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ON JACOBI FORMS OF REAL WEIGHTS AND INDICES

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Abstract
In this paper, we investigate weak Jacobi forms of real weights and indices, and show that they have a very simple structure theorem even when their weights and indices are not integral. By using this structure theorem, we can determine possible weights of Siegel paramodular forms.

1. Introduction

On number theory, modular forms of integral or half-integral weights play very important roles. Then how about modular forms whose weights are neither integral nor half-integral? In case of elliptic modular forms, in 1999, Bannai, Koike, Munemasa and Sekiguchi [4] investigated these modular forms. They showed that the ring of modular forms of weights $k/5$ belonging to the principal congruence subgroup of level 5 is generated by two elements of weights $1/5$. This work has some connection with Klein's work in the 19th century. And then in 2000, Ibukiyama [9] rewrote their theory in a more general context, including the connection between unitary reflection groups and covering groups. We remark that in case of elliptic modular forms we can construct a modular form of real weight by using the real power of the Dedekind eta function.

However, in case of modular forms of several variables, as far as the author knows, there is no explicit example of modular forms whose weights are neither integral nor half-integral. In this paper, we investigate weak Jacobi forms of real weights, which are a kind of modular forms of two variables. Weak Jacobi forms are one of the basic and important examples of modular forms of several variables and they have a very simple structure theorem when their weights and indices are integers [3, 6]. Here we will show that the space of weak Jacobi forms has a very simple structure even when their weights and indices are real numbers.

2. Elliptic modular forms

2.1. Elliptic modular forms of integral weights. First of all, we review elliptic modular forms of integral weights shortly. The group $G := \mathrm{SL}(2, \mathbb{R})$ acts on the complex upper half plane

$$\mathbb{H} := \{ \tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0 \}$$

by fractional linear transformations:

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$$\mathbb{H} \ni \tau \mapsto \gamma\langle\tau\rangle := \frac{a\tau + b}{c\tau + d} \in \mathbb{H}, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G.$$

The automorphic factor of weight $k \in \mathbb{Z}$ is defined by

$$j_k(\gamma; \tau) := (c\tau + d)^k.$$

This j_k satisfies the cocycle condition

$$j_k(\gamma_1\gamma_2; \tau) = j_k(\gamma_1; \gamma_2\langle\tau\rangle) j_k(\gamma_2; \tau).$$

For a holomorphic function f on $\mathbb{H}$, we define

$$(f|_k\gamma)(\tau) := j_k(\gamma; \tau)^{-1} f(\gamma\langle\tau\rangle).$$

Then we have $f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2$, namely, for each $k \in \mathbb{Z}$, $G = \mathrm{SL}(2, \mathbb{R})$ acts on the set of all holomorphic functions on $\mathbb{H}$.

Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Let χ be a character of Γ, namely, a group homomorphism $\chi : \Gamma \to S^1 := \{t \in \mathbb{C} \mid |t| = 1\}$. Now we suppose that a holomorphic function f on $\mathbb{H}$ satisfies the condition

$$f|_k\gamma = \chi(\gamma) f$$

for any $\gamma \in \Gamma$. Let $\gamma_0 \in \mathrm{SL}(2, \mathbb{Z})$. As $\gamma_0^{-1}\Gamma\gamma_0$ is a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$, there exists $b(\gamma_0) \in \mathbb{N}$ such that $\gamma_0' := \gamma_0 \left(\begin{smallmatrix} 1 & b(\gamma_0) \\ 0 & 1 \end{smallmatrix}\right) \gamma_0^{-1} \in \Gamma$. Hence $f|_k\gamma_0$ satisfies the condition

$$(f|_k\gamma_0)(\tau + b(\gamma_0)) = \chi\left(\gamma_0'\right)(f|_k\gamma_0)(\tau).$$

Therefore, $f|_k\gamma_0$ has a Fourier expansion

$$(f|_k\gamma_0)(\tau) = \sum_n c_{\gamma_0}(n)\mathbf{e}(n\tau),$$

where $\mathbf{e}(*) := \exp(2\pi i *)$ $(i := \sqrt{-1})$ and n runs over the discrete set $\{\, n \in \mathbb{R} \mid \mathbf{e}(nb(\gamma_0)) = \chi(\gamma_0')\}$.

Definition. We say a holomorphic function f on $\mathbb{H}$ is an elliptic modular form of weight k with respect to Γ and its character χ if f satisfies the following two conditions:
(1) For any $\gamma \in \Gamma$, $f|_k\gamma = \chi(\gamma) f$.
(2) For any $\gamma_0 \in \mathrm{SL}(2, \mathbb{Z})$, $c_{\gamma_0}(n) = 0$ unless $n \geq 0$.
We denote by $\mathbb{M}_k(\Gamma; \chi)$ the $\mathbb{C}$-vector space of all elliptic modular forms of weight k with respect to Γ and its character χ. When χ is the trivial character, the identity map to $1 \in S^1$, simply we denote this space by $\mathbb{M}_k(\Gamma) := \mathbb{M}_k(\Gamma; \mathbf{1}_\Gamma)$.

Our interest is the structure of $\mathbb{M}_k(\Gamma; \chi)$ or $\mathbb{M}_k(\Gamma)$. By the general theory of modular forms, for any Γ, $\bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k(\Gamma)$ is a graded ring. Especially, we have $\mathbb{M}_k(\Gamma) = \{0\}$ for $k < 0$ and $\mathbb{M}_0(\Gamma) = \mathbb{C}$. For some Γ, the structure of $\mathbb{M}_k(\Gamma)$ is well known. For example, when $\Gamma = \mathrm{SL}(2, \mathbb{Z})$, the structure is given by

$$\bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k(\mathrm{SL}(2, \mathbb{Z})) = \mathbb{C}[e_4, e_6],$$

where

$$e_4(\tau) := 1 + 240 \sum_{n=1}^{\infty} \left(\sum_{d|n} d^3 \right) \mathbf{e}(n\tau) \in \mathbb{M}_4(\mathrm{SL}(2,\mathbb{Z}))$$

and

$$e_6(\tau) := 1 - 504 \sum_{n=1}^{\infty} \left(\sum_{d|n} d^5 \right) \mathbf{e}(n\tau) \in \mathbb{M}_6(\mathrm{SL}(2,\mathbb{Z}))$$

are Eisenstein series of weights 4 and 6, which are algebraically independent over $\mathbb{C}$.

2.2. Elliptic modular forms of real weights. Next we discuss about elliptic modular forms of real weights. Here let $k \in \mathbb{R}$ be a real number. In this case, to apply the previous definition of elliptic modular forms, we need to fix the branch of $j_k(\gamma;\tau) = (c\tau+d)^k$. Namely, we define $j_k(\gamma;\tau) := (c\tau + d)^k := \exp(k \log(c\tau + d))$, where $\log(c\tau + d)$ takes the principal value of logarithm. However, we remark that this j_k does not satisfy the cocycle condition, if k is not integral. Let

$$G_k := \left\{ \widetilde{\gamma} = (\gamma, J) \mid \gamma \in G,\ J(\tau) = t\, j_k(\gamma;\tau),\ t \in S^1 \right\}$$

be a group with its multiplication $(\gamma, J) = (\gamma_1, J_1)(\gamma_2, J_2)$ defined by $\gamma := \gamma_1\gamma_2$ and $J(\tau) = J_1(\gamma_2\langle\tau\rangle)J_2(\tau)$. This G_k is a central extension of G by S^1. For a holomorphic function f on $\mathbb{H}$, we define

$$(f|\widetilde{\gamma})(\tau) := J(\tau)^{-1} f(\gamma\langle\tau\rangle).$$

Then the group G_k acts on the set of all holomorphic functions on $\mathbb{H}$.

