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POLYLOGARITHMIC ANALOGUE OF THE COLEMAN-IHARA FORMULA, I

Dedicated to Professor Yasutaka Ihara
on the occasion of his 77 th birthday

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Abstract

The Coleman-Ihara formula expresses Soule's p-adic characters restricted to p-local Galois group as the Coates-Wiles homomorphism multiplied by p-adic L-values at positive integers. In this paper, we show an analogous formula that ℓ-adic polylogarithmic characters for $\ell = p$ restrict to the Coates-Wiles homomorphism multiplied by Coleman's p-adic polylogarithms at any roots of unity of order prime to p.

1. Introduction

Let p be an odd prime. In his Annals article [12], Yasutaka Ihara introduced the universal power series for Jacobi sums $F_\sigma \in \mathbb{Z}_p[\![u,v]\!]^\times$ ($\sigma \in G_\mathbb{Q} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$) and showed a beautiful (local) formula ([12] Theorem C) relating its coefficient characters with p-adic L-values $L_p(m, \omega^{1-m})$ ($m \geq 3$: odd) multiplied by the m-th Coates-Wiles homomorphisms when σ lies in the p-local subgroup of $G_{\mathbb{Q}(\mu_{p^\infty})}$. In the last part of [12] (p.105, (Col2)) documented is that Robert Coleman proved that these coefficient characters are nothing but the restrictions of Soule's cyclotomic elements in $H^1(G_\mathbb{Q}, \mathbb{Z}_p(m))$. This motivated later works by Anderson [1], Coleman [7], Ihara-Kaneko-Yukinari [14] toward the explicit (global) formula of F_σ for all $\sigma \in G_\mathbb{Q}$ (see [13]). Their formula presents particularly a remarkable symmetric form on the main part $G_{\mathbb{Q}(\mu_{p^\infty})}$ of $G_\mathbb{Q}$ as:

$$F_\sigma(u,v) = \exp\left(\sum_{\substack{m\geq 3 \\ odd}} \frac{\chi_m(\sigma)}{p^{m-1}-1} \sum_{\substack{i+j=m \\ i,j\geq 1}} \frac{U^i V^j}{i!\,j!} \right) \qquad (\sigma \in G_{\mathbb{Q}(\mu_{p^\infty})}), \tag{1.1}$$

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where $1+u=e^U$, $1+v=e^V$ and $\chi_m : G_{\mathbb{Q}(\mu_{p^\infty})} \to \mathbb{Z}_p(m)$ is the m-th Soule character ([28]) defined by the properties:

$$(1.2) \qquad \Big(\prod_{\substack{1\le a<p^n \\ p\nmid a}} (1-\zeta_{p^n}^a)^{a^{m-1}}\Big)^{\frac{1}{p^n}(\sigma-1)} = \zeta_{p^n}^{\chi_m(\sigma)} \quad (n\ge 1).$$

This, together with the above mentioned Ihara's local formula [12] Theorem C, implies Coleman's formula presented in [12] p.105 in the form:

$$(1.3) \qquad \frac{\chi_m(\mathrm{rec}(\epsilon))}{(p^{m-1}-1)} = L_p(m,\omega^{1-m})\phi_m^{CW}(\epsilon) \qquad (\epsilon\in\mathcal{U}_\infty)$$

for $m\ge 3$: odd. Let us quickly explain the notation used here: For each $n\ge 1$, we denote by $\mathcal{U}_n$ the group of principal units of $\mathbb{Q}_p(\mu_{p^n})$ and by $\mathcal{U}_\infty = \varprojlim_n \mathcal{U}_n$ their norm limit. Let Ω_p be the maximal abelian pro-p extension of $\mathbb{Q}(\mu_{p^\infty})$ unramified outside p. Then, Ihara's power series $\sigma\mapsto F_\sigma(u,v)$ factors through $\mathrm{Gal}(\Omega_p/\mathbb{Q})$. Now, fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$ and a coherent system of p-power roots of unity $\{\zeta_{p^n}\}_{n\ge 1}$ to identify $\mathbb{Z}_p$ with $\mathbb{Z}_p(m)$. This embedding and the local class field theory induce the canonical homomorphism $\mathrm{rec} : \mathcal{U}_\infty \hookrightarrow \mathrm{Gal}(\Omega_p/\mathbb{Q}(\mu_{p^\infty}))$ called the reciprocity map. On the other side, the system $\{\zeta_{p^n}\}_n$ determines, for $m\ge 1$, the Coates-Wiles homomorphism $\phi_m^{CW} : \mathcal{U}_\infty \to \mathbb{Z}_p$. The coefficient $L_p(m,\omega^{1-m})$ is the Kubota-Leopoldt p-adic L-value at m with respect to the power of the Teichmüller character ω.

Indeed, Coleman's paper [7] proves (1.3) by applying his theory on Hilbert norm residue symbols ([3], [4], [6]) to Jacobi sums which are special values $F_\sigma(\zeta_{p^n}^a-1,\zeta_{p^n}^b-1)$ at Frobenius elements σ over various primes in $\mathbb{Q}(\mu_{p^n})$ not dividing p. Especially, it relies on the Tchebotarev density. Consequently, the formula (1.3) results from combination of [7] and (1.1), relying on global arithmetic nature of $F_\sigma(u,v)$. The global proof certainly enables us to highlight (1.3) in contexts enriched with many important materials of Iwasawa theory (see, e.g., [11] §3-1). However, (1.3) itself is essentially of local nature, concerning the ratio between the Coates-Wiles homomorphism and Soule's character restricted on the local Galois group; Passing through the Jacobi sum interpolation properties of $F_\sigma(u,v)$ to derive (1.3) should look rather roundabout.

The purpose of this paper is to give an alternative direct proof of (1.3) and its polylogarithmic variants, where the Soule's characters χ_m in LHS are generalized to the (l-adic) Galois polylogarithms $\ell i_m(z,\gamma)$ (for the case $l=p$) introduced in [30]-[31]. They are defined as certain coefficients of Galois transforms of a defining path γ from $\overrightarrow{01}$ to z on $\mathbb{P}^1\setminus\{0,1,\infty\}$ (cf. § (2.4)). The values $L_p(m,\omega^{1-m})$ in RHS have obvious generalization to $\mathrm{Li}_m^{p\text{-adic}}(z)$, the p-adic polylogarithms of Coleman (see [4]). As a corollary of our main formula (Theorem 4.4), we obtain:

Theorem 1.1 (Corollary 4.6, Proposition 5.1, Remark 4.7)**.** *Let p be an odd prime, and F a finite unramified extension of $\mathbb{Q}_p$ with the Frobenius substitution $\sigma_F\in\mathrm{Gal}(F/\mathbb{Q}_p)$. Let $F_\infty := F(\mu_{p^\infty})$ and denote by $\phi_{m,F}^{CW} : G_{F_\infty}\to F\otimes\mathbb{Z}_p(m)$ the m-th Coates-Wiles homomorphism for the local field F (cf. Definition 3.5). Then, for any root of unity z contained in F, there is* *a standard specific path* $\overrightarrow{01} \rightsquigarrow z$ *such that*

$$\ell i_m(z,\gamma)(\sigma) = \frac{-1}{(m-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^m}\right)\mathrm{Li}_m^{p\text{-adic}}(z)\right\}\phi^{CW}_{m,F}(\sigma)\right) \tag{1.4}$$

holds for $m \geq 1$ *and for* $\sigma \in G_{F_\infty}$.

Our approach taken in this Part I is to apply the theory of Coleman power series (in a direct but different way from [7]) to the so called Koblitz measure that produces $\mathrm{Li}_m^{p\text{-adic}}(z)$ ([19]) on one hand, and on another hand, to the explicit formula of $\ell i_m(z)$ ([23]) generalizing the above Soule's characters (1.2). After the Introduction, in §2, we shall recall basic setup for Galois polylogarithms and p-adic polylogarithms, and in §3, we introduce and study a special family of Coleman power series that bridges these two kinds of polylogarithms through Coleman's reciprocity law. In §4, we present a general formula for p-adic polylogarithmic characters on the image of $\mathcal{U}_\infty(F) \xrightarrow{\text{rec}} G^{\text{ab}}_{F_\infty}$ (Theorem 4.4) and prove Theorem 1.1. A (direct) proof of the original Coleman-Ihara formula (1.3) is also obtained as a special case of $z = 1$, $F = \mathbb{Q}_p$ (Remark 4.5).

In a subsequent Part II [22], we study a generalization of the above formula to the case of more general $z \in (\mathbb{P}^1 \setminus \{0,1,\infty\})(\mathbb{Z}_p^{ur})$. For the generalization, we switch over to a view on the polylogarithmic torsors of paths in Deligne's Tannakian approach [De], and apply a non-commutative lift of Bloch-Kato's explicit reciprocity law that reverses logarithmic mapping of torsors studied by M.Kim [18], M.Olsson [25]. Here, we partially rely on [26] for some technical details.

We note that Kurihara [21] and Gros [17, II] §IV gave an alternative local approach to the essentially same formula as our above Theorem 1.1 by using syntomic cohomology. See [21] (2.11), (2.12) for the case of $m < p-2$ and a comment in (2.15) on extension to general case of $m > 1$ (in (2.12) of loc. cit., "$\mathrm{Tr}_{F/\mathbb{Q}_p}$" seems lacking in print). Compared to the Gros-Kurihara formulation, our approach in the present paper is of more elementary nature and may be useful to find a source reason behind the formula (1.4) in certain explicit Coleman power series $f_{z,c}(T)$ given in §3 below. Combined with illustrations in loc. cit. and [20] (in particular, p.425), our above result suggests that the l-adic Galois polylogarithm $\ell i_m(z,\gamma)$ stands nearby a shadow of Beilinson's cyclotomic element in $K_{2m-1}(F) \otimes \mathbb{Q}$ at least when z is a root of unity of order prime to p.

