Sung, C.
Osaka J. Math.
53 (2016), 1055–1061

SOME EXOTIC ACTIONS OF FINITE GROUPS ON SMOOTH 4-MANIFOLDS

CHANYOUNG SUNG

(Received January 19, 2015, revised October 20, 2015)

Abstract

Using G-monopole invariants, we produce infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite number of $S^2 \times S^2$, $\mathbb{C}P_2$, $\overline{\mathbb{C}P}_2$, and $K3$ surfaces, where $k \geq 2$, and H is any nontrivial finite group acting freely on S^3.

1. Introduction

The purpose of this paper is to present exotic, i.e. C^0-equivalent but smoothly inequivalent smooth actions of finite groups on some smooth 4-manifolds. We say that two smooth group actions G_1 and G_2 on a smooth manifold M is C^m-equivalent for $m = 0, 1, \ldots, \infty$, if there exists a C^m-homeomorphism $f : M \to M$ such that

$$G_1 = f \circ G_2 \circ f^{-1}.$$

Such exotic smooth actions of finite groups on smooth 4-manifolds have been found abundantly, for e.g., [7, 4, 9, 2, 10, 22, 8]. Ue showed that for any nontrivial finite group G there exists a smooth closed 4-manifold with infinitely many free G-actions which are all C^0-equivalent but mutually smoothly inequivalent. And Fintushel, Stern, and Sunukjian constructed infinite families of exotic actions of finite cyclic groups on smooth closed 4-manifolds with nontrivial Seiberg–Witten invariant. All these examples are either free or cyclic actions.

In this article we use G-monopole invariants to detect infinitely many non-free non-cyclic exotic group actions on certain connected sums of 4-manifolds with vanishing Seiberg–Witten invariant. For example, for $k \geq 2$ and any nontrivial finite group H acting freely on S^3, there exist infinitely many exotic non-free actions of $\mathbb{Z}_k \oplus H$ on some connected sums of finite numbers of $S^2 \times S^2$, $\mathbb{C}P_2$, $\overline{\mathbb{C}P}_2$, and $K3$ surfaces.

2. Preliminaries on G-monopole invariant

Let M be a smooth closed oriented 4-manifold. Suppose that a finite group G acts on M smoothly preserving the orientation, and this action lifts to an action on a Spin^c structure $\mathfrak{s}$ of M. For a G-invariant Riemannian metric and G-invariant perturbation ε,

2010 Mathematics Subject Classification. 57R57, 57M60, 57R50.

we consider a G-monopole moduli space $\mathfrak{X}$ defined as the set of G-invariant solutions (A, Φ) of (perturbed) Seiberg–Witten equations

$$D_A\Phi = 0, \quad F_A^+ = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}\mathrm{Id} + \varepsilon$$

modulo the group $\mathcal{G}^G = Map(M, S^1)^G$ of G-invariant gauge transformations. As shown in [19, 20], $\mathfrak{X}$ for a generic ε is a smooth compact orientable finite-dimensional manifold, if the dimension $b_2^+(M)^G$ of the space of G-invariant self-dual harmonic 2-forms on M is bigger than 0. In fact, it is a subset of the ordinary Seiberg–Witten moduli space.

The intersection theory on $\mathfrak{X}$ using the universal cohomology classes of the ordinary Seiberg–Witten moduli space gives various *G-monopole invariants* defined first by Y. Ruan [17]. Considering gauge equivalence classes of G-invariant solutions under a based G-invariant gauge transformation group $\mathcal{G}_o^G = \{g \in \mathcal{G}^G \mid g(o) = 1\}$ for a fixed base point $o \in M$, we get a based G-monopole moduli space which is the principal S^1-bundle over $\mathfrak{X}$ induced by $\mathcal{G}^G/\mathcal{G}_o^G$ action. Let's denote its first Chern class by μ, which is independent of choice of the base point by the connectedness of M. We define a G-monopole invariant $SW_{M,\mathfrak{s}}^G$ as $\langle \mu^{(\dim \mathfrak{X})/2}, [\mathfrak{X}]\rangle$. (When $\dim \mathfrak{X}$ is odd, $SW_{M,\mathfrak{s}}^G$ is just set to be 0.)

