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Osaka J. Math.
53 (2016), 731–750

PARTIALLY ORDERED SETS OF NON-TRIVIAL NILPOTENT π-SUBGROUPS

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(Received July 3, 2015)

Abstract

In this paper, we introduce a subposet $\mathcal{L}_\pi(G)$ of a poset $\mathcal{N}_\pi(G)$ of all non-trivial nilpotent π-subgroups of a finite group G. We examine basic properties of subgroups in $\mathcal{L}_\pi(G)$ which contain the notion of both radical p-subgroups and centric p-subgroups of G. It is shown that $\mathcal{L}_\pi(G)$ is homotopy equivalent to $\mathcal{N}_\pi(G)$. As examples, we investigate in detail the case where symmetric groups.

1. Introduction

Let G be a finite group, and $\mathrm{Sgp}(G)$ the totality of subgroups of G. We regard $\mathrm{Sgp}(G)$ as a partially ordered set (poset for short) with respect to the inclusion-relation $\leq$. Then any subset $\mathcal{X} \subseteq \mathrm{Sgp}(G)$ can be thought of a subposet of $(\mathrm{Sgp}(G), \leq)$ which is identified with the associated order complex. Let $p \in \pi(G)$. Denote by $\mathcal{S}_p(G)$ the totality of non-trivial p-subgroups of G. A p-subgroup complex $\mathcal{X} \subseteq \mathcal{S}_p(G)$ itself is studied well by many authors (see [9] and various references in it). On the other hand, for distinct $p, q \in \pi(G)$, it is also quite important to investigate $\mathcal{X} \subseteq \mathcal{S}_p(G)$ and $\mathcal{Y} \subseteq \mathcal{S}_q(G)$ simultaneously. In order to do so, we focus on nilpotent subgroups, and actually deal with a poset $\mathcal{N}_\pi(G)$ of all non-trivial nilpotent π-subgroups of G where $\pi \subseteq \pi(G)$. In particular, we introduce a subposet $\mathcal{L}_\pi(G)$ of $\mathcal{N}_\pi(G)$, and show that they are homotopy equivalent each other. It is worth mentioning that a subgroup in $\mathcal{L}_\pi(G)$ contains the notion of both radical p-subgroups and centric p-subgroups of G.

The paper is organized as follows: In Section 2, we establish some notations, and prepare a number of standard posets of subgroups like $\mathcal{N}_\pi(G)$. In Section 3, we introduce a new poset $\mathcal{L}_\pi(G)$ consisting of certain nilpotent π-subgroups of G. We give another description of $\mathcal{L}_\pi(G)$ which is different from the form of the definition. Furthermore some tools for determining $\mathcal{L}_\pi(G)$ are developed. Then by using those results, we classify subgroups in $\mathcal{L}_\pi(G)$ for some groups G as examples. In Section 4, we provide homotopy equivalences among $\mathcal{L}_\pi(G)$ and the other standard posets of subgroups. Relations with known p-subgroup posets are examined. In Section 5, we investigate in detail the case where the symmetric group $\mathfrak{S}_n$ of degree n. In particular, we give a strategy to determine $\mathcal{L}_\pi(\mathfrak{S}_n)$ which is focused on irreducible subgroups (see Definition 5.5). Then, as

2010 Mathematics Subject Classification. Primary 20E15; Secondary 20D15.

examples, we classify subgroups in $\mathcal{L}_\pi(\mathfrak{S}_n)$ for $n \leq 6$ by using our method.

Finally, this work is derived from a series of our papers [5, 6, 7].

2. Preliminaries

In this section, we establish some notations which will be used in this paper. Let G be a finite group with the identity element e. Denote by $\pi(G)$ the set of all prime divisors of the order of G. Let π be a subset of $\pi(G)$. A subgroup H of G is called a π-subgroup if $\pi(H) \subseteq \pi$. The notation $\mathrm{Sgp}(G)$ stands for the totality of subgroups of G. Note that $\mathrm{Sgp}(G)$ is regarded as a poset together with the usual inclusion-relation $\leq$. We define the following subposets of $(\mathrm{Sgp}(G), \leq)$:

$$\mathcal{N}_\pi(G) := \{U \in \mathrm{Sgp}(G) \mid U \text{ is a non-trivial nilpotent } \pi\text{-subgroup of } G\},$$

$$\mathcal{A}\mathrm{b}_\pi(G) := \{U \in \mathrm{Sgp}(G) \mid U \text{ is a non-trivial abelian } \pi\text{-subgroup of } G\}.$$

Furthermore let $\mathcal{A}_\pi(G)$ be a subposet consisting of all non-trivial direct products of elementary abelian p-subgroups of G where p runs over primes in π. Then we have three posets $\mathcal{A}_\pi(G) \subseteq \mathcal{A}\mathrm{b}_\pi(G) \subseteq \mathcal{N}_\pi(G)$ on which the group G acts by conjugation. The set of all maximal elements in $(\mathcal{N}_\pi(G), \leq)$ is denoted by $\mathcal{N}_\pi(G)^{\max}$. For $\pi = \{p_1, \ldots, p_k\} \subseteq \pi(G)$, we sometimes write $\mathcal{N}_{p_1,\ldots,p_k}(G)$ in place of $\mathcal{N}_\pi(G)$. The ways of writing $\mathcal{N}_\pi(G)^{\max}$ and $\mathcal{N}_{p_1,\ldots,p_k}(G)$ are applied to the other posets. Let $p \in \pi(G)$. Denote by $\mathcal{S}_p(G)$ the totality of non-trivial p-subgroups of G. Then we note that $\mathcal{N}_p(G) = \mathcal{S}_p(G)$.

Denote by $Z(G)$ and $O_\pi(G)$ respectively the center of G, and the largest normal π-subgroup of G. For $A \in \mathcal{A}\mathrm{b}_\pi(G)$, suppose that $A = A_1 \times \cdots \times A_k$ is the direct product of Sylow p_i-subgroups A_i $(1 \leq i \leq k)$ of A. Then denote by $\Omega_1(A) := \Omega_1(A_1) \times \cdots \times \Omega_1(A_k) \in \mathcal{A}_\pi(G)$ where $\Omega(A_i) \in \mathcal{A}_{p_i}(G)$ is a subgroup generated by all elements in A_i of order p_i. For a subgroup $H \leq G$, if $O_\pi(Z(H)) \neq \{e\}$ then $O_\pi(Z(H)) \in \mathcal{A}\mathrm{b}_\pi(G)$ and $\Omega_1(O_\pi(Z(H))) \in \mathcal{A}_\pi(G)$. We express these subgroups as $O_\pi Z(H)$ and $\Omega_1 O_\pi Z(H)$ for short. In this way, we frequently omit parentheses of the composition of group operators throughout this paper.

Let $(\mathcal{P}, \leq)$ be a poset. For $z \in \mathcal{P}$, put $\mathcal{P}_{\leq z} := \{x \in \mathcal{P} \mid x \leq z\}$. Similarly, we define $\mathcal{P}_{<z}$, $\mathcal{P}_{\geq z}$, and $\mathcal{P}_{>z}$.

3. Subposets of $\mathcal{N}_\pi(G)$

Let G be a finite group, and $\pi \subseteq \pi(G)$. We introduce subposets of $(\mathcal{N}_\pi(G), \leq)$ as follows:

$$\mathcal{L}_\pi(G) := \{U \in \mathcal{N}_\pi(G) \mid U \geq O_\pi Z N_G(U)\},$$

$$\mathcal{L}^*_\pi(G) := \{U \in \mathcal{N}_\pi(G) \mid U \geq \Omega_1 O_\pi Z N_G(U)\}.$$

Both families are closed under G-conjugation. In this section, we study basic properties of $\mathcal{L}_\pi(G) \subseteq \mathcal{L}^*_\pi(G)$, and provide some examples. Note that, for a subgroup U of G,

$U \geq O_\pi Z N_G(U)$ if and only if $Z(U) \geq O_\pi Z N_G(U)$.

REMARK 3.1 (p-radicals and p-centrics). Let $p \in \pi(G)$.
(1) Denote by $\mathcal{B}_p(G)$ the totality of non-trivial p-subgroups U of G satisfying $O_p N_G(U) = U$. A subgroup in $\mathcal{B}_p(G)$ is called a radical p-subgroup (or just p-radical) of G. The poset $\mathcal{B}_p(G)$ is a generalized object of the Tits building, and it plays an important role in the area of group geometry. For a p-radical $U \in \mathcal{B}_p(G)$, we have that $U \geq Z(U) = Z O_p N_G(U) \geq O_p Z N_G(U)$. It follows that $\mathcal{B}_p(G) \subseteq \mathcal{L}_p(G)$, and thus, a subgroup in $\mathcal{L}_\pi(G)$ contains the notion of p-radicals. Furthermore, we see later in Remark 4.9 that $\mathcal{B}_p(G)$ is homotopy equivalent to $\mathcal{L}_p(G)$.
(2) A centric p-subgroup (or just p-centric) U of G is defined as a subgroup in $\mathcal{S}_p(G)$ such that any p-element in $C_G(U)$ is contained in U. This is also important in the area of group geometry or representation theory. Then it is now easy to check that a condition $U \geq O_p Z N_G(U)$ holds for a p-centric U. Thus $\mathcal{L}_p(G)$ includes all p-centrics.

Lemma 3.2. *Suppose that $p \in \pi$. Then $\mathcal{L}_\pi(G) \cap \mathcal{N}_p(G) \subseteq \mathcal{L}_p(G)$, and $\mathcal{L}^*_\pi(G) \cap \mathcal{N}_p(G) \subseteq \mathcal{L}^*_p(G)$.*

Proof. For any $U \in \mathcal{L}_\pi(G) \cap \mathcal{N}_p(G)$, we have that $U \geq O_\pi Z N_G(U)$. But U is a p-subgroup, so that, $O_\pi Z N_G(U) = O_p Z N_G(U)$. Thus $U \in \mathcal{L}_p(G)$. The second assertion similarly holds. □

Lemma 3.3. *For $U \in \mathcal{N}_\pi(G)$, put $K_U := O_\pi Z N_G(U)$. Then the product $U K_U$ is a member of $\mathcal{L}_\pi(G)$.*

Proof. Since U and K_U are nilpotent π-subgroups such that $[U, K_U] = \{e\}$, so is the product $U K_U$. Set $H := Z N_G(U K_U)$. Since $U \leq N_G(U) \leq N_G(U K_U)$, we have that $H \leq C_G(U) \leq N_G(U)$. It follows that H is contained in $Z N_G(U)$. Thus $O_\pi(H) \leq O_\pi Z N_G(U) = K_U \leq U K_U$. This shows that $U K_U \in \mathcal{L}_\pi(G)$. □

Below is a description of $\mathcal{L}_\pi(G)$ by using $U K_U$.