Let Γ be a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$ and we put

$$\Gamma_k := \{ \widetilde{\gamma} = (\gamma, J) \in G_k \mid \gamma \in \Gamma \}.$$

Let χ be a character of Γ_k, namely, a group homomorphism $\chi : \Gamma_k \to S^1$. Now we suppose that a holomorphic function f on $\mathbb{H}$ satisfies the condition $f|\widetilde{\gamma} = \chi(\widetilde{\gamma})f$ for any $\widetilde{\gamma} \in \Gamma_k$. In the same manner as described in the previous subsection, for each $\widetilde{\gamma_0} \in \mathrm{SL}(2,\mathbb{Z})_k$. we can show that $f|\widetilde{\gamma_0}$ has a Fourier expansion

$$(f|\widetilde{\gamma_0})(\tau) = \sum_n c_{\widetilde{\gamma_0}}(n)\mathbf{e}(n\tau).$$

DEFINITION. We say a holomorphic function f on $\mathbb{H}$ is an elliptic modular form with respect to Γ_k and its character χ if f satisfies the following two conditions:
(1) For any $\widetilde{\gamma} \in \Gamma_k$, $f|\widetilde{\gamma} = \chi(\widetilde{\gamma})f$.
(2) For any $\widetilde{\gamma_0} \in \mathrm{SL}(2,\mathbb{Z})_k$, $c_{\widetilde{\gamma_0}}(n) = 0$ unless $n \geq 0$.
We denote by $\mathbb{M}(\Gamma_k;\chi)$ the $\mathbb{C}$-vector space of all elliptic modular forms with respect to Γ_k and its character χ.

The group Γ_k is generated by two types of its elements:

$$(\gamma, j_k(\gamma;\tau)) \quad (\gamma \in \Gamma) \quad \text{and} \quad (E_2, t) \quad (t \in S^1).$$

If $\mathbb{M}(\Gamma_k;\chi) \neq \{0\}$, the character χ should satisfy the condition $\chi(E_2, t) = t^{-1}$. Hence, from now on, we assume this condition holds. We remark that this definition is a generalization of the definition of elliptic modular forms of integral weights, which is defined in the previous subsection. Namely, if $k \in \mathbb{Z}$, we have $\mathbb{M}_k(\Gamma;\chi) = \mathbb{M}(\Gamma_k;\chi')$, where χ' is defined by $\chi'(\gamma, j_k(\gamma;\tau)) := \chi(\gamma)$ and $\chi'(E_2, t) := t^{-1}$. This definition also contains the usual definition of elliptic modular forms of half-integral weights and the definition of elliptic modular forms of rational weights given by Bannai, Koike, Munemasa, Sekiguchi and Ibukiyama [4, 9].

One of the most important example of elliptic modular forms of real weights is the Dedekind eta function

$$\eta(\tau) := \mathbf{e}\left(\frac{1}{24}\tau\right)\prod_{n=1}^{\infty}(1-\mathbf{e}(n\tau)),$$

whose weight is $\frac{1}{2}$. This function η is holomorphic on $\mathbb{H}$, has no zero on $\mathbb{H}$ and satisfies two conditions

$$\eta(\tau+1) = \mathbf{e}\left(\frac{1}{24}\right)\eta(\tau) \quad \text{and} \quad \frac{1}{\sqrt{\tau}}\,\eta\left(-\frac{1}{\tau}\right) = \mathbf{e}\left(-\frac{1}{8}\right)\,\eta(\tau),$$

where we choose $0 < \arg\left(\sqrt{\tau}\right) < \frac{\pi}{2}$. From this eta function, we can construct a modular form of an arbitrary real weight. Namely, we can define $\log\eta$ as a single valued function on $\mathbb{H}$ by

$$\log\eta(\tau) := \frac{\pi i\tau}{12} - \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{d|n}d\right)\mathbf{e}(n\tau),$$

and then for any $k \in \mathbb{R}$, we can define $\eta^{2k}(\tau) := \exp(2k\log\eta(\tau))$. This η^{2k} is holomorphic on $\mathbb{H}$, has no zero on $\mathbb{H}$, has a Fourier expansion

$$\eta^{2k}(\tau) = \mathbf{e}\left(\frac{k}{12}\tau\right) - 2k\mathbf{e}\left(\left(1+\frac{k}{12}\right)\tau\right) + \cdots$$

and satisfies two conditions:

$$\eta^{2k}(\tau+1) = \mathbf{e}\left(\frac{k}{12}\right)\eta^{2k}(\tau) \quad \text{and} \quad \frac{1}{\tau^k}\eta^{2k}\left(-\frac{1}{\tau}\right) = \mathbf{e}\left(-\frac{k}{4}\right)\,\eta^{2k}(\tau),$$

where $\tau^k := \exp(k\log\tau)$ on which we choose $0 < \mathrm{Im}(\log\tau) < \pi$. Now, for any $\widetilde{\gamma} \in \mathrm{SL}(2,\mathbb{Z})_k$, we define $\chi_k(\widetilde{\gamma}) := \frac{(\eta^{2k}|\widetilde{\gamma})(\tau)}{\eta^{2k}(\tau)}$. Then, this χ_k is a character of $\mathrm{SL}(2,\mathbb{Z})_k$, because

$$\begin{aligned}\chi_k(\widetilde{\gamma_1}\widetilde{\gamma_2}) =& \frac{\left(\left(\eta^{2k}|\widetilde{\gamma_1}\right)|\widetilde{\gamma_2}\right)(\tau)}{\eta^{2k}(\tau)} = \frac{\left(\left(\chi_k(\widetilde{\gamma_1})\,\eta^{2k}\right)|\widetilde{\gamma_2}\right)(\tau)}{\eta^{2k}(\tau)} \\ =&\chi_k(\widetilde{\gamma_1})\,\frac{\left(\eta^{2k}|\widetilde{\gamma_2}\right)(\tau)}{\eta^{2k}(\tau)} = \chi_k(\widetilde{\gamma_1})\,\chi_k(\widetilde{\gamma_2}).\end{aligned}$$

Consequently, we have $\eta^{2k} \in \mathbb{M}(\mathrm{SL}(2,\mathbb{Z})_k;\chi_k)$.

Here, at the end of this section, we mention the structure of the space of all elliptic modular forms of real weights when $\Gamma = \mathrm{SL}(2,\mathbb{Z})$. Let χ be a character of $\mathrm{SL}(2,\mathbb{Z})_k$, where we fix a real weight $k \in \mathbb{R}$. As mentioned above, we assume $\chi(E_2, t) = t^{-1}$. We label two elements of $\mathrm{SL}(2,\mathbb{Z})_k$:

$$S_k := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tau^k \right) \quad \text{and} \quad T := \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Easily we can see

$$S_k^2 = (S_k T)^3 = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{e}\left(\frac{k}{2}\right) \right),$$

and then we have $\chi(S_k) = \chi(T)^{-3}$ and $\chi(T)^6 = \mathbf{e}\left(\frac{k}{2}\right)$. This means that χ is determined only from the value of $\chi(T)$. Let $f \in \mathbb{M}(\mathrm{SL}(2,\mathbb{Z})_k; \chi)$ be a nonzero modular form and we choose $k' \in \mathbb{R}$ such that $0 \leq k' < 12$ and $\chi(T) = \mathbf{e}\left(\frac{k'}{12}\right)$. We remark that $k - k' \in 2\mathbb{Z}$, because $\chi(T)^6 = \mathbf{e}\left(\frac{k}{2}\right)$. As $f(\tau+1) = \mathbf{e}\left(\frac{k'}{12}\right) f(\tau)$, f has a Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} c(n) \mathbf{e}\left(\left(n + \frac{k'}{12}\right)\tau\right)$. Hence f can be divided by $\eta^{2k'}$ and its quotient is an usual elliptic modular forms of weight $k - k' \in 2\mathbb{Z}$ with respect to $\mathrm{SL}(2,\mathbb{Z})$ with trivial character. Namely, we have the following theorem.