Notation. In this paper, we let p be a fixed odd prime. We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_p}$, which determines a coherent system of roots of unity $\zeta_n \in \mu_n \subset \overline{\mathbb{Q}}$ with $\zeta_n = \exp(2\pi i/n)$. Write $\mathbb{Z}_p^{ur}$ for the ring of integers of the maximal unramified extension $\mathbb{Q}_p^{ur}$ of $\mathbb{Q}_p$. The group of roots of unity $\mu(\mathbb{Z}_p^{ur})$ is isomorphic to $\bigcup_{p \nmid N} \mu_N$.

For any unramified extension F of $\mathbb{Q}_p$, we denote the arithmetic Frobenius substitution on F by $\sigma_F : F \to F$. We shall use the notation F_∞ to denote $F(\mu_{p^\infty})$, while we prefer in this paper to keep "F_n" unused to avoid confusion (with its custom usage '$F_n = F(\mu_{p^{n+1}})$' in Iwasawa theory.) We define $\mathcal{U}_\infty(F)$ to be the the norm limit of the group of principal units of $F(\mu_{p^n})$. The Galois group $\mathrm{Gal}(F_\infty/F)$ will be written as G_∞.

For any local field $F/\mathbb{Q}_p$, we write $\mathcal{O}_F$, k_F for the ring of integers and its residue field respectively. For any field F, we denote by G_F the absolute Galois group of F, and by χ_{cyc} the p-adic cyclotomic character $G_F \to \mathbb{Z}_p^\times$.

The Bernoulli polynomials and Bernoulli numbers are given by

$$\sum_{n=1}^{\infty} B_n(X)\frac{t^n}{n!} = \frac{te^{Xt}}{(e^t-1)}$$

and $B_n := B_n(0)$.

2. Review of Galois and p-adic polylogarithms

2.1. Galois polylogarithms. We review ℓ-adic Galois polylogarithms ([30]-[31]) in the case $\ell = p$. Let $\overrightarrow{01}$ be the unit tangential base point on $\mathbb{P}^1 - \{0,1,\infty\}$. We denote by π_1 the maximal pro-p quotient of the etale fundamental group $\pi_1^{\mathrm{et}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,1,\infty\}, \overrightarrow{01})$ and identify $\mathbb{Z}_p(1)$ with that of $\pi_1^{\mathrm{et}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0,\infty\}, \overrightarrow{01})$. We will always regard these fundamental groups as equipped with the actions of $G_{\mathbb{Q}}$ determined by $\overrightarrow{01}$.

Let $\mathbf{p} : \pi_1 \to \mathbb{Z}_p(1)$ be the projection homomorphism. We shall focus on the structure of the quotient group $\pi_1^{\mathrm{pol}} := \pi_1/[\mathrm{Ker}\,\mathbf{p}, \mathrm{Ker}\,\mathbf{p}]$ (the pro-p polylogarithmic quotient) that has the induced projection $\mathbf{p}' : \pi_1^{\mathrm{pol}} \to \mathbb{Z}_p(1)$. By construction, $\mathrm{Ker}\,\mathbf{p}'$ is abelian, hence has a conjugate action of $\mathrm{Im}(p) = \mathbb{Z}_p(1)$. In fact, as discussed in [10, Section 16.11-14], it forms a free $\mathbb{Z}_p[[\mathbb{Z}_p(1)]]$-module generated by a generator y of the inertia subgroup over the puncture 1: There arises a Galois equivariant exact sequence:

$$0 \to \mathbb{Z}_p[[\mathbb{Z}_p(1)]]\cdot y \to \pi_1^{\mathrm{pol}} \xrightarrow{\mathbf{p}'} \mathbb{Z}_p(1) \to 0. \tag{2.1}$$

The sequence turns out to split, as the image $\mathrm{Im}(\mathbf{p}') = \mathbb{Z}_p(1)$ can be lifted to a Galois-invariant inertia subgroup over 0 along $\overrightarrow{01}$. We shall take standard generators x, y of those respective inertia subgroups over 0, 1 so as to correspond to our choice of $\{\zeta_{p^n}\}_n \in \mu_{p^\infty}$. It follows then that π_1^{pol} is isomorphic to a semi-direct product:

$$\pi_1^{\mathrm{pol}} = x^{\mathbb{Z}_p} \ltimes (\mathbb{Z}_p[[x^{\mathbb{Z}_p}]]\cdot y) \cong \mathbb{Z}_p(1) \ltimes (\mathbb{Z}_p[[\mathbb{Z}_p(1)]](1)). \tag{2.2}$$

Here, the action in the last semi-direct product is given by the translation of $\mathbb{Z}_p(1)$. Let $\mathscr{P}^{\mathrm{et}}(\overrightarrow{01})$ be the pro-unipotent completion of π_1^{pol} (which we call the *p-adic etale polylogarithmic group*), and let $\log : \mathscr{P}^{\mathrm{et}}(\overrightarrow{01}) \to \mathrm{Lie}(\mathscr{P}^{\mathrm{et}}(\overrightarrow{01})) = \mathbb{Q}_p(1) \oplus \prod_{k=1}^{\infty} \mathbb{Q}_p(k)$ its logarithm map. We have then the following Lie expansion map (also denoted log):

$$\log : \pi_1^{\mathrm{pol}} \hookrightarrow \mathscr{P}^{\mathrm{et}}(\overrightarrow{01}) \longrightarrow \mathrm{Lie}(\mathscr{P}^{\mathrm{et}}(\overrightarrow{01})) = \mathbb{Q}_p(1) \oplus \prod_{k=1}^{\infty} \mathbb{Q}_p(k). \tag{2.3}$$

In practice, both of $\mathscr{P}^{\mathrm{et}}(\overrightarrow{01})$ and $\mathrm{Lie}(\mathscr{P}^{\mathrm{et}}(\overrightarrow{01}))$ are realized as subsets of the non-commutative power series ring $\mathbb{Q}_p\langle\langle X, Y\rangle\rangle$ in $X = \log(x)$, $Y = \log(y)$ modulo the ideal I_Y generated by the words having Y twice or more.

Now, take any point $z \in (\mathbb{P}^1 \setminus \{0,1,\infty\})(\overline{\mathbb{Q}_p})$ and consider the set $\pi_1(\overrightarrow{01}, z)$ of the pro-p etale paths from $\overrightarrow{01}$ to z. Set $F := \mathbb{Q}_p(z)$. Then $\pi_1(\overrightarrow{01}, z)$ forms a π_1-torsor equipped with a canonical action of G_F. In the well-known manner, the reduction map $\pi_1 \twoheadrightarrow \pi_1^{\mathrm{pol}}$ gives rise to a π_1^{pol}-torsor $\pi_1^{\mathrm{pol}}(\overrightarrow{01}, z)$ with G_F-action on it.

Choose any path $\gamma \in \pi_1^{\mathrm{pol}}(\overrightarrow{01}, z)$ and call it the *defining path*. Then, for each $\sigma \in G_F$, there is a unique element $\mathfrak{f}_\sigma \in \pi_1^{\mathrm{pol}}$ such that γ is written as $\mathfrak{f}_\sigma^z * \sigma(\gamma)$. The p-adic Galois polylogarithms (associated to γ) are defined as coefficients of the expansion of the Lie series $\log(\mathfrak{f}_\sigma^z)$ in $\mathbb{Q}_p[[X, Y]]/I_Y$ as:

$$\log(\mathfrak{f}_\sigma^z) \equiv -\kappa_z(\sigma)X - \sum_{k=1}^{\infty} \ell i_k(z)(\sigma) \cdot ad(X)^{k-1}(Y). \tag{2.4}$$

The above first coefficient $\kappa_z : G_F \to \mathbb{Z}_p$ is the Kummer 1-cocycle

$$\zeta_{p^n}^{\kappa_z(\sigma)} = (z^{1/p^n})^{\sigma-1} := \frac{\sigma(z^{1/p^n})}{z^{1/p^n}} \tag{2.5}$$

over the system $\{z^{1/p^n}\}_{n=1}^{\infty}$ which is determined by the specialization homomorphism $\bigcup_{n=1}^{\infty} \overline{F}[t^{1/p^n}] \to \overline{F}$ along the path γ. The second coefficient $\ell i_1(z) : G_F \to \mathbb{Z}_p$ is the Kummer 1-cocycle κ_{1-z} along the composition of the standard path $0 \rightsquigarrow 1$ with $\bar{\gamma} : 1 \rightsquigarrow 1 - z$, where $\bar{\gamma}$ is the obvious reflection of γ. The other coefficients $\ell i_k(z)(= \ell i_k(z, \gamma))$ $(k \geq 2)$ are in general only 1-cochains $G_F \to \mathbb{Q}_p$.

The following lemma is crucial to understand nature of the LHS of our main statement of Introduction.