As in the ordinary case, $SW_{M,\mathfrak{s}}^G$ is independent of the choice of a G-invariant metric and a G-invariant perturbation ε, if $b_2^+(M)^G > 1$. Thus we get a (smooth) topological invariant of a G-manifold M generalizing the ordinary Seiberg–Witten invariant $SW_{M,\mathfrak{s}}$, which is now $SW_{M,\mathfrak{s}}^{\{1\}}$ for the trivial group $\{1\}$. Also generalizing the Seiberg–Witten polynomial SW_M of M, the G-monopole polynomial of M is defined as

$$W_M^G\, SW_M^G := \sum_{\mathfrak{s}} SW_{M,\mathfrak{s}}^G\, PD(c_1(\mathfrak{s})) \in \mathbb{Z}[H_2(M;\mathbb{Z})^G],$$

where the summation is over the set of all G-equivariant Spin^c structures. Note that G-monopole invariants may change when a homotopically different lift of the G-action to the Spin^c structure is chosen. In a previous paper, we computed some examples of G-monopole invariants, which will be used as an essential tool in this paper:

Theorem 2.1 ([20]). *Let M and N be smooth closed oriented connected* 4-*manifolds satisfying $b_2^+(M) > 1$ and $b_2^+(N) = 0$, and $\bar{M}_k$ for any $k \geq 2$ be the connected sum $M \# \cdots \# M \# N$ where there are k summands of M.*

Suppose that a finite group G with $|G| = k$ acts effectively on N in a smooth orientation-preserving way such that it is free or has at least one fixed point, and that N admits a Riemannian metric of positive scalar curvature invariant under the G-action and a G-equivariant Spin^c *structure $\mathfrak{s}_N$ with $c_1^2(\mathfrak{s}_N) = -b_2(N)$.*

Define a G-action on $\bar{M}_k$ induced from that of N permuting the k summands of M glued along a free orbit in N, and let $\bar{\mathfrak{s}}$ be the Spin^c *structure on $\bar{M}_k$ obtained by* *gluing $\mathfrak{s}_N$ and a* Spin^c *structure $\mathfrak{s}$ of M.*

Then for any G-action on $\bar{\mathfrak{s}}$ covering the above G-action on $\bar{M}_k$,

$$SW^G_{\bar{M}_k,\bar{\mathfrak{s}}} \equiv SW_{M,\mathfrak{s}} \mod 2,$$

if the dimension $b_1(N)^G$ of the vector space consisting of G-invariant elements of $H_1(N;\mathbb{R})$ is zero.

Note that if a smooth closed manifold X has a smooth effective action of a compact Lie group G, then the fixed-point set X^g under $g \in G$ is either empty or an embedded submanifold each component of which has positive codimension. Thus N in the above theorem always has a free orbit under G. When $b_1(N)^G \neq 0$, we also obtained a mod 2 equality relating those two invariants, but we omit it here for simplicity. The examples of such N with $G = \mathbb{Z}_k$ regardless of $b_1(N)^{\mathbb{Z}_k}$ are as follows:

Theorem 2.2 ([20])**.** *Let X be one of*

$$S^4, \quad \overline{\mathbb{C}P}_2, \quad S^1 \times (L_1 \# \cdots \# L_n), \quad \textit{and} \quad \widehat{S^1 \times L}$$

where each L_i and L are quotients of S^3 by free actions of finite groups, and $\widehat{S^1 \times L}$ is the manifold obtained from the surgery on $S^1 \times L$ along an $S^1 \times \{pt\}$.

Then for any integer $l \geq 0$ and any smooth closed oriented 4*-manifold Z with $b_2^+(Z) = 0$ admitting a metric of positive scalar curvature,*

$$X \# klZ$$

satisfies the properties of N in Theorem 2.1 *with $G = \mathbb{Z}_k$, where the* Spin^c *structure of $X \# klZ$ is given by gluing any* Spin^c *structure $\mathfrak{s}_X$ on X and any* Spin^c *structure $\mathfrak{s}_Z$ on Z satisfying $c_1^2(\mathfrak{s}_X) = -b_2(X)$ and $c_1^2(\mathfrak{s}_Z) = -b_2(Z)$ respectively.*

3. Exotic group actions

Following [12], we say that a simply connected 4-manifold *dissolves* if it is diffeomorphic to either

$$n\mathbb{C}P_2 \# m\overline{\mathbb{C}P}_2$$

or

$$\pm(n(S^2 \times S^2) \# mK3)$$

for some $n, m \geq 0$ according to its homeomorphism type. We also use the term mod 2 *basic class* to mean the first Chern class of a Spin^c structure with nonzero mod 2 Seiberg–Witten invariant.

