that

$$\frac{\mu}{2} I_\omega^\alpha(t) \le \int_0^t \omega e^{-\omega(t-s)} \int_\Omega \{a - f(u)\} dxds = \int_0^t \omega e^{-\omega(t-s)} \left\{ a|\Omega| - \frac{d}{ds}\|u\|_{L_1} \right\} ds$$
$$\le a|\Omega|(1 - e^{-\omega t}) + \omega e^{-\omega t}\|u_0\|_{L_1} + \int_0^t \omega^2 e^{-\omega(t-s)}\|u\|_{L_1} ds$$
$$\le (a + a_1\omega)|\Omega| + \omega\|u_0\|_{L_1},$$

which yields (29). □

Lemma 8. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1 \quad \textit{and} \quad 0 < \beta \le \frac{\alpha}{2}.$$

Then, for any exponent $2 \le q \le \alpha/\beta$,

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q \left(|\Omega| + \|u_0\|_{L_1}\right) \tag{30}$$

holds with some positive constants C_q *and* δ_q.

Proof. When $q = 2$ (see [17, Proposition 4.4]), multiplying the second equation of (E) by $-\Delta v + v$ and integrating it over Ω, we see that

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right) dx \le -\frac{1}{2}\int_\Omega (\Delta v)^2 dx - 2\int_\Omega |\nabla v|^2 dx - \frac{1}{2}\int_\Omega v^2 dx + \int_\Omega (1+u)^{2\beta} dx,$$

that is,

$$\tau\frac{d}{dt}\int_\Omega \left(|\nabla v|^2 + v^2\right) dx + \int_\Omega \left(|\nabla v|^2 + v^2\right) dx + \int_\Omega (\Delta v)^2 dx + 3\int_\Omega |\nabla v|^2 dx \le 2\int_\Omega (1+u)^{2\beta} dx. \tag{31}$$

Thus, by Lemma 6 again, we verify

$$\|v\|_{H_2^1}^2 \le e^{-t/\tau}\|v_0\|_{H_2^1}^2 + 2\int_0^t e^{-(t-s)/\tau}\int_\Omega (1+u)^{2\beta} dxds.$$

When $q > 2$ (just in the same way as the second half part of [18, Lemma 5]), we utilize the semigroup $e^{-tA_0/\tau}$ of $A_0 = -\Delta + 1$, Δ be the Laplace operator with Neumann boundary condition. Then the second equation of (E) gives

$$v(t) = e^{-tA_0/\tau}v_0 + \frac{1}{\tau}\int_0^t e^{-(t-s)A_0/\tau} g(u(s))ds. \tag{32}$$

Operating $A_0^{1/2}$ to this equality and applying (17) and (18), we have

$$
\begin{aligned}
\|v\|_{H^1_q} &\le C_q \|A_0^{1/2} v\|_{L_q} \\
&\le C_q \|A_0^{1/2} e^{-tA_0/\tau} v_0\|_{L_q} + \frac{1}{\tau}\int_0^t C_q \|A_0^{1/2} e^{-(t-s)A_0/\tau} g(u)\|_{L_q} ds \\
&\le C_q e^{-\delta_0 t/\tau} \|A_0^{1/2} v_0\|_{L_q} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|g(u)\|_{L_q} ds \\
&\le C_q e^{-\delta_0 t/\tau} \|v_0\|_{H^1_q} + \int_0^t C_q (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds.
\end{aligned}
$$

The last term can be estimated as

$$
\begin{aligned}
&\int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|(1+u)^\beta\|_{L_q} ds \\
&\qquad = \int_0^t (t-s)^{-1/2} e^{-\delta_0(t-s)/\tau} \|1+u\|^\beta_{L_{q\beta}} ds \\
&\qquad \le \left(\int_0^t (t-s)^{-q'/2} e^{-\delta_0(t-s)/\tau} ds \right)^{1/q'} \left(\int_0^t e^{-\delta_0(t-s)/\tau} \|1+u\|^{q\beta}_{L_{q\beta}} ds \right)^{1/q},
\end{aligned}
$$

where $q' = q/(q-1)$. Here we notice that $q'/2 < 1$ and the singular integral converges. Hence we have

$$
\|v\|_{H^1_q} \le C_q e^{-\delta_0 t/\tau} \|v_0\|_{H^1_q} + C_q \left(\int_0^t e^{-\delta_0(t-s)/\tau} \int_\Omega (1+u)^{q\beta} dx ds \right)^{1/q}.
$$

