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A TWISTED FIRST HOMOLOGY GROUP OF THE HANDLEBODY MAPPING CLASS GROUP

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Abstract
Let H_g be a 3-dimensional handlebody of genus g. We determine the twisted first homology group of the mapping class group of H_g with coefficients in the first integral homology group of the boundary surface ∂H_g for $g \geq 2$.

1. Introduction

Let H_g be a 3-dimensional handlebody of genus g, and Σ_g the boundary surface ∂H_g. We denote by $\mathcal{H}_g$ and $\mathcal{M}_g$ the mapping class group of H_g and the boundary surface Σ_g, respectively. These are the groups of isotopy classes of orientation preserving homeomorphisms of Σ_g and H_g. Let D be a closed 2-disk in the boundary Σ_g of the handlebody, and pick a point $*$ in $\operatorname{Int} D$. Let us denote by $\mathcal{H}_g^*$ and $\mathcal{H}_{g,1}$ the groups of the isotopy classes of orientation preserving homeomorphisms of H_g fixing $*$ and D pointwise, respectively. We also denote by $\mathcal{M}_g^*$ and $\mathcal{M}_{g,1}$ the groups of the isotopy classes of orientation preserving homeomorphisms of Σ_g fixing $*$ and D pointwise, respectively. We use integral coefficients for homology groups unless specified throughout the paper.

In the cases of the mapping class groups $\mathcal{M}_g^*$ and $\mathcal{M}_g$ of a surface Σ_g, Morita [17, Corollary 5.4] determined the first homology group with coefficients in the first integral homology group of the surface. Morita [18, Remark 6.3] extended the first Johnson homomorphism to a crossed homomorphism $\mathcal{M}_g^* \to \frac{1}{2}\Lambda^3(H_1(\Sigma_g))$, and showed that the contraction of this crossed homomorphism gives isomorphisms $H_1(\mathcal{M}_g^*; H_1(\Sigma_g)) \cong \mathbb{Z}$ and $H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$ when $g \geq 2$. For twisted first homology groups of the mapping class groups of nonorientable surfaces, see Stukow [25]. In the cases of the automorphism group $\operatorname{Aut} F_n$ and the outer automorphism group $\operatorname{Out} F_n$ of a free group F_n of rank n, Satoh [23] computed the twisted first homology groups $H_1(\operatorname{Aut} F_n; H^1(F_n))$ and $H_1(\operatorname{Out} F_n; H^1(F_n))$ for $n \geq 2$. Kawazumi [12] extended the first Andreadakis-Johnson homomorphism to a crossed homomorphism $\operatorname{Aut} F_n \to H^1(F_n) \otimes H_1(F_n)^{\otimes 2}$, and its contraction induces isomorphisms $H_1(\operatorname{Aut} F_n; H^1(F_n)) \cong \mathbb{Z}$ and $H_1(\operatorname{Out} F_n; H^1(F_n)) \cong \mathbb{Z}/(n-1)\mathbb{Z}$.

In this paper, we compute the twisted first homology groups of $\mathcal{H}_g$ and $\mathcal{H}_g^*$ with coefficients in the first integral homology group of the boundary surface Σ_g. Note that the restrictions of homeomorphisms of H_g to Σ_g induce an injective homomorphism $\mathcal{H}_g \to \mathcal{M}_g$, and we treat the group $\mathcal{H}_g$ as a subgroup of $\mathcal{M}_g$. The followings are main theorems in this paper.

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Theorem 1.1.

$$H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z}/(2g-2)\mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 3, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2. \end{cases}$$

Furthermore, when $g \geq 4$, the homomorphism $H_1(\mathcal{H}_g; H_1(\Sigma_g)) \to H_1(\mathcal{M}_g; H_1(\Sigma_g))$ induced by the inclusion is an isomorphism. When $g = 2, 3$, this homomorphism is surjective and the kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.2.

$$H_1(\mathcal{H}_{g,1}; H_1(\Sigma_g)) \cong H_1(\mathcal{H}_g^*; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2, 3. \end{cases}$$

Furthermore, when $g \geq 4$, the homomorphism $H_1(\mathcal{H}_g^*; H_1(\Sigma_g)) \to H_1(\mathcal{M}_g^*; H_1(\Sigma_g))$ induced by the inclusion is an isomorphism. When $g = 2, 3$, this homomorphism is surjective and the kernel is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

In this paper, we also study relationships between the second homology groups of $\mathcal{H}_g$, $\mathcal{H}_g^*$, and $\mathcal{H}_{g,1}$. The second homology group of $\mathcal{M}_g$ is calculated by Harer [4] when $g \geq 5$. It contains some minor mistakes and these are corrected in [5] later. For surfaces with an arbitrary number of punctures and boundary components, see Korkmaz-Stipsicz [13]. See also Benson-Cohen [1] for genus 2, and Stein [24], Sakasai [22], and Pitsch [19] for genus 3. There are some results which imply that the cohomology group of the handlebody mapping class group $\mathcal{H}_g$ is similar to that of $\mathcal{M}_g$. Morita [16, Proposition 3.1] showed that the rational cohomology group of any subgroup of the mapping class group decomposes into a direct sum. Later, Kawazumi and Morita [11, Proposition 5.2] generalized it to cohomology groups with coefficients in $A = \mathbb{Z}[1/(2g-2)]$ of fiber products of some groups and $\mathcal{M}_g^*$. In particular, if we apply their proposition to the natural inclusion $\mathcal{H}_g \to \mathcal{M}_g$, the fiber product $\mathcal{H}_g \times_{\mathcal{M}_g} \mathcal{M}_g^*$ coincides with the once punctured handlebody mapping class group $\mathcal{H}_g^*$, and its cohomology group decomposes as

$$H^n(\mathcal{H}_g^*; A) \cong H^n(\mathcal{H}_g; A) \oplus H^{n-1}(\mathcal{H}_g; H^1(\Sigma_g; A)) \oplus H^{n-2}(\mathcal{H}_g; A).$$

Hatcher and Wahl [8] showed that the integral cohomology groups of the mapping class groups of 3-manifolds stabilize in more general settings. Hatcher [7, Corollary in p.8] also announced that the rational stable cohomology group coincides with the polynomial ring generated by the even Morita-Mumford classes. However, as far as we know, even the second integral homology group of handlebody mapping class groups has not been computed yet.

Here is the outline of our paper:

In Section 2, we review some classical facts on homological algebra and group homology which we use in Sections 3, 6, and 7.

In Section 3, we investigate the relationship between the second integral homology group of the handlebody mapping class group fixing a point or a 2-disk in Σ_g pointwise with that of $\mathcal{H}_g$ using Theorem 1.1.

In Section 4, we compute the twisted first homology group $H_1(\mathcal{H}_g; H_1(\Sigma_g))$ to prove Theorem 1.1 in the case when $g \geq 4$. We also compute the twisted first homology groups of $\mathcal{H}_g$ with coefficients in $\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g))$ and $H_1(H_g)$.

Let $\mathcal{L}_g$ denote the kernel of the homomorphism $\mathcal{H}_g \to \mathrm{Out}(\pi_1 H_g)$. The exact sequence

$$1 \longrightarrow \mathcal{L}_g \longrightarrow \mathcal{H}_g \longrightarrow \mathrm{Out}(\pi_1 H_g) \longrightarrow 1$$

induces exact sequences between their first homology groups with coefficients in $\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g))$ and $H_1(H_g)$. We call a nonseparating disk properly embedded in H_g a meridian disk. Luft [14] showed that the group $\mathcal{L}_g$ coincides with the twist group, which is generated by Dehn twists along meridian disks. Actually, Dehn twists along separating disks are also contained in his generating set, but using the lantern relation, they can be written as products of Dehn twists along nonseparating disks. Satoh [23] determined the twisted first homology groups $H_1(\mathrm{Out}\, F_n; H_1(F_n))$ and $H_1(\mathrm{Out}\, F_n; H^1(F_n))$. Applying Luft's and Satoh's results to the exact sequences, we can determine $H_1(\mathcal{H}_g; H_1(\Sigma_g))$ when $g \geq 4$.

In Section 5, we review a finite presentation of the handlebody mapping class group $\mathcal{H}_g$ given by Wajnryb [26].

In Section 6, we compute the twisted first homology group $H_1(\mathcal{H}_g; H_1(\Sigma_g))$, using the Wajnryb's presentation of the handlebody mapping class group $\mathcal{H}_g$ to prove Theorem 1.1 in the case when $g = 2, 3$.

In Section 7, we prove Theorem 1.2 and also compute the twisted first homology groups of $\mathcal{H}_g^*$ with coefficients in $\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g))$ and $H_1(H_g)$.

2. Review on homology of groups

In Sections 3, 4, and 7, we use some classical facts on homological algebra and group homology. Although they are well-known and there are many references for them, we arrange them for the better readability.

2.1. Homology of groups. Let G be a group, and M a left $\mathbb{Z}G$-module. A $\mathbb{Z}G$-module is also called a G-module in this paper. The module M is also considered as a right G-module by the action $m \cdot g = g^{-1} \cdot m$ for $m \in M$ and for $g \in G$. Choose a projective resolution

$$\cdots \to X_2 \to X_1 \to X_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ over $\mathbb{Z}G$. The chain and cochain complexes of G with coefficients in M are defined by

$$C_n(G; M) = M \otimes_{\mathbb{Z}G} X_n, \text{ and } C^n(G; M) = \mathrm{Hom}_{\mathbb{Z}G}(X_n, M),$$

for $n \geq 0$. We call the homology and cohomology group of the above chain complexes the homology and cohomology of the group G with coefficients in M, respectively.

The homology and cohomology groups do not depend on the choice of projective resolutions. In Section 3, we consider the normalized bar resolution as a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ which is defined as follows. Let Y_n denote a free $\mathbb{Z}G$-module generated by the direct product G^n for $n \geq 1$, and $Y_0 = \mathbb{Z}G$. We denote the basis of Y_n by $[g_1|g_2|\cdots|g_n]$ which corresponds to the element $(g_1, g_2, \ldots, g_n) \in G^n$. Let us also denote by D_n the $\mathbb{Z}G$-submodule of Y_n generated by the set

$$\{[g_1|g_2|\cdots|g_n] \mid g_1, \ldots, g_n \in G, g_i = 1 \text{ for some } 1 \leq i \leq n\},$$

for $n \geq 1$ and $D_0 = 0$. For $n \geq 1$, Let $d_n : Y_n \to Y_{n-1}$ be a $\mathbb{Z}G$-homomorphism defined by

$$\begin{aligned} d_n([g_1|g_2|\cdots|g_n]) =& g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] \\ &+ (-1)^n[g_1|g_2|\cdots|g_{n-1}], \end{aligned}$$

and let $d_0 : \mathbb{Z}G \to \mathbb{Z}$ denote the augmentation map. Then the map d_n satisfies $d_{n-1} \circ d_n = 0$ for $n \geq 1$, and it induces a homomorphism $\bar{d}_n : Y_n/D_n \to Y_{n-1}/D_{n-1}$.

The free $\mathbb{Z}G$-resolution

$$\cdots Y_2/D_2 \xrightarrow{\bar{d}_2} Y_1/D_1 \xrightarrow{\bar{d}_1} Y_0/D_0 \xrightarrow{\bar{d}_0} \mathbb{Z} \to 0$$

of $\mathbb{Z}$ is called the normalized bar resolution. For more details, see Brown [2, Section I.5].

REMARK 2.1. The cochain complex of the normalized bar resolution is written as

$$\begin{aligned} C^n(G;M) &= \mathrm{Hom}_{\mathbb{Z}G}(Y_n/D_n, M) \\ &= \{f : G^n \to M \mid f(g_1,\ldots,g_n) = 0 \text{ when } g_i = 1 \text{ for some } 1 \leq i \leq n\}. \end{aligned}$$

For $m \in M$, let us denote by $f_m : G \to M$ the crossed homomomorphism defined by $f_m(h) = hm - m$. Then the set of 1-cocycles coincides with the set of crossed homomorphisms $G \to M$, and the set of 1-coboundaries is the image of the coboundary map $\delta : M \to C^1(G;M)$ defined by $\delta(m) = f_m$. See [3, Section 2.3] for details.

We will use the following basic facts on group homology and homological algebra in Sections 3, 4, and 7.

Proposition 2.2 (See, for example, Brown [2, Proposition 6.1]). (1) *There is a natural isomorphism*

$$H_0(G;M) \cong M_G,$$

where M_G is a coinvariant of the action of G on M, that is,

$$M_G = M/\{gm - m \mid g \in G,\ m \in M\}.$$

(2) *For any exact sequence $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{j} M_3 \to 0$ of left G-modules and any $n \geq 1$, there is a natural map $\partial_* : H_n(G;M_3) \to H_{n-1}(G;M_1)$ and $\delta^* : H^n(G;M_3) \to H^{n+1}(G;M_1)$ such that the sequence*

$$\cdots \to H_n(G;M_1) \xrightarrow{i_*} H_n(G;M_2) \xrightarrow{j_*} H_n(G;M_3) \xrightarrow{\partial_*} H_{n-1}(G;M_1) \to \cdots,$$

$$\cdots \to H^n(G;M_1) \xrightarrow{i_*} H^n(G;M_2) \xrightarrow{j_*} H^n(G;M_3) \xrightarrow{\delta^*} H^{n+1}(G;M_1) \to \cdots,$$

are exact.

As in the case of cohomology groups on spaces, we also have the universal coefficient theorem:

Theorem 2.3 (the universal coefficient theorem). *Let G be a group and A an abelian group. Suppose that M is a free $\mathbb{Z}$-module with left G-action. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G;M),A) \longrightarrow H^n(G;M^*) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(H_n(G;M),A) \longrightarrow 0,$$

where $M^* = \mathrm{Hom}_{\mathbb{Z}}(M,A)$.

Proof. This is obtained from a general form of the universal coefficient theorem. Let $\{(K_n,d_n)\}_{n=0}^{\infty}$ be a complex of free $\mathbb{Z}$-modules. Then there exists an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(K_*),A) \to H^n(\mathrm{Hom}_{\mathbb{Z}}(K_*,A)) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(K_*),A) \to 0$$

called the universal coefficient theorem. See, for example, [21, Theorem 7.59]. Since M is a free $\mathbb{Z}$-module, $C_n(G;M)$ is also a free $\mathbb{Z}$-module. If we put $K_n = C_n(G;M)$, we have

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(G;M),A) \to H^n(\mathrm{Hom}_{\mathbb{Z}}(C_n(G;M),A)) \to \mathrm{Hom}_{\mathbb{Z}}(H_n(G;M),A) \to 0.$$

The homomorphism

$$\Phi : \mathrm{Hom}_{\mathbb{Z}}(C_n(G;M),A) = \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}G} X_n, A) \cong \mathrm{Hom}_{\mathbb{Z}G}(X_n, \mathrm{Hom}_{\mathbb{Z}}(M,A))$$

defined by $\Phi(f)(x) = f(x,*)$ for $f \in \mathrm{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}G} X_*, A)$ is clearly an isomorphism, and commutes with the boundary maps. Thus we obtain $H^n(\mathrm{Hom}_{\mathbb{Z}}(C_n(G;M),A)) \cong H^n(G;M^*)$. □

2.2. Lyndon-Hochschild-Serre spectral sequences. Let

$$1 \longrightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \longrightarrow 1$$

be an exact sequence of groups and M a left G-module. Then we have a Lyndon-Hochschild-Serre spectral sequence between their homology groups. It is a spectral sequence whose E^2-term is $E^2_{p,q} = H_p(Q;H_q(K;M))$, and converges into $H_{p+q}(G;M)$. For details, see Brown [2, Section VII.6], Rotman [21, Section 10.7], and Weibel [27, Section 6.8]. See also Hochschild-Serre [9] for spectral sequences of group cohomology, and McCleary [15, Theorem 8^{bis}.12] for general spectral sequences. Here, we review some facts on this spectral sequence. In the spectral sequence, the homology group $H_n(G;M)$ has a filtration

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_{n-1} \subset F_n = H_n(G;M),$$

and there is an isomorphism $E^{\infty}_{p,n-p} \cong F_p/F_{p-1}$ for $0 \leq p \leq n$. The composition map of the quotient map $H_n(G;M) \to F_n/F_{n-1} \cong E^{\infty}_{n,0}$ and the natural inclusion $E^{\infty}_{n,0} \to E^2_{n,0} = H_n(Q;H_0(K;M))$ is called the edge map, and it is described as follows. See Weibel [27, Section 6.8.2] for details.