Theorem 1. *Let $f \in \mathbb{M}(\mathrm{SL}(2,\mathbb{Z})_k; \chi)$ be a nonzero modular form of real weight. We choose $k' \in \mathbb{R}$ such that $0 \leq k' < 12$ and $\chi(T) = \mathbf{e}\left(\frac{k'}{12}\right)$. Then this k' satisfies the condition $k - k' \in 2\mathbb{Z}$ and we have*

$$f \in \eta^{2k'} \mathbb{M}_{k-k'}(\mathrm{SL}(2,\mathbb{Z})).$$

Remark. The author has not yet seen this theorem in another literature, however, the author considers this theorem has already known for some experts on modular forms. Actually, Ibukiyama [9, Proposition 2.4] determined the the structure of the space of elliptic modular forms of weight $k \in \frac{2}{7}\mathbb{Z}$. Freitag [7, Theorem 5.1] gave the dimension formula of modular forms of any real weight, and we easily can induce this theorem from his dimension formula.

3. Jacobi forms

3.1. Jacobi forms of integral weights and indices. Jacobi forms of integral weights and indices were first studied by Eichler and Zagier in their book [6]. In their book, they mainly treated Jacobi forms with respect to the full modular group $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}$. According to their way, easily we can generalize most part of their results to any finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}$, as far as we treat integral weights and indices.

We define a group

$$G^{\mathrm{J}} := \left\{ g = (\gamma, \mathbf{x}, u) \mid \gamma \in G,\ \mathbf{x} \in \mathbb{R}^2,\ u \in \mathbb{R} \right\}$$

with its multiplication

$$(\gamma_1, \mathbf{x}_1, u_1)(\gamma_2, \mathbf{x}_2, u_2) := (\gamma_1\gamma_2, \mathbf{x}_1\gamma_2 + \mathbf{x}_2, u_1 + u_2 + \det \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}),$$

where we regard $\mathbf{x}$ as a row vector. This group acts on $\mathbb{H} \times \mathbb{C}$ by

$$\mathbb{H} \times \mathbb{C} \ni w = (\tau, z) \mapsto g\langle w \rangle := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right) \in \mathbb{H} \times \mathbb{C},$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \quad \mathbf{x} = (\lambda, \mu) \in \mathbb{R}^2.$$

The automorphic factor of weight $k \in \mathbb{Z}$ and index $m \in \mathbb{Z}$ is defined by

$$j_{k,m}(g; w) := (c\tau + d)^k \, \mathbf{e}\left(-m \left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu + u \right) \right).$$

This $j_{k,m}$ satisfies the cocycle condition

$$j_{k,m}(g_1 g_2; w) = j_{k,m}(g_1; g_2\langle w\rangle) j_{k,m}(g_2; w).$$

For a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$, we define

$$(\varphi|_{k,m} g)(w) := j_{k,m}(g; w)^{-1} \varphi(g\langle w\rangle).$$

Then we have $\varphi|_{k,m}(g_1 g_2) = (\varphi|_{k,m} g_1)|_{k,m} g_2$, namely, for each $k, m \in \mathbb{Z}$, the group G^{J} acts on the set of all holomorphic functions on $\mathbb{H}$.

Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Then

$$\Gamma^{\mathrm{J}} := \left\{ g = (\gamma, \mathbf{x}, u) \mid \gamma \in \Gamma, \ \mathbf{x} \in \mathbb{Z}^2, \ u \in \mathbb{Z} \right\}$$

is a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}$. Let χ be a character of Γ^{J}. We suppose that a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ satisfies the condition $\varphi|_{k,m} g = \chi(g)\varphi$ for any $g \in \Gamma^{\mathrm{J}}$. In the same manner as described in the previous section, for each $g_0 \in \mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}$, $\varphi|_{k,m} g$ has a Fourier expansion

$$(\varphi|_{k,m} g_0)(\tau, z) = \sum_{n,l} c_{g_0}(n, l) \mathbf{e}(n\tau + lz).$$

Definition. We say a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ is a weak Jacobi form of weight k and index m with respect to Γ^{J} and its character χ if φ satisfies the following two conditions:
(1) For any $g \in \Gamma^{\mathrm{J}}$, $\varphi|_{k,m} g = \chi(g)\varphi$.
(2) For any $g_0 \in \mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}$, $c_{g_0}(n, l) = 0$ unless $n \geq 0$.
We denote by $\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^{\mathrm{J}}; \chi)$ the $\mathbb{C}$-vector space of all weak Jacobi forms of weight k and index m with respect to Γ^{J} and its character $\widetilde{\chi}$. When χ is the trivial character, simply we denote this space by $\mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^{\mathrm{J}}) := \mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^{\mathrm{J}}; \mathbf{1}_{\Gamma^{\mathrm{J}}})$.

In the book of Eichler and Zagier [6], they constructed nonzero weak Jacobi forms $\varphi_{-2,1} \in \mathbb{J}^{\mathrm{w}}_{-2,1}(\Gamma^{\mathrm{J}})$, $\varphi_{0,1} \in \mathbb{J}^{\mathrm{w}}_{0,1}(\Gamma^{\mathrm{J}})$ and $\varphi_{-1,2} \in \mathbb{J}^{\mathrm{w}}_{-1,2}(\Gamma^{\mathrm{J}})$. (In their book, they denoted by $\tilde{\phi}_{-2,1}$, $\tilde{\phi}_{0,1}$ and $\tilde{\phi}_{-1,2}$.) The structure of weak Jacobi forms of integral weights and indices has already been known.

Theorem 2 ([6, Theorem 9.4], [3, Proposition 6.1])**.** *Let Γ be a finite index subgroup of* $\mathrm{SL}(2, \mathbb{Z})$*. Then we have*

$$\bigoplus_{k,m \in \mathbb{Z}} \mathbb{J}^{\mathrm{w}}_{k,m}(\Gamma^{\mathrm{J}}) = \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k(\Gamma) \right) [\varphi_{0,1}, \varphi_{-2,1}] \oplus \varphi_{-1,2} \left(\bigoplus_{k \in \mathbb{Z}} \mathbb{M}_k(\Gamma) \right) [\varphi_{0,1}, \varphi_{-2,1}] \, .$$

3.2. Jacobi forms of real weights and indices. Here we define Jacobi forms of arbitrary real weight and index in the same manner as elliptic modular forms.

Like the definition of elliptic modular forms of real weights, let

$$G^{\mathrm{J}}_{k,m} := \left\{ \widetilde{g} = (g, J) \mid g \in G^{\mathrm{J}},\ J(w) = t\, j_{k,m}(g; w),\ t \in S^1 \right\}$$

be a group with its multiplication $(g, J) = (g_1, J_1)(g_2, J_2)$ defined by $g := g_1 g_2$ and $J(w) = J_1(g_2\langle w\rangle)J_2(w)$. This $G^{\mathrm{J}}_{k,m}$ is a central extension of G^{J} by S^1. For a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$, we define

$$(\varphi|\widetilde{g})\,(w) := J(w)^{-1}\varphi(g\langle w\rangle).$$

Then the group $G^{\mathrm{J}}_{k,m}$ acts on the set of all holomorphic functions on $\mathbb{H} \times \mathbb{C}$.