Proposition 3.4. *Under the notation in* Lemma 3.3, $\mathcal{L}_\pi(G) = \{U K_U \mid U \in \mathcal{N}_\pi(G)\}$.

Proof. By Lemma 3.3, it is enough to show that a map $f : \mathcal{N}_\pi(G) \to \mathcal{L}_\pi(G)$ defined by $f(U) := U K_U$ is surjective. Indeed, for any $X \in \mathcal{L}_\pi(G) \subseteq \mathcal{N}_\pi(G)$, we have that $X \geq O_\pi Z N_G(X) =: K_X$ by the definition of X. Thus $X = X K_X = f(X)$ as desired. □

From here, we want to develop some tools for determining $\mathcal{L}_\pi(G)$.

Lemma 3.5. *The followings hold.*
(1) *$\mathcal{N}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$ and $\mathcal{A}_\pi(G)^{\max} \subseteq \mathcal{L}^*_\pi(G)$.*

(2) *For* $U \in \mathcal{A}\mathrm{b}_\pi(G)^{\max}$, $\mathcal{N}_\pi(G)_{\geq U} \subseteq \mathcal{L}_\pi(G)$. *In particular,* $\mathcal{A}\mathrm{b}_\pi(G)^{\max} \subseteq \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$.
(3) $\mathcal{A}\mathrm{b}_\pi(G)^{\max} = (\mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G))^{\max}$.

Proof. (1) For $U \in \mathcal{N}_\pi(G)^{\max}$, put $K_U := O_\pi Z N_G(U)$. Since $U \leq U K_U \in \mathcal{N}_\pi(G)$ and the maximality of U, we have that $U K_U = U$ and $U \geq K_U$. Thus $U \in \mathcal{L}_\pi(G)$. On the other hand, for $V \in \mathcal{A}_\pi(G)^{\max}$, put $K_V^* := \Omega_1 O_\pi Z N_G(V) \in \mathcal{A}_\pi(G)$. Since $V \leq V K_V^* \in \mathcal{A}_\pi(G)$, we have the second assertion by the same way.

(2) For $U \in \mathcal{A}\mathrm{b}_\pi(G)^{\max}$, take $V \in \mathcal{N}_\pi(G)_{\geq U}$. Since $U \leq V \leq N_G(V)$, any element $t \in K_V := O_\pi Z N_G(V)$ commutes with U. Thus $U \leq \langle t \rangle U \in \mathcal{A}\mathrm{b}_\pi(G)$. By the maximality of U, we have that $t \in U \leq V$, and so $K_V \leq V$ as desired.

(3) Set $\mathcal{L}_\pi^{\mathrm{ab}}(G) := \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$. For $U \in \mathcal{A}\mathrm{b}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi^{\mathrm{ab}}(G)$, there exists $R \in \mathcal{L}_\pi^{\mathrm{ab}}(G)^{\max} \subseteq \mathcal{A}\mathrm{b}_\pi(G)$ such that $U \leq R$. Then by the maximality of U, $U = R \in \mathcal{L}_\pi^{\mathrm{ab}}(G)^{\max}$. The converse inclusion similarly holds. □

Proposition 3.6. *For* $V \leq U \in \mathcal{L}_\pi(G)$, *suppose that* $Z(U) \leq V \leq U$ *and* $N_G(U) \leq N_G(V)$. *Then* $V \in \mathcal{L}_\pi(G)$.

Proof. Take any $x \in Z N_G(V)$. Since $N_G(U) \leq N_G(V)$, we have that $[x, N_G(U)] = \{e\}$. This yields that $x \in Z N_G(U)$ and $Z N_G(V) \leq Z N_G(U)$. Thus $O_\pi Z N_G(V) \leq O_\pi Z N_G(U) \leq Z(U) \leq V$ as wanted. □

DEFINITION 3.7. For subgroups $A \leq B \leq G$, A is said to be weakly closed in B with respect to G if $A^g \leq B$ for some $g \in G$ implies $A^g = A$. In particular, $N_G(B) \leq N_G(A)$ holds.

The next result is an immediate consequence of Proposition 3.6

Proposition 3.8. *For* $V \leq U \in \mathcal{L}_\pi(G)$, *suppose that* $Z(U) \leq V \leq U$.
(1) *If* V *is weakly closed in* U *with respect to* G *then* $V \in \mathcal{L}_\pi(G)$.
(2) *If* V *is a characteristic subgroup of* U *then* $V \in \mathcal{L}_\pi(G)$. *In particular,* $Z(U) \in \mathcal{L}_\pi(G)$, *and that* $O_\pi Z N_G Z(U) \leq Z(U)$ *holds.*

Before giving examples, we recall some notations. For a subgroup $H \leq G$, we set $H^G := \{g^{-1} H g \mid g \in G\}$. For an integer $n \geq 2$, the symmetric and alternating group of degree n are denoted by S_n and A_n. The notation C_n means the cyclic group of order n.

EXAMPLE 3.9 (Solvable group S_4). Let $G = S_4$ of order $2^3 \cdot 3$, and $\pi := \pi(G) = \{2, 3\}$. We determine $\mathcal{L}_\pi(G)$. By Lemma 3.5 (1), $D_8 \cong U \in \mathrm{Syl}_2(G) \subseteq \mathcal{N}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$. Since any subgroup V of U containing $Z(U)$ is weakly closed in U with respect to G, we have that $V \in \mathcal{L}_\pi(G)$ by Proposition 3.8 (1). Let $W := \langle (12) \rangle$ be a remaining

2-subgroup of G. Since $N_G(W) = \langle (12), (34) \rangle$, we have that $O_\pi Z N_G(W) = \langle (12), (34) \rangle \not\leq W$, so that, $W \notin \mathcal{L}_\pi(G)$. Finally, by Lemma 3.5 (1), $\mathrm{Syl}_3(G) \subseteq \mathcal{N}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$. Therefore, we get

$$\mathcal{L}^*_{2,3}(G) = \mathcal{L}_{2,3}(G) = \mathcal{N}_{2,3}(G) \setminus \langle (12) \rangle^G = (\mathcal{S}_2(G) \setminus \langle (12) \rangle^G) \cup \mathrm{Syl}_3(G).$$

EXAMPLE 3.10 (Non-solvable group S_5). Let $G = S_5$ of order $2^3 \cdot 3 \cdot 5$, and $\pi := \{2, 3\} \subseteq \pi(G)$. We determine $\mathcal{L}_\pi(G)$. By the same way as in Example 3.9, we have that $\mathcal{S}_2(G) \setminus \langle (12) \rangle^G \subseteq \mathcal{L}_\pi(G)$. Let $W := \langle (12) \rangle$ be a remaining 2-subgroup of G. Since $N_G(W) = \langle (12) \rangle \times L$ where L is the symmetric group on $\{3, 4, 5\}$, we have that $O_\pi Z N_G(W) = W$, so that, $W \in \mathcal{L}_\pi(G)$. Let $X := \langle (123) \rangle \in \mathrm{Syl}_3(G) \subseteq \mathcal{N}_\pi(G)$. Since $N_G(X) = \langle (123), (12), (45) \rangle$, we have that $O_\pi Z N_G(X) = \langle (45) \rangle \not\leq X$. Thus $X \notin \mathcal{L}_\pi(G)$. Finally, by Lemma 3.5 (2), $C_6 \cong \langle (123)(45) \rangle \in \mathcal{A}\mathrm{b}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$. Therefore, we get

$$\mathcal{L}^*_{2,3}(G) = \mathcal{L}_{2,3}(G) = \mathcal{N}_{2,3}(G) \setminus \langle (123) \rangle^G = \mathcal{S}_2(G) \cup \langle (123)(45) \rangle^G.$$

EXAMPLE 3.11 (Simple group J_1). Let $G = J_1$ be the Janko simple group of order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, and $\pi := \{2, 3, 5\} \subseteq \pi(G)$. We determine $\mathcal{L}_\pi(G)$ referring [2, p. 36]. There is a unique class of involutions with a representative z. Set $U = \langle z \rangle$. Since $N_G(U) \cong U \times A_5$, we have that $O_\pi Z N_G(U) = U$, so that, $U \in \mathcal{L}_\pi(G)$. By Lemma 3.5 (1), $C_2 \times C_2 \times C_2 \cong V \in \mathrm{Syl}_2(G) \subseteq \mathcal{N}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$. Since $N_G(V) \cong V \rtimes (C_7 \rtimes C_3)$, all subgroups of order 2^2 are G-conjugate each other. Take the four group $C_2 \times C_2 \cong W < A_4 < A_5 < U \times A_5 \cong N_G(U)$. Then $N_G(W) \cong U \times A_4$ and $O_\pi Z N_G(W) = U \not\leq W$. Thus $W \notin \mathcal{L}_\pi(G)$. By looking at the normalizers, we see that $\mathrm{Syl}_3(G) \cup \mathrm{Syl}_5(G) \subseteq \mathcal{L}_\pi(G)$. Finally, by Lemma 3.5 (2), subgroups isomorphic to C_6 or C_{10} are in $\mathcal{A}\mathrm{b}_\pi(G)^{\max} \subseteq \mathcal{L}_\pi(G)$. Therefore, we get

$$\begin{aligned}\mathcal{L}^*_{2,3,5}(G) &= \mathcal{L}_{2,3,5}(G) = \mathcal{N}_{2,3,5}(G) \setminus W^G \\ &= (\mathcal{S}_2(G) \setminus W^G) \cup \mathrm{Syl}_3(G) \cup \mathrm{Syl}_5(G) \cup (C_6)^G \cup (C_{10})^G.\end{aligned}$$

4. Homotopy equivalences

Let $(\mathcal{P}, \leq)$ be a poset. Denote by $\mathsf{O}(\mathcal{P}) = \mathsf{O}(\mathcal{P}, \leq)$ the order complex of $\mathcal{P}$, which is a simplicial complex defined by all inclusion-chains $(x_0 < \cdots < x_k)$, where $x_i \in \mathcal{P}$, as simplices. We identify a poset $\mathcal{P}$ with the associated order complex $\mathsf{O}(\mathcal{P})$. We write $\mathcal{P} \simeq \mathcal{Q}$ when posets $\mathcal{P}$ and $\mathcal{Q}$ (namely, complexes $\mathsf{O}(\mathcal{P})$ and $\mathsf{O}(\mathcal{Q})$) are homotopy equivalent. Now any subset $\mathcal{X} \subseteq \mathrm{Sgp}(G)$ is thought of a subposet of $(\mathrm{Sgp}(G), \leq)$. Thus we can consider homotopy properties of $\mathcal{X}$. In this section, we give homotopy equivalences among $\mathcal{L}_\pi(G)$ and the other standard posets of subgroups. Relations with known p-subgroup posets are also investigated. The next lemma is fundamental in the theory of subgroup complexes.