Proposition 2.4 (edge maps)**.** *For $n \geq 0$, the homomorphism $\pi_* : H_n(G;M) \to H_n(Q;M_K)$ induced by the projection $\pi : G \to Q$ and the quotient map $M \to M_K$ coincides with the composition map*

$$H_n(G;M) \to E^{\infty}_{n,0} \to E^2_{n,0} = H_n(Q;M_K).$$

Remark 2.5. From Proposition 2.4, we have

$$\mathrm{Ker}(H_n(G;M) \to E^{\infty}_{n,0}) = \mathrm{Ker}(H_n(G;M) \to H_n(Q;M_K)).$$

The following exact sequence is called the five-term exact sequence, which is obtained from a diagram chasing in the E^2-term of the spectral sequence. See, for example, Brown [2, Corollary 6.4 in Section VII].

Theorem 2.6 (the five term exact sequence)**.** *Let* $1 \to K \xrightarrow{\iota} G \xrightarrow{\pi} Q \to 1$ *be an exact sequence between groups and* M *a left* G*-module. Then there exists an exact sequence*

$$H_2(G; M) \xrightarrow{\pi_*} H_2(Q; M_K) \to H_1(K; M)_Q \xrightarrow{\iota_*} H_1(G; M) \xrightarrow{\pi_*} H_1(Q; M) \to 0,$$

where $H_1(K; M)_Q$ *denotes the coinvariant of the action of* Q *on* $H_1(K; M)$ *induced by the conjugacy action of* G*.*

Remark 2.7. Let $X_n = Y_n/D_n$ be the normalized bar resolution of K in Section 2.1. Then the action of Q on $H_n(K; M)$ appeared in Theorem 2.6 is induced by the conjugacy action of G on $C_n(K; M) = M \otimes_{\mathbb{Z}K} X_n$ defined by

$$h(m \otimes [k_1|k_2|\cdots|k_n]) = (hm) \otimes [hk_1h^{-1}|hk_2h^{-1}|\cdots|hk_nh^{-1}]$$

for $h \in G$.

We will also use a Gysin exact sequence in Section 3. Suppose that the group K has the same homology group with a m-sphere, and the conjugacy action of G to $H_m(K)$ is trivial. Then we have:

Theorem 2.8 (the Gysin exact sequence)**.** *There is an exact sequence*

$$\cdots \to H_{n+m+1}(G) \xrightarrow{\pi_*} H_{n+m+1}(Q) \to H_n(Q) \xrightarrow{\pi^!} H_{n+m}(G) \to \cdots.$$

The proof of Theorem 2.8 is obtained from a diagram chasing in the E^{m+1}-term of the spectral sequence, and it goes in analogous to the case of Serre spectral sequences of fibrations between topological spaces, which is written in Hatcher [6, Section 4.D] and Weibel [27, Application 5.3.7]. In Section 3, we will use this exact sequence only in the case when $K = \mathbb{Z}$ and $m = 1$.

Remark 2.9. Consider the case when $K = \mathbb{Z}$ and is in the center of G. In the normalized bar resolution, the map $\pi^! : H_n(Q) \xrightarrow{\pi^!} H_{n+1}(G)$ can be written explicitly. Let $X_* \to \mathbb{Z} \to 0$ be the normalized bar resolution of Q, and $\sigma = \sum_{i=1}^{l} a_i[q_i^1|q_i^2|\cdots|q_i^n] \in C_n(Q) = \mathbb{Z} \otimes_{\mathbb{Z}Q} X_n$ be a n-cycle of Q. Then $\pi^!(\sigma)$ is written as

$$\pi^!(\sigma) = \sum_{i=1}^{l}\left(a_i \sum_{j=0}^{n}(-1)^j[\tilde{q}_i^1|\cdots|\tilde{q}_i^j|k|\tilde{q}_i^{j+1}|\cdots|\tilde{q}_i^n]\right) \in C_{n+1}(G),$$

where $\tilde{q}_i^j \in G$ is an element of the inverse image of q_i^j under $\pi : G \to Q$, and k is a generator of $K = \mathbb{Z}$.

In the general case when $H_*(K) = H_*(S^m)$ and G acts on $H_m(K)$ trivially, for $\sigma \in H_n(Q)$ the image $\pi^!(\sigma) \in H_{n+m}(G)$ is also written as a twisted shuffle product of a lift of σ and a m-cycle which represents a generator of $H_m(K)$. We omit the proof of this description, but we can prove it by introducing a filtration $\{A^i\}_{i\in\mathbb{Z}}$ on the chain complex $C_*(G)$, and constructing an isomorphism between $E^1_{p,q} = C_p(Q; H_q(K))$ and $H_{p+q}(A^p/A^{p-1})$ in analogous to the cohomological case written in Hochschild-Serre [9, Theorems 1 and 2].

3. On the second homology of the handlebody mapping class groups fixing a point or a 2-disk pointwise

In this section, we introduce some corollaries of Theorem 1.1 which give relationships between the second homology groups of $\mathcal{H}_g$, $\mathcal{H}_g^*$ and $\mathcal{H}_{g,1}$.

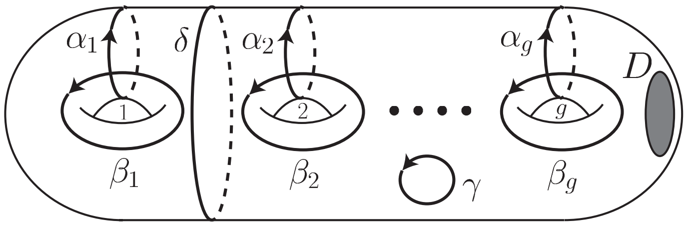

FIG. 1. a 2-disk D and simple closed curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma$

Let $U\Sigma_g$ denote the unit tangent bundle of Σ_g. Let $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ be oriented smooth simple closed curves as in Figure 1, and denote their homology classes in $H_1(\Sigma_g)$ by $x_1 = [\alpha_1], x_2 = [\alpha_2], \ldots, x_g = [\alpha_g], y_1 = [\beta_1], y_2 = [\beta_2], \ldots, y_g = [\beta_g]$. We also denote by γ a null-homotopic smooth simple closed curve in Figure 1. There are natural liftings of $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma$ to $U\Sigma_g$, and let us denote their homology classes in $H_1(U\Sigma_g)$ by $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_g$, $\tilde{y}_1, \tilde{y}_2, \ldots, \tilde{y}_g$, z, respectively. For a group G and a left G-module M, let us denote by M_G its coinvariant, that is, the quotient of M by the submodule spanned by the set $\{gm - m \mid m \in M, g \in G\}$.

Lemma 3.1. *For $g \geq 2$,*

$$H_1(U\Sigma_g)_{\mathcal{H}_g} = 0.$$

Proof. For a simple closed curve c in Σ_g, we denote by t_c the Dehn twist along c. As in [10, Theorem 1B], we have $t_{\alpha_i}(\tilde{y}_i) = \tilde{y}_i + \tilde{x}_i$ for $i = 1, \ldots, g$. Note that our $\tilde{c}$ is denoted by $\vec{c}$ in [10], and is different from what is denoted by $\tilde{c}$ in [10]. Hence we have $\tilde{x}_1 = \cdots = \tilde{x}_g = 0 \in H_1(U\Sigma_g)_{\mathcal{H}_g}$. Let δ'_i and α'_i be simple closed curves as depicted in Figure 2 for $1 \leq i \leq g-1$. Let us denote $h_i = t_{\delta'_i}^{-1} t_{\beta_i} t_{\alpha_{i+1}} \in \mathcal{M}_g$. Since $h_i(\alpha_l) = \alpha_l$ when $l \neq i$ and $h_i(\alpha_i) = \alpha'_i$, the mapping class h_i is actually an element of the handlebody mapping class

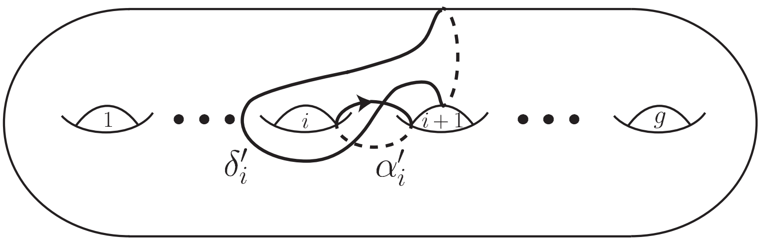

FIG. 2. simple closed curves δ' and α'_i

group $\mathcal{H}_g$. We obtain

$$h_i(\tilde{x}_i) = \tilde{x}_i - \tilde{x}_{i+1} - z \text{ and } h_i(\tilde{y}_{i+1}) = \tilde{y}_i + \tilde{y}_{i+1} - z$$

for $i = 1, \ldots, g-1$. Thus we have $z = \tilde{y}_1 = \cdots = \tilde{y}_{g-1} = 0 \in H_1(U\Sigma_g)_{\mathcal{H}_g}$. Since the rotation $r \in \mathcal{H}_g$ of the surface Σ_g about a vertical line by 180 degrees maps $\tilde{y}_g$ to $-\tilde{y}_1$, we also obtain $\tilde{y}_g = 0$. □

Using the mapping classes h_i and r, we can also show:

Lemma 3.2. *For $g \geq 2$,*

$$\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g))_{\mathcal{H}_g} = 0.$$

Proposition 3.3. *When $g \geq 4$,*

$$H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_g) \oplus \mathbb{Z}.$$

Proof. Let us denote the Lyndon-Hochschild-Serre spectral sequences of the forgetful exact sequences

$$1 \longrightarrow \pi_1\Sigma_g \longrightarrow \mathcal{M}_g^* \longrightarrow \mathcal{M}_g \longrightarrow 1,$$

$$1 \longrightarrow \pi_1\Sigma_g \longrightarrow \mathcal{H}_g^* \longrightarrow \mathcal{H}_g \longrightarrow 1$$

by $\{E^r_{p,q}\}$ and $\{\bar{E}^r_{p,q}\}$, respectively. Recall that $E^2_{p,q} = H_p(\mathcal{M}_g; H_q(\Sigma_g))$ and $\bar{E}^2_{p,q} = H_p(\mathcal{H}_g; H_q(\Sigma_g))$. By Lemma 3.1, we have

$$H_0(\mathcal{M}_g; H_1(\Sigma_g)) = H_1(\Sigma_g)_{\mathcal{M}_g} = 0, \text{ and } H_0(\mathcal{H}_g; H_1(\Sigma_g)) = H_1(\Sigma_g)_{\mathcal{H}_g} = 0.$$

Thus the E^∞ terms of both spectral sequences are written as follows.

$E^\infty_{0,2}$	$*$	$*$
0	$E^\infty_{1,1}$	$*$
$\mathbb{Z}$	$H_1(\mathcal{M}_g)$	$H_2(\mathcal{M}_g)$

$\bar{E}^\infty_{0,2}$	$*$	$*$
0	$\bar{E}^\infty_{1,1}$	$*$
$\mathbb{Z}$	$H_1(\mathcal{H}_g)$	$H_2(\mathcal{H}_g)$

Since the filtrations of $H_2(\mathcal{H}_g^*)$ and $H_2(\mathcal{M}_g^*)$ are compatible with the homomorphism $H_2(\mathcal{H}_g^*) \to H_2(\mathcal{M}_g^*)$, we have a morphism between two exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{E}^\infty_{0,2} & \longrightarrow & \mathrm{Ker}(H_2(\mathcal{H}_g^*) \to \bar{E}^\infty_{2,0}) & \longrightarrow & \bar{E}^\infty_{1,1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E^\infty_{0,2} & \longrightarrow & \mathrm{Ker}(H_2(\mathcal{M}_g^*) \to E^\infty_{2,0}) & \longrightarrow & E^\infty_{1,1} & \longrightarrow & 0. \end{array}$$

Note that by Remark 2.5, we have

$$\mathrm{Ker}(H_2(\mathcal{H}_g^*) \to \bar{E}^\infty_{2,0}) = \mathrm{Ker}(H_2(\mathcal{H}_g^*) \to H_2(\mathcal{H}_g)),$$
$$\mathrm{Ker}(H_2(\mathcal{M}_g^*) \to E^\infty_{2,0}) = \mathrm{Ker}(H_2(\mathcal{M}_g^*) \to H_2(\mathcal{M}_g)).$$

As explained in [13, Propositions 1.4 and 1.5], there exist isomorphisms $\mathrm{Ker}(H_2(\mathcal{M}_g^*) \to H_2(\mathcal{M}_g)) \cong \mathbb{Z}$ and $E^\infty_{0,2} = E^2_{0,2} \cong \mathbb{Z}$ when $g \geq 4$. It is also true when $g = 3$ as in [22, Corollary 4.9]. Moreover, there exists a surjective homomorphism $S_1 : H_2(\mathcal{M}_g^*) \to \mathbb{Z}$ defined in [4, Section 0] which maps the fundamental class $[\Sigma_g] \in H_2(\Sigma_g) = E^\infty_{0,2}$ to $(2g-2)$-times a generator and whose restriction to $\mathrm{Ker}(H_2(\mathcal{M}_g^*) \to H_2(\mathcal{M}_g))$ is surjective. These facts show that $E^\infty_{1,1}$ is a cyclic group of order $2g-2$. Since Morita [17] showed $E^2_{1,1} = H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$ when $g \geq 2$, we obtain $E^2_{1,1} = E^\infty_{1,1}$.

When $g \geq 4$, this fact and the isomorphism $H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong H_1(\mathcal{M}_g; H_1(\Sigma_g))$ show that in the commutative diagram

$$\begin{tikzcd} \bar{E}^2_{1,1} \arrow[r] \arrow[d] & \bar{E}^\infty_{1,1} \arrow[d] \\ E^2_{1,1} \arrow[r] & E^\infty_{1,1}, \end{tikzcd}$$

we have an isomorphism $\bar{E}^\infty_{1,1} \cong E^\infty_{1,1}$. As a conclusion, we obtain

$$\mathrm{Ker}(H_2(\mathcal{H}_g^*) \to H_2(\mathcal{H}_g)) \cong \mathrm{Ker}(H_2(\mathcal{M}_g^*) \to H_2(\mathcal{M}_g)) \cong \mathbb{Z}.$$

Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & H_2(\mathcal{H}_g^*) \arrow[r] \arrow[d] & H_2(\mathcal{H}_g) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & H_2(\mathcal{M}_g^*) \arrow[r] & H_2(\mathcal{M}_g) \arrow[r] & 0. \end{tikzcd}$$

Since the lower exact sequence splits, we obtain $H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_g) \oplus \mathbb{Z}$. □

When $g \geq 2$, we have $H_1(U\Sigma_g)_{\mathcal{H}_g} = H_1(\Sigma_g)_{\mathcal{H}_g} = 0$ by Lemma 3.1. By the five-term exact sequences (Theorem 2.6) induced by the exact sequences

$$1 \longrightarrow \pi_1\Sigma_g \longrightarrow \mathcal{H}_g^* \longrightarrow \mathcal{H}_g \longrightarrow 1,$$
$$1 \longrightarrow \pi_1 U\Sigma_g \longrightarrow \mathcal{H}_{g,1} \longrightarrow \mathcal{H}_g \longrightarrow 1,$$

we have:

Lemma 3.4. *When $g \geq 2$,*

$$H_1(\mathcal{H}_{g,1}) \cong H_1(\mathcal{H}_g^*) \cong H_1(\mathcal{H}_g).$$

Remark 3.5. By the Wajnryb's presentation which we review in Section 5.1, we can compute the abelianization as follows:

$$H_1(\mathcal{H}_g) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 1, \\ \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } g \geq 3. \end{cases}$$

We can also see that it is generated by $s_1 = t_{\beta_1} t_{\alpha_1}^2 t_{\beta_1}$ when $g \geq 3$. Note that Wajnryb made a mistake in his calculation of the abelianization in [26, Theorem 20] when $g = 2$.