Let Γ be a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})$. Then

$$\Gamma^{\mathrm{J}}_{k,m} := \left\{ \widetilde{g} = (g, J) \in G^{\mathrm{J}}_{k,m} \mid g = (\gamma, \mathbf{x}, u),\ \gamma \in \Gamma,\ \mathbf{x} \in \mathbb{Z}^2,\ u \in \mathbb{Z} \right\}$$

is a finite index subgroup of $\mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}_{k,m}$. Let χ be a character of $\Gamma^{\mathrm{J}}_{k,m}$. We suppose that a holomorphic function φ on $\mathbb{H} \times \mathbb{C}$ satisfies the condition $\varphi|\widetilde{g} = \chi(\widetilde{g})\varphi$ for any $\widetilde{g} \in \Gamma^{\mathrm{J}}_{k,m}$. In the same manner as described in the previous section, for any $\widetilde{g} \in \mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}_{k,m}$, $\varphi|\widetilde{g}$ has a Fourier expansion

$$(\varphi|\widetilde{g})\,(\tau, z) = \sum_{n,l} c_{\widetilde{g}}(n, l)\mathbf{e}(n\tau + lz).$$

DEFINITION. We say a holomorphic function φ on $\mathbb{H}\times\mathbb{C}$ is a weak Jacobi form with respect to $\Gamma^{\mathrm{J}}_{k,m}$ and its character χ if φ satisfies the following two conditions:
(1) For any $\widetilde{g} \in \Gamma^{\mathrm{J}}_{k,m}$, $\varphi|\widetilde{g} = \chi(\widetilde{g})\varphi$.
(2) For any $\widetilde{g} \in \mathrm{SL}(2, \mathbb{Z})^{\mathrm{J}}_{k,m}$, $c_{\widetilde{g}}(n, l) = 0$ unless $n \geq 0$.
We denote by $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m}; \chi)$ the $\mathbb{C}$-vector space of all weak Jacobi forms with respect to $\Gamma^{\mathrm{J}}_{k,m}$ and its character χ.

REMARK. We say a weak Jacobi forms φ is a Jacobi form if $c_{\widetilde{g}}(n, l) = 0$ unless $4nm - l^2 \geq 0$. Our definition of (weak) Jacobi forms is a generalization of the definition of (weak) Jacobi forms of integral weights and indices.

The aim of this paper is to give a basis of the $\mathbb{C}$-vector space $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m}; \chi)$ explicitly. In this paper, we assume $-E_2 \in \Gamma$.

3.3. Examples. Theta functions are important examples of Jacobi forms. Let

$$\theta_{ab}(\tau, z) := \sum_{n\in\mathbb{Z}} \mathbf{e}\left(\frac{1}{2}\left(n + \frac{1}{2}a\right)^2 \tau + \left(n + \frac{1}{2}a\right)\left(z + \frac{1}{2}b\right)\right),$$

be the Jacobi theta function, where we choose $a, b \in \{0, 1\}$. It is well known that Jacobi theta function satisfies the following translation formulas:

$$\theta_{00}(\tau, z) = \theta_{00}(\tau, -z), \qquad \theta_{01}(\tau, z) = \theta_{01}(\tau, -z),$$
$$\theta_{10}(\tau, z) = \theta_{10}(\tau, -z), \qquad \theta_{11}(\tau, z) = -\theta_{11}(\tau, -z),$$

$$\theta_{00}(\tau,z)=\theta_{00}(\tau,z+1), \qquad \theta_{01}(\tau,z)=\theta_{01}(\tau,z+1),$$
$$\theta_{10}(\tau,z)=-\theta_{10}(\tau,z+1), \qquad \theta_{11}(\tau,z)=-\theta_{11}(\tau,z+1),$$

$$\theta_{00}(\tau,z)=\mathbf{e}\left(\frac{\tau}{2}+z\right)\theta_{00}(\tau,z+\tau), \qquad \theta_{01}(\tau,z)=-\mathbf{e}\left(\frac{\tau}{2}+z\right)\theta_{01}(\tau,z+\tau),$$
$$\theta_{10}(\tau,z)=\mathbf{e}\left(\frac{\tau}{2}+z\right)\theta_{10}(\tau,z+\tau), \qquad \theta_{11}(\tau,z)=-\mathbf{e}\left(\frac{\tau}{2}+z\right)\theta_{11}(\tau,z+\tau),$$

$$\theta_{00}(\tau,z)=\theta_{01}(\tau+1,z), \qquad \theta_{01}(\tau,z)=\theta_{00}(\tau+1,z),$$
$$\theta_{10}(\tau,z)=\mathbf{e}\left(-\frac{1}{8}\right)\theta_{10}(\tau+1,z), \qquad \theta_{11}(\tau,z)=\mathbf{e}\left(-\frac{1}{8}\right)\theta_{11}(\tau+1,z),$$

$$\theta_{00}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\sqrt{\tau}\,\mathbf{e}\left(\frac{z^2}{2\tau}-\frac{1}{8}\right)\theta_{00}(\tau,z), \qquad \theta_{01}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\sqrt{\tau}\,\mathbf{e}\left(\frac{z^2}{2\tau}-\frac{1}{8}\right)\theta_{10}(\tau,z),$$
$$\theta_{10}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\sqrt{\tau}\,\mathbf{e}\left(\frac{z^2}{2\tau}-\frac{1}{8}\right)\theta_{01}(\tau,z), \qquad \theta_{11}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\sqrt{\tau}\,\mathbf{e}\left(\frac{z^2}{2\tau}-\frac{3}{8}\right)\theta_{11}(\tau,z).$$

Hence these four theta functions $\theta_{00},\theta_{01},\theta_{10}$ and θ_{11} are (weak) Jacobi forms of weights $\frac{1}{2}$ and indices $\frac{1}{2}$ with respect to suitable finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ with characters.

From these theta functions, here we construct weak Jacobi forms of lower weights. On the values of these theta functions on $z=0$, the following formulas hold:

$$\theta_{00}(\tau,0)=\frac{\eta(\frac{\tau+1}{2})^2}{\mathbf{e}\left(\frac{1}{24}\right)\eta(\tau)}, \qquad \theta_{01}(\tau,0)=\frac{\eta(\frac{\tau}{2})^2}{\eta(\tau)},$$
$$\theta_{10}(\tau,0)=\frac{2\eta(2\tau)^2}{\eta(\tau)}, \qquad \theta_{11}(\tau,0)=0, \qquad \frac{1}{2\pi i}\frac{\partial}{\partial\tau}\theta_{11}(\tau,0)=\mathbf{e}\left(\frac{1}{4}\right)\eta(\tau)^3.$$

Hence, as the eta function does not vanish on $\mathbb{H}$, we can define four holomorphic functions on $\mathbb{H}\times\mathbb{C}$ by

$$\varphi_A(\tau,z):=\frac{2\theta_{00}(\tau,z)}{\theta_{00}(\tau,0)}, \qquad \varphi_B(\tau,z):=\frac{2\theta_{01}(\tau,z)}{\theta_{01}(\tau,0)},$$
$$\varphi_C(\tau,z):=\frac{2\theta_{10}(\tau,z)}{\theta_{10}(\tau,0)} \quad \text{and} \quad \varphi_D(\tau,z):=\frac{\theta_{11}(\tau,z)}{\mathbf{e}\left(\frac{1}{4}\right)\eta(\tau)^3}.$$