Combining both cases when $q = 2$ and when $q > 2$, we have

$$
\|v\|^q_{H^1_q} \le C_q e^{-\delta_q t/\tau} \|v_0\|^q_{H^1_q} + C_q \int_0^t e^{-\delta_q(t-s)/\tau} \int_\Omega (1+u)^{q\beta} dx ds \tag{33}
$$

for $q \ge 2$ with some positive constants C_q and δ_q. Applying (29), we prove (30) for $2 \le q \le \alpha/\beta$. □

Lemma 9. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$
\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).
$$

Then, for any $2 \le q \le \alpha/\beta$ *satisfying* $q > 2n\alpha/[(n+2)(\alpha-1)]$*, and for any exponent* $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$*,*

$$
\|1+u\|^\theta_{L_\theta} \le e^{-qt/(2\tau)} \|1+u_0\|^\theta_{L_\theta} + \psi_{\theta,q}\left(\|1+u_0\|_{L_1} + \|v_0\|_{H^1_q} \right) \tag{34}
$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$*. In addition, for an arbitrary constant* $\omega > 0$*,*

$$
\begin{aligned}
I^{\alpha+\theta-1}_\omega(t) \le \frac{4}{\mu} \Bigg\{ \left(1 + \frac{\omega}{\theta}\right) \psi_{\theta,q}\left(\|u_0\|_{L_1} + \|v_0\|_{H^1_q} \right) \\
+ \omega \left(\frac{1}{\theta} \|1+u_0\|^\theta_{L_\theta} + \zeta \frac{2\tau}{q} \|v_0\|^q_{H^1_q} \right) \Bigg\} \equiv \bar{I}^{\alpha+\theta-1}_\omega
\end{aligned}
\tag{35}
$$

holds with some constant $\zeta > 0$*.*

Proof. We describe the proof in several steps.

Step 1. Multiplying the first equation of (E) by $(1+u)^{\theta-1}$ and integrating it over Ω, we see that

$$\frac{1}{\theta}\frac{d}{dt}\int_\Omega (1+u)^\theta dx = -(\theta-1)\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + \chi(\theta-1)\int_\Omega u(1+u)^{\theta-2}\nabla u\cdot\nabla v dx + \int_\Omega (1+u)^{\theta-1} f(u)dx$$
$$\le -\frac{\theta-1}{2}\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + \frac{\chi^2(\theta-1)}{2}\int_\Omega (1+u)^\theta |\nabla v|^2 dx + \int_\Omega (1+u)^{\theta-1} f(u)dx.$$

For the second term on the right-hand side, using (12), we note that

$$\frac{\chi^2(\theta-1)}{2}\int_\Omega (1+u)^\theta|\nabla v|^2 dx \le \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\left\||\nabla v|^2\right\|_{L_\kappa}$$
$$= \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\left\||\nabla v|^{q/2}\right\|^{4/q}_{L_{4\kappa/q}}$$
$$\le \frac{\chi^2(\theta-1)}{2}\left\|(1+u)^\theta\right\|_{L_{\kappa/(\kappa-1)}}\cdot C_q\left\||\nabla v|^{q/2}\right\|^{2/\kappa}_{H_2^1}\left\||\nabla v|^{q/2}\right\|^{(4/q)-(2/\kappa)}_{L_2}$$
$$\le C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|^2_{H_2^1} + \eta\left\|\nabla v\right\|^{(2\kappa-q)/(\kappa-1)}_{L_q}\int_\Omega (1+u)^{\theta\kappa/(\kappa-1)}dx$$

with $\kappa = q(n+2)/(2n)$ and an arbitrary $\eta > 0$. Hence we have

$$\frac{1}{\theta}\frac{d}{dt}\int_\Omega (1+u)^\theta dx \le -\frac{\theta-1}{2}\int_\Omega (1+u)^{\theta-2}|\nabla u|^2 dx + C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|^2_{H_2^1} + \int_\Omega\left[\eta\left\|\nabla v\right\|^{(2\kappa-q)/(\kappa-1)}_{L_q}(1+u)^{\theta\kappa/(\kappa-1)} + (1+u)^{\theta-1}f(u)\right]dx. \tag{36}$$