Lemma 4.1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be posets. Let $\varphi\colon \mathcal{P} \to \mathcal{P}$ and $\psi\colon \mathcal{P} \to \mathcal{Q}$ be poset maps.*
(1) (*cf.* Lemma 3.3.3 *in* [9]) *If there exists $x_0 \in \mathcal{P}$ such that $\varphi(x) \geq x$ and $\varphi(x) \geq x_0$ for any $x \in \mathcal{P}$ (that is, $\mathcal{P}$ is conically contractible) then $\mathcal{P}$ is contractible.*
(2) (*cf.* Proposition 3.1.12 (2) *in* [9]) *Suppose that $\varphi(x) \leq x$ for any $x \in \mathcal{P}$. Then for any subset* $\operatorname{Im}\varphi \subseteq \mathcal{R} \subseteq \mathcal{P}$, *we have that $\mathcal{P} \simeq \mathcal{R}$. (And dually for $\varphi(x) \geq x$.)*
(3) (*Quillen's fiber theorem*; *cf.* Theorem 4.2.1 *in* [9]) *Suppose that $\psi^{-1}(\mathcal{Q}_{\leq z})$ is contractible for any $z \in \mathcal{Q}$. Then $\mathcal{P} \simeq \mathcal{Q}$. (And dually for $\mathcal{Q}_{\geq z}$.)*
(4) (*cf.* Theorem 4.3.2 *in* [9]) *Suppose that $\mathcal{P}$ is finite. Let*

$$\mathcal{P}^{<} := \{z \in \mathcal{P} \mid \mathcal{P}_{<z} \text{ is not contractible}\},$$
$$\mathcal{P}^{>} := \{z \in \mathcal{P} \mid \mathcal{P}_{>z} \text{ is not contractible}\}.$$

Then for any subset $\mathcal{P}^{<} \subseteq \mathcal{R} \subseteq \mathcal{P}$, we have that $\mathcal{P} \simeq \mathcal{R}$. (And dually for $\mathcal{P}^{>}$.)

Proposition 4.2. *The inclusions $\mathcal{A}_\pi(G) \hookrightarrow \mathcal{N}_\pi(G)$ and $\mathcal{A}\mathrm{b}_\pi(G) \hookrightarrow \mathcal{N}_\pi(G)$ induce homotopy equivalences.*

Proof. Let $f\colon \mathcal{A}_\pi(G) \hookrightarrow \mathcal{N}_\pi(G)$ be the inclusion map. Then by Lemma 4.1 (3), it is enough to show that $f^{-1}(\mathcal{N}_\pi(G)_{\leq U}) = \{E \in \mathcal{A}_\pi(G) \mid E \leq U\} = \mathcal{A}_\pi(U)$ is contractible for any $U \in \mathcal{N}_\pi(G)$. Express $U = U_1 \times \cdots \times U_m$ as the direct product of Sylow subgroups U_i $(1 \leq i \leq m)$ of U. Then $A := \Omega_1 Z(U) = \Omega_1 Z(U_1) \times \cdots \times \Omega_1 Z(U_m) \neq \{e\}$ is a member of $\mathcal{A}_\pi(U)$. Let $\varphi\colon \mathcal{A}_\pi(U) \to \mathcal{A}_\pi(U)$ be a poset map defined by $\varphi(E) := AE$ for $E \in \mathcal{A}_\pi(U)$, which satisfies $\varphi(E) \geq E$ and $\varphi(E) \geq A$. This yields that $\mathcal{A}_\pi(U)$ is contractible by Lemma 4.1 (1).

By the same way, we obtain $\mathcal{A}\mathrm{b}_\pi(G) \simeq \mathcal{N}_\pi(G)$ although we may replace $A := \Omega_1 Z(U)$ with just $Z(U)$ in the above discussion. □

Proposition 4.3. *$\mathcal{N}_\pi(G)^{>} \subseteq \mathcal{L}_\pi(G) \subseteq \mathcal{L}^*_\pi(G) \subseteq \mathcal{N}_\pi(G)$ holds. In particular, $\mathcal{N}_\pi(G)$, $\mathcal{L}_\pi(G)$, and $\mathcal{L}^*_\pi(G)$ are homotopy equivalent each other by* Lemma 4.1 (4).

Proof. It is enough to show that $\mathcal{N}_\pi(G)^{>} \subseteq \mathcal{L}_\pi(G)$. For $U \in \mathcal{N}_\pi(G)$, we have that $\mathcal{N}_\pi(G)_{>U} \simeq \mathcal{N}_\pi(N_G(U))_{>U}$. Indeed, for any $V \in \mathcal{N}_\pi(G)_{>U}$, $N_V(U) > U$ as V is nilpotent. Then a poset map

$$f\colon \mathcal{N}_\pi(G)_{>U} \to \mathcal{N}_\pi(G)_{>U}$$

defined by $V \mapsto N_V(U) \leq V$ provides us $\mathcal{N}_\pi(G)_{>U} \simeq \operatorname{Im} f = \mathcal{N}_\pi(N_G(U))_{>U}$ by Lemma 4.1 (2).

Set $K_U := O_\pi Z N_G(U)$. Since U and K_U are normal nilpotent π-subgroups of $N_G(U)$, we have that $UK_U \in \mathcal{N}_\pi(N_G(U))$. Suppose that $U \not\geq K_U$, that is, $U \notin \mathcal{L}_\pi(G)$.

Then $UK_U \in \mathcal{N}_\pi(N_G(U))_{>U}$. Furthermore, for $X \in \mathcal{N}_\pi(N_G(U))_{>U}$, we have that $[X, K_U] = \{e\}$. This yields that $\mathcal{N}_\pi(N_G(U))_{>U} \ni XK_U = X(UK_U)$, and that a poset map

$$\varphi\colon \mathcal{N}_\pi(N_G(U))_{>U} \to \mathcal{N}_\pi(N_G(U))_{>U}$$

defined by $X \mapsto X(UK_U)$ induces contractibility of $\mathcal{N}_\pi(N_G(U))_{>U}$ by Lemma 4.1 (1). It follows that $\mathcal{N}_\pi(G)^{>} \subseteq \mathcal{L}_\pi(G)$. $\square$

REMARK 4.4. The converse inclusion $\mathcal{N}_\pi(G)^{>} \supseteq \mathcal{L}_\pi(G)$ is not necessarily established. For example, let $G = M_{12}$ be the Mathieu group of degree 12 of order $2^6 \cdot 3^3 \cdot 5 \cdot 11$, and $\pi := \{2\} \subseteq \pi(G)$. Referring [2, p. 33], there exists a subgroup $U \cong C_4 \times C_4$ of G with $N_G(U) \cong U \rtimes D_{12}$ and $O_2 Z N_G(U) = \{e\} \leq U$. Thus $U \in \mathcal{L}_2(G)$. However, $\mathcal{N}_2(N_G(U))_{>U} \cong \mathcal{N}_2(D_{12}) = \mathcal{S}_2(D_{12})$ is contractible since $O_2(D_{12}) \cong C_2$. This shows that $U \notin \mathcal{N}_2(G)^{>}$.

Proposition 4.5. *The followings hold.*
(1) $\mathcal{A}\mathrm{b}_\pi(G)^{>} \subseteq \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G) \subseteq \mathcal{A}\mathrm{b}_\pi(G)$.
(2) $\mathcal{A}_\pi(G)^{>} \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}^*_\pi(G) \subseteq \mathcal{A}_\pi(G)$.
In particular, we have homotopy equivalences $\mathcal{A}\mathrm{b}_\pi(G) \simeq \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$ *and* $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}^*_\pi(G)$ *by* Lemma 4.1 (4).

Proof. For $U \in \mathcal{A}\mathrm{b}_\pi(G)$, set $K_U := O_\pi Z N_G(U)$. Since $[U, K_U] = \{e\}$, we have that $UK_U \in \mathcal{A}\mathrm{b}_\pi(G)$. Suppose that $U \ngeq K_U$, that is, $U \notin \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$. Then $UK_U \in \mathcal{A}\mathrm{b}_\pi(G)_{>U}$. Furthermore, for $X \in \mathcal{A}\mathrm{b}_\pi(G)_{>U}$, we have that $X \leq C_G(U) \leq N_G(U)$, and thus $[X, K_U] = \{e\}$. This yields that $\mathcal{A}\mathrm{b}_\pi(G)_{>U} \ni XK_U = X(UK_U)$, and that a poset map

$$\varphi\colon \mathcal{A}\mathrm{b}_\pi(G)_{>U} \to \mathcal{A}\mathrm{b}_\pi(G)_{>U}$$

defined by $X \mapsto X(UK_U)$ induces contractibility of $\mathcal{A}\mathrm{b}_\pi(G)_{>U}$ by Lemma 4.1 (1). It follows that $\mathcal{A}\mathrm{b}_\pi(G)^{>} \subseteq \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$.

By the same way, we obtain $\mathcal{A}_\pi(G)^{>} \subseteq \mathcal{A}_\pi(G) \cap \mathcal{L}^*_\pi(G) \subseteq \mathcal{A}_\pi(G)$ by using $K^*_U := \Omega_1 O_\pi Z N_G(U)$ in place of $K_U := O_\pi Z N_G(U)$ in the above discussion. $\square$

Summarizing Propositions 4.2, 4.3, and 4.5, we obtain the next.

Proposition 4.6. *The following homotopy equivalences hold.*
(α) $\mathcal{N}_\pi(G) \simeq \mathcal{L}_\pi(G) \simeq \mathcal{L}^*_\pi(G) \simeq \mathcal{A}\mathrm{b}_\pi(G) \simeq \mathcal{A}_\pi(G)$.
(β) $\mathcal{A}\mathrm{b}_\pi(G) \simeq \mathcal{A}\mathrm{b}_\pi(G) \cap \mathcal{L}_\pi(G)$.
(γ) $\mathcal{A}_\pi(G) \simeq \mathcal{A}_\pi(G) \cap \mathcal{L}^*_\pi(G)$.

Note that equivalences in Proposition 4.6 can be extended to G-homotopy equivalences (see [9, Section 3.5] or [11]).