Let

$$\Gamma_A:=\left\{\gamma\in\mathrm{SL}(2,\mathbb{Z})\;\middle|\;\gamma\equiv E_2\text{ or }\gamma\equiv\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)\pmod 2\right\},$$
$$\Gamma_B:=\left\{\gamma\in\mathrm{SL}(2,\mathbb{Z})\;\middle|\;\gamma\equiv E_2\text{ or }\gamma\equiv\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\pmod 2\right\},\quad\text{and}$$
$$\Gamma_C:=\left\{\gamma\in\mathrm{SL}(2,\mathbb{Z})\;\middle|\;\gamma\equiv E_2\text{ or }\gamma\equiv\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\pmod 2\right\}\big(=\Gamma_0(2)\big)$$

be finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$. Then easily we can see these four functions $\varphi_A,\varphi_B,\varphi_C$ and φ_D are weak Jacobi forms of weights $0,0,0$ and -1 and indices $\frac{1}{2}$ with respect to $\Gamma_A,\Gamma_B,\Gamma_C$ and $\mathrm{SL}(2,\mathbb{Z})$ with characters, respectively. We define characters χ_A,χ_B,χ_C and χ_D of $(\Gamma_A)^{\mathrm{J}}_{0,\frac{1}{2}},(\Gamma_B)^{\mathrm{J}}_{0,\frac{1}{2}},(\Gamma_C)^{\mathrm{J}}_{0,\frac{1}{2}}$ and $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}_{-1,\frac{1}{2}}$ by

$$\varphi_A \in \mathbb{J}^{\mathrm{w}}((\Gamma_A)^{\mathrm{J}}_{0,\frac{1}{2}};\chi_A), \qquad \varphi_B \in \mathbb{J}^{\mathrm{w}}((\Gamma_B)^{\mathrm{J}}_{0,\frac{1}{2}};\chi_B),$$
$$\varphi_C \in \mathbb{J}^{\mathrm{w}}((\Gamma_C)^{\mathrm{J}}_{0,\frac{1}{2}};\chi_C) \quad \text{and} \quad \varphi_D \in \mathbb{J}^{\mathrm{w}}(\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}_{-1,\frac{1}{2}};\chi_D)$$

respectively.

Weak Jacobi forms $\varphi_{0,1}, \varphi_{-2,1}$ and $\varphi_{-1,2}$ in Theorem 2 are given by

$$\varphi_{0,1} = \varphi_A^2 + \varphi_B^2 + \varphi_C^2, \quad \varphi_{-2,1} = \varphi_D^2 \quad \text{and} \quad \varphi_{-1,2} = \frac{1}{4}\varphi_A\varphi_B\varphi_C\varphi_D.$$

As φ_A^2, φ_B^2 and φ_C^2 are weak Jacobi forms of weights 0 and indices 1 with respect to Γ_A, Γ_B and Γ_C with trivial characters, these three weak Jacobi forms can be represented by $\varphi_{0,1}, \varphi_{-2,1}$ and $\varphi_{-1,2}$ (Theorem 2). Well-known formulas

$$\begin{aligned}
\theta_{00}(\tau,z)^2\theta_{00}(\tau,0)^2 &= \theta_{01}(\tau,z)^2\theta_{01}(\tau,0)^2 + \theta_{10}(\tau,z)^2\theta_{10}(\tau,0)^2,\\
\theta_{11}(\tau,z)^2\theta_{00}(\tau,0)^2 &= \theta_{01}(\tau,z)^2\theta_{10}(\tau,0)^2 - \theta_{10}(\tau,z)^2\theta_{01}(\tau,0)^2
\end{aligned}$$

induce two relations

$$\begin{aligned}
\varphi_A(\tau,z)^2\theta_{00}(\tau,0)^4 &= \varphi_B(\tau,z)^2\theta_{01}(\tau,0)^4 + \varphi_C(\tau,z)^2\theta_{10}(\tau,0)^4,\\
4\varphi_D(\tau,z)^2\eta(\tau)^6\theta_{00}(\tau,0)^2 &= \left(\varphi_C(\tau,z)^2 - \varphi_B(\tau,z)^2\right)\theta_{01}(\tau,0)^2\theta_{10}(\tau,0)^2.
\end{aligned}$$

Hence, by using another well-known formulas

$$\begin{aligned}
\theta_{00}(\tau,0)^4 &= \theta_{01}(\tau,0)^4 + \theta_{10}(\tau,0)^4,\\
\theta_{00}(\tau,0)\theta_{01}(\tau,0)\theta_{10}(\tau,0) &= 2\eta(\tau)^3 = -\frac{1}{\pi}\frac{\partial}{\partial\tau}\theta_{11}(\tau,0),
\end{aligned}$$

we have

$$\begin{aligned}
3\varphi_A(\tau,z)^2 &= \varphi_{0,1}(\tau,z) - \left(\theta_{01}(\tau,0)^4 - \theta_{10}(\tau,0)^4\right)\varphi_{-2,1}(\tau,z),\\
3\varphi_B(\tau,z)^2 &= \varphi_{0,1}(\tau,z) - \left(\theta_{00}(\tau,0)^4 + \theta_{10}(\tau,0)^4\right)\varphi_{-2,1}(\tau,z),\\
3\varphi_C(\tau,z)^2 &= \varphi_{0,1}(\tau,z) + \left(\theta_{00}(\tau,0)^4 + \theta_{01}(\tau,0)^4\right)\varphi_{-2,1}(\tau,z).
\end{aligned}$$

4. Space of Jacobi forms

4.1. Characters. In this section we investigate possible characters of $\Gamma^{\mathrm{J}}_{k,m}$. Here, for simplicity, we label some elements of $\Gamma^{\mathrm{J}}_{k,m}$:

$$U := ((E_2, \mathbf{0}, 1), \mathbf{e}(-m)), \qquad W := ((-E_2, \mathbf{0}, 0), 1),$$
$$X := ((E_2, (1,0), 0), \mathbf{e}(-m(\tau+2z))), \qquad Y := ((E_2, (0,1), 0), 1)$$

and

$$V(\gamma) := ((\gamma, \mathbf{0}, 0), j_{k,m}), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We denote the identity of $G^{\mathrm{J}}_{k,m}$ by

$$I := V(E_2) = ((E_2, \mathbf{0}, 0), 1).$$

Let χ be a character of $\Gamma^{\mathrm{J}}_{k,m}$ such that $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \neq \{0\}$. We denote by

$$\epsilon_0 := \chi(U), \quad \epsilon_1 := \chi(W), \quad \epsilon_2 := \chi(X) \quad \text{and} \quad \epsilon_3 := \chi(Y).$$

Proposition 3. $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$.

Proof. As $W^2 = I$, we have $\chi(W)^2 = 1$. Hence $\epsilon_1 = \chi(W) \in \{1, -1\}$. In the same way, as $(WX)^2 = I$ and $(WY)^2 = I$, we have $\epsilon_2, \epsilon_3 \in \{1, -1\}$. As $XY = YXU^2$, we have $\epsilon_0 \in \{1, -1\}$. □

Proposition 4. $\epsilon_0 = \mathbf{e}(m)$. *Hence m should be integral or half-integral.*

Proof. Let $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$ be a nonzero function. As $\varphi|U = \chi(U)\varphi$, and $(\varphi|U)(\tau, z) = \mathbf{e}(m)\varphi(\tau, z)$, we have $\epsilon_0 = \chi(U) = \mathbf{e}(m)$. As $\epsilon_0 \in \{1, -1\}$, m should be integral or half-integral. □

Proposition 5. *If* $\epsilon_2 = \epsilon_3 \neq \epsilon_0$, Γ *should be a subset of* Γ_A. *If* $\epsilon_0 = \epsilon_2 \neq \epsilon_3$, Γ *should be a subset of* Γ_B. *If* $\epsilon_0 = \epsilon_3 \neq \epsilon_2$, Γ *should be a subset of* Γ_C.