Step 2. We present the differential inequality on $\|v\|^q_{H_q^1}$ for $q \ge 2$. For the present assume $q > 2$. Firstly, multiplying the second equation of (E) by v^{q-1} and integrating it over Ω, we see that

$$\frac{\tau}{q}\frac{d}{dt}\int_\Omega v^q dx = -(q-1)\int_\Omega v^{q-2}|\nabla v|^2 dx - \int_\Omega v^q dx + \int_\Omega v^{q-1}g(u) \le -(q-1)\int_\Omega v^{q-2}|\nabla v|^2 dx - \frac{1}{2}\int_\Omega v^q dx + \frac{C_q'}{2}\int_\Omega (1+u)^{q\beta}dx. \tag{37}$$

Next, differentiating the second equation of (E), we have

$$\tau\frac{\partial}{\partial t}|\nabla v|^2 = 2\tau\nabla v\cdot\nabla v_t = 2\nabla v\cdot\nabla\Delta v - 2|\nabla v|^2 + 2\nabla v\cdot\nabla g(u).$$

Noting that $\Delta|\nabla v|^2 = 2|D^2v|^2 + 2\nabla v\cdot\nabla\Delta v$ and $(\Delta v)^2 \le n|D^2 v|^2$, where $|D^2v|^2 = \sum_{i,j}\left|D_iD_jv\right|^2$, we see

$$\tau \frac{\partial}{\partial t} |\nabla v|^2 \le \Delta |\nabla v|^2 - \frac{2}{n} (\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v \cdot \nabla g(u).$$

Multiplying this inequality by $|\nabla v|^{q-2}$, integrating it over Ω and applying (19), we obtain

$$\begin{aligned}
\frac{2\tau}{q}\frac{d}{dt}\int_\Omega |\nabla v|^q dx &\le \int_\Omega |\nabla v|^{q-2}\left\{\Delta|\nabla v|^2 - \frac{2}{n}(\Delta v)^2 - 2|\nabla v|^2 + 2\nabla v\cdot\nabla g(u)\right\} dx\\
&= \int_{\partial\Omega} |\nabla v|^{q-2}\,\frac{\partial |\nabla v|^2}{\partial \nu} dx - \int_\Omega \nabla |\nabla v|^{q-2}\cdot \nabla|\nabla v|^2 dx\\
&\quad - \int_\Omega |\nabla v|^{q-2}\left\{\frac{2}{n}(\Delta v)^2 + 2|\nabla v|^2\right\} dx + \int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u) dx\\
&\le 2\kappa_\Omega \int_{\partial\Omega} |\nabla v|^q dx - \int_\Omega \frac{q-2}{2}|\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2 dx - \int_\Omega \frac{2}{n}|\nabla v|^{q-2}(\Delta v)^2 dx\\
&\qquad - \int_\Omega 2|\nabla v|^q dx + \int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u) dx.
\end{aligned}$$

For the first term on the right-hand side, applying (20) and (6) with any $1/2 < s < 1$ and $\varepsilon > 0$, we see that

$$\begin{aligned}
2\kappa_\Omega \int_{\partial\Omega} |\nabla v|^q dx &= 2\kappa_\Omega \left\| |\nabla v|^{q/2}\right\|^2_{L_2(\partial\Omega)} \le C\left\||\nabla v|^{q/2}\right\|^2_{H_2^s(\Omega)}\\
&\le C\left\||\nabla v|^{q/2}\right\|^{2s}_{H_2^1}\left\||\nabla v|^{q/2}\right\|^{2(1-s)}_{L_2} \le \varepsilon\left\|\nabla\left(|\nabla v|^{q/2}\right)\right\|^2_{L_2} + C_\varepsilon\left\||\nabla v|^{q/2}\right\|^2_{L_2}.
\end{aligned}$$