REMARK 4.7 (The whole $\pi(G)$ case). In the case of $\pi = \pi(G)$, our equivalence (α) in Proposition 4.6 gives $\mathcal{N}(G) \simeq \mathcal{A}\mathrm{b}(G) \simeq \mathcal{A}(G)$ where these three posets are respectively the totality of non-trivial nilpotent subgroups, abelian subgroups, and direct products of elementary abelian subgroups of G. This result coincides with a part of [8, Proposition 1.2].

Like Lemma 4.1, posets $\mathcal{S}_p(G)$, $\mathcal{A}_p(G)$, and $\mathcal{B}_p(G)$ (see Remark 3.1) are also fundamental in the theory of subgroup complexes. In particular, those three posets are homotopy equivalent each other (cf. [9, p. 165]). Below is an immediate consequence of Proposition 4.6 with $\pi = \{p\}$. In particular, equivalences related to $\mathcal{L}_p(G)$ should be new.

Corollary 4.8. *The following homotopy equivalences hold.*

$$\mathcal{S}_p(G) = \mathcal{N}_p(G) \simeq \mathcal{A}\mathrm{b}_p(G) \simeq \mathcal{A}_p(G) \simeq \mathcal{L}_p(G) \simeq \mathcal{L}_p^*(G),$$
$$\mathcal{A}\mathrm{b}_p(G) \simeq \{U \in \mathcal{A}\mathrm{b}_p(G) \mid U \geq O_p Z N_G(U)\},$$
$$\mathcal{A}_p(G) \simeq \{U \in \mathcal{A}_p(G) \mid U \geq \Omega_1 O_p Z N_G(U)\}.$$

REMARK 4.9. (1) Recall that a poset $\mathcal{Z}_p(G) := \{U \in \mathcal{A}_p(G) \mid \Omega_1 O_p Z C_G(U) = U\}$ is introduced by Benson (see [1, p. 226]). It is known that $\mathcal{A}_p(G)^{>} \subseteq \mathcal{Z}_p(G)$ (cf. [9, Remark 4.3.5]), so that, $\mathcal{A}_p(G) \simeq \mathcal{Z}_p(G)$. But this equivalence of $\mathcal{A}_p(G)$ is different from $\mathcal{A}_p(G) \simeq \mathcal{A}_p(G) \cap \mathcal{L}_p(G)$ in Corollary 4.8.
(2) As mentioned in Remark 3.1, $\mathcal{B}_p(G)$ is included in $\mathcal{L}_p(G)$. Thus a relation $\mathcal{B}_p(G) = \mathcal{B}_p(G) \cap \mathcal{L}_p(G)$ holds. Furthermore, we have that $\mathcal{B}_p(G) \simeq \mathcal{S}_p(G) \simeq \mathcal{L}_p(G)$ by Corollary 4.8.

REMARK 4.10. We investigated $\mathcal{N}_\pi(G)^{>}$ in Proposition 4.3, and also $\mathcal{A}\mathrm{b}_\pi(G)^{>}$ and $\mathcal{A}_\pi(G)^{>}$ in Proposition 4.5. On the other hand, it is known (cf. [9, p. 152]) that $\mathcal{S}_p(G)^{<} = \mathcal{A}_p(G)$ and $\mathcal{S}_p(G)^{>} \subseteq \mathcal{B}_p(G)$ in general. Furthermore the equality $\mathcal{S}_p(G)^{>} = \mathcal{B}_p(G)$ holds assuming Quillen conjecture which is saying that if $\mathcal{S}_p(G)$ is contractible then $O_p(G)$ is non-trivial. From this viewpoint, a subgroup in $\mathcal{N}_\pi(G)^{>} \subseteq \mathcal{L}_\pi(G)$ might be a candidate of "π-radicals". In addition, we already saw in Remark 3.1 that a subgroup in $\mathcal{L}_\pi(G)$ contains the notion of p-radicals.

REMARK 4.11. Suppose that $O_p(G) \neq \{e\}$. Then a relation $U \leq U O_p(G) \geq O_p(G)$ for any $U \in \mathcal{S}_p(G)$ gives us (conical) contractibility of $\mathcal{S}_p(G)$. The converse is Quillen conjecture. How about $\mathcal{N}_\pi(G)$? Let G be the symmetric group S_4 of degree 4, and $\pi := \pi(G) = \{2, 3\}$. Then $\mathcal{N}_\pi(G) = \mathcal{S}_2(G) \cup \mathcal{S}_3(G)$ is disconnected (i.e. non-contractible) even if $O_\pi(G) = G \neq \{e\}$ or $O_\pi F(G) = F(G) \cong C_2 \times C_2 \neq \{e\}$ where $F(G)$ is the Fitting subgroup of G.

5. Investigations on $\mathcal{L}_\pi(\mathfrak{S}_n)$

For a positive integer n, denote by $\mathfrak{S}(\Omega) = \mathfrak{S}_n$ the symmetric group on a set $\Omega := \{1, 2, \dots, n\}$. In this section, we investigate subgroups in $\mathcal{L}_\pi(\mathfrak{S}(\Omega))$. It is shown that the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))$ can be reduced to the case where H is irreducible (see Definition 5.5) such that there is no fixed point of H on Ω. Then focusing on the irreducibility of subgroups, we provide a strategy to determine $\mathcal{L}_\pi(\mathfrak{S}_n)$. As examples, we classify subgroups in $\mathcal{L}_\pi(\mathfrak{S}_n)$ for $n \le 6$ by using our method.

For a family $\mathcal{H} \subseteq \mathrm{Sgp}(\mathfrak{S}_n)$ of subgroups closed under $\mathfrak{S}_n$-conjugation, denote by $\mathcal{H}/{\sim_{\mathfrak{S}_n}}$ a set of $\mathfrak{S}_n$-conjugate representatives of $\mathcal{H}$.

5.1. The symmetric group. We establish some notations on $\mathfrak{S}(\Omega)$. For $x, y \in \mathfrak{S}(\Omega)$, the composition $xy \in \mathfrak{S}(\Omega)$ is read from left to right, and denote by $\alpha^x \in \Omega$ the image of $\alpha \in \Omega$ under x. Let $e \in \mathfrak{S}(\Omega)$ be the identity element. The notation $E := \{e\}$ stands for the trivial subgroup of $\mathfrak{S}(\Omega)$. For a subgroup $H \le \mathfrak{S}(\Omega)$, as in [3, p. 19], the set of fixed points and support of H are defined by

$$\mathrm{fix}(H) := \{\alpha \in \Omega \mid \alpha^h = \alpha \text{ for all } h \in H\},$$
$$\mathrm{supp}(H) := \Omega \setminus \mathrm{fix}(H) = \{\alpha \in \Omega \mid \alpha^h \ne \alpha \text{ for some } h \in H\}.$$

It is clear that $H = E$ if and only if $\mathrm{supp}(H) = \emptyset$.

NOTATION 5.1. For an H-invariant subset $\Gamma \subseteq \Omega$, denote by $H|_\Gamma \le \mathfrak{S}(\Omega)$ the group of permutations which agree with an element of H on Γ and are the identity on $\Omega \setminus \Gamma$. In other words, for an element $h \in H$, we identify a bijective restriction map $h|_\Gamma \colon \Gamma \to \Gamma$ with a permutation on Ω which is the identity on $\Omega \setminus \Gamma$. Then the group $H|_\Gamma$ is defined by $\{h|_\Gamma \mid h \in H\} \le \mathfrak{S}(\Gamma) \hookrightarrow \mathfrak{S}(\Omega)$.

A subset $\mathrm{supp}(H) \subseteq \Omega$ is $N_{\mathfrak{S}(\Omega)}(H)$-invariant, and H is identified with $H|_{\mathrm{supp}(H)} \le \mathfrak{S}(\mathrm{supp}(H))$. For any H-invariant subset $\Gamma \subseteq \Omega$, it is clear that $\mathrm{supp}(H|_\Gamma) = \mathrm{supp}(H) \cap \Gamma$.

5.2. Reduction to the fixed point free case. In this section, we show that the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))$ can be reduced to the case where H has no fixed points in Ω. Put

$$\mathcal{L}_\pi(\mathfrak{S}(\Omega))^0 := \{H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega)) \mid \mathrm{fix}(H) = \emptyset\}.$$

Lemma 5.2. *Let $H \le \mathfrak{S}(\Omega)$ be a non-trivial subgroup.*
(1) *Suppose $2 \notin \pi$. Then $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))$ if and only if $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega \setminus \mathrm{fix}(H)))^0$.*
(2) *Suppose $2 \in \pi$. Then $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))$ if and only if $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega \setminus \mathrm{fix}(H)))^0$ and $|\mathrm{fix}(H)| \ne 2$.*

Proof. Set $G := \mathfrak{S}(\Omega)$, $\Omega_+ := \mathrm{supp}(H)$, and $\Omega_0 := \mathrm{fix}(H)$. Recall that H is identified with $H_+ := H|_{\mathrm{supp}(H)}$. In order to prove this lemma, it is enough to show that $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))$ if and only if $H_+ \in \mathcal{L}_\pi(\mathfrak{S}(\Omega_+))^0$, and $|\Omega_0| \neq 2$ or $2 \notin \pi$. Now since $N_G(H)$ acts on both Ω_0 and Ω_+, we have that $N_G(H) \leq \mathfrak{S}(\Omega_0) \times \mathfrak{S}(\Omega_+)$. Hence

$$N_G(H) = N_{\mathfrak{S}(\Omega_0)\times\mathfrak{S}(\Omega_+)}(H_+) = \mathfrak{S}(\Omega_0) \times N_{\mathfrak{S}(\Omega_+)}(H_+),$$
$$O_\pi Z N_G(H) = O_\pi Z(\mathfrak{S}(\Omega_0)) \times O_\pi Z(N_{\mathfrak{S}(\Omega_+)}(H_+)).$$

Suppose that $H \in \mathcal{L}_\pi(G)$, that is, $H_+ = H \geq O_\pi Z N_G(H)$. Then $O_\pi Z(\mathfrak{S}(\Omega_0)) = E$ and $H_+ \geq O_\pi Z(N_{\mathfrak{S}(\Omega_+)}(H_+))$. Thus $H_+ \in \mathcal{L}_\pi(\mathfrak{S}(\Omega_+))^0$. Furthermore $Z(\mathfrak{S}(\Omega_0))$ is non-trivial if and only if $|\Omega_0| = 2$. This yields that $O_\pi Z(\mathfrak{S}(\Omega_0)) = E$ if and only if $|\Omega_0| \neq 2$ or $2 \notin \pi$. The converse is now clear. The proof is complete. $\square$

The following result is a consequence of Lemma 5.2.