Lemma 2.1 ([30] Theorem 5.3.1)**.** *The fixed field H_m of the intersection of the kernels of χ_{cyc}, κ_z and of $\ell i_1(z, \gamma), \ldots, \ell i_{m-1}(z, \gamma)$ is independent of the choice of the defining path $\gamma : \overrightarrow{01} \rightsquigarrow z$. Moreover, for $\sigma \in G_{H_m}$, the value $\ell i_m(z, \gamma) \in \mathbb{Q}_p$ is independent of the choice of γ.* □

Explicit formulas for $\ell i_k(z)$ have been given in [23] for all $\sigma \in G_F$. For simplicity, we present the formula only for $\sigma \in G_F$ with $\chi_{\mathrm{cyc}}(\sigma) = 1$ and $\kappa_z(\sigma) = 0$ in the following

Proposition 2.2 ([23] §3 Corollary)**.** *For $m \geq 1$ and $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$, we have*

$$\ell i_m(z)(\sigma) = (-1)^{m-1} \frac{\tilde{\chi}_m^z(\sigma)}{(m-1)!} \tag{2.6}$$

where $\tilde{\chi}_m^z(\sigma) \in \mathbb{Z}_p$ is defined by the Kummer properties

$$\zeta_{p^n}^{\tilde{\chi}_m^z(\sigma)} = \left(\prod_{a=0}^{p^n-1} (1 - \zeta_{p^n}^a z^{1/p^n})^{\frac{a^{m-1}}{p^n}} \right)^{\sigma-1} \qquad (n \geq 1). \quad \square \tag{2.7}$$

Indeed, in [23], where an embedding $\overline{F} \hookrightarrow \mathbb{C}$ is fixed and $\gamma : \vec{01} \rightsquigarrow z$ is taken to be a continous curve on $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$, we constructed a measure $\hat{\boldsymbol{\kappa}}_{z,\gamma}(\sigma) \in \hat{\mathbb{Z}}[[\hat{\mathbb{Z}}]]$ called the (adelic) *Kummer-Heisenberg measure*, after specifying standard branches of power roots of $(1 - \zeta_n^a z^{1/n})$ along γ. (In fact, in loc.cit., we assumed $z \in F$ a number field, but the argument goes similarly for any subfield F of $\mathbb{C}$.) The $\mathbb{Z}_p$-valued measure $\boldsymbol{\kappa}_{z,\gamma}(\sigma) \in \mathbb{Z}_p[[\mathbb{Z}_p]]$ obtained as the image of $\hat{\kappa}_{z,\gamma}(\sigma)$ is given by

$$\zeta_{p^r}^{\kappa_{z,\gamma}(\sigma)(a+p^n\mathbb{Z}_p)} = \frac{\sigma((1 - \zeta_{p^n}^{\chi_{\mathrm{cyc}}(\sigma)^{-1}a} z^{1/p^n})^{\frac{1}{p^r}})}{(1 - \zeta_{p^n}^a \sigma(z^{1/p^n}))^{\frac{1}{p^r}}} \qquad (a \in \mathbb{Z}_p,\ r \geq 1) \tag{2.8}$$

for each $\sigma \in G_F$. The *p-adic polylogarithmic character* $\tilde{\chi}^z_m : G_F \to \mathbb{Z}_p$ defined in [23] can be written as the moment integral

$$\tilde{\chi}^z_m(\sigma) = \int_{\mathbb{Z}_p} x^{m-1} d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(x) \quad (\sigma \in G_F). \tag{2.9}$$

In this paper, we shall also consider a restricted version of the above moment integral to $\mathbb{Z}_p^\times$, and introduce the *restricted p-adic polylogarithmic character* $\chi^z_m : G_F \to \mathbb{Z}_p$ by

$$\chi^z_m(\sigma) = \int_{\mathbb{Z}_p^\times} x^{m-1} d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(x) \quad (\sigma \in G_F). \tag{2.10}$$

Then, it is easy to see from Proposition 2.2 that the value $\chi^z_m(\sigma) \in \mathbb{Z}_p$ for $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$ is characterized by the Kummer properties

$$\zeta_{p^n}^{\chi^z_m(\sigma)} = \Bigg(\prod_{\substack{1 \le a \le p^n \\ p \nmid a}} (1 - \zeta^a_{p^n} z^{1/p^n})^{\frac{a^{m-1}}{p^n}} \Bigg)^{\sigma - 1} \qquad (n \ge 1). \tag{2.11}$$

When $z = 1$ and $F = \mathbb{Q}$, this is nothing but what is called the m-th Soule character.

Remark 2.3. Our considering the measure $\hat{\boldsymbol{\kappa}}_{z,\gamma}(\sigma)$ should be traced back partly to an old idea of O. Gabber (as documented in [23] §3 Remark 1 and [24] Acknowledgments). In [32], it is generalized to a sequence of measures '$K_r(z)$' on $\mathbb{Z}_p^r$ ($r \ge 1$) that encodes all coefficients of $\log(\mathfrak{f}^z_\sigma)$ (which correspond to the multiple polylogarithms) as integrals over $\mathbb{Z}_p^r$.

A simple connection between $\tilde{\chi}^z_m$ and χ^z_m can be obtained by comparing (2.7) and (2.11). The following generalizes [23] §2 Remark 2:

Lemma 2.4 ([32] Proposition 5.1(v))**.** *For any continous path $\gamma : \overrightarrow{01} \rightsquigarrow z$ with respect to an embedding $\overline{F} \hookrightarrow \mathbb{C}$, any $\sigma \in G_F$, and $m \ge 2$, we have*

$$\tilde{\chi}^z_m(\sigma) = \chi^z_m(\sigma) + p^{m-1}\chi^{z^{1/p}}_m(\sigma) + \cdots = \sum_{k=0}^{\infty} p^{k(m-1)} \chi^{z^{1/p^k}}_m(\sigma).$$

Proof. Denote the path $\overrightarrow{01} \rightsquigarrow z^{1/p^k}$ induced from $\gamma : \overrightarrow{01} \rightsquigarrow z$ by the same symbol γ. Then, it is easy to see from (2.8) that $\boldsymbol{\kappa}_{z,\gamma}(\sigma)(p^k a + p^{n+k}\mathbb{Z}_p) = \boldsymbol{\kappa}_{z^{1/p^k},\gamma}(\sigma)(a + p^n\mathbb{Z}_p)$ for all $k \ge 0$. Then, decomposing the integral to infinite pieces as

$$\begin{aligned} \int_{\mathbb{Z}_p} x^{m-1} d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(x) &= \sum_{k=0}^{\infty} \int_{p^k\mathbb{Z}_p - p^{k+1}\mathbb{Z}_p} x^{m-1} d\boldsymbol{\kappa}_{z,\gamma}(\sigma)(x) \\ &= \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^\times} (p^k x)^{m-1} d\boldsymbol{\kappa}_{z^{1/p^k},\gamma}(\sigma)(x) \end{aligned}$$

immediately yields the desired formula. $\square$

Remark 2.5. As noted above (after (2.5)), every etale path $\gamma : \overrightarrow{01} \rightsquigarrow z$ determines the value $z^{1/p^n} \in \overline{F}$, hence the above Kummer quantity in (2.8) makes sense for $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$ with no need to mention the choice of $\overline{F} \hookrightarrow \mathbb{C}$. So, in what follows, we will consider both versions of p-adic polylogarithmic characters $\tilde{\chi}^z_m, \chi^z_m$ for arbitrary $\gamma \in \pi_1^{\mathrm{pol}}(\overrightarrow{01}, z)$ but only for $\sigma \in G_{F(\zeta_{p^\infty}, z^{1/p^\infty})}$. Formulas in Proposition 2.2 and Lemma 2.4 also hold for them.

2.2. Koblitz measure and p-adic polylogarithms. Let $|\,|_p$ denote the standard norm on $\mathbb{C}_p$ with $|p|_p = p^{-1}$ and write $\mathcal{O} := \mathcal{O}_{\mathbb{C}_p} = \{z \in \mathbb{C}_p; |z|_p \leq 1\}$. For $z \in \mathbb{C}_p$ with $|1 - z|_p \geq 1$, Neal Koblitz introduced in [19, p. 457], an $\mathcal{O}$-valued measure μ_z on $\mathbb{Z}_p$ by

$$\mu_z(a + p^n\mathbb{Z}_p) = \frac{z^a}{1 - z^{p^n}} \text{ for all } n \in \mathbb{Z},\ 1 \leq a \leq p^n.$$

Note here that $|1 - z^{p^n}|_p = |z^{p^n}|_p$ if $|z|_p > 1$ and that $|1 - z^{p^n}|_p \geq 1$ if $|z|_p \leq 1$ for all $n \geq 0$ under our assumption on z.

Lemma 2.6. *Let $\mathscr{F}_z(T)$ be the element of the Iwasawa algebra $\mathcal{O}[\![T]\!]$ that corresponds to μ_z, and let $\mu_z^{(p)}$ be the restriction of the measure μ_z to $\mathbb{Z}_p^\times$. Then,*

$$\text{(1)} \qquad \mathscr{F}_z(T) = \frac{1}{1 - z(1 + T)} \in \mathcal{O}[\![T]\!],$$

$$\text{(2)} \qquad \mathscr{F}_z^{(p)}(T) = \frac{1}{1 - z(1 + T)} - \frac{1}{1 - z^p(1 + T)^p}.$$

Proof. The first formula is a consequence of the congruence

$$\mathscr{F}_z(T) = \frac{1}{1 - z^{p^n}}\left(\frac{1 - z^{p^n}}{1 - z(1 + T)}\right) \equiv \sum_{a=0}^{p^n-1} \frac{z^a}{1 - z^{p^n}}(1 + T)^a$$

modulo the ideal $((1 + T)^{p^n} - 1)$ in $\mathcal{O}[\![T]\!]$ for every n. For the second one, recalling a formula [9] §3.4 for the restriction of a measure on $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$, we obtain $\mathscr{F}_z^{(p)} = \mathscr{F}_z(T) - \frac{1}{p}\sum_{\xi\in\mu_p} \mathscr{F}_z(\xi(1 + T) - 1)$. Apply then a general formula $\sum_{\xi\in\mu_n} \frac{1}{1-\xi Y} = \frac{n}{1-Y^n}$. □

Note that there is an equality

$$\text{(2.12)} \qquad \mathscr{F}_z^{(p)}(T) = \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^\times} x^n d\mu_z(x)\right)\frac{X^n}{n!}$$

in $\mathbb{C}_p[\![X]\!]$ with $1 + T = e^X$.