For the last term on the right-hand side, we see

$$\begin{aligned}
\int_\Omega 2|\nabla v|^{q-2}\nabla v\cdot\nabla g(u) dx &= -\int_\Omega \left\{(q-2)|\nabla v|^{q-4}\nabla|\nabla v|^2\cdot\nabla v + 2|\nabla v|^{q-2}\Delta v\right\} g(u) dx\\
&\le \int_\Omega (q-2)|\nabla v|^{q-3}\left|\nabla|\nabla v|^2\right|(1+u)^\beta dx + \int_\Omega 2|\nabla v|^{q-2}|\Delta v|\,(1+u)^\beta dx\\
&\le \frac{q-2}{4}\int_\Omega |\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2 dx + \frac{1}{n}\int_\Omega |\nabla v|^{q-2}(\Delta v)^2 dx + \int_\Omega |\nabla v|^q dx\\
&\qquad + C''_q\,(n+q-2)^{q/2}\int_\Omega (1+u)^{q\beta} dx.
\end{aligned}$$

Hence, noting that $\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 = (q^2/16)|\nabla v|^{q-4}\left|\nabla|\nabla v|^2\right|^2$, we have

$$\begin{aligned}
(38)\quad \frac{2\tau}{q}\frac{d}{dt}\int_\Omega |\nabla v|^q dx &+ \frac{4(q-2)}{q^2}\int_\Omega \left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 dx\\
&+ \frac{1}{n}\int_\Omega |\nabla v|^{q-2}(\Delta v)^2 dx + \int_\Omega |\nabla v|^q dx\\
\le \varepsilon\int_\Omega \left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2 dx &+ C_\varepsilon\int_\Omega |\nabla v|^q dx + C''_q\,(n+q-2)^{q/2}\int_\Omega (1+u)^{q\beta} dx.
\end{aligned}$$

Adding (38) to (37) and taking $\varepsilon = 2(q-2)/q^2$, we see

$$(39)\quad \frac{2\tau}{q}\frac{d}{dt}\int_\Omega(|\nabla v|^q+v^q)dx+\int_\Omega(|\nabla v|^q+v^q)dx \\ +\frac{2(q-2)}{q^2}\int_\Omega\left|\nabla\left(|\nabla v|^{q/2}\right)\right|^2dx+\frac{1}{n}\int_\Omega|\nabla v|^{q-2}(\Delta v)^2dx+\frac{8(q-1)}{q^2}\int_\Omega\left|\nabla\left(v^{q/2}\right)\right|^2dx \\ \le C\int_\Omega|\nabla v|^qdx+C_q\int_\Omega(1+u)^{q\beta}dx.$$

This inequality holds also for $q=2$ (see (31)). The right-hand side is bounded in terms of $\|u_0\|_{L_1}$ and $\|v_0\|_{H^1_q}$ in view of Lemmas 7 and 8 since $q\beta\le\alpha$.

Step 3. Adding (39) multiplied by some weight $\zeta>0$ to (36) , we see

$$(40)\quad \frac{1}{\theta}\frac{d}{dt}\int_\Omega(1+u)^\theta dx+\zeta\left(\frac{2\tau}{q}\frac{d}{dt}\|v\|^q_{H^1_q}+\|v\|^q_{H^1_q}+\frac{2(q-2)}{q^2}\left\||\nabla v|^{q/2}\right\|^2_{H^1_2}\right) \\ \le C_q\eta^{-\kappa+1}\chi^{2\kappa}(\theta-1)^\kappa\left\||\nabla v|^{q/2}\right\|^2_{H^1_2}+C\|\nabla v\|^q_{L_q} \\ +\int_\Omega\left[\eta\|\nabla v\|^{(2\kappa-q)/(\kappa-1)}_{L_q}(1+u)^{\theta\kappa/(\kappa-1)}+(1+u)^{\theta-1}f(u)+\zeta C_q(1+u)^{q\beta}\right]dx.$$

Since $q\beta\le\alpha<\alpha+\theta-1$ and $\theta\kappa/(\kappa-1)\le\alpha+\theta-1$ from the assumptions, suitable choice of η and ζ yields

$$(41)\quad \frac{d}{dt}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)+\frac{q}{2\tau}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right) \\ \le\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)-\frac{\mu}{4}\int_\Omega u^{\alpha+\theta-1}dx,$$

and hence, by Lemma 6,

$$\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\le e^{-qt/(2\tau)}\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right) \\ +\int_0^t e^{-q(t-s)/(2\tau)}\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)ds.$$

Application of (30) to the right-hand side of this inequality leads to (34).