Theorem 3.1. *Let M be a smooth closed oriented connected 4-manifold and $\{M_i \mid i \in \mathfrak{I}\}$ be a family of smooth 4-manifolds such that every M_i is homeomorphic to M and the numbers of* mod 2 *basic classes of M_i's are all mutually different, but each $M_i \# l_i(S^2 \times S^2)$ is diffeomorphic to $M \# l_i(S^2 \times S^2)$ for an integer $l_i \geq 1$.*

If $l_{\max} := \sup_{i\in\mathfrak{I}} l_i < \infty$, then for any integers $k \geq 2$ and $l \geq l_{\max} + 1$,

$$klM \# (l-1)(S^2 \times S^2)$$

admits an $\mathfrak{I}$-family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ where H is any group of order l acting freely on S^3.

Proof. Think of $klM \# (l-1)(S^2 \times S^2)$ as

$$klM_i \# (l-1)(S^2 \times S^2),$$

and our H action is defined as the deck transformation map of the l-fold covering map onto

$$\bar{M}_{i,k} := kM_i \# \widehat{S^1 \times L}$$

where $\widehat{S^1 \times L}$ for $L = S^3/H$ is defined as in Theorem 2.2. To define a $\mathbb{Z}_k$-action, note that $\bar{M}_{i,k}$ has a $\mathbb{Z}_k$-action coming from the $\mathbb{Z}_k$-action of $\widehat{S^1 \times L}$ defined in Theorem 2.2, which is basically a rotation along the S^1-direction. This $\mathbb{Z}_k$ action is obviously lifted to the above l-fold cover, and it commutes with the above defined H action. Thus we have an $\mathfrak{I}$-family of $\mathbb{Z}_k \oplus H$ actions on $klM\#(l-1)(S^2\times S^2)$, which are all topologically equivalent by using the homeomorphism between each M_i and M.

Recall from Theorem 2.2 and its proof in [20] that all the Spin^c structures on a spin manifold $\widehat{S^1 \times L}$ are $\mathbb{Z}_k$-equivariant with $c_1^2 = -b_2(\widehat{S^1 \times L}) = 0$, and hence $\mathbb{Z}_k$-equivariant Spin^c structures on $\bar{M}_{i,k}$ are parametrized by

$$H_2(\bar{M}_{i,k}; \mathbb{Z})^{\mathbb{Z}_k} \cong H_2(M_i; \mathbb{Z}) \oplus H_2(\widehat{S^1 \times L}; \mathbb{Z}).$$

By Theorem 2.1 and the fact that $b_1(\widehat{S^1 \times L}) = 0$, for any Spin^c structure $\mathfrak{s}_i$ on M_i,

$$SW^{\mathbb{Z}_k}_{\bar{M}_{i,k},\bar{\mathfrak{s}}_i} \equiv SW_{M_i,\mathfrak{s}_i} \mod 2,$$

and hence

$$SW^{\mathbb{Z}_k}_{\bar{M}_{i,k}} \equiv SW_{M_i} \sum_{[\alpha]\in H_2(\widehat{S^1\times L};\mathbb{Z})} [\alpha]$$

modulo 2. Therefore $SW^{\mathbb{Z}_k}_{\bar{M}_{i,k}}$ mod 2 for all i have mutually different numbers of monomials, and hence the $\mathfrak{I}$-family of $\mathbb{Z}_k \oplus H$ actions on $klM \# (l-1)(S^2 \times S^2)$ cannot be smoothly equivalent, completing the proof. □

Corollary 3.2. *Let H be a finite group of order $l \geq 2$ acting freely on S^3. For any $k \geq 2$, there exists an infinite family of topologically equivalent but smoothly distinct non-free actions of $\mathbb{Z}_k \oplus H$ on*

$$(klm + l - 1)(S^2 \times S^2),$$
$$(kl(n-1) + l - 1)(S^2 \times S^2) \# klnK3,$$
$$(kl(2n'-1) + l - 1)\mathbb{C}P_2 \# (kl(10n' + m' - 1) + l - 1)\overline{\mathbb{C}P}_2$$

for infinitely many m, and any $m' \geq 1$, $n, n' \geq 2$.