In [7], Coleman introduced the p-adic polylogarithm function $\mathrm{Li}_k^{p\text{-adic}}(z)$ ($k \geq 1$) and its companion function $\mathrm{Li}_k^{(p)}(z) := \mathrm{Li}_k^{p\text{-adic}}(z) - p^{-k}\mathrm{Li}_k^{p\text{-adic}}(z^p)$ (In his notation, $\mathrm{Li}_k^{p\text{-adic}}(z) := \ell_k(z)$, $\mathrm{Li}_k^{(p)}(z) := \ell_k^{(p)}(z)$). The latter function is given by the Koblitz measure as follows:

$$\text{(2.13)} \qquad \mathrm{Li}_k^{(p)}(z) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_z(x) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_z^{(p)}(x)$$

(see [7, Lemma 7.2]). In other words, if $|1-z|_p \geq 1$, we have the equality

$$\mathrm{Li}_k^{(p)}(z) = \lim_{n\to\infty}\left(\sum_{a=1,\, p\nmid a}^{p^n} \frac{z^a a^{-k}}{1-z^{p^n}}\right). \tag{2.14}$$

Remark 2.7. It is worthwhile to recall the following: Coleman [5] showed that $\mathrm{Li}_k^{p\text{-adic}}(z)$ has "analytic continuation along Frobenius" to $\mathbb{P}^1(\mathbb{C}_p)-\{1,\infty\}$ depending on branch parameter of Iwasawa logarithm (cf. e.g., [20, p.425]). This result has been extended to the p-adic multiple polylogarithms by H. Furusho [16]. See also Remark 2.3 for comparable progress in the theory of Galois polylogarithms.

Bernoulli distribution and Kubota-Leopoldt L-function

Before closing this subsection, we quickly review classically known facts about the Bernoulli distribution for the case $z = 1$. Let c be an integer $\neq 1$ prime to p. Then a measure E_c on $\mathbb{Z}_p$ is defined as follows (cf. [29] §12.2): For each $n \geq 1$, choose any integer $\bar{c} = \bar{c}_n$ with $c\bar{c} \equiv 1 \bmod p^n$, and define

$$E_c(a+p^n\mathbb{Z}_p) := B_1\left(\left\{\frac{a}{p^n}\right\}\right) - cB_1\left(\left\{\frac{\bar{c}a}{p^n}\right\}\right), \tag{2.15}$$

where $\{*\}$ means the fractional part. This is independent of the choice of $\bar{c}$, and forms a $\mathbb{Z}_p$-valued measure E_c on $\mathbb{Z}_p$. A well-known calculation ($e^X = 1+T$):

$$\frac{1}{T} - \frac{c}{(1+T)^c-1} = \sum_{n=1}^{\infty}(1-c^n)\frac{B_n}{n}\cdot\frac{X^{n-1}}{(n-1)!} \tag{2.16}$$

as well as the formula $L_p(1-n,\omega^n) = -(1-p^{n-1})\frac{B_n}{n}$ give

$$\int_{\mathbb{Z}_p^\times} x^{m-1}dE_c(x) = (1-c^m)(1-p^{m-1})\frac{B_m}{m} = (c^m-1)L_p(1-m,\omega^m). \tag{2.17}$$

(Cf. proofs of [29] Cor. 5.13, Cor. 12.3). A formula of the p-adic Mellin transformation reads then as follows:

$$\int_{\mathbb{Z}_p^\times} x^{-m}dE_c(x) = (c^{1-m}-1)L_p(m,\omega^{1-m}). \tag{2.18}$$

(Cf. [29] Theorem 12.2 with $d = 1$, $\chi = \omega^{1-m}$, $s = -m$.) Finally, we remark that the power series $\mathscr{F}_z^{(p)}$ of Lemma 2.6 (2) for $z = 1$ has a pole at $T = 0$. However, its usual 'c-correction' cancel the pole and has an expansion in the form:

$$\mathscr{F}_1^{(p)}(T) - c\mathscr{F}_1^{(p)}((1+T)^c-1) = \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^\times} x^n dE_c(x)\right)\frac{X^n}{n!} \tag{2.19}$$

in $\mathbb{Q}_p[\![X]\!]$ with $1+T = e^X$. (Note LHS=$Ug(T)$ of [29] §12 p.251-252.)

3. Special family of Coleman series

3.1. Basic setup and $f_{z,c}(T)$. In this subsection, after reviewing some basic notions, we shall introduce a special class of Coleman power series that play important roles in our proof of Theorem 1.1.

Let F be a finite unramified extension of $\mathbb{Q}_p$. In this section, we study Galois and p-adic polylogarithms $\ell i_m(z)$ and $\mathrm{Li}_m^{p\text{-adic}}(z)$ for $z \in F \cap \mu(\mathbb{Z}_p^{ur})$, a root of unity of order prime to p. We will introduce and observe behaviors of certain special power series $f_{z,c}(T) \in \mathcal{O}_F[\![T]\!]$ closely related to $\mathscr{F}_z^{(p)}(T)$ of the previous section.

We begin by setting up basic operations on power series. Set $F_\infty := F(\mu_{p^\infty})$ and denote by G_∞ the Galois group of F_∞/F. Note that, the p-adic cyclotomic character induces the canonical isomorphism

$$\bar{\chi}_{\mathrm{cyc}} : G_\infty \xrightarrow{\sim} \mathbb{Z}_p^\times$$

as F is unramified over $\mathbb{Q}_p$. For each $a \in \mathbb{Z}_p^\times$, we define $\sigma_a \in G_\infty$ to be $\bar{\chi}_{\mathrm{cyc}}^{-1}(a)$.

We set $\mathcal{O}_F[\![T]\!]^{\times 1} := \mathrm{Ker}(\mathcal{O}_F[\![T]\!]^\times \xrightarrow{\mathrm{aug}} \mathcal{O}_F^\times \to k_F^\times)$. Here, the first map is the augmentation map and k_F the residue field of F. Define the action of G_∞ on $\mathcal{O}_F[\![T]\!]$ by

$$\sigma_a f(T) = f((1+T)^a - 1).$$

that restricts to the action on $\mathcal{O}_F[\![T]\!]^{\times 1}$. Since $\mathcal{O}_F[\![T]\!]$ and $\mathcal{O}_F[\![T]\!]^{\times 1}$ are compact (additive and multiplicative) $\mathbb{Z}_p$-modules , the complete group ring $\mathbb{Z}_p[\![G_\infty]\!]$ acts on both topological abelian groups. We regard $\mathcal{O}_F[\![T]\!]$ (resp. $\mathcal{O}_F[\![T]\!]^{\times 1}$) as a left (resp. right) $\mathbb{Z}_p[\![G_\infty]\!]$-module.

Remark 3.1. We remark the followings:

1. The $\mathbb{Z}_p$-module structure on $\mathcal{O}_F[\![T]\!]^{\times 1}$ is given by $f(T)^c = \sum_{n=0}^{\infty} \binom{c}{n} (f(T)-1)^n$ for $c \in \mathbb{Z}_p$, $f \in \mathcal{O}_F[\![T]\!]^{\times 1}$, where $\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!} \in \mathbb{Z}_p$.
2. Since $\mathcal{O}_F[\![T]\!]$ has a canonical structure of (additive) $\mathcal{O}_F$-module, the complete group ring $\mathcal{O}_F[\![G_\infty]\!]$ acts on it. However, $\mathcal{O}_F[\![T]\!]^{\times 1}$ does not have a canonical $\mathcal{O}_F$-module structure, i.e., there does not exist a canonical action of $\mathcal{O}_F[\![G_\infty]\!]$ on it.
3. A finite element $\sum_{a \in \mathbb{Z}_p^\times} c_a \sigma_a$ ($c_a = 0$ for all but finitely many $a \in G_\infty$) acts on $f(T)$ in the following forms: $(\sum_a c_a \sigma_a) \cdot f(T) = \sum_a c_a f((1+T)^a - 1)$, and $f(T)^{\sum_a c_a \sigma_a} = \prod_a f((1+T)^a - 1)^{c_a}$.

Special Coleman series. For $c \in \mathbb{Z}$ ($c \neq 1$, $p \nmid c$) and for $z \in F \cap \mu(\mathbb{Z}_p^{ur})$, let us introduce power series $f_{z,c} \in \mathcal{O}_F[\![T]\!]$ by

$$f_{z,c}(T) := \begin{cases} c \cdot \dfrac{(1+T)^{-\frac{1}{2}} - (1+T)^{\frac{1}{2}}}{(1+T)^{-\frac{c}{2}} - (1+T)^{\frac{c}{2}}}, & (z = 1); \\[2ex] \dfrac{(1+T)^{-\frac{1}{2}} - z(1+T)^{\frac{1}{2}}}{(1+T)^{-\frac{c}{2}} - z(1+T)^{\frac{c}{2}}}, & (z \in \mu(\mathbb{Z}_p^{ur}) \setminus \{1\}). \end{cases} \tag{3.1}$$

We denote by $\sigma_F : F \to F$ the Frobenius automorphism of F. It acts on each element of $F[\![T]\!]$ by the action on coefficients. Define the *integral logarithm* $\mathcal{L} : \mathcal{O}_F[\![T]\!]^\times \to$

$\mathcal{O}_F[\![T]\!]$ by the formula

$$\mathcal{L}(f(T)) := \frac{1}{p}\log\left(\frac{f(T)^p}{(\sigma_F f)((1+T)^p-1)}\right).$$

Further, we define the differential operator D on $F[\![T]\!]$ to be $(1+T)\frac{d}{dT}$ and the ring homomorphism $[p]:\mathcal{O}_F[\![T]\!]\to\mathcal{O}_F[\![T]\!]$ by $([p]f)(T)=f((1+T)^p-1)$.