Step 4. The proof of (35) is very similar to that of (29), as follows: using (41),

$$\frac{\mu}{4}I^{\alpha+\theta-1}_\omega(t)\le\int_0^t\omega e^{-\omega(t-s)}\left\{\psi_{\theta,q}\left(\|v\|_{H^1_q}\right)-\frac{d}{dt}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)\right\}ds \\ \le\psi_{\theta,q}\left(\sup_{t\ge0}\|v\|_{H^1_q}\right)+\omega e^{-\omega t}\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right) \\ +\int_0^t\omega^2e^{-\omega(t-s)}\left(\frac{1}{\theta}\|1+u\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v\|^q_{H^1_q}\right)ds \\ \le\left(1+\frac{\omega}{\theta}\right)\psi_{\theta,q}\left(\sup_{t\ge0}\|v\|_{H^1_q}\right)+\omega\left(\frac{1}{\theta}\|1+u_0\|^\theta_{L_\theta}+\zeta\frac{2\tau}{q}\|v_0\|^q_{H^1_q}\right).$$

Thus we complete the proof of the lemma. □

Lemma 10. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Suppose that for some exponent $\sigma > 1$ *and* $r > 2n\alpha/[(n+2)(\alpha-1)]$ *the integral* $I_\omega^{\alpha+\sigma-1}(t)$ *is bounded by*

$$I_\omega^{\alpha+\sigma-1}(t) \le (1+\omega)\,\psi_{\sigma,r}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \equiv \bar{I}_\omega^{\alpha+\sigma-1} \tag{42}$$

for an arbitrary constant $\omega > 0$*. Then, for any exponent* $q \ge 2$ *satisfying* $r \le q \le (\alpha+\sigma-1)/\beta$,

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q\,\psi_{\sigma,r}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \tag{43}$$

holds with some positive constants C_q *and* δ_q*. Moreover, for any exponent* $\sigma \le \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$,

$$\|1+u\|_{L_\theta}^\theta \le e^{-qt/(2\tau)}\|1+u_0\|_{L_\theta}^\theta + \psi_{\theta,q}\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_q^1}\right) \tag{44}$$

holds with some increasing function $\psi_{\theta,q}(\cdot)$*. In addition, for an arbitrary constant* $\omega > 0$*, it holds that*

$$I_\omega^{\alpha+\theta-1}(t) \le (1+\omega)\,\psi_{\theta,q}\left(\|1+u_0\|_{L_\theta} + \|v_0\|_{H_q^1}\right) \equiv \bar{I}_\omega^{\alpha+\theta-1}. \tag{45}$$

Proof. We can prove the lemma in the similar argument as in Lemmas 8 and 9.

Firstly, the inequality (33) holds also in this case. Since $q\beta \le \alpha + \sigma - 1$, by Lemma 6 again, we verify

$$\|v\|_{H_q^1}^q \le C_q e^{-\delta_q t}\|v_0\|_{H_q^1}^q + C_q\, b_{q,\alpha+\sigma-1}\left(|\Omega| + I_{q/2\tau}^{\alpha+\sigma-1}(t)\right).$$

By (42), we obtain (43).

The estimate (44) is verified from the inequality (41) together with (43), since $q\beta \le \alpha + \sigma - 1 \le \alpha + \theta - 1$ and $\theta\kappa/(\kappa - 1) \le \alpha + \theta - 1$ with $\kappa = q(n+2)/(2n)$. The proof of (45) is just the same as that of (35).

Thus we complete the proof of the lemma. □

For obtaining the final a priori estimate, we apply Lemma 10 iteratively. We then show the following a priori estimate.