In the following, we choose a 2-disk D in the boundary Σ_g so that it is disjoint from the simple closed curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ as in Figure 1 and pick a point $*$ in $\mathrm{Int}\, D$.

Lemma 3.6. *When $g \geq 3$,*

$$H_2(\mathcal{H}_g^*) \cong H_2(\mathcal{H}_{g,1}) \oplus \mathbb{Z}.$$

Proof. Let $\pi : \mathcal{H}_{g,1} \to \mathcal{H}_g^*$ denote the forgetful map. The Gysin exact sequence explained in Theorem 2.8 of the central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{H}_{g,1} \xrightarrow{\pi} \mathcal{H}_g^* \longrightarrow 1.$$

is written as

$$\cdots \to H_1(\mathcal{H}_g^*) \xrightarrow{\pi^!} H_2(\mathcal{H}_{g,1}) \xrightarrow{\pi_*} H_2(\mathcal{H}_g^*) \to H_0(\mathcal{H}_g^*) \xrightarrow{\pi^!} H_1(\mathcal{H}_{g,1}) \to \cdots .$$

As explained in Remark 2.9, the Gysin homomorphism $\pi^! : H_1(\mathcal{H}_g^*) \to H_2(\mathcal{H}_{g,1})$ maps $[h]$ to $[\tilde{h}|t_{\partial D}] - [t_{\partial D}|\tilde{h}]$ in the bar resolution for $h \in \mathcal{H}_g^*$, where $\tilde{h} \in \mathcal{H}_{g,1}$ is an elenent in the inverse image of h under π. By Lemma 3.4 and [26, Theorem 20], $H_1(\mathcal{H}_g^*)$ is the cyclic group of order 2 generated by s_1 when $g \geq 3$.

We can prove that the Gysin homomorphism $\pi^! : H_1(\mathcal{H}_g^*) \to H_2(\mathcal{H}_{g,1})$ is the zero map as follows. Let us choose a representing diffeomorphism of s_1 whose support is in a genus 1 subsurface S of $\Sigma_g - \operatorname{Int} D$. Then we have a 2-chain which has its support is in $(\Sigma_g - \operatorname{Int} D) - S$ and bounds $[t_{\partial D}] \in C_1(\mathcal{H}_{g,1})$ as follows. Let $t_1, t_2, t_3, t_1', t_2', t_3'$ be the Dehn twists along simple closed curves as in Figure 3. By the Lantern relation, we have $t_3' t_3^{-1} t_2' t_2^{-1} t_1' t_1^{-1} = t_{\partial D}$. In the

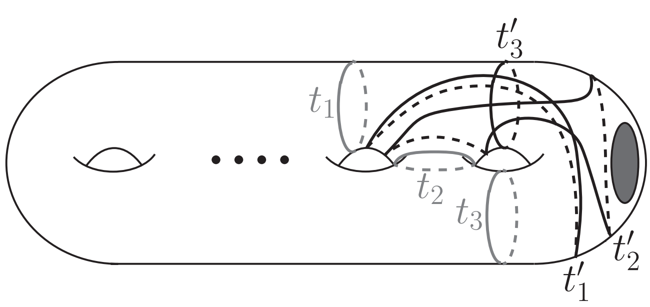

Fig. 3. the lantern relation

handlebody mapping class group, Dehn twists along two meridian disks are conjugate. Thus there exists $\varphi_i \in \mathcal{H}_{g,1}$ such that $\varphi_i t_i \varphi_i^{-1} = t_i'$ for $i = 1, 2, 3$. Moreover, we can represent φ_i by a diffeomorphism whose support is in $(\Sigma_g - \operatorname{Int} D) - S$. Define a 2-chain $\sigma \in C_2(\mathcal{H}_{g,1})$ by

$$\begin{aligned}\sigma = &-\sum_{i=1}^{3}([\varphi_i|t_i] + [\varphi_i t_i|\varphi_i^{-1}] + [\varphi_i t_i \varphi_i^{-1}|t_i^{-1}] - [t_i|t_i^{-1}] - [\varphi_i|\varphi_i^{-1}])\\ &- [t_3' t_3^{-1}|t_2' t_2^{-1}] - [t_3' t_3^{-1} t_2' t_2^{-1}|t_1' t_1^{-1}].\end{aligned}$$

Then we obtain $\partial\sigma = [t_{\partial D}] \in C_1(\mathcal{H}_{g,1})$. For simplicity, we denote $\sigma = \sum_{i=1}^{17} \epsilon_i [x_i|y_i]$, where $\epsilon_i = \pm 1$ and $\epsilon_i [x_i|y_i]$ are terms appeared above. Let $\tilde{s}_1 = t_{\beta_1} t_{\alpha_1}^2 t_{\beta_1} \in \mathcal{H}_{g,1}$. Since $\tilde{s}_1$ commutes with x_i and y_i, the 3-chain $\tau = \sum_{i=1}^{17} \epsilon_i([x_i|y_i|\tilde{s}_1] - [x_i|\tilde{s}_1|y_i] + [\tilde{s}_1|x_i|y_i])$ bounds $[t_{\partial D}|\tilde{s}_1] - [\tilde{s}_1|t_{\partial D}] \in C_2(\mathcal{H}_{g,1})$, and the Gysin homomorphism is the zero map.

Another Gysin homomorphism $\pi^! : H_0(\mathcal{H}_g^*) \to H_1(\mathcal{H}_{g,1})$ maps a generator of $H_0(\mathcal{H}_g^*) \cong \mathbb{Z}$ to $[t_{\partial D}]$. Since

$$[t_{\partial D}] = \sum_{i=1}^{3}([t_i'] - [t_i]) = \sum_{i=1}^{3}([\varphi_i t_i \varphi_i^{-1}] - [t_i]) = 0 \in H_1(\mathcal{H}_{g,1}),$$

the homomorphism $\pi^! : H_0(\mathcal{H}_g^*) \to H_1(\mathcal{H}_{g,1})$ is also trivial. Thus we obtain the exact sequence

$$0 \to H_2(\mathcal{H}_{g,1}) \to H_2(\mathcal{H}_g^*) \to \mathbb{Z} \to 0.$$

□

Both of the direct sum decompositions of $H_2(\mathcal{H}_g^*)$ in Proposition 3.3 and Lemma 3.6 are induced by the composition of the natural homomorphism $H_2(\mathcal{H}_g^*) \to H_2(\mathcal{M}_g^*)$ and $S_1 : H_2(\mathcal{M}_g^*) \to \mathbb{Z}$ defined in [4, Section 4] up to sign. Thus we obtain:

Corollary 3.7. *When $g \geq 4$,*

$$H_2(\mathcal{H}_{g,1}) \cong H_2(\mathcal{H}_g).$$

4. Proof of Theorem 1.1 for $g \geq 4$

In the rest of this paper, we write H for $H_1(\Sigma_g)$ and denote by L the kernel of the homomorphism $H_1(\Sigma_g) \to H_1(H_g)$ induced by the inclusion for simplicity. Note that $H_1(H_g)$ is isomorphic to H/L as an $\mathcal{H}_g$-module. In this section, we prove Theorem 1.1 when $g \geq 4$. Luft's result on $\mathrm{Ker}(\mathcal{H}_g \to \mathrm{Out}\, F_g)$ and Satoh's result on $\mathrm{Out}\, F_g$ make it much easier to determine the first homology $H_1(\mathcal{H}_g; H)$ when $g \geq 4$ than when $g = 2, 3$.

Lemma 4.1. *Let $g \geq 2$, and G a subgroup of the mapping class group $\mathcal{M}_g$. When the induced map $H_1(U\Sigma_g)_G \to H_G$ by the natural projection is injective, there exists a surjective homomorphism*

$$H_1(G; H) \to \mathbb{Z}/(2g-2)\mathbb{Z}.$$

Proof. Considering the Serre spectral sequence of the bundle map $\varpi : U\Sigma_g \to \Sigma_g$, we obtain the Gysin exact sequence

$$H_2(\Sigma_g) \xrightarrow{\Phi} H_0(\Sigma_g) \xrightarrow{\varpi^!} H_1(U\Sigma_g) \xrightarrow{\varpi_*} H_1(\Sigma_g) \to 0.$$

It is a classical fact that Φ maps the fundamental class $[\Sigma_g]$ to the Euler characteristic $e([\Sigma_g]) = 2 - 2g$ under the isomorphism $H_0(\Sigma_g) \cong \mathbb{Z}$. For example, the description of the cohomological Gysin exact sequences of oriented sphere bundles can be found in Hatcher [6, pp.437–438]. Thus we obtain the exact sequence

$$0 \longrightarrow \mathbb{Z}/(2g-2)\mathbb{Z} \xrightarrow{\varpi^!} H_1(U\Sigma_g) \xrightarrow{\varpi_*} H \to 0,$$

where $\varpi^!$ maps $1 \in \mathbb{Z}/(2g-2)\mathbb{Z}$ to the homology class $H_1(U\Sigma_g)$ represented by a fiber circle. Applying Proposition 2.2 to this sequence, we have an exact sequence

$$H_1(G; H) \longrightarrow (\mathbb{Z}/(2g-2)\mathbb{Z})_G \xrightarrow{\varpi^!} H_1(U\Sigma_g)_G \xrightarrow{\varpi_*} H_G.$$

Since $H_1(U\Sigma_g)_G \to H_G$ is injective, and G acts on $\mathbb{Z}/(2g-2)\mathbb{Z}$ trivially, we obtain the surjective homomorphism $H_1(G; H) \to \mathbb{Z}/(2g-2)\mathbb{Z}$. □

Remark 4.2. The homomorphism $H_1(G; H) \to \mathbb{Z}/(2g-2)\mathbb{Z}$ is written in [17, Section 6] explicitly. This coincide with the mod $(2g-2)$-reduction of the contraction of the twisted homomorphism called the first Johnson homomorphism. In particular, the homomorphism $H_1(G; H) \to H_1(\mathcal{M}_g; H)$ induced by the inclusion is surjective. Note that the handlebody mapping class group $\mathcal{H}_g$ satisfies the assumption of Lemma 4.1 because of Lemma 3.1.

By Lemma 3.1, we obtain a lower bound on the order of $H_1(\mathcal{H}_g; H)$. For a simple closed curve c in Σ_g, we denote by $\mathcal{H}_g(c)$ the subgroup of $\mathcal{H}_g$ which preserves the curve c setwise.

Lemma 4.3. *Let M be an $\mathcal{H}_g$-module on which $\mathcal{L}_g$ acts trivially. Then there is a surjective homomorphism*

$$M_{\mathcal{H}_g(\alpha_1)} \to H_1(\mathcal{L}_g; M)_{\mathcal{H}_g}.$$

Proof. Since $\mathcal{L}_g$ acts on M trivially, we have $H_1(\mathcal{L}_g; M)_{\mathcal{H}_g} = (M \otimes H_1(\mathcal{L}_g))_{\mathcal{H}_g}$. Here, by Remark 2.7, the action of $\mathcal{H}_g$ on $M \otimes H_1(\mathcal{L}_g)$ is described as

$$\varphi(m \otimes [l]) = \varphi(m) \otimes [\varphi l \varphi^{-1}],$$

where we denote the homology class of $l \in \mathcal{L}_g$ by $[l]$. Luft [14, Corollary 2.4] proved that $\mathcal{L}_g$ is normally generated by the Dehn twists along the curves α_1 and δ in Figure 4. When

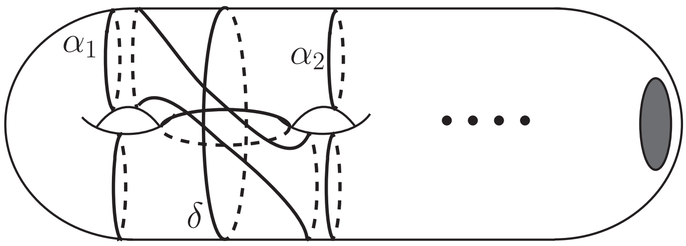

FIG. 4. the curve δ and the Lantern relation

$g \geq 2$, the lantern relation implies that the Dehn twist t_δ can be written as a product of right and left Dehn twists along the boundary curves of the meridian disks depicted in Figure 4, each of which is conjugate to t_{α_1} or $t_{\alpha_1}^{-1}$. Thus $\mathcal{L}_g$ is normally generated by the Dehn twist t_{α_1}, and $H_1(\mathcal{L}_g; M)_{\mathcal{H}_g}$ is generated by $\{m \otimes [t_{\alpha_1}] \mid m \in M\}$. Therefore, we obtain the surjective homomorphism $M \to H_1(\mathcal{L}_g; M)_{\mathcal{H}_g}$ defined by $m \mapsto m \otimes [t_{\alpha_1}]$, and it factors through $M_{\mathcal{H}_g(\alpha_1)}$. □

Lemma 4.4. (1) *$H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$ when $g \geq 2$.*
(2) *$H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} = 0$ when $g \geq 3$, and $H_1(\mathcal{L}_2; L)_{\mathcal{H}_2} \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$ when $g = 2$.*

Proof. The Luft group $\mathcal{L}_g$ is generated by Dehn twists along meridian disks, and it acts on L and H/L trivially. Thus we have the isomorphism $H_1(\mathcal{L}_g; M) = M \otimes H_1(\mathcal{L}_g)$ for $M = H/L, L$. By applying Lemma 4.3, we obtain the surjective homomorphism $M_{\mathcal{H}_g(\alpha_1)} \to H_1(\mathcal{L}_g; M)_{\mathcal{H}_g}$. We compute the order of the image of this homomorphism for each of the cases $M = H/L$ and $M = L$.

(1) There exists a mapping class $r_{1,j} \in \mathcal{H}_g$ for $2 \leq j \leq g$ which preserves α_1 setwise and satisfies

$$r_{1,j}(x_l) = \begin{cases} -x_1 - x_2 - \cdots - x_j & \text{if } l = j, \\ x_l & \text{otherwise,} \end{cases}$$

$$r_{1,j}(y_l) = \begin{cases} x_1 + x_2 + \cdots + x_j + y_l - y_j & \text{if } 1 \leq l \leq j-1, \\ x_1 + x_2 + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j, \\ y_l & \text{otherwise.} \end{cases}$$

See Lemma 5.3 for details. Let us denote by $\bar{y}_i$ the image of y_i under the natural homomorphism $H \to H_1(H_g) \cong H/L$ induced by the inclusion $\Sigma_g = \partial H_g \to H_g$. Since $r_{1,j}$ commutes with t_{α_1}, we have

$$r_{1,j}(\bar{y}_1 \otimes [t_{\alpha_1}]) = r_{1,j}(\bar{y}_1) \otimes [r_{1,j} t_{\alpha_1} r_{1,j}^{-1}] = (\bar{y}_1 - \bar{y}_j) \otimes [t_{\alpha_1}] \in H_1(\mathcal{L}_g; H/L).$$

Thus we obtain $\bar{y}_j \otimes [t_{\alpha_1}] = 0 \in H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g}$ for $j = 2, \ldots, g$. Since the mapping class $(t_{\beta_1} t_{\alpha_1})^3$ preserves each α_i setwise for $i = 1, 2, \ldots, g$, it is an element in $\mathcal{H}_g$, and it satisfies $(t_{\beta_1} t_{\alpha_1})^3(\bar{y}_1) = -\bar{y}_1$. Thus we have

$$(t_{\beta_1} t_{\alpha_1})^3(\bar{y}_1 \otimes [t_{\alpha_1}]) = -\bar{y}_1 \otimes [t_{\alpha_1}],$$

and $2\bar{y}_1 \otimes [t_{\alpha_1}] = 0 \in H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g}$. Since $\mathrm{Im}(M_{\mathcal{H}_g(\alpha_1)} \to (M \otimes H_1(\mathcal{L}_g))_{\mathcal{H}_g})$ is generated by $\{\bar{y}_i \otimes [t_{\alpha_1}]\}_{i=1}^{g}$, we obtain $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} = 0$ or $\mathbb{Z}/2\mathbb{Z}$.