Let $\mathcal{N}:\mathcal{O}_F(\!(T)\!)^\times\to\mathcal{O}_F(\!(T)\!)^\times$ (resp. $\overline{\mathcal{S}}:\mathcal{O}_F[\![T]\!]\to\mathcal{O}_F[\![T]\!]$) be the norm operator (resp. the 'reduced' trace operator) introduced by Coleman (cf. [3], [4, p.386]). They are characterized by

$$([p]\mathcal{N}f)(T)=\prod_{\xi\in\mu_p}f(\xi(1+T)-1),\quad ([p]\overline{\mathcal{S}}f)(T)=\frac{1}{p}\sum_{\xi\in\mu_p}f(\xi(1+T)-1).$$

Lemma 3.2. *Let F be a finite unramified extension of $\mathbb{Q}_p$ containing $z\in\mu(\mathbb{Z}_p^{ur})$. Then,*

$$f_{z,c}(T)\in 1+T\mathcal{O}_F[\![T]\!]. \tag{1}$$

$$\mathcal{N}(f_{z,c}(T))=(\sigma_F f_{z,c})(T). \tag{2}$$

$$D\mathcal{L}f_{z,c}(T)=-\left(\mathscr{F}_z^{(p)}(T)-c\mathscr{F}_z^{(p)}((1+T)^c-1)\right) \tag{3}$$

Proof. (1) This is just claiming $f_{z,c}(0)=1$. Use d'Hôspital rule when $z=1$. (2) For $a\in\mathbb{Z}_p^\times$, set $k_{z,a}(T):=(1+T)^{-\frac{a}{2}}-z(1+T)^{\frac{a}{2}}$. Then, by the definition of the norm operator, we have:

$$\begin{aligned}([p]\mathcal{N}k_{z,a})(T) &= \prod_{\xi\in\mu_p}\left((\xi(1+T))^{-\frac{a}{2}}-z(\xi(1+T))^{\frac{a}{2}}\right)\\ &= \prod_{\xi\in\mu_p}\left((1+T)^{-\frac{a}{2}}-z\xi^a(1+T)^{\frac{a}{2}}\right)\\ &= (1+T)^{-\frac{pa}{2}}-z^p(1+T)^{\frac{pa}{2}}=([p]k_{z^p})(T).\end{aligned}$$

As $[p]$ is injective, it follows that $\mathcal{N}k_{z,a}=k_{z^p,a}$. Since σ_F acts on $\mu(\mathbb{Z}_p^{ur})$ by p-power and the operator $\mathcal{N}$ is multiplicative, the assertion (2) follows. (3) also follows from a simple calculation

$$D\mathcal{L}(k_{z,a}(T))=a\left(-\frac{1}{1-z(1+T)^a}+\frac{1}{1-\sigma_F(z)(1+T)^{ap}}\right)$$

with Lemma 2.6 (2), including the case $z=1$ where $\mathscr{F}_1^{(p)}(T)-c\mathscr{F}_1^{(p)}((1+T)^c-1)\in\mathbb{Z}_p[\![T]\!]$ is discussed in (2.19). (See also Remark 3.3 below.) □

REMARK 3.3. (1) The integral logarithm $\mathcal{L}$ on $\mathcal{O}_F[\![T]\!]^{\times 1}$ preserves the action of $\mathbb{Z}_p[\![G_\infty]\!]$, namely, it holds that $\mathcal{L}(f^\lambda)=\lambda\cdot\mathcal{L}(f)$ for any $\lambda\in\mathbb{Z}_p[\![G_\infty]\!]$ and $f\in\mathcal{O}_F[\![T]\!]^{\times 1}$. By a simple argument, one can easily see that $\mathcal{N}(f)=\sigma_F(f)$ implies $\overline{\mathcal{S}}(\mathcal{L}f)=0$. Namely, $\mathcal{L}$ maps $(\mathcal{O}_F[\![T]\!]^\times)^{\mathcal{N}=\sigma_F}$ into $\mathcal{O}_F[\![T]\!]^{\overline{\mathcal{S}}=0}$.

(2) If $f\in\mathcal{O}_F[\![T]\!]^\times$ is not necessarily in $\mathcal{O}_F[\![T]\!]^{\times 1}$, then $\log(f)$ has no obvious sense and the convergence (and stay) of $\mathcal{L}(f)$ in $\mathcal{O}_F[\![T]\!]$ involves technical estimate of

coefficients (see [9] Lemma 2.5.1). We have $\mathcal{L}(fg) = \mathcal{L}(f) + \mathcal{L}(g)$ as long as both $f, g \in \mathcal{O}_F [\![T]\!]^\times$. But, in practical computation, it could happen that we know the existence of $\mathcal{L}(fg)$ but not of individual $\mathcal{L}(f)$ or $\mathcal{L}(g)$. A possible way to remedy such a computational difficulty (which occurs also in the proof of the above Lemma 3.2 (3) when $z = 1$) is to consider the logarithmic derivative $D\mathcal{L}(f) = (1 - [p]\sigma_F) \cdot (Df/f)$ (whose existence is often easier to see) and to use $D\mathcal{L}(fg) = D\mathcal{L}(f) + D\mathcal{L}(g)$.

(3) As remarked in [6], Corollary of Theorem 3, the differential operator D gives an isomorphism $D : \mathcal{O}_F [\![T]\!]^{\overline{S}=0} \xrightarrow{\sim} \mathcal{O}_F [\![T]\!]^{\overline{S}=0}$. (The proof given in loc. cit. for $\mathbb{Z}_p [\![T]\!]$ works also for $\mathcal{O}_F [\![T]\!]$ with no obstructions.) This defines the inverse map

$$D^{-1} : \mathcal{O}_F [\![T]\!]^{\overline{S}=0} \xrightarrow{\sim} \mathcal{O}_F [\![T]\!]^{\overline{S}=0}.$$

Lemma 3.4. *For a power series $f(T) \in \mathcal{O}_F [\![T]\!]$, denote by μ_f the corresponding $\mathcal{O}_F$-valued measure on $\mathbb{Z}_p$. Suppose $f(T) \in \mathcal{O}_F [\![T]\!]^{\overline{S}=0}$. Then, for $k \in \mathbb{Z}$, it holds that*

$$D^{-k}(f)(T) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p^\times} x^{n-k} d\mu_f(x) \right) \frac{X^n}{n!}$$

in $F [\![X]\!]$ with $e^X = 1 + T$. In particular, we have

$$D^{-k}(f)(0) = \int_{\mathbb{Z}_p^\times} x^{-k} d\mu_f(x).$$

Proof. The case of $k \leq 0$ is well known (e.g., [9] Lemma 3.3.5). So, assume $k > 0$. Consider a linear functional L on the $\mathcal{O}_F$-valued continuous functions on $\mathbb{Z}_p$ defined by $h \mapsto \int_{\mathbb{Z}_p} hx^{-k} d\mu_f(x)$. This is well defined, as $\overline{S}(f) = 0$ insures the support of the measure μ_f is on $\mathbb{Z}_p^\times$. Since L is bounded with support in $\mathbb{Z}_p^\times$, it follows that L is of the form $h \mapsto \int_{\mathbb{Z}_p} h\, d\mu_g(x)$ for a unique $g(T) \in \mathcal{O}_F [\![T]\!]^{\overline{S}=0}$ that, by the construction, should have the same expansion as RHS of the lemma (cf. [9] p.35 and Lemma 3.3.5). Since $D = (1 + T)\frac{d}{dT} = \frac{d}{dX}$, we find $D^k(g) = f$. The uniqueness of g (cf. [6] Corollary of Theorem 3) in $\mathcal{O}_F [\![T]\!]^{\overline{S}=0}$ concludes $D^{-k}(f) = g$. □

3.2. Coleman's reciprocity law.

Definition 3.5 (Coates-Wiles homomorphism). Let $\mathcal{U}_\infty(F)$ be the norm limit of principal units of $\{F(\mu_{p^n})\}_n$ and denote by

$$[\mathrm{Col}] : \mathcal{U}_\infty(F) \to \mathcal{O}_F [\![T]\!]^{\times 1} \cap (\mathcal{O}_F [\![T]\!]^\times)^{\mathcal{N}=\sigma_F}, \ \epsilon = (\epsilon_n) \mapsto g_\epsilon(T)$$

the Coleman map which is characterized by

$$(\sigma_F^{-n} g_\epsilon)(T)|_{T=\zeta_{p^n}-1} = \epsilon_n. \tag{3.2}$$

The Coates-Wiles homomorphism $\phi_{m,F}^{CW} : \mathcal{U}_\infty(F) \to \mathcal{O}_F$ is defined by the equality

$$\log(g_\epsilon(T)) = \sum_{m=0}^{\infty} \frac{\phi^{CW}_{m,F}(\epsilon)}{m!} X^m \tag{3.3}$$

in $F[\![X]\!]$ with $1 + T = \exp(X)$.

Let $\mathrm{rec}_n : F(\mu_{p^n})^\times \to G^{\mathrm{ab}}_{F(\mu_{p^n})}$ be the reciprocity map of local class field theory ($n \leq \infty$). When $n = \infty$, the reciprocity map induces an embedding $\mathrm{rec}_\infty : \mathcal{U}_\infty(F) \hookrightarrow G^{\mathrm{ab}}_{F_\infty}$ (recall $F_\infty = F(\mu_{p^\infty})$, as defined in Notation of Introduction). The above Coates-Wiles homomorphism $\phi^{CW}_{m,F}$ extends uniquely to a $G_\infty = \mathrm{Gal}(F_\infty/F)$-homomorphism from $G^{\mathrm{ab}}_{F_\infty}$ into $F(m) := F \otimes \mathbb{Z}_p(m)$. This extension and its image in $\mathrm{Hom}_{G_\infty}(G^{\mathrm{ab}}_{F_\infty}, F(m)) \cong H^1(F, F(m))$ will also be denoted by $\phi^{CW}_{m,F}$ (cf. Bloch-Kato [2] Section 2).