Proposition 11. *Let* (u, v) *be a local solution to* (E)*, and assume that*

$$\alpha > 1, \quad 0 < \beta \le \frac{\alpha}{2} \quad \text{and} \quad \beta < \frac{n+2}{2n}(\alpha - 1).$$

Then, for any exponent $p > 2$*, it holds that*

$$\|1+u\|_{L_p}^p + \|v\|_{H_p^1}^p \le Ce^{-pt/(2\tau)}\left(\|1+u_0\|_{L_p}^p + \|v_0\|_{H_p^1}^p\right) + \psi_p\left(\|1+u_0\|_{L_\sigma} + \|v_0\|_{H_r^1}\right) \tag{46}$$

with some exponents $1 < \sigma < p$, $\alpha/\beta < r < p$ *and some increasing function* $\psi_p(\cdot)$.

Proof. The proof is given by induction. Firstly we have estimates (28) on $\|u\|_{L_1}$. Let

$$\theta_0 = 1.$$

Secondly we have (30) on $\|v\|_{H^1_q}$ for $2 \le q \le \alpha/\beta$ by Lemma 8 and (34) on $\|1+u\|_{L_\theta}$ for $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$ by Lemma 9. Let

$$q_1 = \frac{\theta_0 + \alpha - 1}{\beta} = \frac{\alpha}{\beta}, \quad \theta_1 = \left(\frac{n+2}{2n} q_1 - 1\right)(\alpha - 1).$$

For each integer k and given θ_k, we can obtain by Lemma 10 the estimates (43) on $\|v\|_{H^1_q}$ for $2 < q \le (\theta_k + \alpha - 1)/\beta$ and (44) on $\|1+u\|_{L_\theta}$ for $1 < \theta \le \{q(n+2)/(2n) - 1\}(\alpha - 1)$. Define

$$q_{k+1} = \frac{\theta_k + \alpha - 1}{\beta} = \frac{(n+2)(\alpha - 1)}{2n\beta} q_k, \quad \theta_{k+1} = \left(\frac{n+2}{2n} q_{k+1} - 1\right)(\alpha - 1).$$

Since $(n+2)(\alpha - 1)/(2n\beta) > 1$ by assumption, we can easily see that

$$q_k \to \infty \quad \text{and} \quad \theta_k \to \infty \quad \text{as} \quad k \to \infty.$$

Hence, for any $p > 1$, there exists a finite integer k_0 such that $q_{k_0} > p$ and $\theta_{k_0} > p$, and the desired estimates are obtained. □

By using the a priori estimates shown above, we prove the main theorem for the global existence of the solutions.

Proof of Theorem 1. From Theorem 5 for each pair of nonnegative initial functions (u_0, v_0) there exists a unique nonnegative local solution (u, v) on the interval $[0, T]$ with the estimate (27), and the existence time T depends only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$. In addition, from Proposition 11, the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p}$, $0 \le t \le T$, is estimated from above by a uniform constant C_{U_0} also depending only on the norm $\|u_0\|_{L_p} + \|v_0\|_{H^1_p}$. Hence, the interval can be extended to $[0, T + \tilde{T}]$, where the extended time $\tilde{T}$ and the norm $\|u(t)\|_{L_p} + \|v(t)\|_{H^1_p}$, $0 \le t \le T + \tilde{T}$, are estimated by the same constant C_{U_0}. The existence interval can be again extended, to $[0, T + 2\tilde{T}]$. Repeating this procedure proves the global existence theorem with the estimate (5). □

Appendix . On the domains of fractional powers of Laplace operators in L_p-spaces

Here we discuss the characterization of the domains of definition of fractional powers of Laplace operator $A_0 = -\Delta + 1$ with Neumann boundary condition on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, as a closed operator in $L_p(\Omega)$ for each $1 < p < \infty$.

We have already known the following facts.

Theorem A.1 ([2, Theorem 2.4.1.3], [24, Theorem 5.3.4], [30, Theorem 2.15])**.** *For each $1 < p < \infty$, A_0 is considered as a closed operator in $L_p(\Omega)$, the domain of which is $H^2_{p,N}(\Omega)$. If we denote $A_p = A_0|_{L_p}$, then it holds that $\mathcal{D}(A_p) = H^2_{p,N}(\Omega)$ with norm equivalence.*