Proposition 5.3. *For positive integers $n \geq 3$ and $2 \leq k \leq n-1$, set $[k] := \{1, \dots, k\} \subseteq \Omega$. Then we have that*

$$\mathcal{L}_\pi(\mathfrak{S}(\Omega))/{\sim_{\mathfrak{S}(\Omega)}} = \begin{cases} \left(\bigcup_{k=2}^{n-1} \mathcal{L}_\pi(\mathfrak{S}([k]))^0/{\sim_{\mathfrak{S}([k])}}\right) \cup \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0/{\sim_{\mathfrak{S}(\Omega)}} & \text{if} \quad 2 \notin \pi, \\ \left(\bigcup_{\substack{k=2 \\ k\neq n-2}}^{n-1} \mathcal{L}_\pi(\mathfrak{S}([k]))^0/{\sim_{\mathfrak{S}([k])}}\right) \cup \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0/{\sim_{\mathfrak{S}(\Omega)}} & \text{if} \quad 2 \in \pi. \end{cases}$$

By Proposition 5.3 together with the inductive argument, the determination of $\mathcal{L}_\pi(\mathfrak{S}(\Omega))$ can be reduced to that of $\mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$.

5.3. Reduction to components. In this section, we introduce the irreducibility of a subgroup of $\mathfrak{S}(\Omega)$, and show that any non-trivial subgroup H of $\mathfrak{S}(\Omega)$ can be uniquely decomposed into irreducible subgroups of H. Using such a decomposition of H, the notion of components of H comes out. Then we show that the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$ can be reduced to the case where H itself is a component of H.

NOTATION 5.4. If a direct product subgroup $H = H_1 \times H_2 \leq \mathfrak{S}(\Omega)$ satisfies $\mathrm{supp}(H_1) \cap \mathrm{supp}(H_2) = \emptyset$, then we denote it by $H = H_1 \perp H_2$. In this case, we have a disjoint union $\mathrm{supp}(H) = \mathrm{supp}(H_1) \uplus \mathrm{supp}(H_2)$. Furthermore, we recursively define $H_1 \perp H_2 \perp \cdots \perp H_l$ for any finite number of subgroups $H_i \leq \mathfrak{S}(\Omega)$ by $(H_1 \perp \cdots \perp H_{l-1}) \perp H_l$.

DEFINITION 5.5. Let $H \leq \mathfrak{S}(\Omega)$ be a subgroup. H is said to be reducible if there exist non-trivial subgroups $H_1, H_2 \leq H$ such that $H = H_1 \perp H_2$. On the other hand, we call H irreducible if $H \neq E$ and H is not reducible, that is, whenever $H = K \perp L$ for subgroups $K, L \leq H$ then $K = E$ or $L = E$.

Lemma 5.6. (1) *For a subgroup $H = H_1 \perp H_2 \leq \mathfrak{S}(\Omega)$ and an H-invariant subset $\Gamma \subseteq \Omega$, we have that $H|_\Gamma = H_1|_\Gamma \perp H_2|_\Gamma$.*
(2) *Suppose that $A \perp B = A \perp C \leq \mathfrak{S}(\Omega)$. Then $B = C$.*

Proof. (1) Straightforward.

(2) Set $D := A \perp B$. Then $\Gamma_B := \mathrm{supp}(B) = \mathrm{supp}(D) \setminus \mathrm{supp}(A) = \mathrm{supp}(C) =: \Gamma_C$. For a D-invariant subset $\Gamma_B = \Gamma_C$, we have by (1) that

$$D|_{\Gamma_B} = (A \perp B)|_{\Gamma_B} = A|_{\Gamma_B} \perp B|_{\Gamma_B} = E \perp B = B,$$
$$D|_{\Gamma_C} = (A \perp C)|_{\Gamma_C} = A|_{\Gamma_C} \perp C|_{\Gamma_C} = E \perp C = C.$$

Thus $B = C$ as wanted. □

Proposition 5.7. *Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup. Then H is decomposed as*

$$H = H_1 \perp \cdots \perp H_l$$

where the $H_i \leq H$ are irreducible and unique up to order.

Proof. We proceed by induction on $|\mathrm{supp}(H)| > 0$. For the existence, we may assume that H is reducible. Then there exist non-trivial subgroups $H_1, H_2 \leq H$ such that $H = H_1 \perp H_2$. Since the supports of H_1 and H_2 are strictly contained in $\mathrm{supp}(H)$, we have that each H_i can be decomposed into irreducible subgroups by induction. This shows the existence of the decomposition.

Suppose next that $H = H_1 \perp \cdots \perp H_l = K_1 \perp \cdots \perp K_m$ for some irreducible subgroups $H_i, K_j \leq \mathfrak{S}(\Omega)$. Since $\Gamma := \mathrm{supp}(H_1) \subseteq \mathrm{supp}(H) = \bigcup_{j=1}^m \mathrm{supp}(K_j)$, we may assume that $\Gamma \cap \Lambda \neq \emptyset$ for $\Lambda := \mathrm{supp}(K_1)$. Then $\mathrm{supp}(K_1|_\Gamma) = \mathrm{supp}(K_1) \cap \Gamma = \Lambda \cap \Gamma \neq \emptyset$ and $K_1|_\Gamma \neq E$. Now

$$H_1 = H|_\Gamma = (K_1 \perp \cdots \perp K_m)|_\Gamma = K_1|_\Gamma \perp \cdots \perp K_m|_\Gamma.$$

By the irreducibility of H_1, $H_1 = K_1|_\Gamma$ and $\Gamma = \mathrm{supp}(H_1) = \mathrm{supp}(K_1|_\Gamma) \subseteq \Lambda$. Exchanging roles of Γ and Λ, we can obtain that $\Lambda \subseteq \Gamma$, so that, $\Gamma = \Lambda$. This yields that $H_1 = K_1|_\Gamma = K_1|_\Lambda = K_1$. Then by Lemma 5.6, $H' := H_2 \perp \cdots \perp H_l = K_2 \perp \cdots \perp K_m$. Since the support of H' is strictly contained in $\mathrm{supp}(H)$, the uniqueness also holds by induction. □

Corollary 5.8. *Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup, and let $H = H_1 \perp \cdots \perp H_l$ be a decomposition of H as in* Proposition 5.7. *Set $\Gamma_i := \mathrm{supp}(H_i)$ for $1 \leq i \leq l$.*

Suppose that $\mathrm{supp}(H) = \Omega$. *Then we have that if* $H_i \in \mathcal{L}_\pi(\mathfrak{S}(\Gamma_i))^0$ *for all* $1 \le i \le l$ *then* $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$.

Proof. Any element $g \in O_\pi Z N_{\mathfrak{S}(\Omega)}(H)$ commutes with H_i for all $1 \le i \le l$. So Γ_i is $\langle g\rangle$-invariant. Since $\mathrm{supp}(H) = \Omega$, we have that $g = \prod_{i=1}^{l} g|_{\Gamma_i}$ which is contained in $\prod_{i=1}^{l} O_\pi Z N_{\mathfrak{S}(\Gamma_i)}(H_i)$. Thus

$$O_\pi Z N_{\mathfrak{S}(\Omega)}(H) \le \prod_{i=1}^{l} O_\pi Z N_{\mathfrak{S}(\Gamma_i)}(H_i),$$

and this completes the proof. □

We establish the situation once more here. Set $G := \mathfrak{S}(\Omega)$, and let $H \le \mathfrak{S}(\Omega)$ be a non-trivial subgroup. Suppose that $H = H_1 \perp \cdots \perp H_l$ be a decomposition of H into irreducible subgroups H_i $(1 \le i \le l)$ as in Proposition 5.7. Then a set $\mathcal{X}_H := \{H_1, \ldots, H_l\}$ is uniquely determined by H. Let $\{K_1, \ldots, K_t\} \subseteq \mathcal{X}_H$ be a set of representatives of G-conjugate classes in $\mathcal{X}_H$. For each K_i, denote by $[K_i] := \{H_j \in \mathcal{X}_H \mid H_j \sim_G K_i\}$ the class containing K_i. We set $[K_i] = \{K_i^{(1)}, K_i^{(2)}, \ldots, K_i^{(m_i)}\}$, and define a subgroup

$$M(K_i) := \langle K \mid K \in [K_i]\rangle = K_i^{(1)} \perp K_i^{(2)} \perp \cdots \perp K_i^{(m_i)} \le H.$$

Then $H = M(K_1) \perp M(K_2) \perp \cdots \perp M(K_t)$. We call each subgroup $M(K_i)$ a "component" of H. Put

$$X_i := \mathrm{supp}(M(K_i)) = \bigcup_{j=1}^{m_i} \mathrm{supp}(K_i^{(j)}), \quad G_i := \mathfrak{S}(X_i) \le G.$$

Proposition 5.9. *With the above notations, suppose that* $\mathrm{supp}(H) = \Omega$. *Then we have that*

(1) $N_G(H) = N_{G_1}(M(K_1)) \perp N_{G_2}(M(K_2)) \perp \cdots \perp N_{G_t}(M(K_t))$.

(2) $H \in \mathcal{L}_\pi(G)^0$ *if and only if* $M(K_i) \in \mathcal{L}_\pi(G_i)^0$ *for all* $1 \le i \le t$.

Proof. (1) For any $g \in N_G(H)$, $H = H^g = H_1^g \perp \cdots \perp H_l^g$. Since $\mathcal{X}_H$ is uniquely determined by H by Proposition 5.7, we have that $\langle g\rangle$ acts on $\mathcal{X}_H$ and $[K_i]$ for any $1 \le i \le t$. This yields that X_i is $\langle g\rangle$-invariant, and thus $g|_{X_i} \in N_{G_i}(M(K_i))$. Since $\mathrm{supp}(H) = \Omega$, we have that $g = \prod_{i=1}^{t} g|_{X_i}$ which is contained in $N_{G_1}(M(K_1)) \perp \cdots \perp N_{G_t}(M(K_t))$. The converse inclusion is trivial.

(2) Straightforward from (1). □

By Proposition 5.9 (2), the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$ can be reduced to the case where H itself is a component of H, that is, all subgroups in $\mathcal{X}_H$ are $\mathfrak{S}(\Omega)$-conjugate each other.