Remark 3.6. In (3.2), we employ Bloch-Kato's normalization ([2], Theorem 2.2) on powers of σ_F, which differs from that in [4]. The constant term of g_ϵ is $\equiv 1 \bmod p$, but may not be 1. This causes our summation in (3.3) to start from $m = 0$ which modifies [2] (p.344).

The Hilbert norm residue symbol

$$(\ ,\ )_{p^n} : F(\mu_{p^n})^\times \times F(\mu_{p^n})^\times \to \mu_{p^n}$$

is defined by the formula

$$(a, b)_{p^n} = (a^{1/p^n})^{\mathrm{rec}_n(b)-1}.$$

We shall make use of Coleman's explicit reciprocity law on Hilbert norm residue symbols in the following form: Recall that Coleman [4] introduces a continuous linear functional $\int_n : F(\!(T)\!)_1 \to F$ (where $F(\!(T)\!)_1$ denotes the ring of power series which converge on the unit open ball on $\mathbb{C}_p$) by

$$\int_n f := \frac{1}{p^n} \sum_{\zeta \in \mu_{p^n}} f(\zeta - 1).$$

Theorem 3.7 (Coleman [4],[7]). *Let $f(T) \in 1 + T\mathcal{O}_F[\![T]\!]$ satisfy $\mathcal{N}(f) = \sigma_F(f)$, and $g = g_\epsilon(T) = [\mathrm{Col}](\epsilon)$ be the Coleman power series associated to $\epsilon = (\epsilon_n) \in \mathcal{U}_\infty$. Then,*

$$(f(\zeta_{p^n} - 1), \epsilon_n)_{p^n} = \zeta_{p^n}^{\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\int_n \mathcal{L}(f)\cdot D\mathcal{L}(\sigma_F^{-n} g_\epsilon)\right)}.$$

Proof. For reader's convenience, we shall show how to derive this formula from Coleman's work: A direct application of the formula in [4] Theorem 1 tells that the exponent of ζ_{p^n} in RHS is

$$\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\int_n \mathcal{L}(f) \cdot D\log(\sigma_F^{-n} g_\epsilon)\right).$$

We remark that there exists no error term in the sense of Coleman (cf. loc.cit.) as $\mathcal{N}(g_\epsilon) = \sigma_F(g_\epsilon)$ (hence $k(0) = 0$ in his notation). Since $\frac{Dg}{g} - D\mathcal{L}(g) = [p]\sigma_F(\frac{Dg}{g})$ (Remark 3.3 (2)), it suffices to show $\int_n \mathcal{L}(f) \cdot [p](\sigma_F \frac{Dg}{g}) = 0$. This follows from [7] (4.4), as $\overline{S}(\mathcal{L}(f)) = 0$ when $\mathcal{N}(f) = \sigma_F(f)$ (cf. Remark 3.3 (1)). $\square$

4. Proof of Main formula

In this section, we fix a root of unity z in $F \cap \mu(\mathbb{Z}_p^{ur})$ (i.e., of order prime to p). Note then that $F(\mu_{p^\infty}, z^{1/p^\infty}) = F_\infty$, so that the p-adic polylogarithmic characters $\tilde{\chi}_m^z(\sigma)$, $\chi_m^z(\sigma)$ are defined for $\sigma \in G_{F_\infty}$ and for arbitrary etale paths $\gamma \in \pi_1^{\mathrm{pol}}(\overrightarrow{01}, z)$ as in Remark 2.5. Since the mod p reduction gives an isomorphism $\mu(\mathbb{Z}_p^{ur}) \xrightarrow{\sim} \overline{\mathbb{F}}_p^\times$, we can pick an integer $d \geq 1$ such that $z^{p^d} = z$.

Lemma 4.1. *There is an etale path $\gamma : \overrightarrow{01} \rightsquigarrow z$ on $\mathbb{P}^1_{\overline{\mathbb{Q}}_p} - \{0, 1, \infty\}$ that determines a compatible branch z^{1/p^n} $(n = 1, 2, \dots)$ inside $\mu(\mathbb{Z}_p^{ur})$ with period d, i.e., $z^{1/p^n} = z^{1/p^{n+d}}$.*

Proof. If one changes the choice of γ by composition with x^b $(b \in \mathbb{Z}_p)$, then the induced branch changes from z^{1/p^n} to $z^{1/p^n}\zeta_{p^n}^b$. As there is only one element in $(z^{1/p^n} \cdot \mu_{p^n}) \cap \mu(\mathbb{Z}_p^{ur})$ under the assumption, one may find a correct class b (mod p^n) for each $n \geq 1$, and hence get a correct b as their limit. □

We define the "weight accelerator" homomorphism

$$\mathcal{O}_F[\![G_\infty]\!] \to \mathcal{O}_F[\![G_\infty]\!]; \quad \omega \mapsto \omega(k)$$

for $k \in \mathbb{Z}$ to be the $\mathcal{O}_F$-linear extension of the mapping $\sigma_a \mapsto a^k \sigma_a$ $(a \in \mathbb{Z}_p^\times)$. It is not difficult to see that, for every $\omega \in \mathcal{O}_F[\![G_\infty]\!]$,

$$D \cdot \omega = \omega(1) \cdot D \tag{4.1}$$

holds as operators on $\mathcal{O}_F[\![T]\!]$. Let us also introduce the basic element

$$\omega_n := \sum_{1 \leq i \leq p^n,\, p \nmid i} \sigma_i \in \mathbb{Z}_p[\![G_\infty]\!].$$

Proposition 4.2. *Let $z \in \mu(\mathbb{Z}_p^{ur})$ and let z^{1/p^n} $(n = 1, 2, \dots)$ be the compatible sequence taken inside $\mu(\mathbb{Z}_p^{ur})$. Let F be a finite unramified extension of $\mathbb{Q}_p$ containing z and hence all of z^{1/p^n}. Then, for $\epsilon = (\epsilon_n) \in \mathcal{U}_\infty(F)$ and for any positive integer m, we have:*

$$\begin{aligned}&\Big(((f_{z^{1/p^n},c})^{\omega_n(m-1)})(\zeta_{p^n} - 1), \epsilon_n\Big)_{p^n}\\ &\quad= \zeta_{p^n}^{(-1)^{m-1}(c^{1-m}-1)\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\mathrm{Li}_m^{(p)}(z)\cdot(1-p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\epsilon)\right)}.\end{aligned}$$

Here, if $z = 1$, we understand $\mathrm{Li}_m^{(p)}(1)$ represents $L_p(m, \omega^{1-m})$.

Proof. We denote the left hand side by $\zeta_{p^n}^\alpha$. Then, according to Theorem 3.7 and [7, Lemma (4.6)], we have

$$\begin{aligned}\alpha &\equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\int_n \omega_n(m-1)\mathcal{L}(f_{z^{1/p^n},c}(T)) \cdot D\mathcal{L}(\sigma_F^{-n} g_\epsilon)\right)\\ &\equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\Big((-1)^{m-1}(D^{-(m-1)}\mathcal{L}f_{z^{1/p^n},c})(0) \cdot (D^m\mathcal{L}(\sigma_F^{-n}g_\epsilon))(0)\Big) \qquad (\mathrm{mod}\ p^n).\end{aligned}$$

Observe from (3.3) that

$$(D^m\mathcal{L}(\sigma_F^{-n}g_\epsilon))(0) = D^m\mathcal{L}(\sigma_F^{-n}g_\epsilon)|_{T=0} = \sigma_F^{-n}(1 - p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\epsilon).$$

On the other hand, noting that $(1-c\sigma_c)(-m) = 1 - c^{1-m}\sigma_c$ and hence $D^{-m}(1-c\sigma_c) = (1-c^{1-m}\sigma_c)D^{-m}$ by (4.1), we obtain for $z \neq 1$,

$$\begin{aligned} D^{-m}D\mathcal{L}\left(f_{z^{1/p^n},c}(T)\right) &= D^{-m}(1-c\sigma_c)(-\mathscr{F}^{(p)}_{z^{1/p^n}}(T)) \quad \text{(Lemma 3.2 (3))} \\ &= -(1-c^{1-m}\sigma_c)D^{-m}(\mathscr{F}^{(p)}_{z^{1/p^n}}(T)) \\ &= -(1-c^{1-m}\sigma_c)\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^\times} x^{n-m}d\mu_{z^{1/p^n}}(x)\right)\frac{X^n}{n!}, \end{aligned}$$

where Lemma 3.4 is used in the last equality. Hence

$$(D^{-m+1}\mathcal{L}f_{z^{1/p^n},c})(0) = (c^{1-m}-1)\mathrm{Li}_m^{(p)}(z^{1/p^n}) = (c^{1-m}-1)(\sigma_F^{-n}\mathrm{Li}_m^{(p)}(z))$$

according to our choice of z^{1/p^n}. Therefore

$$\begin{aligned} \alpha &\equiv (c^{1-m}-1)\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\sigma_F^{-n}\mathrm{Li}_m^{(p)}(z)\cdot\sigma_F^{-n}(1-p^{m-1}\sigma_F)\phi^{CW}_{m,F}(\epsilon)\right) \\ &= (c^{1-m}-1)\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\mathrm{Li}_m^{(p)}(z)\cdot(1-p^{m-1}\sigma_F)\phi^{CW}_{m,F}(\epsilon)\right) \end{aligned}$$

as desired. For $z = 1$, we obtain similarly

$$\begin{aligned} D^{-m}D\mathcal{L}\left(f_{1,c}(T)\right) &= D^{-m}(1-c\sigma_c)(-\mathscr{F}^{(p)}_{1}(T)) \quad \text{(Lemma 3.2 (3))} \\ &= D^{-m}\sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^\times} x^{n}dE_c(x)\right)\frac{X^n}{n!} \quad (2.19) \\ &= \sum_{n=0}^{\infty}\left(\int_{\mathbb{Z}_p^\times} x^{n-m}dE_c(x)\right)\frac{X^n}{n!} \quad \text{(Lemma 3.4)}, \end{aligned}$$

hence, by (2.18), $(D^{-m+1}\mathcal{L}f_{1,c})(0) = (c^{1-m}-1)L_p(m,\omega^{1-m})$. This completes the proof of the proposition. □