Theorem A.2 ([2, Theorem 2.5.1.1], [24, Theorems 5.3.4 and 5.4.1])**.** *Let k be a positive integer and $1 < p < \infty$. Then $u \in H^{k+2}_p(\Omega) \cap H^2_{p,N}(\Omega)$ yields $A_0 u \in H^k_p(\Omega)$. Moreover, if $u \in H^2_{p,N}(\Omega)$ satisfies $A_0 u \in H^k_p(\Omega)$, then $u \in H^{k+2}_p(\Omega)$. That means, as the first example, if we denote $\mathfrak{A}_p = A_0|_{H^1_p}$, that the identity $\mathcal{D}(\mathfrak{A}_p) = H^3_{p,N}(\Omega)$ holds with norm equivalence.*

To interpolate these results between $k = 0$ and $k = 2$, we apply the theory of bounded H_∞ functional calculus in $L_p(\Omega)$ given by Yagi [30, Sec.16.1.2].

Theorem A.3. *For the operator $A_p = A_0|_{L_p}$, the identity*

$$\mathcal{D}(A_p^\theta) = [L_p(\Omega), H^2_{p,N}(\Omega)]_\theta = \begin{cases} H^{2\theta}_p(\Omega) & \textit{for } 0 \le \theta < \frac{1}{2} + \frac{1}{2p} \\ H^{2\theta}_{p,N}(\Omega) & \textit{for } \frac{1}{2} + \frac{1}{2p} < \theta \le 1 \end{cases} \tag{A.1}$$

holds with norm equivalence.

Proof. Firstly, it is obvious that $L_p(\Omega)$ is a reflexive Banach space and A_p is a sectorial operator in $L_p(\Omega)$ with angle $\omega_A = 0$. Hence, we can directly verify the following condition given in [30, Theorem 16.5] with $A = A_p$, $X = L_p(\Omega)$, $X^* = L_{p'}(\Omega)$ and $\langle\cdot,\cdot\rangle$ their duality product:

(H) For every angle $\omega_A < \omega < \pi$ and every exponent $0 < \theta < 1$, the integrable condition along the V-shaped contour $\Gamma_\omega : \lambda = \rho e^{\pm i\omega}$ $(0 \le \rho < \infty)$

$$I_{\omega,\theta} = \int_{\Gamma_\omega} |\lambda|^{2\theta-1} \left|\langle A^{2(1-\theta)}(\lambda - A)^{-2}F, G\rangle\right| |d\lambda| \le C_{\omega,\theta}\|F\|\|G\|_*, \quad F \in X, G \in X^*, \tag{A.2}$$

holds with some constant $C_{\omega,\theta} > 0$.

We omit the detail here. Then, by [30, Theorem 16.5] it is verified that A_p has a bounded H_∞ functional calculus in $L_p(\Omega)$. Again by [30, Theorem 16.5], we have the first identity of (A.1). The rest part of the theorem has been already shown in [30, Theorem 16.11]. □

The next theorem shows the interpolation result between $k = 1$ and $k = 3$.

Theorem A.4. *For the operator $\mathfrak{A}_p = A_0|_{H^1_p}$, the identity*

$$\mathcal{D}(\mathfrak{A}_p^\theta) = [H^1_p(\Omega), H^3_{p,N}(\Omega)]_\theta = \begin{cases} H^{1+2\theta}_p(\Omega) & \textit{for } 0 \le \theta < \frac{1}{2p} \\ H^{1+2\theta}_{p,N}(\Omega) & \textit{for } \frac{1}{2p} < \theta \le 1 \end{cases} \tag{A.3}$$

holds with norm equivalence.

Proof. It is also obvious that $H^1_p(\Omega)$ is a reflexive Banach space with duality product $\langle\langle\cdot,\cdot\rangle\rangle$ of $H^1_p(\Omega) \times H^1_{p'}(\Omega)$, and that $\mathfrak{A}_p$ is a sectorial operator in $H^1_p(\Omega)$ with angle $\omega_A = 0$. Hence, again we can prove the condition (H) above by direct calculation to see that the Laplace operator $\mathfrak{A}_p = A_0|_{H^1_p}$ has a bounded H_∞ functional calculus in $H^1_p(\Omega)$. Then, we have the first identity of (A.3) again by [30, Theorem 16.5]. For the proof of the rest part of the theorem, we must follow carefully the proof of [30, Theorem 16.11].