(2) Since $r_{1,j}$ commutes with t_{α_1} for $j = 2, 3, \ldots, g$, we have

$$r_{1,j}(x_j \otimes [t_{\alpha_1}]) = -(x_1 + x_2 + \cdots + x_j) \otimes [t_{\alpha_1}],$$

and $(x_1 + x_2 + \cdots + 2x_j) \otimes [t_{\alpha_1}] = 0 \in H_1(\mathcal{L}_g; L)_{\mathcal{H}_g}$. For $j = 1, 2, \ldots, g$, the mapping class $s_j = t_{\beta_j} t_{\alpha_j}^2 t_{\beta_j} \in \mathcal{H}_g$ also preserves α_1 setwise, and satisfies $s_j(x_j) = -x_j$. Thus we also have

$$s_j(x_j \otimes [t_{\alpha_1}]) = -x_j \otimes [t_{\alpha_1}],$$

and $2x_j \otimes [t_{\alpha_1}] = 0 \in H_1(\mathcal{L}_g; L)_{\mathcal{H}_g}$. Consequently, we obtain

$$x_1 \otimes [t_{\alpha_1}] = x_2 \otimes [t_{\alpha_1}] = \cdots = x_{g-1} \otimes [t_{\alpha_1}] = 2x_g \otimes [t_{\alpha_1}] = 0,$$

and it implies $H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} \cong 0$ or $\mathbb{Z}/2\mathbb{Z}$.

Now suppose $g \geq 3$. Then there exists a mapping class t_{g-1} (see Lemma 5.3) which preserves α_1 setwise and satisfies $t_{g-1}(x_g) = x_{g-1}$. Thus we also obtain $(x_g - x_{g-1}) \otimes [t_{\alpha_1}] = 0$ when $g \geq 3$, and it implies $H_1(\mathcal{L}_g; L)_{\mathcal{H}_g} = 0$. □

Note that $H^1(H_g)$ is naturally isomorphic to the kernel $\mathrm{Ker}(H_1(\Sigma_g) \to H_1(H_g)) = L$ of the natural surjection. This is shown by using Poincaré duality $H^1(H_g) \cong H_2(H_g, \Sigma_g)$ and the cohomology exact sequence between spaces (H_g, Σ_g), which is written as $0 \to H_2(H_g, \Sigma_g) \to H_1(\Sigma_g) \to H_1(H_g) \to 0$. Since $\pi_1(H_g)$ is the free group F_g of rank g, we identify their cohomology and homology groups. That is, $H^1(F_g) = H^1(H_g) = L$ and $H_1(F_g) = H_1(H_g) = H/L$.

From the five-term exact sequence (Theorem 2.6) induced by the exact sequence $1 \to \mathcal{L}_g \to \mathcal{H}_g \to \mathrm{Out}\, F_g \to 1$, we have

$$H_1(\mathcal{L}_g; M)_{\mathcal{H}_g} \to H_1(\mathcal{H}_g; M) \to H_1(\mathrm{Out}\, F_g; M) \to 0$$

for $M = H/L, L$. Here, since $\mathcal{L}_g$ acts trivially on M, we have $M_{\mathcal{L}_g} = M$. Lemma 4.4 implies:

Lemma 4.5. *When $g \geq 2$, we have an exact sequence*

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow H_1(\mathrm{Out}\, F_g; H_1(F_g)) \longrightarrow 0,$$

where $\mathbb{Z}/2\mathbb{Z} \to H_1(\mathcal{H}_g; H/L)$ *denotes the zero map if* $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} = 0$. *When* $g \geq 3$, *we also have an isomorphism*

$$H_1(\mathcal{H}_g; L) \cong H_1(\operatorname{Out} F_g; H^1(F_g)).$$

The twisted first homology groups of $\operatorname{Out} F_n$ with coefficients in $H_1(F_n)$ and $H^1(F_n)$ were computed by Satoh [23, Theorem 1 (2)] as follows.

Theorem 4.6 (Satoh [23, Theorem 1 (2)])**.**

$$H_1(\operatorname{Out} F_n; H^1(F_n)) \cong \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z} & \textit{when } n \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \textit{when } n = 3, \\ \mathbb{Z}/2\mathbb{Z} & \textit{when } n = 2, \end{cases}$$

$$H_1(\operatorname{Out} F_n; H_1(F_n)) \cong \begin{cases} 0 & \textit{when } n \geq 4, \\ \mathbb{Z}/2\mathbb{Z} & \textit{when } n = 2, 3. \end{cases}$$

For a finite set S, let us denote its order by $|S|$. Recall that $|H_1(\mathcal{H}_g; H)|$ is at least $2g-2$ by Lemma 4.1. The homology exact sequence (Proposition 2.2 (2)) induced by the exact sequence $0 \to L \to H \to H/L \to 0$ implies that $|H_1(\mathcal{H}_g; H/L)|$ is at least $(2g-2)/|H_1(\mathcal{H}_g; L)|$. By Lemma 4.5 and Theorem 4.6, we obtain:

Lemma 4.7.

$$H_1(\mathcal{H}_g; L) \cong \begin{cases} \mathbb{Z}/(g-1)\mathbb{Z} & \textit{if } g \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \textit{if } g = 3, \end{cases} \qquad H_1(\mathcal{H}_g; H/L) \cong \mathbb{Z}/2\mathbb{Z} \textit{ if } g \geq 4.$$

Remark 4.8. Theorem 4.6 and Lemma 4.7 show $\operatorname{Ker}(H_1(\mathcal{H}_g; H/L) \to H_1(\operatorname{Out} F_g; H_1(F_g))) \cong \mathbb{Z}/2\mathbb{Z}$ when $g \geq 4$. Thus Lemma 4.4 (1) implies $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong \mathbb{Z}/2\mathbb{Z}$ when $g \geq 4$.

Remark 4.9. By Lemma 4.5 and Theorem 4.6, we see that the order of $H_1(\mathcal{H}_g; H/L)$ for $g = 2, 3$ is at most 4. In Propositions 6.9 and 6.19, we will show $H_1(\mathcal{H}_2; L) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_1(\mathcal{H}_g; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2$ for $g = 2, 3$. By Lemma 4.4 (1) and Theorem 4.6, it also follows that $H_1(\mathcal{L}_g; H/L)_{\mathcal{H}_g} \cong \mathbb{Z}/2\mathbb{Z}$ for $g = 2, 3$.

By Lemma 3.2, $H_0(\mathcal{H}_g; L) = L_{\mathcal{H}_g} = 0$ for $g \geq 2$. Thus the homology exact sequence (Proposition 2.2 (2)) induced by the sequence $0 \to L \to H \to H/L \to 0$ is written as

$$H_1(\mathcal{H}_g; L) \longrightarrow H_1(\mathcal{H}_g; H) \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow 0. \tag{4.1}$$

Lemma 4.7 and the exact sequence (4.1) give an upper bound on the order of $H_1(\mathcal{H}_g; H)$. Comparing this with the lower bound obtained in Lemma 4.1, we complete the proof of Theorem 1.1 for $g \geq 4$.

Remark 4.10. In the proof of Theorem 1.1 above, we also see the sequence

$$0 \longrightarrow H_1(\mathcal{H}_g; L) \longrightarrow H_1(\mathcal{H}_g; H) \longrightarrow H_1(\mathcal{H}_g; H/L) \longrightarrow 0$$

is exact when $g \geq 4$.

5. The Wajnryb's presentation of the handlebody mapping class group

In this section, we review the Wajnryb's presentation of the handlebody mapping class group $\mathcal{H}_g$ and compute the action of the handlebody mapping class group $\mathcal{H}_g$ to the first homology $H_1(\Sigma_g)$. This is for preparing to calculate the twisted first homology $H_1(\mathcal{H}_g; H)$ when $g = 2, 3$ in Section 6.

5.1. A presentation of the handlebody mapping class group. Let $g \geq 2$. We identify the surface in Figure 1 with that in Figure 5. Let ϵ_i be a simple closed curve in Figure 5 for $i = 1, \ldots, g-1$. By cutting the surface Σ_g along the simple closed curves $\alpha_1, \ldots, \alpha_g$,

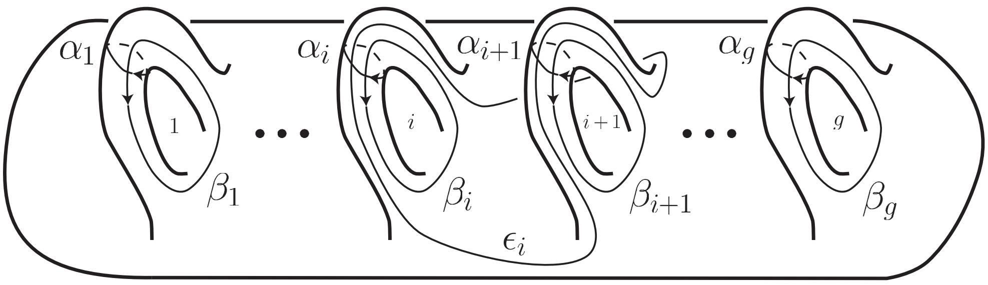

FIG. 5. the surface Σ_g

we obtain a $(2g)$-holed sphere with boundary components $\{\partial_{-i}, \partial_i\}_{i=1}^{g}$ as in Figure 6, where α_i and β_i correspond to the boundary components $\partial_{-i} \amalg \partial_i$ and the path from ∂_i to ∂_{-i}, respectively. For integers i, j satisfying $1 \leq i < j \leq g$, we denote by $\delta_{-j,-i}$ and $\delta_{i,j}$ the simple

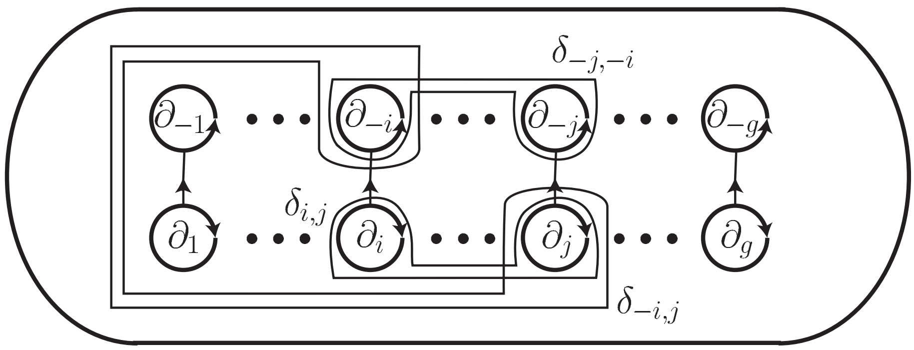

FIG. 6. the $(2g)$-holed sphere

closed curves in Figure 6. For integers i, j satisfying $1 \leq i \leq g$ and $1 \leq j \leq g$, we also denote by $\delta_{-i,j}$ the simple closed curve in Figure 6. For simplicity, we denote by $a_i, b_i, e_i, d_{1,2}$ the Dehn twists along the curves $\alpha_i, \beta_i, \epsilon_j, \delta_{1,2}$, respectively. Let us denote

$$\begin{aligned}
I_0 &= \{-g, -(g-1), \ldots, -2, -1, 1, 2 \ldots, g-1, g\}, \\
s_1 &= b_1 a_1^2 b_1, \\
t_i &= e_i a_i a_{i+1} e_i, \text{ for } i = 1, \ldots, g-1.
\end{aligned}$$

Since t_i permutes the simple closed curves α_i and α_{i+1} and fixes other α_j, we also have $t_i \in \mathcal{H}_g$. In the following, we denote $\varphi * \psi = \varphi\psi\varphi^{-1}$ for $\varphi, \psi \in \mathcal{H}_g$. For $i, j \in I_0$ satisfying

$i < j$, we denote

$$
\begin{aligned}
d_{i,j} &= (t_{i-1}t_{i-2}\cdots t_1 t_{j-1}t_{j-2}\cdots t_2) * d_{1,2} \text{ if } i > 0,\\
d_{i,j} &= (t_{-i-1}^{-1}t_{-i-2}^{-1}\cdots t_1^{-1}s_1^{-1}t_{j-1}t_{j-2}\cdots t_2) * d_{1,2} \text{ if } i < 0 \text{ and } i+j > 0,\\
d_{i,j} &= (t_{-i-1}^{-1}t_{-i-2}^{-1}\cdots t_1^{-1}s_1^{-1}t_j t_{j-1}\cdots t_2) * d_{1,2} \text{ if } j > 0 \text{ and } i+j < 0,\\
d_{i,j} &= (t_{-j-1}^{-1}t_{-j-2}^{-1}\cdots t_1^{-1}t_{-i-1}^{-1}t_{-i-2}^{-1}\cdots t_2^{-1}s_1^{-1}t_1^{-1}s_1^{-1}) * d_{1,2} \text{ if } j < 0,\\
d_{i,j} &= (t_{j-1}^{-1}d_{j-1,j}t_{j-2}^{-1}d_{j-2,j-1}\cdots t_1^{-1}d_{1,2}) * (s_1^2 a_1^4), \text{ if } i+j = 0.
\end{aligned}
$$

Here, $d_{i,j}$ is actually the Dehn twist along $\delta_{i,j}$ in Figure 6 as explained in [26, p. 211]. However, to give a presentation of $\mathcal{H}_g$ with a small generating set, we treat $d_{i,j}$ as the products above. We also denote

$$
\begin{aligned}
d_I &= (d_{i_1,i_2}d_{i_1,i_3}\cdots d_{i_1,i_n}d_{i_2,i_3}\cdots d_{i_2,i_n}d_{i_3,i_4}\cdots d_{i_{n-1},i_n})(a_{i_1}\cdots a_{i_n})^{2-n},\\
&\quad \text{where } I = \{i_1,\dots,i_n\} \subset I_0 \text{ and } i_1 < \cdots < i_n,\\
c_{i,j} &= d_I, \text{ where } I = \{k \in I_0 \mid i \le k \le j\} \text{ for } i \le j.
\end{aligned}
$$

Here, d_I and $c_{i,j}$ are the Dehn twists along simple closed curves which enclose $\{\partial_{i_1},\dots,\partial_{i_n}\}$ and $\{\partial_i,\dots,\partial_j\}$, respectively. See [26, p. 211] for details. Let us denote

$$\tilde{I} = \{(i,j) \in I_0^2 \mid i = 1, 1 < j\} \cup \{(i,j) \in I_0^2 \mid i < 0, -i < j \le g+i\},$$

and

$$
\begin{aligned}
r_{i,j} &= b_j a_j c_{i,j} b_j, \text{ for } (i,j) \in \tilde{I},\\
k_j &= a_j a_{j+1} t_j d_{j,j+1}^{-1} \text{ for } j = 1,\dots,g-1,\\
s_j &= (k_{j-1}k_{j-2}\cdots k_1) * s_1 \text{ for } j = 2,\dots,g,\\
z &= a_1 a_2\cdots a_g(s_1t_1t_2\cdots t_{g-1})(s_1t_1\cdots t_{g-2})\cdots(s_1t_1)s_1 d_I, \text{ where } I = \{1,\dots,g\},\\
z_j &= k_{j-1}k_{j-2}\cdots k_{g+1-j}z \text{ for } j > \frac{g}{2}.
\end{aligned}
$$

Here, $r_{i,j}$ also lies in $\mathcal{H}_g$ as is explained in [26, p. 211]. For $\varphi,\psi \in \mathcal{H}_g$, let us denote their commutator by $[\varphi,\psi] = \varphi\psi\varphi^{-1}\psi^{-1}$. Note that the elements defined here can be written as a product of $a_1,\dots,a_g$, $d_{1,2}$, s_1, $t_1,\dots,t_{g-1}$, and $r_{i,j}$ for $(i,j) \in \tilde{I}$.

Theorem 5.1 ([26, Theorem 18]). *The handlebody mapping class group of genus g admits the following presentation: The set of generators consists of $a_1,\dots,a_g$, $d_{1,2}$, s_1, $t_1,\dots,t_{g-1}$, and $r_{i,j}$ for $(i,j) \in \tilde{I}$. The set of defining relations is:*

(P1) $[a_i,a_j] = 1$, $[a_i,d_{j,k}] = 1$, *for all* $i,j,k \in I_0$,

(P2) *Let* $i,j,r,s \in I_0$.