Lemma 4.3. *Let m be a fixed positive integer. Then, there is an integer N_m such that for every $n \geq 1$, p^n divides $N_m \sum_{\substack{1\leq a\leq p^n \\ p\nmid a}} a^m$.*

Proof. In fact, from the classical Bernoulli formula of power sums, it follows that

$$\sum_{\substack{1\leq a\leq p^n \\ p\nmid a}} a^m = \frac{p^n}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k\cdot(p^{n(m-k)} - p^{(n-1)(m-k)+m}).$$

Thus, any common multiple of $m+1$ and of the denominators of Bernoulli numbers $B_0, \ldots,$ B_m will do the role of N_m. □

Now, we shall show our main formula:

Theorem 4.4. *For $z \in \mu(\mathbb{Z}_p^{\mathrm{ur}})$, let $\gamma : \overrightarrow{01} \rightsquigarrow z \in \mu(\mathbb{Z}_p^{ur})$ be a specific path of Lemma 4.1, and let F be a finite unramified extension of $\mathbb{Q}_p$ containing z. Suppose that $\sigma \in G_F$ lies in the image of $\mathcal{U}_\infty(F) \xrightarrow{\mathrm{rec}_\infty} G_{F_\infty}^{\mathrm{ab}}$. Then, for $m \geq 1$,*

$$\chi_m^z(\sigma) = (-1)^m \operatorname{Tr}_{F/\mathbb{Q}_p}(\mathrm{Li}_m^{(p)}(z) \cdot (1 - p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\sigma)). \tag{1}$$

Moreover, if $m \geq 2$, then,

$$\tilde{\chi}_m^z(\sigma) = (-1)^m \mathrm{Tr}_{F/\mathbb{Q}_p}(Li_m^{(p)}(z)\phi_{m,F}^{CW}(\sigma)). \tag{2}$$

Here, if $z = 1$, we understand $\mathrm{Li}_m^{(p)}(1) = L_p(m, \omega^{1-m})$.

Proof. (1) Choose $c \in \mathbb{Z}$ so that $p \nmid c$ and $c \neq 1$. When $z = 1$, we moreover impose $c \in (1 + p\mathbb{Z}_p)^{N_m}$ for some N_m as in Lemma 4.3. For $n \geq 1$, pick an integer $\bar{c} = \bar{c}_n \in \mathbb{Z}$ such that $c\bar{c} \equiv 1 \bmod p^n$. Then, by simple computation, it follows that

$$((f_{z^{1/p^n},c})^{\omega_n(m-1)})(\zeta_{p^n} - 1) \equiv \xi \cdot \prod_{\substack{1 \leq a \leq p^n \\ p \nmid a}} (1 - z^{1/p^n}\zeta_{p^n}^a)^{a^{m-1}(1-\bar{c}^{m-1})}$$

modulo $F(\zeta_{p^n})^{\times p^n}$ for some constant ξ that lies in $\mu_{p^n} \cdot (1+p\mathbb{Z}_p)^{p^n}$. Putting this into Proposition 4.2, we obtain

$$\begin{aligned} &(1 - \bar{c}^{m-1})\chi_m^z(\sigma) \\ &\equiv (-1)^{m-1}(c^{1-m} - 1)\mathrm{Tr}_{F/\mathbb{Q}_p}(\mathrm{Li}_m^{(p)}(z) \cdot (1 - p^{m-1}\sigma_F)\phi_{m,F}^{CW}(\sigma)) \end{aligned} \tag{4.2}$$

mod p^n, where, if $z = 1$, $\mathrm{Li}_m^{(p)}(z)$ stands for $L_p(m, \omega^{1-m})$. Replacing $(1 - \bar{c}^{m-1})$ by $(1 - c^{1-m})$ in LHS of (4.2), and then passing over $n \to \infty$, we obtain the formula of proposition.

(2) Suppose $z^{p^d} = z$ for $d \in \mathbb{Z}_{>0}$. Then, our specific choice of γ makes the sequence $\{z^{1/p^k}\}_{k \in \mathbb{Z}}$ cyclic of period d. By using Lemma 2.4, we can relate $\tilde{\chi}_m^z(\sigma)$ to $\chi_m^z(\sigma)$ in such a way that $\tilde{\chi}_m^z(\sigma) = \sum_{k=0}^{d-1} \frac{p^{(m-1)k}}{1-p^{(m-1)d}} \chi_m^{z^{1/p^k}}(\sigma)$ (cf. also [32] Proposition 5.3 (i)). Putting the formula (1) into this, and replacing $\mathrm{Li}_m^{(p)}(z^{1/p^k}) \cdot \sigma_F(\phi_{m,F}^{CW}(\sigma))$ by $\sigma_F(\mathrm{Li}_m^{(p)}(z^{1/p^{k+1}}) \cdot \phi_{m,F}^{CW}(\sigma))$, we may rewrite $\tilde{\chi}_m^z(\sigma)$ with $\alpha_k := p^{(m-1)k}(\mathrm{Li}_m^{(p)}(z^{1/p^k}) \cdot \phi_{m,F}^{CW}(\sigma))$ as follows:

$$\begin{aligned} \tilde{\chi}_m^z(\sigma) &= \frac{(-1)^m}{1 - p^{(m-1)d}} \sum_{k=0}^{d-1} \mathrm{Tr}_{F/\mathbb{Q}_p}(\alpha_k - \sigma_F(\alpha_{k+1})) \\ &= \frac{(-1)^m}{1 - p^{(m-1)d}} \sum_{k=0}^{d-1} (\mathrm{Tr}_{F/\mathbb{Q}_p}(\alpha_k) - \mathrm{Tr}_{F/\mathbb{Q}_p}(\alpha_{k+1})) \\ &= \frac{(-1)^m}{1 - p^{(m-1)d}} \mathrm{Tr}_{F/\mathbb{Q}_p}(\alpha_0 - \alpha_d). \end{aligned}$$

But since $z^{1/p^d} = z$, we have $\alpha_d = p^{(m-1)d}\alpha_0$. This enables us to cancel the denominator $(1 - p^{(m-1)d})^{-1}$ in the above expression, and hence to conclude $\tilde{\chi}_m^z(\sigma) = (-1)^m \mathrm{Tr}_{F/\mathbb{Q}_p}(\alpha_0)$ as asserted in (2). □

Remark 4.5. The original Coleman-Ihara formula (1.3) in Introduction results from the above Theorem 4.4 (1) with $z = 1$ and $F = \mathbb{Q}_p$. Note $(-1)^m = -1$ for odd m.

We are arriving at the main Theorem 1.1 of Introduction:

Corollary 4.6. *Let $\gamma : \overrightarrow{01} \rightsquigarrow z \in \mu(\mathbb{Z}_p^{ur})$ be a specific path of Lemma 4.1, and let F be a finite unramified extension of $\mathbb{Q}_p$ containing z. Suppose that $\sigma \in G_F$ lies in the image of $\mathcal{U}_\infty(F) \xrightarrow{\mathrm{rec}_\infty} G^{\mathrm{ab}}_{F_\infty}$. Then, for $m \geq 2$,*

$$\ell i_m(z,\gamma)(\sigma) = \frac{-1}{(m-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^m}\right)\mathrm{Li}_m^{p\text{-adic}}(z)\right\}\phi_{m,F}^{CW}(\sigma)\right).$$

Here, if $z = 1$, we understand $(1 - \frac{\sigma_F}{p^m})\mathrm{Li}_m^{p\text{-adic}}(1) = L_p(m, \omega^{1-m})$.

Proof. Theorem 4.4 (2) and Proposition 2.2 enable us to derive:

$$\begin{aligned}\ell i_m(z)(\sigma) &= \frac{(-1)^{m-1}}{(m-1)!}\tilde{\chi}_m^z(\sigma) \\ &= \frac{-1}{(m-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}(\mathrm{Li}_m^{(p)}(z)\cdot\phi_{m,F}^{CW}(\sigma)).\end{aligned}$$

Recalling $\mathrm{Li}_m^{(p)}(z) = \mathrm{Li}_m^{p\text{-adic}}(z) - \frac{1}{p^m}\mathrm{Li}_m^{p\text{-adic}}(z^p)$ by definition, where now $z^p = \sigma_F(z)$ for $z \in \mu(\mathbb{Z}_p^{ur})$, we conclude the corollary. □

Remark 4.7. Let $P := G_F^{\mathrm{ab}}$ and U the image of $\mathcal{U}_\infty(F)$ in P. Then, as in [2, p.342], P/U is isomorphic to $\hat{\mathbb{Z}}$ acted on trivially by $G := \mathrm{Gal}(F(\zeta_{p^\infty})/F)$ so as to fit in the canonical identification

$$H^1(F, F(m)) \cong \mathrm{Hom}_G(P, F(m)) \cong \mathrm{Hom}_G(U, F(m))$$

for $F(m) := F\otimes\mathbb{Z}_p(m)$ $(m \geq 1)$. From this, we obtain a canonical extension of the RHS of the above corollary to a G-homomorphism from P to $\mathbb{Q}_p(m)$. On the other hand, if we choose a path $\gamma : \overrightarrow{01} \rightsquigarrow z$ so that $\ell i_m(z,\gamma) \in \mathrm{Hom}_G(P, \mathbb{Q}_p(m))$ (this is the case for γ in Lemma 4.1) then, it annihilates an inverse image u_0 of $1 \in \hat{\mathbb{Z}}$ in P. Then, these two extended homomorphisms on P turn out to coincide with each other, as they both coincide on U and kill u_0.