Proof. The way of the proof is just same as previous two propositions. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have two equations $XV(\gamma) = V(\gamma)Y^b X^a U^{ab}$ and $YV(\gamma) = V(\gamma)Y^d X^c U^{cd}$. Hence we have $\epsilon_2^{a-1}\epsilon_3^b = \epsilon_0^{ab}$ and $\epsilon_2^c\epsilon_3^{d-1} = \epsilon_0^{cd}$, which induce the statement of the proposition. □

We recall that χ be a character of $\Gamma^{\mathrm{J}}_{k,m}$ such that $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \neq \{0\}$. As $\Gamma^{\mathrm{J}}_{k,m}$ is generated by U, W, X, Y and $V(\gamma)$ $(\gamma \in \Gamma)$, the character χ is determined by its value on $V(\gamma)$ and four parameters $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$. Here we classify characters under 16 kinds by the parameters $\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3 \in \{1, -1\}$. For example, $\varphi_{-2,1}$ and $\varphi_{0,1}$ belong to the case $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) = (1, 1, 1, 1)$ and $\varphi_{-1,2}$ belongs to the case $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3) = (1, -1, 1, 1)$. $\varphi_A, \varphi_B, \varphi_C$ and φ_D given in §3.3 belong to the case $(-1, 1, 1, 1), (-1, 1, -1, 1), (-1, 1, 1, -1)$ and $(-1, -1, -1, -1)$, respectively. The following Table 1 is a list of examples of weak Jacobi forms of each kind of characters. On all of the 17 examples on the list, $\chi(V(\gamma)) = 1$ for any $\gamma \in \Gamma$.

4.2. Estimation. In this subsection, our story proceeds just similar to the case of integral weights and integral indices, given in the book of Eichler and Zagier [6, Section 9].

Let χ be a character of $\Gamma^{\mathrm{J}}_{k,m}$ such that $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \neq \{0\}$.

Proposition 6 (c.f. [6, Theorem 1.2])**.** *Let* $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$ *be a nonzero weak Jacobi form of real weight and index. Then for a fixed* $\tau \in \mathbb{H}$, *the function* $z \mapsto \varphi(\tau, z)$ *has exactly* $2m$ *zeros* (*counting multiplicity*) *in any fundamental domain for the action of the lattice* $\mathbb{Z} + \tau\mathbb{Z}$ *on* $\mathbb{C}$.

Proof. Regardless of the values of ϵ_2 and ϵ_3, we have

$$\mathbf{e}(4m(\tau + z))\varphi(\tau, z + 2\tau) = \varphi(\tau, z) \quad \text{and} \quad \varphi(\tau, z + 2) = \varphi(\tau, z).$$

Hence we have

$$\frac{1}{2\pi i}\oint_{\partial P}\frac{\frac{\partial}{\partial z}\varphi(\tau,z)}{\varphi(\tau,z)}dz = 8m,$$

where P is a fundamental domain for $\mathbb{C}/2\mathbb{Z}+2\tau\mathbb{Z}$. Namely, the function $z \mapsto \varphi(\tau,z)$ has exactly $8m$ zeros in a fundamental domain for $\mathbb{C}/2\mathbb{Z}+2\tau\mathbb{Z}$. Because

$$\epsilon_2\mathbf{e}(m(\tau+z))\varphi(\tau,z+\tau) = \varphi(\tau,z) \quad \text{and} \quad \epsilon_3\varphi(\tau,z+1) = \varphi(\tau,z),$$

the number of zeros in a fundamental domain for $\mathbb{C}/2\mathbb{Z}+2\tau\mathbb{Z}$ is just four times that in a fundamental domain for $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. Therefore the function $z \mapsto \varphi(\tau,z)$ has exactly $2m$ zeros in a fundamental domain for $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$. □

Table 1.

ϵ_0	ϵ_1	ϵ_2	ϵ_3	example	weight	index	Γ
1	1	1	1	$\varphi_{0,1} := \varphi_A^2+\varphi_B^2+\varphi_C^2$	0	1	$\mathrm{SL}(2,\mathbb{Z})$
				$\varphi_{-2,1} := \varphi_D^2$	-2	1	$\mathrm{SL}(2,\mathbb{Z})$
1	1	1	-1	$\varphi_A\varphi_C$	0	1	Γ_B
1	1	-1	1	$\varphi_A\varphi_B$	0	1	Γ_C
1	1	-1	-1	$\varphi_B\varphi_C$	0	1	Γ_A
1	-1	1	1	$\varphi_{-1,2} := \frac{1}{4}\varphi_A\varphi_B\varphi_C\varphi_D$	-1	2	$\mathrm{SL}(2,\mathbb{Z})$
1	-1	1	-1	$\varphi_B\varphi_D$	-1	1	Γ_B
1	-1	-1	1	$\varphi_C\varphi_D$	-1	1	Γ_C
1	-1	-1	-1	$\varphi_A\varphi_D$	-1	1	Γ_A
-1	1	1	1	φ_A	0	1/2	Γ_A
-1	1	1	-1	φ_C	0	1/2	Γ_C
-1	1	-1	1	φ_B	0	1/2	Γ_B
-1	1	-1	-1	$\varphi_A\varphi_B\varphi_C$	0	3/2	$\mathrm{SL}(2,\mathbb{Z})$
-1	-1	1	1	$\varphi_B\varphi_C\varphi_D$	-1	3/2	Γ_A
-1	-1	1	-1	$\varphi_A\varphi_B\varphi_D$	-1	3/2	Γ_C
-1	-1	-1	1	$\varphi_A\varphi_C\varphi_D$	-1	3/2	Γ_B
-1	-1	-1	-1	φ_D	-1	1/2	$\mathrm{SL}(2,\mathbb{Z})$

Here, for $s \in \mathbb{Z}$, we define a character of Γ_{k+s} by

$$(p_s(\chi))(\gamma,J) := \chi((\gamma,\mathbf{0},0),\widetilde{J}\,), \qquad \widetilde{J}(\tau,z) := (c\tau+d)^s J(\tau)\mathbf{e}\left(\frac{mcz^2}{c\tau+d}\right).$$

Proposition 7. *We have*

$$\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) = \{0\} \quad (m<0)$$

and

$$\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,0};\chi) = \begin{cases} \mathbb{M}(\Gamma_k; p_0(\chi)) & (\epsilon_1 = \epsilon_2 = \epsilon_3 = 1), \\ \{0\} & (\mathit{otherwise}). \end{cases}$$

Proof. When $m < 0$, Proposition 6 induces $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) = \{0\}$ immediately. When $m = 0$, Proposition 6 induces that $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,0};\chi)$ does not depend on z. Therefore, since

$$\varphi(\tau, z) = \epsilon_1\varphi(\tau, -z) = \epsilon_2\varphi(\tau, z + \tau) = \epsilon_3\varphi(\tau, z + 1),$$

φ should be zero unless $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. In the case $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, easily we can see $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,0};\chi) = \mathbb{M}(\Gamma_k; p_0(\chi))$. □

For any non-negative integer s, we define an operator D_s by $D_s := \frac{\partial^s}{\partial^s z}|_{z=0}$. Let

$$\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[s] := \left\{\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \mid D_t(\varphi) = 0 \text{ for } 0 \le t < s\right\}.$$

From Proposition 6, we have $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[2m + 1] = \{0\}$. By definition, we have an exact sequence

$$0 \longrightarrow \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[s + 1] \longrightarrow \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[s] \xrightarrow{D_s} \mathbb{M}(\Gamma_{k+s}; p_s(\chi)).$$

The following lemma is immediately induced from $\varphi(\tau, z) = \epsilon_1\varphi(\tau, -z)$.