5.4. Reduction to irreducible subgroups. In this section, we show that the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$ can be reduced to the case where H is irreducible. Set $G := \mathfrak{S}(\Omega)$. By reason of Proposition 5.9 (2), we assume the following Hypothesis 5.10

HYPOTHESIS 5.10. Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup. Suppose that $H = H_1 \perp \cdots \perp H_l$ be a decomposition of H into irreducible subgroups H_i $(1 \leq i \leq l)$ as in Proposition 5.7. Then $H_i \sim_G H_j$ for any $1 \leq i, j \leq l$.

We examine the structure of $N_G(H)$. Set $\Gamma_i := \mathrm{supp}(H_i)$ and $G_i := \mathfrak{S}(\Gamma_i)$ for $1 \leq i \leq l$. By Hypothesis 5.10, for each $2 \leq i \leq l$, there exists $g_i \in G$ such that $H_i = H_1^{g_i} := g_i^{-1} H_1 g_i$ which induces a permutation equivalence $(H_1, \Gamma_1) \simeq (H_i, \Gamma_i)$. In other words, there exist bijections $f_i \colon H_1 \to H_i$ defined by $x \mapsto x^{g_i} := g_i^{-1} x g_i$ for $x \in H_1$, and $\varphi_i : \Gamma_1 \to \Gamma_i$ defined by $\alpha \mapsto \alpha^{g_i}$ for $\alpha \in \Gamma_1$ satisfying $(\alpha^{\varphi_i})^{x^{f_i}} = (\alpha^x)^{\varphi_i}$ for any $x \in H_1$ and $\alpha \in \Gamma_1$. Now we define an involution

$$\sigma_i := \prod_{\alpha \in \Gamma_1} (\alpha, \alpha^{\varphi_i}) \in \mathfrak{S}(\Gamma_1 \cup \Gamma_i) \leq \mathfrak{S}(\Omega) \quad (2 \leq i \leq l)$$

which acts on $\mathcal{X}_H = \{H_1, \ldots, H_l\}$ as a transposition (H_1, H_i). Then $S := \langle \sigma_2, \ldots, \sigma_l \rangle \cong \mathfrak{S}_l$ acts on both $\mathcal{X}_H$ and $\{N_{G_1}(H_1), \ldots, N_{G_l}(H_l)\}$ as $\mathfrak{S}_l$ respectively, and a subgroup $N_{G_1}(H_1) \wr S \cong B \rtimes S \leq N_G(H)$ is defined where $B := N_{G_1}(H_1) \times \cdots \times N_{G_l}(H_l)$.

Proposition 5.11. *Assume* Hypothesis 5.10. *With the above notations, suppose that* $\mathrm{supp}(H) = \Omega$. *Then we have that*
(1) $N_G(H) = B \rtimes S$.
(2) $H \in \mathcal{L}_\pi(G)^0$ *if and only if* $H_1 \in \mathcal{L}_\pi(G_1)^0$.

Proof. (1) For any element $g \in N_G(H)$, $\langle g \rangle$ acts on $\mathcal{X}_H$ as in the proof of Proposition 5.9. Then there exists $\sigma \in S$ such that σ is equal to g as elements of $\mathfrak{S}(\mathcal{X}_H)$. Thus $g\sigma^{-1}$ fixes H_i for all $1 \leq i \leq l$, so that, $(g\sigma^{-1})|_{\Gamma_i} \in N_{G_i}(H_i)$. Since $\mathrm{supp}(H) = \Omega$, we have that $g\sigma^{-1} = \prod_{i=1}^{l} (g\sigma^{-1})|_{\Gamma_i}$ which is contained in B. So $g \in B\sigma \subseteq B \rtimes S$.

(2) Suppose that $H_1 \notin \mathcal{L}_\pi(G_1)^0$, and then we will show that $H \notin \mathcal{L}_\pi(G)^0$. We may assume that $l \geq 2$. Now there exists $z_1 \in O_\pi Z N_{G_1}(H_1) \setminus H_1$. For $2 \leq i \leq l$, put

$$z_i := \sigma_i^{-1} z_1 \sigma_i \in O_\pi Z N_{G_i}(H_i) \setminus H_i, \quad z_0 := \prod_{i=1}^{l} z_i \in N_G(H) \setminus H.$$

Then $[z_0, B] = E$. Furthermore, for each $\sigma_j \in S$ $(2 \leq j \leq l)$, we have that

$$z_0^{\sigma_j} = z_1^{\sigma_j} \times \prod_{\substack{i=2 \\ i \neq j}}^{l} z_1^{\sigma_i \sigma_j} \times z_1^{\sigma_j \sigma_j} = z_1^{\sigma_j} \times \prod_{\substack{i=2 \\ i \neq j}}^{l} z_1^{\sigma_i} \times z_1 = z_0.$$

This implies that $[z_0, S] = E$ and $z_0 \in ZN_G(H)$ by Proposition 5.11 (1). Thus z_0 is in $O_\pi ZN_G(H) \setminus H$, and $H \notin \mathcal{L}_\pi(G)^0$ as desired. The converse follows from Corollary 5.8. □

Summarizing Propositions 5.9 and 5.11, we have the following.

Theorem 5.12. *Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup, and let*

$$H = (H_1^{(1)} \perp \cdots \perp H_1^{(m_1)}) \perp (H_2^{(1)} \perp \cdots \perp H_2^{(m_2)}) \perp \cdots \perp (H_t^{(1)} \perp \cdots \perp H_t^{(m_t)})$$

be a decomposition of H as in Proposition 5.7 *where each $H_i^{(1)} \perp \cdots \perp H_i^{(m_i)}$ is a component of H. Set $\Gamma_i := \operatorname{supp}(H_i^{(1)})$ for $1 \leq i \leq t$. Suppose that* $\operatorname{supp}(H) = \Omega$. *Then we have that $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$ if and only if $H_i^{(1)} \in \mathcal{L}_\pi(\mathfrak{S}(\Gamma_i))^0$ for all $1 \leq i \leq t$.*

By Theorem 5.12, the determination of $H \in \mathcal{L}_\pi(\mathfrak{S}(\Omega))^0$ can be reduced to the case where H is irreducible.

5.5. On intransitive subgroups. In this section, we show that intransitive subgroups of $\mathfrak{S}(\Omega)$ can be described inductively in terms of smaller irreducible subgroups. This idea will be used in Section 5.6. First we recall pullbacks.

REMARK 5.13. (1) Let G and H be groups, and let $\theta\colon G/N \to H/K$ be a group isomorphism between quotient groups. Then the pullback $G \times^\theta H$ of G and H via θ is a subgroup $\{(g,h) \in G \times H \mid (gN)^\theta = hK\}$ of $G \times H$ (cf. [4, Definition 13.11]]). Note that if θ is trivial, that is, G/N is the trivial group, then $G \times^\theta H = G \times H$.
(2) Let $G = K \times L$ be a direct product. Then any subgroup H of G can be realized as the pullback of certain subgroups in K and L. More precisely, there exist subgroups $K \geq K_1 \trianglerighteq K_2$ and $L \geq L_1 \trianglerighteq L_2$, and also a group isomorphism $\theta\colon K_1/K_2 \to L_1/L_2$ such that $H = K_1 \times^\theta L_1$ (cf. [10, (4.19)]).

Let $H \leq \mathfrak{S}(\Omega)$ be a non-trivial subgroup. Suppose that $\operatorname{supp}(H) = \Omega$, and that H acts intransitively on Ω. Let

$$\Omega = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1} \cup \mathcal{O}_m \quad (m \geq 2)$$

be a decomposition of Ω into H-orbits. Set $\Lambda_1 := \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_{m-1}$ and $\Lambda_2 := \mathcal{O}_m$. Then a subgroup $B := H|_{\Lambda_2} \leq \mathfrak{S}(\Lambda_2)$ is transitive on Λ_2, that is, irreducible. On the other hand, a subgroup $H|_{\Lambda_1} \leq \mathfrak{S}(\Lambda_1)$ is decomposed as $H|_{\Lambda_1} = A_1 \perp \cdots \perp A_l$ into irreducible subgroups A_i $(1 \leq i \leq l)$ by Proposition 5.7. It follows that

$$H \leq H|_{\Lambda_1} \times H|_{\Lambda_2} = (A_1 \perp \cdots \perp A_l) \perp B.$$

Since the supports of A_i and B are strictly contained in $\text{supp}(H) = \Omega$, we may assume that a list of irreducible subgroups A_i and B is already known by induction. Thus H can be concretely described as the pullback $H_1 \times^\theta H_2$ of certain subgroups $H_1 \leq A_1 \perp \cdots \perp A_l$ and $H_2 \leq B$ where θ is a group isomorphism between quotients (see Remark 5.13). Note that, if H is irreducible then θ must not be trivial. In the next, we give a result on irreducible pullbacks under the above situation.

Proposition 5.14. *Let $B \leq \mathfrak{S}(\Omega)$ be an irreducible subgroup, and let $A := A_1 \perp \cdots \perp A_l \leq \mathfrak{S}(\Omega)$ where A_i is irreducible for all $1 \leq i \leq l$. Suppose that* $\text{supp}(A) \cap \text{supp}(B) = \emptyset$ *and* $\text{supp}(A \perp B) = \Omega$. *Suppose further that there exists a group isomorphism $\theta\colon A/N_1 \to B/N_2$ $(\neq \bar{E})$ for some $N_1 \trianglelefteq A$ and $N_2 \trianglelefteq B$ such that $A_i \not\leq N_1$ for all $1 \leq i \leq l$. Then the pullback $P := A \times^\theta B = \{(a, b) \in A \times B \mid (aN_1)^\theta = bN_2\}$ is irreducible.*

Proof. Set $\Gamma_i := \text{supp}(A_i)$ $(1 \leq i \leq l)$ and $\Gamma := \text{supp}(B)$. Suppose that P is reducible. Then there exist non-trivial subgroups $K, L \leq P$ such that $P = K \perp L$. Let $\pi_A\colon P \to A$ and $\pi_B\colon P \to B$ be the projections of P on A and B respectively. Both π_A and π_B are surjective. This implies that $P|_{\Gamma_i} = A_i$ $(1 \leq i \leq l)$ and $P|_\Gamma = B$. Since $B = P|_\Gamma = K|_\Gamma \perp L|_\Gamma$ is irreducible, we may assume that

$$K|_\Gamma = B \quad \text{i.e.} \quad \Gamma = \text{supp}(B) \subseteq \text{supp}(K),$$
$$L|_\Gamma = E \quad \text{i.e.} \quad L \leq A = A_1 \perp \cdots \perp A_l.$$