- (a) $d_{r,s}^{-1} * d_{i,j} = d_{i,j}$ *if* $r < s < i < j$ *or* $i < r < s < j$,
- (b) $d_{r,i}^{-1} * d_{i,j} = d_{r,j} * d_{i,j}$ *if* $r < i < j$,
- (c) $d_{i,s}^{-1} * d_{i,j} = (d_{i,j}d_{s,j}) * d_{i,j}$ *if* $i < s < j$,
- (d) $d_{r,s}^{-1} * d_{i,j} = [d_{r,j},d_{s,j}] * d_{i,j}$ *if* $r < i < s < j$,

(P3) $d_{I_0} = 1$,

(P4) $d_{I_k} = a_{|k|}$ *where* $I_k = I_0 - \{k\}$ *for* $k \in I_0$,

(P5) $t_it_{i+1}t_i = t_{i+1}t_it_{i+1}$ *for* $i = 1,\dots,g-2$, *and* $[t_i,t_j] = 1$ *if* $1 \le i < j-1 < g-1$,

(P6) $t_i^2 = d_{i,i+1}d_{-i-1,-i}a_i^{-2}a_{i+1}^{-2}$ *for* $i = 1,\dots,g-1$, (P7) $[s_1, a_i] = 1$ *for* $i = 1, \dots, g$, $t_i * a_i = a_{i+1}$ *for* $i = 1, \dots, g-1$, $[a_i, t_j] = 1$ *for* $i, j \in I_0$ *satisfying* $j \neq i, i-1$, *and* $[t_i, s_1] = 1$ *for* $i = 2, \dots, g-1$,

(P8) $[s_1, d_{2,3}] = 1$, $[s_1, d_{-2,2}] = 1$, $s_1t_1s_1t_1 = t_1s_1t_1s_1$, *and* $[t_i, d_{1,2}] = 1$ *for* $i = 1, 3, \dots, g-1$,

(P9) $r_{i,j}^2 = s_jc_{i,j}s_jc_{i,j}^{-1}$ *for* $(i, j) \in \tilde{I}$,

(P10) *Let* $(i, j) \in \tilde{I}$.

(a) $r_{i,j} * a_j = c_{i,j}$ *and* $[r_{i,j}, a_k] = 1$ *if* $k \neq j$,

(b) $[r_{i,j}, t_k] = 1$ *if* $k \neq |i|, j$ *or* $k = i = 1 < j-1$,

(c) $[r_{i,j}, s_k] = 1$ *if* $k < |i|$, $j < k$ *or* $k = -i$,

(d) $[r_{i,j}, d_{k,m}] = 1$ *if* $k, m \in \{i, \dots, j-1\}$ *or* $k, m \notin \{-j, i, i+1, \dots, j\}$,

(e) $[r_{i,j}, z_j] = 1$ *if* $(i, j) = (1, g)$ *or* $j = i + g$,

(f) $r_{i,j} * d_{i,j} = d_J$ *where* $J = \{k \in I_0; i < k \leq j\}$,

(g) $r_{1,j} * d_{-j,1-j} = (t_{j-2}t_{j-3}\cdots t_1) * c_{-1,j}$,

(h) $r_{i,j} * d_{-j,1-j} = (t_{j-2}t_{j-3}\cdots t_{1-i}) * c_{i-1,j}$ *if* $i < 0$ *and* $j + i > 1$,

(i) $r_{i,j}^{-1} * d_{-j-1,-j} = s_{j+1}^{-1} * c_{i,j+1}$ *if* $j < g$,

(P11) $r_{i,j} * t_{j-1} = t_{j-1}^{-1} * r_{i,j}$ *if* $(i, j) \in \tilde{I}$ *and* $-i + 1 \neq j$,

(P12) (a) *Let* $h_2 = k_{j-1}^{-1}t_{j-2}^{-1}t_{j-3}^{-1}\cdots t_1^{-1}k_{j-1}k_{j-2}\cdots k_2$.

$$r_{1,j} = s_jc_{1,j}s_jc_{1,j}^{-1}k_{j-1}a_jc_{1,j-2}t_{j-1}c_{1,j-1}^{-1}t_{j-1}^{-1}r_{1,j-1}^{-1}s_{j-1}h_2r_{1,2}^{-1}h_2^{-1}k_{j-1}^{-1}$$

for $3 \leq j \leq g$.

(b) *Let* $h_3 = s_1k_{j-1}k_{j-2}\cdots k_2$.

$$r_{-1,j} = h_3r_{1,2}^{-1}h_3^{-1}s_jr_{1,j}^{-1}c_{-1,j-1}^{-1}c_{1,j-1}a_1s_jc_{-1,j}s_jc_{-1,j}^{-1}$$

for $2 \leq j \leq g-1$.

(c) *Let* $h_3 = s_{-i}t_{-1-i}^{-1}t_{-2-i}^{-1}\cdots t_1^{-1}k_{j-1}k_{j-2}\cdots k_3k_2$.

$$r_{i,j} = h_3r_{1,2}^{-1}h_3^{-1}s_jr_{i+1,j}^{-1}c_{i,j-1}^{-1}c_{i+1,j-1}a_{-i}s_jc_{i,j}s_jc_{i,j}^{-1}$$

for $i < -1$ *and* $(i, j) \in \tilde{I}$.

REMARK 5.2. Note that there are some mistakes in the Wajnryb's presentation in [26]. The mapping class z_j is defined as the conjugation of z by $k_{j-1}k_{j-2}\cdots k_{g+1-j}$ in [26]. However, as mentioned in [20], it should be defined as the product $k_{j-1}k_{j-2}\cdots k_{g+1-j}z$. In (P11), the condition $-i+1 \neq j$ is needed. The relations of type (P11) are obtained in the situation when the pair of simple closed curves ∂_k and ∂_{-k} are separated by $\gamma_{i,j}$ for $k = j, j-1$ (see CASE 1 in [26, p.223]), and the equation $r_{-(j-1),j} * t_{j-1} = t_{j-1}^{-1} * r_{-(j-1),j}$ in fact does not hold for any $2 \leq j \leq g$. We also erase the relation $s_1^2 = d_{-1,1}a_1^{-4}$ in (P6) written in [26]. This is because we already defined $d_{-1,1}$ as $s_1^2a_1^4$.

5.2. Action on the first homology $H_1(\Sigma_g)$. Here, we compute the action of the handlebody mapping class group $\mathcal{H}_g$ on the first homology $H_1(\Sigma_g)$ of the boundary surface. Recall that $x_1, \dots, x_g, y_1, \dots, y_g$ are the homology classes represented by the simple closed curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ in Figures 1 and 5.

Lemma 5.3. *For* $1 \leq i \leq g$,

$$a_i(x_l) = x_l, \quad a_i(y_l) = \begin{cases} x_i + y_i & \textit{if } l = i, \\ y_l & \textit{otherwise}, \end{cases}$$

and

$$s_i(x_l) = \begin{cases} -x_i & \textit{if } l = i, \\ x_l & \textit{otherwise}, \end{cases} \qquad s_i(y_l) = \begin{cases} 2x_i - y_i & \textit{if } l = i, \\ y_l & \textit{otherwise}. \end{cases}$$

For each $i, j \in I_0$ *such that* $i < j$,

$$d_{i,j}(x_l) = x_l, \quad d_{i,j}(y_l) = \begin{cases} \varepsilon(i)x_{|i|} + \varepsilon(j)x_{|j|} + y_l & \textit{if } l = |i|, |j|, \\ y_l & \textit{otherwise}, \end{cases}$$

where $\varepsilon(i) = 1$ *if* $i > 0$, *and* $\varepsilon(i) = -1$ *if* $i < 0$.

For $1 \le i \le g - 1$,

$$t_i(x_l) = \begin{cases} x_{i+1} & \textit{if } l = i, \\ x_i & \textit{if } l = i+1, \\ x_l & \textit{otherwise}, \end{cases} \qquad t_i(y_l) = \begin{cases} x_i + y_{i+1} & \textit{if } l = i, \\ x_{i+1} + y_i & \textit{if } l = i+1, \\ y_l & \textit{otherwise}, \end{cases}$$

and

$$k_i(x_l) = \begin{cases} x_{i+1} & \textit{if } l = i, \\ x_i & \textit{if } l = i+1, \\ x_l & \textit{otherwise}, \end{cases} \qquad k_i(y_l) = \begin{cases} y_{i+1} & \textit{if } l = i, \\ y_i & \textit{if } l = i+1, \\ y_l & \textit{otherwise}. \end{cases}$$

For $1 < j \le g$,

$$r_{1,j}(x_l) = \begin{cases} -x_1 - \cdots - x_j & \textit{if } l = j, \\ x_l & \textit{otherwise}, \end{cases}$$

$$r_{1,j}(y_l) = \begin{cases} x_1 + \cdots + x_j + y_l - y_j & \textit{if } 1 \le l \le j-1, \\ x_1 + \cdots + x_{j-1} + 2x_j - y_j & \textit{if } l = j, \\ y_l & \textit{otherwise}, \end{cases}$$

and for $(i, j) \in \tilde{I}$ *such that* $i < 0$,

$$r_{i,j}(x_l) = \begin{cases} -x_{-i+1} - \cdots - x_j & \textit{if } l = j, \\ x_l & \textit{otherwise}, \end{cases}$$

$$r_{i,j}(y_l) = \begin{cases} x_{-i+1} + \cdots + x_j + y_l - y_j & \textit{if } -i+1 \le l \le j-1, \\ x_{-i+1} + \cdots + x_{j-1} + 2x_j - y_j & \textit{if } l = j, \\ y_l & \textit{otherwise}. \end{cases}$$

Proof. The equations for the mapping classes a_i and $d_{i,j}$ are obvious because a_i and $d_{i,j}$ are Dehn twists along α_i and $\delta_{i,j}$ respectively. Similarly we have

$$b_i(x_l) = \begin{cases} x_i - y_i & \text{if } l = i, \\ y_l & \text{otherwise}, \end{cases} \qquad b_i(y_l) = y_l,$$

for $1 \le i \le g$ and

$$e_i(x_l) = \begin{cases} x_i - y_i + y_{i+1} & \text{if } l = i, \\ x_{i+1} + y_i - y_{i+1} & \text{if } l = i+1, \\ x_l & \text{otherwise}, \end{cases} \qquad e_i(y_l) = y_l,$$

for $1 \leq i \leq g-1$.

Since $t_i = e_i a_i a_{i+1} e_i$, we have

$$\begin{aligned} t_i(x_i) &= (e_i a_i a_{i+1})(x_i - y_i + y_{i+1}) = e_i(x_{i+1} - y_i + y_{i+1}) = x_{i+1}, \\ t_i(x_{i+1}) &= (e_i a_i a_{i+1})(x_{i+1} + y_i - y_{i+1}) = e_i(x_i + y_i - y_{i+1}) = x_i, \\ t_i(y_i) &= (e_i a_i a_{i+1})(y_i) = e_i(x_i + y_i) = x_i + y_{i+1}, \\ t_i(y_{i+1}) &= (e_i a_i a_{i+1})(y_{i+1}) = e_i(x_{i+1} + y_{i+1}) = x_{i+1} + y_i \end{aligned}$$

and t_i acts trivially on other x_l's and y_l's.

Since $k_i = a_i a_{i+1} t_i d_{i,i+1}^{-1}$, we have

$$\begin{aligned} k_i(x_i) &= (a_i a_{i+1} t_i)(x_i) = (a_i a_{i+1})(x_{i+1}) = x_{i+1}, \\ k_i(x_{i+1}) &= (a_i a_{i+1} t_i)(x_{i+1}) = (a_i a_{i+1})(x_i) = x_i, \\ k_i(y_i) &= (a_i a_{i+1} t_i)(-x_i - x_{i+1} + y_i) = (a_i a_{i+1})(-x_{i+1} + y_{i+1}) = y_{i+1}, \\ k_i(y_{i+1}) &= (a_i a_{i+1} t_i)(-x_i - x_{i+1} + y_{i+1}) = (a_i a_{i+1})(-x_i + y_i) = y_i, \end{aligned}$$

and k_i acts trivially on other x_l's and y_l's.

Since $s_1 = b_1 a_1^2 b_1$, we have

$$\begin{aligned} s_1(x_1) &= (b_1 a_1^2)(x_1 - y_1) = b_1(-x_1 - y_1) = -x_1, \\ s_1(y_1) &= (b_1 a_1^2)(y_1) = b_1(2x_1 + y_1) = 2x_1 - y_1, \end{aligned}$$

and s_1 acts trivially on other x_l's and y_l's. The elements s_i's are inductively defined by the recurrence relation $s_{i+1} = k_i s_i k_i^{-1}$. The element k_i replaces x_i and x_{i+1} with each other and y_i and y_{i+1} also. Hence the equation for s_i follows by induction.

Lastly, we verify the equations for $r_{i,j}$. In the case $0 < i < j$, we have

$$c_{i,j}(x_l) = x_l, \quad c_{i,j}(y_l) = \begin{cases} x_i + \cdots + x_j + y_l & \text{if } i \leq l \leq j, \\ y_l & \text{otherwise.} \end{cases}$$

Since $r_{i,j} = b_j a_j c_{i,j} b_j$, we have

$$\begin{aligned} r_{1,j}(x_j) &= (b_j a_j c_{1,j})(x_j - y_j) \\ &= (b_j a_j)(-x_1 - \cdots - x_{j-1} - y_j) \\ &= -x_1 - \cdots - x_j, \end{aligned}$$

and for $1 \leq l \leq j$

$$\begin{aligned} r_{1,j}(y_l) &= (b_j a_j c_{1,j})(y_l) \\ &= (b_j a_j)(x_1 + \cdots + x_j + y_l) \\ &= \begin{cases} x_1 + \cdots + x_j + y_l - y_j & \text{if } 1 \leq l \leq j-1, \\ x_1 + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j. \end{cases} \end{aligned}$$

The element $r_{1,j}$ acts trivially on other x_l's and y_l's.

In the case $(i, j) \in \tilde{I}$ and $i < 0$, we have

$$c_{i,j}(x_l) = x_l, \quad c_{i,j}(y_l) = \begin{cases} x_{-i+1} + \cdots + x_j + y_l & \text{if } -i+1 \leq l \leq j, \\ y_l & \text{otherwise.} \end{cases}.$$

Hence we have

$$\begin{aligned} r_{i,j}(x_j) &= (b_j a_j c_{i,j})(x_j - y_j) \\ &= (b_j a_j)(-x_{-i+1} - \cdots - x_{j-1} - y_j) \\ &= -x_{-i+1} - \cdots - x_j, \end{aligned}$$

and for $-i+1 \leq l \leq j$

$$\begin{aligned} r_{i,j}(y_l) &= (b_j a_j c_{i,j})(y_l) \\ &= (b_j a_j)(x_{-i+1} + \cdots + x_j + y_l) \\ &= \begin{cases} x_{-i+1} + \cdots + x_j + y_l - y_j & \text{if } -i+1 \leq l \leq j-1, \\ x_{-i+1} + \cdots + x_{j-1} + 2x_j - y_j & \text{if } l = j. \end{cases} \end{aligned}$$

The element $r_{i,j}$ acts trivially on other x_l's and y_l's. □

6. Proof of Theorem 1.1 for $g = 2, 3$

In this section, we prove Theorem 1.1 for $g = 2, 3$. We denote by A the ring $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for an integer $n \geq 2$, and $H_A = H_1(\Sigma_g; A)$. Recall that, for a group G and a left G-module M, a map $d : G \to M$ is called a crossed homomorphism if it satisfies $d(hh') = d(h) + hd(h')$ for $h, h' \in G$. As explained in Remark 2.1, if we consider the normalized bar resolution of G, the set of 1-cocycles $Z^1(G; M)$ is identified with the set of crossed homomorphisms from G to M, and the set of 1-coboundaries $B^1(G; M)$ is identified with the image of the coboundary map $\delta : M \to Z^1(G; M)$ defined by $\delta(m)(h) = hm - m$ for $m \in M$.