Lemma 8. *The image of D_s satisfies the following properties. When $\epsilon_1 = 1$, the image $D_s\left(\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[s]\right) = \{0\}$ if s is odd. When $\epsilon_1 = -1$, the image $D_s\left(\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[s]\right) = \{0\}$ if s is even.*

This lemma induces an upper bound of the dimension of the space of weak Jacobi forms. We have

$$\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \le \begin{cases} \displaystyle\sum_{t=0}^{\lfloor m \rfloor} \dim_{\mathbb{C}} \mathbb{M}(\Gamma_{k+2t}; p_{2t}(\chi)) & (\epsilon_1 = 1), \\ \displaystyle\sum_{t=0}^{\lfloor m-\frac{1}{2} \rfloor} \dim_{\mathbb{C}} \mathbb{M}(\Gamma_{k+2t+1}; p_{2t+1}(\chi)) & (\epsilon_1 = -1) \end{cases}$$

where $\lfloor m \rfloor$ means the largest integer not greater than m.

The above estimation seems to be rough, however, surprisingly, in the case of $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$, this coincides with the true dimension of the space of weak Jacobi forms. As $D_0(\varphi_{0,1}) = 12$ and $D_0(\varphi_{-2,1}) = 0$, two weak Jacobi forms $\varphi_{0,1}$ and $\varphi_{-2,1}$ are algebraically independent on the ring of all holomorphic functions on $\mathbb{H}$. Hence we have the following theorem.

Theorem 9. *Let $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$ and we assume $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. As $\epsilon_0 = 1$, here $m \in \mathbb{Z}$. Then we have the following:*
(1) *If $m < 0$, $\varphi = 0$.*
(2) *If $m = 0$, φ does not depend on z and $\varphi \in \mathbb{M}(\Gamma_k; p_0(\chi))$.*
(3) *If $m > 0$, there exist $f_j \in \mathbb{M}(\Gamma_{k+2j}; p_{2j}(\chi))$ $(j = 0, 1, \ldots, m)$ such that*

$$\varphi = \sum_{j=0}^{m} f_j \varphi_{0,1}^{m-j} \varphi_{-2,1}^{j}.$$

REMARK. This theorem is a generalization of Theorem 2, which gives the structure of weak Jacobi forms of integral weights and indices.

In another 15 cases, we need a bit more investigation to obtain the true dimension of the space of weak Jacobi forms. Again let $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$. As

$$\begin{aligned}\varphi(\tau,z) &= \epsilon_1\varphi(\tau,-z),\\ \varphi(\tau,z) &= \epsilon_2\mathbf{e}(m(\tau+2z))\varphi(\tau,z+\tau),\\ \varphi(\tau,z) &= \epsilon_3\varphi(\tau,z+1),\end{aligned}$$

and $\epsilon_0 = \mathbf{e}(m)$, we have the following properties:
(1) If $\epsilon_1\epsilon_2 = -1$, then φ vanishes at $z = \frac{\tau}{2}$.
(2) If $\epsilon_1\epsilon_3 = -1$, then φ vanishes at $z = \frac{1}{2}$.
(3) If $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 = -1$, then φ vanishes at $z = \frac{1+\tau}{2}$.
These conditions decrease the possible multiplicity of zero of φ at $z = 0$. Then we have a bit more sharp upper bound of the dimension of the space of weak Jacobi forms.

To avoid complication, for a while we treat only the case $\epsilon_0 = \epsilon_1 = \epsilon_2 = -1$ and $\epsilon_3 = 1$. In this case, $\varphi \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$ vanishes at $z = \frac{\tau}{2}, \frac{1}{2}, \frac{1+\tau}{2}$. Hence, from Proposition 5 and Proposition 6, we have

$$\begin{cases} \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)[2m-2] = \{0\} & (\Gamma \subset \Gamma_B),\\ \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) = \{0\} & (\text{otherwise}).\end{cases}$$

Therefore, when $\Gamma \subset \Gamma_B$, we have

$$\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) \leq \sum_{t=0}^{m-3/2} \dim_{\mathbb{C}} \mathbb{M}(\Gamma_{k+2t+1}; p_{2t+1}(\chi)).$$

Now, we recall that $\varphi_A\varphi_C\varphi_D \in \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{-1,\frac{3}{2}};\chi_A\chi_C\chi_D)$ is an example of weak Jacobi forms of this case, given in the previous section. Since

$$\begin{aligned}\dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) &\leq \sum_{t=0}^{m-3/2} \dim_{\mathbb{C}} \mathbb{M}(\Gamma_{k+2t+1}; p_{2t+1}(\chi))\\ &= \dim_{\mathbb{C}} \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k+1,m-3/2};\chi\,(\chi_A\chi_C\chi_D)^{-1}),\end{aligned}$$

we have

$$\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) = \begin{cases} \varphi_A\varphi_C\varphi_D\, \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k+1,m-3/2};\chi\,(\chi_A\chi_C\chi_D)^{-1}) & (\Gamma \subset \Gamma_B),\\ \{0\} & (\text{otherwise}).\end{cases}$$

Here we remark that the character $\chi\,(\chi_A\chi_C\chi_D)^{-1}$ has a parameter $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. Therefore, the structure of $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k+1,m-3/2};\chi\,(\chi_A\chi_C\chi_D)^{-1})$ has already been given in Theorem 9.

Not only in this case, but also in all another cases, we can determine the structure of $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi)$ in the same manner. Consequently, we have the following our main theorem.

Theorem 10. *Let Γ be a finite index subgroup of* $\mathrm{SL}(2,\mathbb{Z})$ *and χ be a character of $\Gamma^{\mathrm{J}}_{k,m}$. We assume the character $\widetilde{\chi}$ is not in the case $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. Then we have*

$$\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k,m};\chi) = \begin{cases} \psi\, \mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k-k_0,m-m_0};\chi\chi_0^{-1}) & (\Gamma \subset \Gamma_0), \\ \{0\} & (\textit{otherwise}). \end{cases}$$

Here ψ and Γ_0 are a weak Jacobi form and a finite index subgroup of $\mathrm{SL}(2,\mathbb{Z})$*, which are given in the following* Table 2 *corresponds to the parameters $\epsilon_0, \epsilon_1, \epsilon_2$ and ϵ_3. k_0, m_0 and χ_0 are the weight, the index and the character of ψ. We remark that the character $\chi\chi_0^{-1}$ has a parameter $\epsilon_0 = \epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. Therefore, the structure of $\mathbb{J}^{\mathrm{w}}(\Gamma^{\mathrm{J}}_{k-k_0,m-m_0};\chi\chi_0^{-1})$ has already been given in* Theorem 9.

Table 2.