Suppose that $\Gamma \subset \text{supp}(K) \subseteq \Omega = \Gamma_1 \cup \cdots \cup \Gamma_l \cup \Gamma$. Then we may assume that $\emptyset \neq \text{supp}(K) \cap \Gamma_1 = \text{supp}(K|_{\Gamma_1})$, so that, $K|_{\Gamma_1} \neq E$. Since $A_1 = P|_{\Gamma_1} = K|_{\Gamma_1} \perp L|_{\Gamma_1}$ is irreducible, we have that

$$K|_{\Gamma_1} = A_1 \quad \text{i.e.} \quad \Gamma_1 = \text{supp}(A_1) \subseteq \text{supp}(K) \quad \text{and} \quad \Gamma \cup \Gamma_1 \subseteq \text{supp}(K),$$
$$L|_{\Gamma_1} = E \quad \text{i.e.} \quad L \leq A_2 \perp \cdots \perp A_l.$$

Repeating this process, we may assume that there exists $t < l$ such that

$$(*) \qquad \begin{aligned} &\text{supp}(K) = \Gamma \cup \Gamma_1 \cup \cdots \cup \Gamma_t, \\ &L \leq A_{t+1} \perp \cdots \perp A_l. \end{aligned}$$

Note that if $t = l$ then $L = E$, a contradiction. Now $\pi_A\colon P = K \perp L \to A$ is surjective. Thus for any $a \in A_l$, there exist $(a_K, b_K) \in K \leq A \times B$ and $(a_L, e) \in L \leq A$ such that

$$a = \pi_A((a_K, b_K) \times (a_L, e)) = a_K a_L.$$

But by the above condition $(*)$, $a_K \in A_1 \perp \cdots \perp A_t$ and $a_L \in A_{t+1} \perp \cdots \perp A_l$. Thus $a_K = e$ and $a = a_L \in L \leq P$. This implies $(a, e) \in P$ and $(aN_1)^\theta = eN_2 = N_2$ by the definition of P. Therefore $A_l \leq N_1$ which contradicts our assumption. The proof is complete. $\square$

5.6. A strategy to determine $\mathcal{L}_\pi(\mathfrak{S}_n)^0$. In this section, we provide a method of determining $\mathcal{L}_\pi(\mathfrak{S}_n)^0$ which is focused on irreducible subgroups. So we introduce the notations

$$\mathsf{IRR}(n)^0 := \{E \neq H \leq \mathfrak{S}(\Omega) \mid H \text{ is irreducible such that } \mathrm{fix}(H) = \emptyset\},$$
$$\mathsf{T}(n) := \{E \neq H \leq \mathfrak{S}(\Omega) \mid H \text{ is transitive on } \Omega\} \subseteq \mathsf{IRR}(n)^0.$$

Then, as in the following, we divide our work of determining $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ into two cases where H is irreducible or not.

A: Determine $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ such that H is not irreducible (see Theorem 5.12).
(Step A1) Give a non-trivial partition $n = (n_1 + \cdots + n_1) + \cdots + (n_t + \cdots + n_t)$ of n such that $n_i \geq 2$ and $n_i > n_{i+1}$.
(Step B2) H is $\mathfrak{S}_n$-conjugate to one of subgroups of the form $(H_1 \perp \cdots \perp H_1) \perp \cdots \perp (H_t \perp \cdots \perp H_t)$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_{n_i})^0$ for $1 \leq i \leq t$.

B: Determine $H \in \mathcal{L}_\pi(\mathfrak{S}_n)^0$ such that H is irreducible.
(Step B1) Make a list of $\mathfrak{S}_n$-conjugate classes in $\mathsf{T}(n)$.
(Step B2) Describe subgroups in $\mathsf{IRR}(n)^0 \setminus \mathsf{T}(n)$, namely intransitive irreducible subgroups H having no fixed points (see Section 5.5). Indeed, we first give a non-trivial partition $n = n_1 + \cdots + n_{r-1} + n_r$ of n such that $n_i \geq 2$. Let $A \leq \mathfrak{S}_{n-n_r}$ and $B \in \mathsf{T}(n_r)$ such that A has $r-1$ orbits of lengths n_i for $1 \leq i \leq r-1$. Calculate an irreducible pullback $H = A_1 \times^\theta B_1$ via a group isomorphism $\theta\colon A_1/A_2 \to B_1/B_2$ ($\neq \bar{E}$) where $A \geq A_1 \trianglerighteq A_2$ and $B \geq B_1 \trianglerighteq B_2$.
(Step B3) By the previous two Steps B1–B2, the set $\mathsf{IRR}(n)^0$ is complete. Then, from $\mathsf{IRR}(n)^0$, pick up subgroups belonging to $\mathcal{L}_\pi(\mathfrak{S}_n)$.

5.7. Examples $\mathcal{L}_\pi(\mathfrak{S}_n)^0$ ($n \leq 6$). According to a strategy introduced in Section 5.6, we determine $\mathcal{L}_\pi(\mathfrak{S}_n)$ for $4 \leq n \leq 6$. Let $\mathfrak{A}(\Omega) = \mathfrak{A}_n$ be the alternating group on $\Omega = \{1, \dots, n\}$. For a prime number p and a positive integer m, denote by p^m, C_m, D_{2m} respectively the elementary abelian p-group of order p^m, cyclic group of order m, dihedral group of order $2m$. Set $\pi := \pi(\mathfrak{S}_n)$.

The cases of $\mathfrak{S}_2$ and $\mathfrak{S}_3$ are trivial as follows:

- $\mathsf{IRR}(2)^0 = \mathsf{T}(2) = \mathcal{L}_\pi(\mathfrak{S}_2)^0 = \{\mathfrak{S}_2 \cong C_2\}$,
- $\mathsf{IRR}(3)^0 = \mathsf{T}(3) = \{\mathfrak{S}_3, \mathfrak{A}_3\}$, and $\mathcal{L}_\pi(\mathfrak{S}_3)^0 = \{\mathfrak{A}_3 \cong C_3\}$.

The case of $\mathfrak{S}_4$:

(Steps A1–A2) A non-trivial partition of 4 not containing 1 as summands is only $4 = 2 + 2$. Then any non-irreducible subgroup H in $\mathcal{L}_\pi(\mathfrak{S}_4)^0$ is conjugate to $H_1 \perp H_2$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_2)^0$. Thus $H \sim_{\mathfrak{S}_2} \langle(1, 2)\rangle \perp \langle(3, 4)\rangle$.

(Step B1) It is easy to see that $\mathsf{T}(4)/{\sim_{\mathfrak{S}_4}} = \{\mathfrak{S}_4, \mathfrak{A}_4, \langle(1, 2, 3, 4), (2, 4)\rangle \cong D_8, V, \langle(1, 2, 3, 4)\rangle \cong C_4\}$ where $V := \langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle$ is the four group. In particular, $\mathsf{T}(4)/{\sim_{\mathfrak{S}_4}} \cap \mathcal{L}_\pi(\mathfrak{S}_4) = \{D_8, V, C_4\}$.

(Step B2) A non-trivial partition of 4 not containing 1 as summands is $4 = 2+2$. There is the unique transitive subgroup $B := \langle(3,4)\rangle \in \mathsf{T}(2)$ on $\{3,4\}$. Then we choose a transitive subgroup $A \in \mathsf{T}(2)$ on $\{1, 2\}$ having a quotient A/N of order 2, namely $(A, N) = (\langle(1, 2)\rangle, E)$. Define a group isomorphism $\theta\colon A/N \to B$. The pullback $A \times^{\theta} B = \langle(1, 2)(3, 4)\rangle \cong C_2$ is irreducible.

(Step B3) By Steps B1–B2, we have that

$$\mathsf{IRR}(4)^0/{\sim_{\mathfrak{S}_4}} = \mathsf{T}(4)/{\sim_{\mathfrak{S}_4}} \cup \{\langle(1, 2)(3, 4)\rangle\}.$$

Then $\mathcal{L}_\pi(\mathfrak{S}_4)^0$ consists of 5-classes whose representatives are as follows:

$H \in \mathcal{L}_\pi(\mathfrak{S}_4)^0/{\sim_{\mathfrak{S}_4}}$	$\cong$	
$\langle(1, 2)\rangle \perp \langle(3, 4)\rangle$	2^2	non-irreducible
$\langle(1, 2, 3, 4), (2, 4)\rangle$	D_8	irreducible and transitive
V	2^2	
$\langle(1, 2, 3, 4)\rangle$	C_4	
$\langle(1, 2)(3, 4)\rangle$	2	irreducible and intransitive

The case of $\mathfrak{S}_5$:

(Steps A1–A2) A non-trivial partition of 5 not containing 1 as summands is only $5 = 3 + 2$. Then any non-irreducible subgroup H in $\mathcal{L}_\pi(\mathfrak{S}_5)^0$ is conjugate to $H_1 \perp H_2$ where $H_1 \in \mathcal{L}_\pi(\mathfrak{S}_3)^0$ and $H_2 \in \mathcal{L}_\pi(\mathfrak{S}_2)^0$. Thus $H \sim_{\mathfrak{S}_5} \langle(1, 2, 3)\rangle \perp \langle(4, 5)\rangle$.

(Step B1) Since the order of a transitive group of degree 5 is divisible by 5, it is easy to see that $\mathsf{T}(5)/{\sim_{\mathfrak{S}_5}} = \{\mathfrak{S}_5, \mathfrak{A}_5, C_5 \rtimes C_4, C_5 \rtimes C_2, C_5\}$. In particular, $\mathsf{T}(5)/{\sim_{\mathfrak{S}_5}} \cap \mathcal{L}_\pi(\mathfrak{S}_5) = \{\langle(1, 2, 3, 4, 5)\rangle \cong C_5\}$.

(Step B2) A non-trivial partition of 5 not containing 1 as summands is $5 = 3+2$. There is the unique transitive subgroup $B := \langle(4,5)\rangle \in \mathsf{T}(2)$ on $\{4,5\}$. Then we choose a transitive subgroup $A \in \mathsf{T}(3)$ on $\{1, 2, 3\}$ having a quotient A/N of order 2, namely $(A, N) = (\mathfrak{S}_3, \mathfrak{A}_3)$. Define a group isomorphism $\theta\colon A/N \to B$. The pullback $A \times^{\theta} B = \langle(1, 2, 3), (1, 2)(4, 5)\rangle \cong \mathfrak{S}_3$ is irreducible.