As written in [26, Theorem 19], the handlebody mapping class group $\mathcal{H}_g$ is generated by a_1, s_1, $r_{1,2}$, t_1, and $u = t_1 t_2 \cdots t_{g-1}$. Therefore, crossed homomorphisms $d : \mathcal{H}_g \to H_A$ are uniquely determined by the values $d(a_1)$, $d(s_1)$, $d(r_{1,2})$, $d(t_1)$, and $d(u)$. Moreover, a 5-tuple of elements in H_A becomes values of a_1, s_1, $r_{1,2}$, t_1, and u under some crossed homomorphism d on $\mathcal{H}_g$ if and only if they are compatible with the relations (P1)-(P12) in Theorem 5.1. The basis $\{x_1, \ldots, x_g, y_1, \ldots, y_g\}$ of H_A induces an isomorphism $H_A \cong A^{2g}$. For $v \in H_A$, we denote its projection to the i-th coordinate of A^{2g} by $v_i \in A$ for $i = 1, 2, \ldots, 2g$.

Lemma 6.1.

$$\begin{aligned} H^1(\mathcal{H}_2; H_A) &\cong \{d \in Z^1(\mathcal{H}_2; H_A); d(r_{1,2})_1 = d(s_1)_3 - d(r_{1,2})_4 = d(u)_2 = d(u)_4 = 0\}, \\ H^1(\mathcal{H}_3; H_A) &\cong \{d \in Z^1(\mathcal{H}_3; H_A); \\ &\qquad d(r_{1,2})_1 = d(s_1)_4 - d(r_{1,2})_5 = d(u)_2 = d(u)_3 = d(u)_5 = d(u)_6 = 0\}. \end{aligned}$$

Proof. Let A, B, and C are modules, and $g : A \to B$ and $f : B \to C$ are homomorphisms such that $f \circ g : A \to C$ is an isomorphism. Then it is easy to show that the composition map of the natural inclusion $\operatorname{Ker} f \to B$ and the projection $B \to B/\operatorname{Im} g$ induces an isomorphism $\operatorname{Ker} f \cong B/\operatorname{Im} g$. We use this fact below.

Let $f_2 \colon Z^1(\mathcal{H}_2; H_A) \to A^4$ and $f_3 \colon Z^1(\mathcal{H}_3; H_A) \to A^6$ be homomorphisms defined by

$$\begin{aligned} f_2(d) &= (d(r_{1,2})_1, d(s_1)_3 - d(r_{1,2})_4, d(u)_2, d(u)_4), \text{ and} \\ f_3(d) &= (d(r_{1,2})_1, d(s_1)_4 - d(r_{1,2})_5, d(u)_2, d(u)_3, d(u)_5, d(u)_6), \end{aligned}$$

respectively. Then the composition maps $f_g \circ \delta : H_A \to A^{2g}$ are written as

$$f_2 \circ \delta(v) = (-v_2 + v_3 + v_4, -v_3 + 2v_4, v_1 - v_2 + v_4, v_3 - v_4),$$
$$f_3 \circ \delta(v) = (-v_2 + v_4 + v_5, -v_4 + 2v_5, v_1 - v_2 + v_6, v_2 - v_3 + v_6, v_4 - v_5, v_5 - v_6),$$

for $v \in H_A$. Since these maps are isomorphisms, we have

$$H^1(\mathcal{H}_g; H_A) = Z^1(\mathcal{H}_g; H_A)/B^1(\mathcal{H}_g; H_A) \cong \operatorname{Ker} f_g$$

for $g = 2, 3$. □

Lemma 6.2. *Suppose $d \in Z^1(\mathcal{H}_g; H_A)$ satisfies $d(u)_2 = \cdots = d(u)_g = d(u)_{g+2} = \cdots = d(u)_{2g} = 0$ as in* Lemma 6.1. *Then,*

(1) $d(a_i) = u^{i-1} d(a_1)$.
(2) $d(t_i) = u^{i-1} d(t_1)$.

Proof. Note that $a_i = u^{i-1} a_1 u^{-(i-1)}$. It can be checked using the relation (P7). Hence we have

$$d(a_{i+1}) = d(u) + ud(a_i) - ua_i u^{-1} d(u) = d(u) + ud(a_i) - a_{i+1} d(u).$$

Since $(a_{i+1}v)_1 = v_1$ and $(a_{i+1}v)_{g+1} = v_{g+1}$ for any $v \in H_A$, we have $a_{i+1}d(u) = d(u)$, and thus $d(a_{i+1}) = ud(a_i)$. By induction on i, we have the equation (1). The equation (2) can be similarly verified. □

Lemma 6.3. *Suppose $d \in Z^1(\mathcal{H}_g; H_A)$ satisfies $d(u)_2 = \cdots = d(u)_g = d(u)_{g+2} = \cdots = d(u)_{2g} = 0$ as in* Lemma 6.1. *Then*

(1) $d(a_1)_{g+1} = \cdots = d(a_1)_{2g} = 0$.
(2) $2d(a_1)_2 = \cdots = 2d(a_1)_g = 0$.
(3) $d(s_1)_{g+2} = \cdots = d(s_1)_{2g} = 0$.
(4) $d(s_1)_{g+1} = -2d(a_1)_1$.
(5) $d(a_1)_2 + d(r_{1,2})_{g+1} = 0$.

Proof. For any i and j,

$$d(a_i a_j) = d(a_i) + a_i d(a_j) = d(a_i) + d(a_j) + d(a_j)_{g+i} x_i.$$

Since a_1 and a_i commute for any $1 \leq i \leq g$ by the relation (P1), it must be $d(a_1 a_i) = d(a_i a_1)$, and thus $d(a_1)_{g+i} = 0$ for any $2 \leq i \leq g$.

Since a_1 and $r_{1,2}$ commute by the relation (P10)(a), it must be

$$(1 - a_1)d(r_{1,2}) = (1 - r_{1,2})d(a_1).$$

Since $((1 - r_{1,2})v)_{g+2} = v_{g+1} + 2v_{g+2}$ for any $v \in H_A$ while $((1 - a_1)v)_{g+2} = 0$, we have $d(a_1)_{g+1} = 0$ and thus the equation (1). Since $((1 - r_{1,2})v)_1 = v_2 - v_{g+1} - v_{g+2}$ for any $v \in H_A$ while $((1 - a_1)v)_1 = -v_{g+1}$, we have the equation (5).

Note that a_i and s_j commute for any $1 \leq i, j \leq g$. It can be verified using the relations (P1) and (P7). Hence it must be

$$(1 - s_j)d(a_i) = (1 - a_i)d(s_j).$$

Suppose $i = 1$ and $2 \leq j \leq g$. Then we have the equation (2) because $((1-s_j)v)_j = 2v_j - 2v_{g+j}$ and $((1-a_1)v)_j = 0$ for any $v \in H_A$. Suppose $2 \leq i \leq g$ and $j = 1$. Then we have the equation

(3) because $((1-s_1)v)_i = 0$ and $((1-a_i)v)_i = -v_{g+i}$ for any $v \in H_A$. Suppose $i = j = 1$. Then we have the equation (4) because $((1-s_1)v)_1 = 2v_1 - 2v_{g+1}$ and $((1-a_1)v)_1 = -v_{g+1}$ for any $v \in H_A$. □

6.1. $H^1(\mathcal{H}_2; H_A)$**.** Here, we assume $g = 2$ and prove that $H^1(\mathcal{H}_2; H_A) \cong \mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^2, A)$ for $A = \mathbb{Z}$ and $\mathbb{Z}/n$. Since $H_A \cong \mathrm{Hom}_{\mathbb{Z}}(H, A)$ as an $\mathcal{H}_2$-module, and $\mathrm{Ext}^1_{\mathbb{Z}}(H_0(\mathcal{H}_2; H_A), A) = 0$, the universal coefficient theorem (Theorem 2.3) implies

$$\mathrm{Hom}_{\mathbb{Z}}(H_1(\mathcal{H}_2; H), A) \cong H^1(\mathcal{H}_2; \mathrm{Hom}_{\mathbb{Z}}(H, A)) \cong \mathrm{Hom}((\mathbb{Z}/2\mathbb{Z})^2, A).$$

By the structure theorem for finitely generated abelian groups, we have $H_1(\mathcal{H}_2; H) \cong (\mathbb{Z}/2\mathbb{Z})^2$, and we complete the proof of Theorem 1.1 when $g = 2$.

Let $d \in Z^1(\mathcal{H}_2; H_A)$ be a crossed homomorphism satisfying the condition $d(r_{1,2})_1 = d(s_1)_3 - d(r_{1,2})_4 = d(u)_2 = d(u)_4 = 0$ as in Lemma 6.1. Note that in this case $u = t_1$. By Lemma 6.3, we can set

$$\begin{aligned} d(a_1) &= w_{1,1}x_1 + w_{1,2}x_2, \\ d(s_1) &= w_{2,1}x_1 + w_{2,2}x_2 + w_{2,3}y_1, \\ d(t_1) &= w_{3,1}x_1 + w_{3,3}y_1, \\ d(r_{1,2}) &= w_{4,2}x_2 + w_{4,3}y_1 + w_{4,4}y_2. \end{aligned}$$

By the condition on d and Lemma 6.3, we also have

$$2w_{1,2} = 0, w_{2,3} = w_{4,4} = -2w_{1,1}, \text{ and } w_{1,2} + w_{4,3} = 0. \tag{6.1}$$

Lemma 6.4.

$$w_{1,2} = w_{4,3} = 0.$$

Moreover, we have

$$d(d_{1,2}) = w_{1,1}(x_1 + x_2).$$

Proof. Since $w_{1,2} + w_{4,3} = 0$, it suffices to prove that $w_{4,3} = 0$. Note that by Lemma 6.2, we have $d(a_2) = t_1 d(a_1) = w_{1,2}x_1 + w_{1,1}x_2$. Since $d_{1,2} = r_{1,2}a_2r_{1,2}^{-1}$ by the relation (P10)(a),

$$\begin{aligned} d(d_{1,2}) &= d(r_{1,2}) + r_{1,2}d(a_2) - r_{1,2}a_2r_{1,2}^{-1}d(r_{1,2}) \\ &= (1 - d_{1,2})d(r_{1,2}) + r_{1,2}d(a_2) \\ &= (-w_{1,1} - w_{4,3} - w_{4,4})(x_1 + x_2) + w_{1,2}x_1. \end{aligned}$$

Since $a_2 = r_{1,2}d_{1,2}r_{1,2}^{-1}$ by the relation (P10)(f),

$$d(a_2) = (1 - a_2)d(r_{1,2}) + r_{1,2}d(d_{1,2}) = w_{1,2}x_1 + (w_{1,1} + w_{4,3})x_2.$$

Thus we obtain $w_{4,3} = 0$. By the equation $w_{4,4} = -2w_{1,1}$ in (6.1), we have $d(d_{1,2}) = w_{1,1}(x_1 + x_2)$. □

Lemma 6.5.

$$w_{2,3} = w_{3,1} = w_{3,3} = w_{4,4} = 0.$$

In particular, we have $d(t_1) = 0$ *and* $2w_{1,1} = 0$.

Proof. Recall that $d_{-2,-1} = (s_1t_1s_1)^{-1}d_{1,2}(s_1t_1s_1)$. In the case $g = 2$, the Dehn twist $d_{-2,-1}$ coincides with $d_{1,2}$. Hence the elements $d_{1,2}$ and $s_1t_1s_1$ commute and it must be $(1 - d_{1,2})d(s_1t_1s_1) = (1 - s_1t_1s_1)d(d_{1,2})$. Since

$$((1 - d_{1,2})d(s_1t_1s_1))_1 = ((1 - d_{1,2})d(s_1t_1s_1))_2 = -2w_{2,3} + w_{3,3} \text{ while}$$
$$((1 - s_1t_1s_1)d(d_{1,2}))_1 = ((1 - s_1t_1s_1)d(d_{1,2}))_2 = 2w_{1,1},$$

we have $2w_{1,1} = -2w_{2,3} + w_{3,3}$. The equation $w_{2,3} = -2w_{1,1}$ in (6.1) shows $w_{2,3} = w_{3,3}$.

Since $w_{2,3} = w_{3,3} = w_{4,4}$, it remains to prove that $w_{3,1} = w_{3,3} = 0$. Note that each of $d(a_1)$, $d(a_2)$, and $d(d_{1,2})$ are in $L_A = \mathrm{Ker}(H_1(\Sigma_2; A) \to H_1(H_2; A))$. By the relation (P6),

$$t_1^2 = d_{1,2}^2 a_1^{-2} a_2^{-2}.$$

Since each of a_1, a_2 and $d_{1,2}$ acts on L_A trivially, we have

$$d(d_{1,2}^2 a_1^{-2} a_2^{-2}) = 2(d(d_{1,2}) - d(a_1) - d(a_2)) = 0.$$

On the other hand, we have

$$d(t_1^2) = d(t_1) + t_1 d(t_1) = w_{3,1}(x_1 + x_2) + w_{3,3}(x_3 + x_4) + w_{3,3}x_1.$$

These equations show $w_{3,1} = w_{3,3} = 0$. □

Lemma 6.6.

$$w_{4,2} = 0.$$

In particular, we have $d(r_{1,2}) = 0$.

Proof. The relation $t_1r_{1,2}t_1 = r_{1,2}t_1r_{1,2}$ in (P11) shows

$$(1 - t_1 + r_{1,2}t_1)d(r_{1,2}) = (1 - r_{1,2} + t_1r_{1,2})d(t_1).$$

By Lemma 6.5, the right hand side is equal to zero. Since

$$(1 - t_1 + r_{1,2}t_1)d(r_{1,2}) = w_{4,2}x_2,$$

we have $w_{4,2} = 0$. □

Lemma 6.7.

$$2w_{2,2} = 0.$$

Proof. The relation $r_{1,2}^2 = s_2d_{1,2}s_2d_{1,2}^{-1}$ in (P9) and Lemma 6.6 show

$$d(s_2d_{1,2}s_2d_{1,2}^{-1}) = d(r_{1,2}^2) = 0.$$

Recall that $k_1 = a_1a_2t_1d_{1,2}^{-1}$ and $s_2 = k_1s_1k_1^{-1}$. Hence we have

$$d(k_1) = d(a_1) + d(a_2) - t_1d(d_{1,2}) = 0, \text{ and}$$
$$d(s_2) = d(k_1) + k_1d(s_1) - s_2d(k_1) = k_1d(s_1).$$

Therefore, we have

$$d(s_2d_{1,2}s_2d_{1,2}^{-1}) = (1 + s_2d_{1,2})d(s_2) + (s_2 - 1)d(d_{1,2}) = 2w_{2,2}x_1$$

and thus $2w_{2,2} = 0$. □

Lemma 6.8.

$$w_{2,1} = w_{2,2}.$$

Proof. Recall that $z = a_1a_2s_1t_1s_1d_{1,2}$ and $z_2 = k_1z$. Hence we have

$$\begin{aligned} d(z) &= d(a_1) + d(a_2) + (1 + s_1t_1)d(s_1) + s_1t_1s_1d(d_{1,2}) \\ &= (w_{2,1} + w_{2,2})(x_1 + x_2), \text{ and} \\ d(z_2) &= d(k_1) + k_1d(z) = d(z). \end{aligned}$$

Hence

$$(1 - r_{1,2})d(z_2) = (w_{2,1} + w_{2,2})(x_1 + 2x_2).$$

Since $r_{1,2}$ and z_2 commute by the relation (P10)(e), it must be $(1-r_{1,2})d(z_2) = (1-z_2)d(r_{1,2}) = 0$. Thus we have $w_{2,1} = w_{2,2}$. □

Summarizing Lemmas 6.4, 6.5, 6.6, 6.7 and 6.8, we have

$$d(a_1) = w_{1,1}x_1, d(s_1) = w_{2,1}(x_1 + x_2) \text{ and } d(t_1) = d(r_{1,2}) = 0,$$

where $2w_{1,1} = 2w_{2,1} = 0$. It can be verified that such d is compatible with the relations (P1)–(P12). Now we have

$$H^1(\mathcal{H}_2; H_A) \cong \operatorname{Ker} f_2 \cong \{(w_{1,1}, w_{2,1}) \in A^2; 2w_{1,1} = 2w_{2,1} = 0\}. \tag{6.2}$$

This completes the proof of Theorem 1.1 in the case $g = 2$.