If moreover $\ell i_m(z,\gamma)$ is a 1-cocycle on G_F, then they should give the same cohomology class in $H^1(F, \mathbb{Q}_p(m))$, as the restriction map from F to $F(\zeta_{p^\infty})$ is injective.

5. Appendix: The Kummer level case

For completeness, we shall examine the case $m = 0, 1$ in Theorem 1.1. We consider the ℓ-adic Galois polylogarithms ℓi_0, ℓi_1 as the Kummer 1-cocycles κ_z, κ_{1-z} respectively as in (2.4), (2.5) (cf. also [24] §5.2). On the other p-adic side, we may regard $\mathrm{Li}_0^{p\text{-adic}}(z) = -\log_p(z)$, $\mathrm{Li}_1^{p\text{-adic}}(z) = -\log_p(1-z)$ (cf. [16]). In fact, we have the following:

Proposition 5.1. *Let p be an odd prime, and F a finite unramified extension of $\mathbb{Q}_p$ with the Frobenius substitution $\sigma_F \in \mathrm{Gal}(F/\mathbb{Q}_p)$. Let $F_\infty := F(\mu_{p^\infty})$ and, for $a \in \mathcal{O}_F^\times$, let $\kappa_a : G_{F_\infty} \to \mathbb{Z}_p(1)$ be the Kummer character for p-power roots of a. Then,*

$$\kappa_a(\sigma) = \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1 - \frac{\sigma_F}{p}\right)\log_p(a)\right\}\phi_{1,F}^{CW}(\sigma)\right) \tag{5.1}$$

holds for $\sigma \in G_{F_\infty}$, where $\phi_{1,F}^{CW} : G_{F_\infty} \to F \otimes \mathbb{Z}_p(1)$ the 1-st Coates-Wiles homomorphism for the local field F (cf. Definition 3.5).

Following the notation system in §3.2, we first consider the case when $\sigma = \mathrm{rec}_\infty(\epsilon)$ for some norm compatible system $\epsilon = (\epsilon_n)_n \in \mathcal{U}_\infty(F)$. Let $g_\epsilon(T) \in \mathcal{O}_F [\![T]\!]^\times$ be the associated Coleman power series such that $\epsilon_n = \sigma_F^{-n} g_\epsilon(\zeta_{p^n} - 1)$ for $n = 1, 2, \ldots$.

The explicit reciprocity law of S. Sen computes the Hilbert norm residue symbol by the formula *

$$(\beta, \alpha)_{p^n} = \zeta_{p^n}^{-\frac{1}{p^m}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\log_p(\alpha)\cdot \mathrm{Tr}_{F'/F}(\delta_m(\beta'))\right)} \tag{5.2}$$

for $\alpha \in 1 + (\zeta_p - 1)^2\mathcal{O}_F$ in [27, Theorem 3 (b)] and more generally for $\alpha \in 1 + (\zeta_p - 1)\mathcal{O}_F$ in [27, II; Theorem 1], where $F' = F(\zeta_{p^m})$ (m :big enough) and $\beta' \in F'$ are taken so that $\alpha^{p^{m-n}} \in 1 + (\zeta_p - 1)^2\mathcal{O}_F$ and $N_{F'/F}(\beta') = \beta$. Finally $\delta_m(\beta')$ means $\frac{\zeta_{p^m}}{g'(\pi)} \cdot \frac{f'(\pi)}{f(\pi)}$ where $f(T)$, $g(T) \in \mathcal{O}_F[T]$ are polynomials with $f(\pi) = \beta$, $g(\pi) = \zeta_{p^m}$ for any prime π of F'. Note that, in our case, we may and do set $\pi = \zeta_{p^m} - 1$ so that $g'(\pi) = 1$ (cf. [4, Cor. 15]).

Now, let us apply (5.2) for $\alpha := a \in 1 + p\mathcal{O}_F$ and $\beta := \epsilon_n$. As $(\alpha, \beta)_{p^n} = (\beta, \alpha)_{p^n}^{-1}$ (cf. [8, p.352]), writing $(a, \epsilon_n)_{p^n} = \zeta_{p^n}^{[a,\epsilon_n]}$ and letting $F' = F(\zeta_{p^m})$ as above, we obtain the following congruence modulo p^n:

$$[a, \epsilon_n] \equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\log_p(a)\mathrm{Tr}_{F'/F}\left(\frac{\zeta_{p^m}}{p^m} \cdot \frac{\sigma_F^{-m} g'_\epsilon(\zeta_p^m - 1)}{\epsilon_m}\right)\right). \tag{5.3}$$

Lemma 5.2. *Notations being as above, we have*

$$\mathrm{Tr}_{F'/F}\left(\frac{\zeta_{p^m}}{p^m} \cdot \frac{\sigma_F^{-m} g'_\epsilon(\zeta_p^m - 1)}{\epsilon_m}\right) = \left(1 - \frac{\sigma_F^{-1}}{p}\right) \cdot \phi_{1,F}^{CW}(\epsilon).$$

Proof. Since $\mathcal{N}(g_\epsilon) = \sigma_F g_\epsilon$, we have $(\sigma_F g_\epsilon)((1 + T)^p - 1) = \prod_{i=0}^{p-1} g_\epsilon(\zeta_p^i(1 + T) - 1)$. Substituting $\zeta_{p^k}^j(1 + T) - 1$ or $(1 + T)^{p^k} - 1$ ($0 \le j \le p - 1$, $k = 1, 2, \ldots$) for T in it, and combining resulted equations in certain multiple ways, one obtains

$$(\sigma_F^k g_\epsilon)((1 + T)^{p^k} - 1) = \prod_{i=0}^{p^k-1} g_\epsilon(\zeta_{p^k}^i(1 + T) - 1) \qquad (k \ge 1). \tag{5.4}$$

*According to [27, II, p.69, line 12-13], the Hilbert symbol in loc. cit. coincides with that in [15]. On the other hand, we employ the Hilbert symbol discussed in Coleman's papers. According to [7, p.59, line -6], it is the inverse of the symbol used in [15]. Therefore, a minus sign is put in the exponent of RHS in (5.2).

Taking derivatives of both sides and putting $T = 0$, we obtain then

$$p^k \phi_{1,F}^{CW}(\epsilon) = \sum_{i=0}^{p^k-1} \zeta_{p^k}^i \frac{\sigma_F^{-k} g_\epsilon'(\zeta_{p^k}^i - 1)}{\sigma_F^{-k} g_\epsilon(\zeta_{p^k}^i - 1)} = \sum_{i=0}^{p^k-1} \zeta_{p^k}^i \frac{\sigma_F^{-k} g_\epsilon'(\zeta_{p^k}^i - 1)}{\epsilon_k}. \tag{5.5}$$

Using (5.5) for $k = m, m-1$, we see that the LHS of Lemma equals to

$$\begin{aligned}
&\frac{1}{p^m} \sum_{\substack{0 \le i < p^m \\ p \nmid i}} \frac{\sigma_F^{-m} g_\epsilon'(\zeta_{p^m}^i - 1)}{\sigma_F^{-m} g_\epsilon(\zeta_{p^m}^i - 1)} \\
=&\frac{1}{p^m}\left(\sum_{i=0}^{p^m} \frac{\sigma_F^{-m} g_\epsilon'(\zeta_{p^m}^i - 1)}{\sigma_F^{-m} g_\epsilon(\zeta_{p^m}^i - 1)} - \sum_{j=0}^{p^{m-1}} \sigma_F^{-1}\Big(\frac{\sigma_F^{-(m-1)} g_\epsilon'(\zeta_{p^m}^{pj} - 1)}{\sigma_F^{-(m-1)} g_\epsilon(\zeta_{p^m}^{pj} - 1)}\Big)\right) \\
=&\phi_{1,F}^{CW}(\epsilon) - \frac{1}{p}\sigma_F^{-1}\Big(\phi_{1,F}^{CW}(\epsilon)\Big).
\end{aligned}$$

This completes the proof of the lemma. □

Plugging this lemma into (5.3), we find

$$\begin{aligned}
[a, \epsilon_n] &\equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\log_p(a)\phi_{1,F}^{CW}(\epsilon) - \frac{\log_p(a)}{p}\sigma_F^{-1}\Big(\phi_{1,F}^{CW}(\epsilon)\Big)\right) \\
&\equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\log_p(a)\phi_{1,F}^{CW}(\epsilon) - \sigma_F^{-1}\left\{\sigma_F\Big(\frac{\log_p(a)}{p}\phi_{1,F}^{CW}(\epsilon)\Big)\right\}\right) \\
&\equiv \mathrm{Tr}_{F/\mathbb{Q}_p}\left(\log_p(a)\Big(1 - \frac{\sigma_F}{p}\Big)\phi_{1,F}^{CW}(\epsilon)\right) \qquad (\mathrm{mod}\ p^n).
\end{aligned}$$

This settles Proposition 5.1 in the case $a \in 1 + p\mathcal{O}_F$ and $\sigma = \mathrm{rec}_\infty(\epsilon)$, since by definition $\kappa_a(\sigma) \equiv [a, \epsilon_n] \bmod p^n$. The general case for $a \in \mathcal{O}^\times$ follows immediately from observing that $\log_p(a) = \log_p(a')$ when $a = \zeta a'$ for any $\zeta \in \mu(\mathbb{Z}_p^{\mathrm{ur}}) \cap F$. The extension of the statement to all $\sigma \in G_{F(\zeta_{p^\infty})}$ follows from Remark 4.7. This settles the proof of Proposition 5.1. □

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