Proof. By the result of B. Hanke, D. Kotschick, and J. Wehrheim [13], $m(S^2 \times S^2)$ for infinitely many m has the property of M in the above theorem with each $l_i = 1$ and $|\mathfrak{I}| = \infty$. The different smooth structures of their examples are constructed by fiber-summing a logarithmic transform of $E(2n)$ and a certain symplectic 4-manifold along a symplectically embedded torus, and different numbers of mod 2 basic classes are due to those different logarithmic transformations. Indeed the Seiberg–Witten polynomial of the multiplicity r logarithmic transform of $E(2n)$ is given by

$$([T]^r - [T]^{-r})^{2n-2}([T]^{r-1} + [T]^{r-3} + \cdots + [T]^{1-r})$$

whose number of terms with coefficients mod 2 can be made arbitrarily large by taking r sufficiently large, and the fiber sum with the other symplectic 4-manifold is performed on a fiber in an $N(2)$ disjoint from the Gompf nucleus $N(2n)$ where the log transform is performed so that all these mod 2 basic classes survive the fiber-summing by the gluing formula of C. Taubes [21]. Therefore $(klm + l - 1)(S^2 \times S^2)$ has desired actions by the above theorem.

For the second example, we use a well-known fact that $E(n)$ for $n \geq 2$ also has the above properties of M in the above theorem with each $l_i = 1$, where its exotica M_i's are $E(n)_K$ for a knot $K \subset S^3$ by the Fintushel–Stern knot surgery. Recall the theorem by S. Akbulut [1] and D. Auckly [3] which says that for any smooth closed simply-connected X with an embedded torus T such that $T \cdot T = 0$ and $\pi_1(X - T) = 0$, a knot-surgered manifold X_K along T via a knot K satisfies

$$X_K \# (S^2 \times S^2) = X \# (S^2 \times S^2).$$

And from the formula

$$SW_{E(n)_K} = \Delta_K([T]^2)([T] - [T]^{-1})^{n-2}$$

where Δ_K is the symmetrized Alexander polynomial of K, one can easily see that the number of mod 2 basic classes of $E(n)_K$ can be made arbitrarily large by choosing K

appropriately. (For example, take K with

$$\Delta_K(t) = 1 + \sum_{j=1}^{2d} (-1)^j (t^{jn} + t^{-jn})$$

for sufficiently large d.) Therefore

$$klE(2n) \# (l-1)(S^2 \times S^2) = klnK3 \# (kl(n-1) + l - 1)S^2 \times S^2$$

has desired actions, where we used the fact that $S \# (S^2 \times S^2)$ dissolves for any smooth closed simply-connected elliptic surface S by the work of R. Mandelbaum [14] and R. Gompf [11].

For the third example, one can take M to be $E(n') \# m'\overline{\mathbb{C}P_2}$ for $n' \geq 2$, $m' \geq 1$, where its exotica M_i's are $E(n')_K \# m'\overline{\mathbb{C}P_2}$ for a knot $K \subset S^3$, because

$$\begin{aligned} SW_{E(n')_K \# m'\overline{\mathbb{C}P_2}} &= SW_{(E(n') \# m'\overline{\mathbb{C}P_2})_K} \\ &= \Delta_K([T]^2)([T] - [T]^{-1})^{n'-2} \prod_{i=1}^{m'} ([E_i] + [E_i]^{-1}), \end{aligned}$$

where E_i's denote the exceptional divisors, and we used the fact that $E(n')$ is of simple type. Since $E(n') \# \overline{\mathbb{C}P_2}$ for any n' is non-spin,

$$kl(E(n') \# m'\overline{\mathbb{C}P_2}) \# (l-1)(S^2 \times S^2) = kl(E(n') \# m'\overline{\mathbb{C}P_2}) \# (l-1)(\mathbb{C}P_2 \# \overline{\mathbb{C}P_2}),$$

and it dissolves into the connected sum of $\mathbb{C}P_2$'s and $\overline{\mathbb{C}P_2}$'s, using the dissolution ([14, 11]) of $E(n') \# \mathbb{C}P_2$ into $2n'\mathbb{C}P_2 \# (10n'-1)\overline{\mathbb{C}P_2}$. □

REMARK. For other combinations of $K3$ surfaces and $(S^2 \times S^2)$'s in the above corollary, one can use B. Hanke, D. Kotschick, and J. Wehrheim's other examples in [13]. One can also construct many other such examples of M with infinitely many exotica which become diffeomorphic after one stabilization by using the knot surgery.