Proposition 6.9.

$$H_1(\mathcal{H}_2; L) \cong \mathbb{Z}/2\mathbb{Z}, H_1(\mathcal{H}_2; H/L) \cong H_1(\mathcal{H}_2; H),$$

and the homomorphism $H_2(\mathcal{H}_2; H/L) \to H_1(\mathcal{H}_2; L)$ *induced by the exact sequence* $0 \to L \to H \to H/L \to 0$ *is surjective.*

Proof. As well as Lemma 6.1, we can verify that

$$H^1(\mathcal{H}_2; H_A/L_A) \cong \{d' \in Z^1(\mathcal{H}_2; H_A/L_A); d'(s_1)_1 - d'(r_{1,2})_2 = d'(u)_2 = 0\}.$$

Since $H_A/L_A = (H/L)\otimes A = \operatorname{Hom}_{\mathbb{Z}}(L, A)$, the universal coefficient theorem (Theorem 2.3) implies that $\operatorname{Hom}(H_1(\mathcal{H}_2; L), A) \cong \operatorname{Hom}(\mathbb{Z}/2\mathbb{Z}, A)$.

Next, we prove that $H_1(\mathcal{H}_2; H/L) \cong H_1(\mathcal{H}_2; H)$ and the homomorphism $H_2(\mathcal{H}_2; H/L) \to H_1(\mathcal{H}_2; L)$ is surjective. Since $H_0(\mathcal{H}_2; L) = L_{\mathcal{H}_2} = 0$ as shown in Lemma 3.2, we have the exact sequence

$$H_2(\mathcal{H}_2; H/L) \longrightarrow H_1(\mathcal{H}_2; L) \longrightarrow H_1(\mathcal{H}_2; H) \longrightarrow H_1(\mathcal{H}_2; H/L) \longrightarrow 0.$$

Thus it suffices to show that the homomorphism $H_1(\mathcal{H}_2; L) \to H_1(\mathcal{H}_2; H)$ is the zero map.

As we saw in Equation (6.2), $\operatorname{Ker}(f_2 \colon Z^1(\mathcal{H}_2; H_A) \to A^4)$ is trivial when $A = \mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ for an odd integer n, and is generated by the crossed homomorphisms $d_1, d_2 : \mathcal{H}_2 \to H_A$

defined by

$$d_1(a_1) = x_1, d_1(s_1) = d_1(t_1) = d_1(r_{1,2}) = 0,$$
$$d_2(s_1) = x_1 + x_2, d_2(a_1) = d_2(s_1) = d_2(t_1) = d_2(r_{1,2}) = 0,$$

when $A = \mathbb{Z}/n\mathbb{Z}$ for an even integer n. Since they are in the image of $Z^1(\mathcal{H}_2; L_A) \to Z^1(\mathcal{H}_2; H_A)$, the homomorphism $H^1(\mathcal{H}_2; L_A) \to H^1(\mathcal{H}_2; H_A)$ induced by the inclusion $L_A \to H_A$ is surjective. The universal coefficient theorem (Theorem 2.3) implies that

$$\mathrm{Hom}(H_1(\mathcal{H}_2; H/L), A) \to \mathrm{Hom}(H_1(\mathcal{H}_2; H), A)$$

is surjective, and if we put $A = H_1(\mathcal{H}_2; H)$, we see that the homomorphism $H_1(\mathcal{H}_2; H) \to H_1(\mathcal{H}_2; H/L)$ is injective. □

6.2. $H^1(\mathcal{H}_3; H_A)$**.** Here, we assume $g = 3$ and prove that $H^1(\mathcal{H}_3; H_A) \cong \mathrm{Hom}(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, A)$. Then the universal coefficient theorem (Theorem 2.3) implies

$$\mathrm{Hom}(H_1(\mathcal{H}_3; H), A) \cong H^1(\mathcal{H}_3; \mathrm{Hom}(H, A)) \cong \mathrm{Hom}(\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, A).$$

By the structure theorem for finitely generated abelian groups, we have $H_1(\mathcal{H}_3; H) \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and complete the proof of Theorem 1.1 when $g = 3$.

Let $d \in Z^1(\mathcal{H}_3; H_A)$ be a crossed homomorphism satisfying the condition $d(r_{1,2})_1 = d(s_1)_4 - d(r_{1,2})_5 = d(u)_2 = d(u)_3 = d(u)_5 = d(u)_6 = 0$ as in Lemma 6.1. By Lemma 6.3, we can set

$$\begin{aligned}
d(a_1) &= w_{1,1}x_1 + w_{1,2}x_2 + w_{1,3}x_3,\\
d(s_1) &= w_{2,1}x_1 + w_{2,2}x_2 + w_{2,3}x_3 + w_{2,4}y_1,\\
d(t_1) &= w_{3,1}x_1 + w_{3,2}x_2 + w_{3,3}x_3 + w_{3,4}y_1 + w_{3,5}y_2 + w_{3,6}y_3,\\
d(r_{1,2}) &= w_{4,2}x_2 + w_{4,3}x_3 + w_{4,4}y_1 + w_{4,5}y_2 + w_{4,6}y_3.
\end{aligned}$$

By the condition on d and Lemma 6.3, we also have

$$2w_{1,2} = 2w_{1,3} = 0, w_{2,4} = w_{4,5} = -2w_{1,1}, \text{ and } w_{1,2} + w_{4,4} = 0. \tag{6.3}$$

Lemma 6.10. (1) $w_{1,2} = w_{4,4} = 0$.
(2) $w_{1,3} = 0$.
(3) $2w_{3,3} = 0$.
(4) $w_{3,4} + w_{3,5} = 0$.
(5) $w_{3,6} = 0$.
(6) $d(d_{1,2}) = w_{1,1}(x_1 + x_2)$.

Proof. Since a_1 and $r_{1,3}$ commute by the relation (P10)(a), it must be $(1 - a_1)d(r_{1,3}) = (1 - r_{1,3})d(a_1)$. Since

$$((1 - r_{1,3})d(a_1))_2 = w_{1,3}, \text{ while } ((1 - a_1)d(r_{1,3}))_2 = 0,$$

we have the equation (2).

Since $d(t_2) = ud(t_1) = t_1t_2d(t_1)$ by Lemma 6.2, we have

$$d(t_2) = (w_{3,3} + w_{3,4} + w_{3,5})x_1 + (w_{3,1} + w_{3,6})x_2 + (w_{3,2} + w_{3,6})x_3 + w_{3,6}y_1 + w_{3,4}y_2 + w_{3,5}y_3.$$

Since a_1 and t_2 commute by the relation (P7), it must be $(1 - a_1)d(t_2) = (1 - t_2)d(a_1)$. Since

$$((1 - t_2)d(a_1))_2 = w_{1,2} \text{ while } ((1 - a_1)d(t_2))_2 = 0,$$

we have the equation (1). Furthermore, since

$$((1 - a_1)d(t_2))_1 = -w_{3,6} \text{ while } ((1 - t_2)d(a_1))_1 = 0,$$

we have the equation (5).

Now we have $d(a_1) = w_{1,1}x_1$. Note that by Lemma 6.2, $d(a_i) = u^{i-1}d(a_1) = w_{1,1}x_i$ for any $1 \leq i \leq 3$. Since $d_{1,2} = r_{1,2}a_2r_{1,2}^{-1}$ by the relation (P10)(a), we have

$$d(d_{1,2}) = (1 - d_{1,2})d(r_{1,2}) + r_{1,2}d(a_2) = (-w_{1,1} - w_{4,5})(x_1 + x_2).$$

Since $w_{4,5} = -2w_{1,1}$, we have the equation (6).

Since $d_{1,2}$ and t_1 commute by the relation (P8), it must be $(1-d_{1,2})d(t_1) = (1-t_1)d(d_{1,2}) = 0$. Since $(1 - d_{1,2})d(t_1) = -(w_{3,4} + w_{3,5})(x_1 + x_2)$, we have the equation (4).

Since s_1 and t_2 commute by the relation (P7), it must be $(1 - s_1)d(t_2) = (1 - t_2)d(s_1)$. Since $((1 - s_1)d(t_2))_1 = 2w_{3,3}$ while $((1 - t_2)d(t_1))_1 = 0$, we have the equation (3). □

Since $((1 - t_2)d(s_1))_2 = w_{2,2} - w_{2,3}$ while $((1 - s_1)d(t_2))_2 = 0$, we have

$$w_{2,2} - w_{2,3} = 0. \tag{6.4}$$

Lemma 6.11.

$$w_{3,2} = w_{3,3} \text{ and } w_{3,4} = w_{3,5} = 0.$$

Proof. Since $d(u) = d(t_1t_2) = d(t_1) + t_1d(t_2)$, a straightforward computation shows

$$d(u)_2 = w_{3,2} + w_{3,3} + w_{3,4}, d(u)_3 = w_{3,2} + w_{3,3}, \text{ and } d(u)_5 = d(u)_6 = w_{3,5}.$$

Since $d(u)_2, d(u)_3, d(u)_5, d(u)_6 = 0$, we have

$$w_{3,2} + w_{3,3} = w_{3,4} = w_{3,5} = 0.$$

Lemma 6.10 (3) implies $w_{3,2} = w_{3,3}$. □

Lemma 6.12.

$$w_{2,2} = w_{2,3}, \quad 2w_{2,2} = 2w_{2,3} = 0.$$

Proof. The equation $w_{2,2} = w_{2,3}$ follows from Equation (6.4).

Next, we prove $2w_{2,2} = 2w_{2,3} = 0$. Since s_1 and $r_{-1,2}$ commute by the relation (P10)(c), it must be $(1 - s_1)d(r_{-1,2}) = (1 - r_{-1,2})d(s_1)$. Since

$$((1 - r_{-1,2})d(s_1))_2 = 2w_{2,2} \text{ while } ((1 - s_1)d(r_{-1,2}))_2 = 0,$$

we obtain $2w_{2,2} = 2w_{2,3} = 0$. □

Lemma 6.13.

$$4w_{1,1} = 2w_{2,4} = 2w_{3,1} = 2w_{4,5} = 0.$$

Proof. By a straightforward calculation,

$$d(r_{1,2}^{-1}d_{-3,-2}r_{1,2}) = 3w_{1,1}(x_1 + x_2 - x_3) \text{ and } d(s_3^{-1}c_{1,3}s_3) = -w_{1,1}(x_1 + x_2 - x_3).$$

Since $r_{1,2}^{-1}d_{-3,-2}r_{1,2} = s_3^{-1}c_{1,3}s_3$ by the relation (P10)(i), we have $4w_{1,1} = 0$. As in (P8), t_1 and $s_1t_1s_1$ commute. Thus it must be $(1 - t_1)d(s_1t_1s_1) = (1 - s_1t_1s_1)d(t_1)$. Since

$$\begin{aligned} d(s_1t_1s_1) &= d(s_1) + s_1d(t_1) + s_1t_1d(s_1) \\ &= (w_{2,1} + w_{2,2} - w_{2,4} - w_{3,1})x_1 + (w_{2,1} + w_{2,2} + w_{3,2})x_2 + w_{3,3}x_3 + w_{2,4}(y_1 + y_2), \end{aligned}$$

we have

$$\begin{aligned} &(1 - t_1)d(s_1t_1s_1) = -(w_{3,1} + w_{3,2})(x_1 - x_2) \text{ while} \\ &(1 - s_1t_1s_1)d(t_1) = (w_{3,1} + w_{3,2})(x_1 + x_2). \end{aligned}$$

Hence we have $2w_{3,1} = 0$. □

Lemma 6.14. (1) *$d(d_{i,j}) = w_{1,1}(x_i + x_j)$ for any $1 \le i < j \le 3$.*
(2) *$d(k_i) = d(t_i)$ for $i = 1, 2$.*
(3) *$d(s_{i+1}) = k_id(s_i)$ for $i = 1, 2$.*

Proof. Since $d_{1,3} = t_2d_{1,2}t_2^{-1}$ and $d_{2,3} = t_1d_{1,3}t_1^{-1}$, we have (1).
Since $k_i = a_ia_{i+1}t_id_{i,i+1}^{-1}$ for $i = 1, 2$, we have

$$\begin{aligned} d(k_i) &= d(a_ia_{i+1}t_id_{i,i+1}^{-1}) \\ &= d(a_i) + a_id(a_{i+1}) + a_ia_{i+1}d(t_i) - k_id(d_{i,i+1}) \\ &= d(t_i). \end{aligned}$$

Since $s_{i+1} = k_is_ik_i^{-1}$ for $i = 1, 2$, we have

$$d(s_{i+1}) = d(k_i) + k_id(s_i) - s_{i+1}d(k_i) = k_id(s_i).$$

□

Lemma 6.15.

$$w_{4,2} = 2w_{4,3} = 0.$$

Proof. By the relation $r_{1,2}^2 = s_2d_{1,2}s_2d_{1,2}^{-1}$ in (P9), we have $d(r_{1,2}^2) = d(s_2d_{1,2}s_2d_{1,2}^{-1})$. First, we have

$$d(r_{1,2}^2) = d(r_{1,2}) + r_{1,2}d(r_{1,2}) = (-w_{4,2} + w_{4,5})x_1 + 2w_{4,3}x_3 + 2w_{4,6}y_3.$$

Next, using Lemma 6.14 we obtain

$$\begin{aligned} d(s_2d_{1,2}s_2d_{1,2}^{-1}) &= (1 + s_2d_{1,2})d(s_2) + s_2(1 - d_{1,2}s_2d_{1,2}^{-1})d(d_{1,2}) \\ &= -2w_{1,1}x_2 + w_{2,4}(x_1 + x_2) \\ &= w_{2,4}x_1. \end{aligned}$$

Comparing $d(r_{1,2}^2)$ and $d(s_2d_{1,2}s_2d_{1,2}^{-1})$, we have $w_{4,2} = 2w_{4,3} = 0$. □

Lemma 6.16.

$$w_{3,1} + w_{3,2} = w_{4,5}, w_{3,3} = w_{4,3}, \textit{ and } w_{4,6} = 0.$$

In particular,

$$2w_{3,1} = 2w_{3,2} = 2w_{3,3} = 2w_{4,3} = 0.$$

Proof. The relation $t_1 r_{1,2} t_1 = r_{1,2} t_1 r_{1,2}$ in (P11) shows

$$(1 - t_1 + r_{1,2}t_1)d(r_{1,2}) = (1 - r_{1,2} + t_1 r_{1,2})d(t_1).$$

A straightforward calculation shows

$$(1 - r_{1,2} + t_1 r_{1,2})d(t_1) = (w_{3,1} + w_{3,2})x_2 + w_{3,3}x_3, \text{ and}$$
$$(1 - t_1 + r_{1,2}t_1)d(r_{1,2}) = w_{4,5}x_2 + w_{4,3}x_3 + w_{4,6}y_3.$$

Thus we have $w_{3,1} + w_{3,2} = w_{4,5}$, $w_{3,3} = w_{4,3}$, and $w_{4,6} = 0$. □

Now we have

$$d(a_1) = w_{1,1}x_1,$$
$$d(s_1) = w_{2,1}x_1 + w_{2,2}x_2 + w_{2,3}x_3 + w_{2,4}y_1,$$
$$d(t_1) = w_{3,1}x_1 + w_{3,2}x_2 + w_{3,3}x_3,$$
$$d(r_{1,2}) = w_{4,3}x_3 + w_{4,5}y_2$$

where

$$4w_{1,1} = 2w_{2,2} = 2w_{2,4} = 2w_{3,1} = 2w_{3,2} = 2w_{4,3} = 0,$$
$$w_{2,2} = w_{2,3}, w_{3,2} = w_{3,3} = w_{4,3}, \text{ and } w_{3,1} + w_{3,2} = w_{4,5}.$$