Step 1: $\mathcal{D}(\mathfrak{A}_p^\theta) \subset H^{1+2\theta}_{p,(N)}(\Omega)$. We can easily see that $[H^1_p(\Omega), H^3_{p,N}(\Omega)]_\theta \subset [H^1_p(\Omega), H^3_p(\Omega)]_\theta = H^{1+2\theta}_p(\Omega)$. To see the boundary condition for $1/(2p) < \theta < 1$, take a sequence $\{u_k\} \subset \mathcal{D}(\mathfrak{A}_p) = H^3_{p,N}(\Omega)$ converging to u in $\mathcal{D}(\mathfrak{A}_p^\theta)$. This implies that every $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$ satisfies the Neumann boundary condition on $\partial\Omega$.

Step 2: $\mathcal{D}(\mathfrak{A}_p^\theta) \supset H^{1+2\theta}_{p,(N)}(\Omega)$. We divide this step into three parts.

(i) $\frac{1}{2} \le \theta \le 1$. Let $u \in H^{1+2\theta}_{p,N}(\Omega)$. Then, for any $v \in H^3_{p',N}(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, similarly in the proof of [30, Theorem 16.11], we see

$$\begin{aligned}\left|\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle\right| &= \left|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta \langle\langle u, (\lambda-\mathfrak{A}_p^*)^{-1}v\rangle\rangle\, d\lambda\right| \\ &\le \|A_0 u\|_{H_p^{2\theta-1}} \left\|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta(\lambda-A_0)^{-1}v\, d\lambda\right\|_{(H_p^{2\theta-1})'} \\ &\le C\|A_0 u\|_{H_p^{2\theta-1}} \left\|A_0^{\theta-\frac{1}{2}}v\right\|_{H_{p'}^{2-2\theta}} \le C\|u\|_{H_p^{1+2\theta}}\|v\|_{H_{p'}^1}.\end{aligned}$$

Here we utilize [30, Theorem 1.43] for the boundedness of $A_0 = 1 - \sum_k D_k^2 : H_p^{2\theta+1}(\Omega) \to H_p^{2\theta-1}(\Omega)$ and Theorem 3 for the boundedness of the fractional powers of $A_0|_{L_p} = A_p$. This inequality yields that, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, the linear form $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle$ is a bounded linear functional of $v \in H_{p'}^1(\Omega)$, that is, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle = \langle\langle w, v\rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(ii) $\frac{1}{2p} < \theta < \frac{1}{2}$. Let $u \in H_{p,N}^{1+2\theta}(\Omega)$. Then, for any $v \in H_{p',N}^3(\Omega) = \mathcal{D}(\mathfrak{A}_p^*)$, by an argument quite similar to (i), we see

$$\left|\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle\right| \le C\|u\|_{H_p^{2\theta+1}} \left\|\frac{1}{2\pi i}\int_\Gamma \lambda^\theta(\lambda-A_0)^{-1}v\, d\lambda\right\|_{H_{p'}^{1-2\theta}} \le C\|u\|_{H_p^{2\theta+1}}\|v\|_{H_{p'}^1}.$$

Here we utilize again [30, Theorem 1.43] for the boundedness of $D_k : H_{p'}^{1-2\theta}(\Omega) \to (H_p^{2\theta}(\Omega))'$. Thus, for each fixed $u \in H_{p,N}^{1+2\theta}(\Omega)$, there exists $w \in H_p^1(\Omega)$ such that $\langle\langle u, (\mathfrak{A}_p^*)^\theta v\rangle\rangle = \langle\langle w, v\rangle\rangle$ for any $v \in H_{p'}^1(\Omega)$. Hence, $w = \mathfrak{A}_p^\theta u$ and $u \in \mathcal{D}(\mathfrak{A}_p^\theta)$.

(iii) $0 < \theta < \frac{1}{2p}$. We can verify that $u \in H_p^{1+2\theta}(\Omega)$ is contained in $\mathcal{D}(\mathfrak{A}_p^\theta)$ by the same argument as (ii) except for the boundary conditions.

Hence we complete the proof. □

As the consequence, we conclude (17) in Section 2.

ACKNOWLEDGEMENTS. The authors are very grateful to Professor Atsushi Yagi for his valuable comments. This work was partially supported by KAKENHI Nos. 26400180 and 23540125.

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