(Step B3) By Steps B1–B2, we have that

$$\mathsf{IRR}(5)^0/{\sim_{\mathfrak{S}_5}} = \mathsf{T}(5)/{\sim_{\mathfrak{S}_5}} \cup \{\langle(1, 2, 3), (1, 2)(4, 5)\rangle\}.$$

Then $\mathcal{L}_\pi(\mathfrak{S}_5)^0$ consists of 2-classes whose representatives are as follows:

$H \in \mathcal{L}_\pi(\mathfrak{S}_5)^0/{\sim_{\mathfrak{S}_5}}$	$\cong$	
$\langle(1, 2, 3)\rangle \perp \langle(4, 5)\rangle$	$C_3 \times C_2$	non-irreducible
$\langle(1, 2, 3, 4, 5)\rangle$	C_5	irreducible and transitive

The case of $\mathfrak{S}_6$:

(Steps A1–A2) Non-irreducible subgroups H in $\mathcal{L}_\pi(\mathfrak{S}_6)^0$ correspond to non-trivial partitions of 6 not containing 1 as summands. Thus those subgroups are determined as follows:

(i) $6 = 4 + 2$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2$ where $H_1 \in \mathcal{L}_\pi(\mathfrak{S}_4)^0$ and $H_2 \in \mathcal{L}_\pi(\mathfrak{S}_2)^0$, and thus

$$H \sim_{\mathfrak{S}_6} D_8 \perp \langle(5, 6)\rangle, \quad V \perp \langle(5, 6)\rangle, \quad C_4 \perp \langle(5, 6)\rangle, \quad \langle(1, 2)(3, 4)\rangle \perp \langle(5, 6)\rangle.$$

(ii) $6 = 3 + 3$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_3)^0$, and thus

$$H \sim_{\mathfrak{S}_6} \langle(1, 2, 3)\rangle \perp \langle(4, 5, 6)\rangle.$$

(iii) $6 = 2 + 2 + 2$: $H \sim_{\mathfrak{S}_6} H_1 \perp H_2 \perp H_3$ where $H_i \in \mathcal{L}_\pi(\mathfrak{S}_2)^0$, and thus

$$H \sim_{\mathfrak{S}_6} \langle(1, 2)\rangle \perp \langle(3, 4)\rangle \perp \langle(5, 6)\rangle.$$

(Step B1) We can find that there are 16-classes of transitive subgroups of $\mathfrak{S}_6$, and representatives are as follows:

$$\begin{aligned}\mathsf{T}(6)/{\sim_{\mathfrak{S}_6}} = \{&\mathfrak{S}_6,\ \mathfrak{A}_6,\ PGL(2, 5) \cong \mathfrak{S}_5, \mathfrak{A}_5,\ \mathfrak{S}_4,\\ &\mathfrak{S}_3 \wr \mathfrak{S}_2 \cong 3^2 \rtimes D_8,\ 3^2 \rtimes C_4, 3^2 \rtimes 2^2,\ 3^2 \rtimes C_2,\ C_3 \times C_2,\ D_{12}, \mathfrak{S}_3,\\ &\mathfrak{S}_2 \wr \mathfrak{S}_3 \cong 2^3 \rtimes S_3, 2^3 \rtimes C_3,\ 2^2 \rtimes C_3,\ \mathfrak{S}_4\}.\end{aligned}$$

In particular, $\mathsf{T}(6)/{\sim_{\mathfrak{S}_6}} \cap \mathcal{L}_\pi(\mathfrak{S}_6) = \{\langle(1, 2, 3, 4, 5, 6)\rangle \cong C_6\}$.

(Step B2) In order to examine intransitive subgroups H in $\mathsf{IRR}(6)^0$, we consider pullbacks associated to non-trivial partitions of 6 not containing 1 as summands as follows:

(i) $6 = 4 + 2$: There is the unique transitive subgroup $B := \langle(5, 6)\rangle \in \mathsf{T}(2)$ on $\{5, 6\}$. Then we choose a transitive subgroup $A \in \mathsf{T}(4)$ on $\{1, 2, 3, 4\}$ having a quotient A/N of order 2, so that, a group isomorphism $\theta\colon A/N \to B$ is defined.

$\theta\colon A/N \to B$	$H = A \times^\theta B$	nilp.	$N_{\mathfrak{S}_6}(H)$
$\mathfrak{S}_4/\mathfrak{A}_4 \to B$	$\langle\mathfrak{A}_4, (1, 2)(5, 6)\rangle \cong \mathfrak{S}_4$	no	
$D_8/C_4 \to B$	$\langle(1, 2, 3, 4), (2, 4)(5, 6)\rangle \cong D_8$	yes	$D^{(1)} \times \langle(5, 6)\rangle$
$D_8/V \to B$	$\langle(1, 2)(3, 4), (1, 3)(2, 4), (2, 4)(5, 6)\rangle$ $= \langle(1, 2, 3, 4)(5, 6), (2, 4)(5, 6)\rangle \cong D_8$	yes	$D^{(1)} \times \langle(5, 6)\rangle$
$D_8/\langle(1, 3), (2, 4)\rangle \to B$	$\langle(1, 3), (2, 4), (1, 2)(3, 4)(5, 6)\rangle$ $= \langle(1, 2, 3, 4)(5, 6), (2, 4)\rangle \cong D_8$	yes	$D^{(1)} \times \langle(5, 6)\rangle$
$V/\langle(1, 2)(3, 4)\rangle \to B$	$\langle(1, 2)(3, 4), (1, 3)(2, 4)(5, 6)\rangle \cong 2^2$	yes	$D^{(2)} \times \langle(5, 6)\rangle$
$C_4/C_2 \to B$	$\langle(1, 3)(2, 4), (1, 2, 3, 4)(5, 6)\rangle \cong C_4$	yes	$D^{(1)} \times \langle(5, 6)\rangle$

where $D^{(1)} := \langle(1, 2, 3, 4), (2, 4)\rangle$ and $D^{(2)} := \langle(1, 3, 2, 4), (1, 2)\rangle$.

(ii) $6 = 3 + 3$: There are three non-trivial quotients A/N of transitive subgroups $A \in \mathsf{T}(3)$, namely $(A, N) = (\mathfrak{S}_3, \mathfrak{A}_3)$, $(\mathfrak{S}_3, E)$, and $(\mathfrak{A}_3, E)$.

$\theta\colon A/N \to A/N$	$H = A \times^{\theta} A$	nilp.	$N_{\mathfrak{S}_6}(H)$
$\mathfrak{S}_3/\mathfrak{A}_3 \to \mathfrak{S}_3/\mathfrak{A}_3$	$\langle(1, 2, 3), (4, 5, 6), (1, 2)(4, 5)\rangle \cong 3^2 \rtimes C_2$	no	
$\mathfrak{S}_3/E \to \mathfrak{S}_3/E$	$\langle(1, 2, 3)(4, 5, 6), (1, 2)(4, 5)\rangle \cong \mathfrak{S}_3$	no	
$\mathfrak{A}_3/E \to \mathfrak{A}_3/E$	$\langle(1, 2, 3)(4, 5, 6)\rangle \cong C_3$	yes	$3^2 \rtimes C_2 \rtimes C_2$

(iii) $6 = (2 + 2) + 2$: There is the unique transitive subgroup $B := \langle(5, 6)\rangle \in \mathsf{T}(2)$ on $\{5, 6\}$. Then we choose an intransitive subgroup $A \leq \mathfrak{S}_4$ on $\{1, 2, 3, 4\}$ which has two orbits of length 2. Namely A is an irreducible subgroup $A_1 = \langle(1, 2)(3, 4)\rangle$ or non-irreducible subgroup $A_2 = \langle(1, 2)\rangle \perp \langle(3, 4)\rangle$. Each A_i has a quotient of order 2.

$\theta\colon A/N \to B$	$H = A \times^{\theta} B$	nilp.	$N_{\mathfrak{S}_6}(H)$
$A_1/E \to B$	$\langle(1, 2)(3, 4)(5, 6)\rangle \cong C_2$	yes	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$A_2/\langle(1, 2)(3, 4)\rangle \to B$	$\langle(1, 2)(3, 4), (1, 2)(5, 6)\rangle \cong 2^2$	yes	$\mathfrak{S}_2 \wr \mathfrak{S}_3$
$A_2/\langle(1, 2)\rangle \to B$	$\langle(1, 2)\rangle \perp \langle(3, 4)(5, 6)\rangle \cong 2^2$	yes	

Note that the last $\langle(1, 2)\rangle \perp \langle(3, 4)(5, 6)\rangle$ is the only non-irreducible subgroup among the above twelve subgroups in Step B2 (compare with Proposition 5.14). Thus there are 11-classes of intransitive subgroups in $\mathsf{IRR}(6)^0$.

(Step B3) By Steps B1–B2, there are $(16 + 11)$-classes of subgroups in $\mathsf{IRR}(6)^0$, and then $\mathcal{L}_\pi(\mathfrak{S}_6)^0$ consists of 9-classes whose representatives are as follows:

$H \in \mathcal{L}_\pi(\mathfrak{S}_6)^0/\sim_{\mathfrak{S}_6}$	$\cong$	
$\langle(1, 2, 3, 4), (2, 4)\rangle \perp \langle(5, 6)\rangle$	$D_8 \times C_2$	non-irreducible
$V \perp \langle(5, 6)\rangle$	2^3	
$\langle(1, 2, 3, 4)\rangle \perp \langle(5, 6)\rangle$	$C_4 \times C_2$	
$\langle(1, 2)(3, 4)\rangle \perp \langle(5, 6)\rangle$	2^2	
$\langle(1, 2, 3)\rangle \perp \langle(4, 5, 6)\rangle$	3^2	
$\langle(1, 2)\rangle \perp \langle(3, 4)\rangle \perp \langle(5, 6)\rangle$	2^3	
$\langle(1, 2, 3, 4, 5, 6)\rangle$	$C_2 \times C_3$	irreducible and transitive
$\langle(1, 2, 3)(4, 5, 6)\rangle$	C_3	irreducible and intransitive
$\langle(1, 2)(3, 4)(5, 6)\rangle$	C_2	

Furthermore, Proposition 5.3 tells us that, since $2 \in \pi$, the whole $\mathcal{L}_\pi(\mathfrak{S}_6)$ is constructed by four parts $\mathcal{L}_\pi(\mathfrak{S}_2)^0$, $\mathcal{L}_\pi(\mathfrak{S}_3)^0$, $\mathcal{L}_\pi(\mathfrak{S}_5)^0$, and $\mathcal{L}_\pi(\mathfrak{S}_6)^0$. Therefore there are $(1 + 1 + 2 + 9)$-classes of subgroups in $\mathcal{L}_\pi(\mathfrak{S}_6)$.

ACKNOWLEDGMENTS. The second author was supported by JSPS KAKENHI Grant Number 25400006.

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