Lemma 6.17.

$$w_{2,1} = 0.$$

Proof. As in (P12), $r_{1,3} = s_3 c_{1,3} s_3 c_{1,3}^{-1} k_2 a_3 a_1 t_2 d_{1,2}^{-1} t_2^{-1} r_{1,2}^{-1} s_2 h_2 r_{1,2}^{-1} h_2^{-1} k_2^{-1}$, where $h_2 = k_2^{-1} t_1^{-1} k_2$. Since

$$d(h_2) = w_{3,1}x_1 + w_{3,2}x_2 + w_{3,3}x_3, \text{ and} \tag{6.5}$$
$$d(s_3 c_{1,3} s_3 c_{1,3}^{-1}) = w_{2,4}(x_1 + x_2),$$

we have

$$d(r_{1,3}) = (-w_{2,1} + w_{2,2})x_1 + (w_{2,3} + w_{3,1})x_2 - w_{2,1}x_3 + w_{4,5}y_3, \text{ and}$$
$$d(r_{1,3}^2) = d(r_{1,3}) + r_{1,3}d(r_{1,3}) = (-w_{2,1} + w_{4,5})x_1 + (w_{2,1} + w_{4,5})x_2$$

by a straightforward calculation. The relation $r_{1,3}^2 = s_3 c_{1,3} s_3 c_{1,2}^{-1}$ in (P9) and Equation (6.5) show that $w_{2,1} = 0$. □

Lemma 6.18.

$$w_{3,2} = w_{3,3} = w_{4,3} = 0.$$

In particular,

$$w_{2,4} = w_{3,1} = w_{4,5} = 2w_{1,1}.$$

Proof. It is sufficient to prove that $w_{3,3} = 0$. Recall that $z = (a_1a_2a_3)s_1t_1t_2s_1t_1s_1c_{1,3}$ and $z_3 = k_2k_1z$. Hence we have

$$d(z) = w_{2,4}(x_1 + x_3 + y_1 + y_2 + y_3) + w_{3,2}x_1 + w_{3,1}x_2 + w_{3,2}x_3,$$

and

$$d(z_3) = w_{2,4}(x_2 + x_3 + y_1 + y_2 + y_3) + w_{3,1}x_1 + w_{3,2}x_2 + w_{3,2}x_3.$$

Since $r_{1,3}$ and z_3 commute by the relation (P10)(e), it must be $(1-r_{1,3})d(z_3) = (1-z_3)d(r_{1,3})$. A straightforward calculation shows

$$(1 - r_{1,3})d(z_3) = w_{3,3}(x_1 + x_2)$$

while $(1 - z_3)d(r_{1,3}) = 0$. Since $2w_{2,4} = 2w_{3,2} = 0$, we obtain $w_{2,4} = w_{3,2} = 0$. □

Summarizing Lemmas 6.10, 6.11, 6.12, 6.13, 6.15, 6.16, 6.17 and 6.18, we have

$$d(a_1) = w_{1,1}x_1, d(s_1) = w_{2,2}(x_2 + x_3) + 2w_{1,1}y_1,$$
$$d(t_1) = 2w_{1,1}x_1, \text{ and } d(r_{1,2}) = 2w_{1,1}y_2$$

where $4w_{1,1} = 2w_{2,2} = 0$. It can be verified that such d is compatible with the relations (P1)–(P12). Now we have

(6.6) $$H^1(\mathcal{H}_3; H_A) \cong \operatorname{Ker} f_3 \cong \{(w_{1,1}, w_{2,2}) \in A^2; 4w_{1,1} = 2w_{2,2} = 0\}.$$

This completes the proof of Theorem 1.1 in the case $g = 3$.

Proposition 6.19.

$$H_1(\mathcal{H}_3; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2,$$

and the image of the homomorphism $H_2(\mathcal{H}_3; H/L) \to H_1(\mathcal{H}_3; L)$ induced by the exact sequence $0 \to L \to H \to H/L \to 0$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. As we saw in Equation (6.6), $\operatorname{Ker} f_3$ is in the image $\operatorname{Im}(Z^1(\mathcal{H}_3; L_A) \to Z^1(\mathcal{H}_3; H_A))$ and the homomorphism $H^1(\mathcal{H}_3; L_A) \to H^1(\mathcal{H}_3; H_A)$ is surjective when $A = \mathbb{Z}/2\mathbb{Z}$. The universal coefficient theorem (Theorem 2.3) implies that the homomorphism

$$\operatorname{Hom}(H_1(\mathcal{H}_3; H/L), A) \to \operatorname{Hom}(H_1(\mathcal{H}_3; H), A).$$

is also surjective, and the image of the above homomorphism contains $(\mathbb{Z}/2\mathbb{Z})^2$. In particular, the order of $H_1(\mathcal{H}_3; H/L)$ is at least 4. On the other hand, the order of $H_1(\mathcal{H}_3; H/L)$ is at most 4 as explained in Remark 4.9. Thus we obtain $H_1(\mathcal{H}_3; H/L) \cong (\mathbb{Z}/2\mathbb{Z})^2$.

Recall that $H_1(\mathcal{H}_3; L) \cong (\mathbb{Z}/2\mathbb{Z})^2$ as shown in Lemma 4.7 and $H_1(\mathcal{H}_3; H) \cong \mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$. Since the coinvariant $H_0(\mathcal{H}_3; L) = L_{\mathcal{H}_3}$ is trivial, the natural homomorphism $H_1(\mathcal{H}_3; H) \to H_1(\mathcal{H}_3; H/L)$ is surjective. Thus we have the exact sequence

$$H_2(\mathcal{H}_3; H/L) \longrightarrow H_1(\mathcal{H}_3; L) \longrightarrow H_1(\mathcal{H}_3; H) \longrightarrow H_1(\mathcal{H}_3; H/L) \longrightarrow 0,$$

and it implies $\mathrm{Im}(H_2(\mathcal{H}_3; H/L) \to H_1(\mathcal{H}_3; L)) = \mathrm{Ker}(H_1(\mathcal{H}_3; L) \to H_1(\mathcal{H}_3; H)) \cong \mathbb{Z}/2\mathbb{Z}$. □

7. Proof of Theorem 1.2

In this section, we prove Theorem 1.2, and calculate $H_1(\mathcal{H}_g^*; L)$ and $H_1(\mathcal{H}_g^*; H/L)$.

Lemma 7.1.

$$(H^{\otimes 2})_{\mathcal{H}_g} \cong (L \otimes H)_{\mathcal{H}_g} \cong \mathbb{Z}.$$

Proof. The actions of $\mathcal{H}_g$ on L and L^* factor through $\mathrm{GL}(g;\mathbb{Z})$, and L is isomorphic to $V = \mathbb{Z}^g$ endowed with the structure of natural left $\mathrm{GL}(g;\mathbb{Z})$-module. Thus the well-known facts that

$$(V \otimes V)_{\mathrm{GL}(g;\mathbb{Z})} = (V^* \otimes V^*)_{\mathrm{GL}(g;\mathbb{Z})} = 0, \text{ and } (V \otimes V^*)_{\mathrm{GL}(g;\mathbb{Z})} \cong \mathbb{Z}$$

imply $(L \otimes L)_{\mathcal{H}_g} = (L^* \otimes L^*)_{\mathcal{H}_g} = 0$ and $(L \otimes L^*)_{\mathcal{H}_g} = \mathbb{Z}$.

Recall that the intersection form on H induces an isomorphism $L^* \cong H/L$, and we have the exact sequences

$$(L \otimes L)_{\mathcal{H}_g} \longrightarrow (L \otimes H)_{\mathcal{H}_g} \longrightarrow (L \otimes L^*)_{\mathcal{H}_g} \longrightarrow 0,$$
$$(L^* \otimes L)_{\mathcal{H}_g} \longrightarrow (L^* \otimes H)_{\mathcal{H}_g} \longrightarrow (L^* \otimes L^*)_{\mathcal{H}_g} \longrightarrow 0.$$

Thus we obtain $(L \otimes H)_{\mathcal{H}_g} \cong (L \otimes L^*)_{\mathcal{H}_g} \cong \mathbb{Z}$. Let $\{x_1^*, x_2^*, \dots, x_g^*\}$ denote the dual basis of $\{x_1, x_2, \dots, x_g\}$. Since the image $\mathrm{Im}((L^* \otimes L)_{\mathcal{H}_g} \to (L^* \otimes H)_{\mathcal{H}_g})$ is generated by $x_1^* \otimes x_1$, and

$$a_1(x_1^* \otimes y_1) - x_1^* \otimes y_1 = x_1^* \otimes x_1,$$

we have $\mathrm{Im}((L^* \otimes L)_{\mathcal{H}_g} \to (L^* \otimes H)_{\mathcal{H}_g}) = 0$. It implies $(L^* \otimes H)_{\mathcal{H}_g} \cong (L^* \otimes L^*)_{\mathcal{H}_g} = 0$. The exact sequence

$$(L \otimes H)_{\mathcal{H}_g} \longrightarrow (H \otimes H)_{\mathcal{H}_g} \longrightarrow (L^* \otimes H)_{\mathcal{H}_g} \longrightarrow 0$$

and the intersection form on H imply $(H^{\otimes 2})_{\mathcal{H}_g} \cong \mathbb{Z}$. □

Proof of Theorem 1.2. By the five-term exact sequence (Theorem 2.6) induced by the short exact sequence $1 \to \mathbb{Z} \to \mathcal{H}_{g,1} \to \mathcal{H}_g^* \to 1$, we have

$$H_1(\mathbb{Z}; H)_{\mathcal{H}_g^*} \longrightarrow H_1(\mathcal{H}_{g,1}; H) \longrightarrow H_1(\mathcal{H}_g^*; H) \longrightarrow 0.$$

Since $H_1(\mathbb{Z}; H)_{\mathcal{H}_g^*} \cong H_{\mathcal{H}_g^*} = 0$, we obtain an isomorphism $H_1(\mathcal{H}_{g,1}; H) \cong H_1(\mathcal{H}_g^*; H)$.

Next, we compute $H_1(\mathcal{H}_g^*; H)$. Morita showed that $H^1(\mathcal{M}_g^*; H) \cong \mathbb{Z}$ in [17, Section 6]. In fact, he described a generator of $H^1(\mathcal{M}_g^*; H)$ as a crossed homomorphism, and showed $H_1(\mathcal{M}_g^*; H) \cong \mathbb{Z}$ in [17, Proof of Proposition 6.4]. The forgetful exact sequence $1 \to \pi_1\Sigma_g \to \mathcal{M}_g^* \to \mathcal{M}_g \to 1$ induces the homology exact sequence (Theorem 2.6)

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_1(\mathcal{M}_g^*; H) \longrightarrow H_1(\mathcal{M}_g; H) \longrightarrow 0$$

when $g \geq 2$. Since the action of $\mathcal{H}_g^*$ on $H_1(\pi_1\Sigma_g; H)$ factors through $\mathcal{H}_g$, Lemma 7.1 implies the isomorphism $H_1(\pi_1\Sigma_g; H)_{\mathcal{H}_g^*} = H_1(\pi_1\Sigma_g; H)_{\mathcal{H}_g} = (H^{\otimes 2})_{\mathcal{H}_g} \cong \mathbb{Z}$, which is induced by the intersection form on H. Thus restricting the exact sequence to $\mathcal{H}_g^*$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*;H) & \longrightarrow & H_1(\mathcal{H}_g;H) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(\mathcal{M}_g^*;H) & \longrightarrow & H_1(\mathcal{M}_g;H) & \longrightarrow & 0. \end{array}$$

By the above diagram, both of the kernels and the cokernels of the homomorphisms $H_1(\mathcal{H}_g^*;H) \to H_1(\mathcal{M}_g^*;H)$ and $H_1(\mathcal{H}_g;H) \to H_1(\mathcal{M}_g;H)$ coincide. By Remark 4.2 and Theorem 1.1, we see that $\mathrm{Coker}(H_1(\mathcal{H}_g^*;H) \to H_1(\mathcal{M}_g^*;H))$ is trivial for $g \geq 2$. We also see that $\mathrm{Ker}(H_1(\mathcal{H}_g^*;H) \to H_1(\mathcal{M}_g^*;H))$ is trivial when $g \geq 4$, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $g = 2, 3$. Thus we can determine $H_1(\mathcal{H}_g^*;H)$. ◻

Proposition 7.2. (1) *When $g \geq 2$, the forgetful homomorphism $\mathcal{H}_g^* \to \mathcal{H}_g$ induces an isomorphism*

$$H_1(\mathcal{H}_g^*;H/L) \cong H_1(\mathcal{H}_g;H/L).$$

In particular, we have

$$H_1(\mathcal{H}_g^*;H/L) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } g \geq 4, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2,3. \end{cases}$$

(2) *When $g \geq 4$, the homomorphism $H_1(\mathcal{H}_g^*;L) \to H_1(\mathcal{H}_g^*;H)$ induced by the inclusion $L \to H$ is injective. When $g = 2, 3$, there exists an isomorphism* $\mathrm{Ker}(H_1(\mathcal{H}_g^*;L) \to H_1(\mathcal{H}_g^*;H)) \cong \mathbb{Z}/2\mathbb{Z}$. *In particular, we have*

$$H_1(\mathcal{H}_g^*;L) \cong \begin{cases} \mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } g = 2,3. \end{cases}$$

Proof. Consider the exact sequences between homology groups with coefficients in L induced by the forgetful exact sequences $1 \to \pi_1\Sigma_g \to \mathcal{M}_g^* \to \mathcal{M}_g \to 1$ and its restriction $1 \to \pi_1\Sigma_g \to \mathcal{H}_g^* \to \mathcal{H}_g \to 1$. Applying Lemma 7.1, we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*;L) & \longrightarrow & H_1(\mathcal{H}_g;L) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_1(\mathcal{H}_g^*;H) & \longrightarrow & H_1(\mathcal{H}_g;H) & \longrightarrow & 0. \end{array}$$

By the above diagram, both of the kernels and the cokernels of the homomorphisms $H_1(\mathcal{H}_g^*;L) \to H_1(\mathcal{H}_g^*;H)$ and $H_1(\mathcal{H}_g;L) \to H_1(\mathcal{H}_g;H)$ coincide. Consider the homology exact sequences induced by the exact sequence $0 \to L \to H \to H/L \to 0$, which are

$$\begin{array}{ccccccc} H_1(\mathcal{H}_g^*;L) & \longrightarrow & H_1(\mathcal{H}_g^*;H) & \longrightarrow & H_1(\mathcal{H}_g^*;H/L) & \longrightarrow & L_{\mathcal{H}_g^*} \\ \downarrow & & \downarrow & & \downarrow & & \| \\ H_1(\mathcal{H}_g;L) & \longrightarrow & H_1(\mathcal{H}_g;H) & \longrightarrow & H_1(\mathcal{H}_g;H/L) & \longrightarrow & L_{\mathcal{H}_g}. \end{array}$$

By Lemma 3.2, $L_{\mathcal{H}_g^*} = L_{\mathcal{H}_g}$ is trivial, and we obtain

$$\begin{aligned} H_1(\mathcal{H}_g^*; H/L) &\cong \mathrm{Coker}(H_1(\mathcal{H}_g^*; L) \to H_1(\mathcal{H}_g^*; H)) \\ &\cong \mathrm{Coker}(H_1(\mathcal{H}_g; L) \to H_1(\mathcal{H}_g; H)) \\ &\cong H_1(\mathcal{H}_g; H/L). \end{aligned}$$

In Remark 4.10 and Propositions 6.9 and 6.19, we see that $\mathrm{Ker}(H_1(\mathcal{H}_g; L) \to H_1(\mathcal{H}_g; H))$ is trivial when $g \geq 4$, and is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $g = 2, 3$. In Lemma 4.7 and Propositions 6.9 and 6.19, we also see that $\mathrm{Coker}(H_1(\mathcal{H}_g; L) \to H_1(\mathcal{H}_g; H))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ when $g \geq 4$, and is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ when $g = 2, 3$. Thus we can determine $H_1(\mathcal{H}_g^*; L)$. □

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