ϵ_0	ϵ_1	ϵ_2	ϵ_3	ψ	k_0	m_0	Γ_0
1	1	1	−1	$\varphi_A\varphi_C$	0	1	Γ_B
1	1	−1	1	$\varphi_A\varphi_B$	0	1	Γ_C
1	1	−1	−1	$\varphi_B\varphi_C$	0	1	Γ_A
1	−1	1	1	$\varphi_{-1,2} := \frac{1}{4}\varphi_A\varphi_B\varphi_C\varphi_D$	−1	2	$\mathrm{SL}(2,\mathbb{Z})$
1	−1	1	−1	$\varphi_B\varphi_D$	−1	1	Γ_B
1	−1	−1	1	$\varphi_C\varphi_D$	−1	1	Γ_C
1	−1	−1	−1	$\varphi_A\varphi_D$	−1	1	Γ_A
−1	1	1	1	φ_A	0	1/2	Γ_A
−1	1	1	−1	φ_C	0	1/2	Γ_C
−1	1	−1	1	φ_B	0	1/2	Γ_B
−1	1	−1	−1	$\varphi_A\varphi_B\varphi_C$	0	3/2	$\mathrm{SL}(2,\mathbb{Z})$
−1	−1	1	1	$\varphi_B\varphi_C\varphi_D$	−1	3/2	Γ_A
−1	−1	1	−1	$\varphi_A\varphi_B\varphi_D$	−1	3/2	Γ_C
−1	−1	−1	1	$\varphi_A\varphi_C\varphi_D$	−1	3/2	Γ_B
−1	−1	−1	−1	φ_D	−1	1/2	$\mathrm{SL}(2,\mathbb{Z})$

By Theorem 9 and Theorem 10, now we know the structure of the space of all weak Jacobi forms of arbitrary real weight and index. If we say above theorem simply like a catchword, the bigraded ring of weak Jacobi forms of real weights and indices are generated by $\varphi_A, \varphi_B, \varphi_C$ and φ_D over the graded ring of elliptic modular forms of real weights.

5. Application

At the end of this paper, we give one application. Jacobi forms appear in the Fourier-Jacobi expansion of Siegel modular forms. To determine the structure of the graded ring of Siegel modular forms, the Fourier-Jacobi expansion is very helpful, because we have already known the structure of (weak) Jacobi forms in many cases (e.g. [1, 3]). In the case of Siegel paramodular forms of degree 2 with its level ≤ 4, the structure has already been determined by Ibukiyama, Poor and Yuen [10]. Using their method (more precisely in [2]), here we will show possible weights of Siegel paramodular forms of degree 2 for arbitrary level and character.

First we review Siegel modular forms of degree 2 shortly. We denote Siegel upper half space of degree 2 by

$$\mathbb{H}_2 := \left\{ Z = {}^tZ = \begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} \in \mathrm{M}(2, \mathbb{C}) \;\middle|\; \mathrm{Im}\, Z > 0 \right\}.$$

The symplectic group

$$\mathrm{Sp}(2, \mathbb{R}) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{M}(4, \mathbb{R}) \;\middle|\; {}^tMJM = J := \begin{pmatrix} O & -E_2 \\ E_2 & O \end{pmatrix} \right\}$$

acts on $\mathbb{H}_2$ by

$$\mathbb{H}_2 \ni Z \longmapsto M\langle Z\rangle := (AZ + B)(CZ + D)^{-1} \in \mathbb{H}_2.$$

The automorphic factor of weight $k \in \mathbb{Z}$ is $\det(CZ + D)^k$ and satisfies the cocycle condition if $k \in \mathbb{Z}$. For a holomorphic function F on $\mathbb{H}_2$, we define

$$(F|_kM)(Z) := \det(CZ + D)^{-k} F(M\langle Z\rangle).$$

The paramodular group of level N is defined by

$$K(N) := \begin{pmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \frac{1}{N}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap \mathrm{Sp}(2, \mathbb{R}).$$

Let

$$V_N := \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -N & 0 \end{pmatrix} \in \mathrm{Sp}(2, \mathbb{R})$$

and we denote by $K^*(N)$ the subgroup of $\mathrm{Sp}(2, \mathbb{R})$ generated by $K(N)$ and V_N. We can easily show that $[K^*(N) : K(N)] = 2$. Let χ be a character of $K^*(N)$.

DEFINITION. We say a holomorphic function F on $\mathbb{H}_2$ is a Siegel paramodular modular form of degree 2 of level N and weight k with character χ if F satisfies the condition $F|_k\gamma = \chi(\gamma)F$ for any $\gamma \in K^*(N)$. We denote by $\mathbb{M}_k(K^*(N); \chi)$ the $\mathbb{C}$-vector space of all Siegel paramodular modular forms of degree 2 of level N and weight k with character χ.

In the same manner as elliptic modular forms and weak Jacobi forms, we can define Siegel paramodular forms of real weight. For $k \in \mathbb{R}$, we denote by $\mathbb{M}(K^*(N)_k; \chi)$ the $\mathbb{C}$-vector space of all Siegel paramodular forms with respect to $K^*(N)_k$ and its character χ.

Let $F \in \mathbb{M}(K^*(N)_k; \chi)$ be a nonzero modular form. In the same manner as elliptic modular forms and weak Jacobi forms, F has a Fourier expansion

$$F(Z) = \sum_{n,l,m} c(n,l,m)\mathbf{e}(n\tau + lz + m\omega). \tag{1}$$

From the translation formula of $\widetilde{V_N} := (V_N, 1)$, we have

$$F\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \chi(\widetilde{V_N})F\begin{pmatrix} N\omega & -z \\ -z & N^{-1}\tau \end{pmatrix}.$$

Hence we have the symmetry of the Fourier coefficients:

$$c(N^{-1}m, -l, Nn) = \chi(\widetilde{V_N})c(n,l,m). \tag{2}$$

On the Fourier-Jacobi expansion

$$F(Z) = \sum_m \varphi_m(\tau,z)\mathbf{e}(m\omega), \quad \varphi_m(\tau,z) := \sum_{n,l} c(n,l,m)\mathbf{e}(n\tau + lz), \tag{3}$$

easily we can see each φ_m is a Jacobi form of weight k and index m, namely, $\varphi_m \in \mathbb{J}^{\mathrm{w}}(\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}_{k,m}; \chi_m)$, where χ_m is a suitable character of $\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}_{k,m}$. As we know the structure of $\mathbb{J}^{\mathrm{w}}(\mathrm{SL}(2,\mathbb{Z})^{\mathrm{J}}_{k,m}; \chi_m)$, in the paper by Ibukiyama, Poor and Yuen [10], they determined the structure of $\mathbb{M}_k(K^*(N); \mathbf{1}_{K^*(N)})$ by using (2) and (3) cleverly for $N \leqq 4$.

Here we consider the general case $N \in \mathbb{N}$. Again let $F \in \mathbb{M}(K^*(N)_k; \chi)$ be a nonzero modular form, where χ is an arbitrary character of $K^*(N)_k$. By Proposition 4, on the Fourier expansion (1), $c(n,l,m) = 0$ unless $2m \in \mathbb{Z}$. Hence, by the symmetry (2), $c(n,l,m) = 0$ unless $2Nn \in \mathbb{Z}$. In Theorem 1, 9 and 10, all generators except the real power of the Dedekind eta function has no Fourier coefficients at $n \notin \mathbb{Z}$. The weight of the real power of the Dedekind eta function with nonzero Fourier coefficients at $n \notin \mathbb{Z}$ should be in $12(n+\mathbb{Z}) \subset \frac{6}{N}\mathbb{Z}$. In Theorem 1, 9 and 10, the weights of all generators except the real power of the Dedekind eta function is integral, therefore, k should be in $\frac{6}{N}\mathbb{Z} + \mathbb{Z} = \frac{(N,6)}{N}\mathbb{Z}$. Namely, we have the following theorem:

Theorem 11. *Possible weights of Siegel paramodular forms of degree* 2 *with level N is in $\frac{(N,6)}{N}\mathbb{Z}$.*

We remark that, on the conference at RIMS (Kyoto) in 2003, Richard Hill [8] stated that there is no modular forms of $\mathrm{Sp}(g,\mathbb{R})$ ($g \geq 2$) whose weights are neither integral nor half-integral. He commented that this was pointed out by Deligne [5]. Our result is another one on possible weights of modular forms of several variables.

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