Any finite group acting freely on S^3 is in fact a subgroup of $SO(4)$ by the well-known result of G. Perelman ([15, 16]), and Theorem 3.1 and Corollary 3.2 can be generalized a little further. (See [18].)

ACKNOWLEDGEMENT. The author would like to express sincere thanks to Prof. Ki-Heon Yun for helpful discussions and supports.

References

[1] S. Akbulut: *Variations on Fintushel–Stern knot surgery on* 4*-manifolds*, Turkish J. Math. **26** (2002), 81–92.

[2] S. Akbulut: *Cappell–Shaneson homotopy spheres are standard*, Ann. of Math. (2) **171** (2010), 2171–2175.

[3] D. Auckly: *Families of four-dimensional manifolds that become mutually diffeomorphic after one stabilization*, Topology Appl. **127** (2003), 277–298.

[4] S.E. Cappell and J.L. Shaneson: *Some new four-manifolds*, Ann. of Math. (2) **104** (1976), 61–72.

[5] W. Chen: *Pseudoholomorphic curves in four-orbifolds and some applications*; in Geometry and Topology of Manifolds, Fields Inst. Commun. **47**, Amer. Math. Soc., Providence, RI, 11–37, 2005.

[6] W. Chen: *Smooth s-cobordisms of elliptic* 3*-manifolds*, J. Differential Geom. **73** (2006), 413–490.

[7] R. Fintushel and R.J. Stern: *An exotic free involution on S^4*, Ann. of Math. (2) **113** (1981), no. 2, 357–365.

[8] R. Fintushel, R.J. Stern and N. Sunukjian: *Exotic group actions on simply connected smooth* 4*-manifolds*, J. Topol. **2** (2009), 769–778.

[9] R.E. Gompf: *Killing the Akbulut–Kirby* 4*-sphere, with relevance to the Andrews–Curtis and Schoenflies problems*, Topology **30** (1991), 97–115.

[10] R.E. Gompf: *More Cappell–Shaneson spheres are standard*, arXiv:0908.1914.

[11] R.E. Gompf: *Sums of elliptic surfaces*, J. Differential Geom. **34** (1991), 93–114.

[12] R.E. Gompf and A.I. Stipsicz: 4-Manifolds and Kirby Calculus, Amer. Math. Soc., Providence, RI, 1999.

[13] B. Hanke, D. Kotschick and J. Wehrheim: *Dissolving four-manifolds and positive scalar curvature*, Math. Z. **245** (2003), 545–555.

[14] R. Mandelbaum: *Decomposing analytic surfaces*; in Geometric Topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York, 147–217, 1979.

[15] J. Morgan and G. Tian: Ricci Flow and the Poincaré Conjecture, Amer. Math. Soc., Providence, RI, 2007.

[16] J. Morgan and G. Tian: *Completion of the proof of the geometrization conjecture*, arXiv:0809.4040.

[17] Y. Ruan: *Virtual neighborhoods and the monopole equations*; in Topics in Symplectic 4-Manifolds (Irvine, CA, 1996), First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 101–116, 1998.

[18] C. Sung: *G-monopole classes, Ricci flow, and Yamabe invariants of* 4*-manifolds*, Geom. Dedicata **169** (2014), 129–144.

[19] C. Sung: *Finite group actions and G-monopole classes on smooth* 4*-manifolds*, arXiv: 1108.3875.

[20] C. Sung: *G-monopole invariants on some connected sums of* 4*-manifolds*, Geom. Dedicata **178** (2015), 75–93.

[21] C.H. Taubes: *The Seiberg–Witten invariants and* 4*-manifolds with essential tori*, Geom. Topol. **5** (2001), 441–519.

[22] M. Ue: *Exotic group actions in dimension four and Seiberg–Witten theory*, Proc. Japan Acad. Ser. A Math. Sci. **74** (1998), 68–70.

Dept. of mathematics education
Korea national university of education
Cheongju
Korea
e-mail: cysung@kias.re.kr