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THE CENTER OF A QUANTIZED ENVELOPING ALGEBRA AT AN EVEN ROOT OF UNITY

To Shin-ichi Kato on his 60th birthday

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Abstract

We will give an explicit description of the center of the De Concini–Kac type specialization of a quantized enveloping algebra at an even root of unity. The case of an odd root of unity was already dealt with by De Concini–Kac–Procesi. Our description in the even case is similar to but a little more complicated than the odd case.

1. Introduction

The representation theory of the De Concini–Kac type specialization of a quantized enveloping algebra at a root of unity was initiated by De Concini–Kac [5]. It is quite different from and much more complicated than the generic parameter case. A special feature at a root of unity is that the center of the quantized enveloping algebra becomes much larger than the generic parameter case. An explicit description of the center of the De Concini–Kac type specialization at a root of unity was given by De Concini–Kac–Procesi [6] when the order of the root of unity is odd. In this paper we give a similar description of the center in the even order case. We point out that there already exists partial results in the even order case in Beck [2].

Let $U_q = U_q(\Delta)$ be the simply-connected quantized enveloping algebra associated to a finite irreducible root system Δ (the Cartan part is isomorphic to the group algebra of the weight lattice). For $z \in \mathbb{C}^\times$ we denote by $U_z = U_z(\Delta)$ the specialization at $q = z$ of the De Concini–Procesi form of U_q. Set $d = 1$ (resp. 2, resp. 3) when Δ is of type A, D, E (resp. B, C, F, resp. G_2). We note that U_z coincides with the specialization of the more standard De Concini–Kac form if $z^{2d} \neq 1$. Let l be a positive integer, and let $\zeta \in \mathbb{C}^\times$ be a primitive l-th root of 1. We assume that the order of ζ^2 is greater than d.

Assume that l is odd. If Δ is of type G_2, we also assume that l is prime to 3. In this case De Concini–Kac–Procesi [6] gave an explicit description of the center $Z(U_\zeta)$

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as explained in the following. Denote by $Z_{\mathrm{Har}}(U_\zeta)$ the subalgebra of $Z(U_\zeta)$ consisting of reductions of central elements of U_q contained in the De Concini–Procesi form. Then we have a Harish-Chandra type isomorphism $Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[2P]^W$, where P is the weight lattice, W is the Weyl group, and the action of W on the group algebra $\mathbb{C}[2P]$ is a twisted one. On the other hand we have a Frobenius homomorphism $F\colon U_1 \to U_\zeta$, which is an injective Hopf algebra homomorphism whose image is contained in $Z(U_\zeta)$. Set $Z_{\mathrm{Fr}}(U_\zeta) = \mathrm{Im}(F)$. Then De Concini–Kac–Procesi proved that the canonical homomorphism

$$Z_{\mathrm{Fr}}(U_\zeta) \otimes_{Z_{\mathrm{Fr}}(U_\zeta)\cap Z_{\mathrm{Har}}(U_\zeta)} Z_{\mathrm{Har}}(U_\zeta) \to Z(U_\zeta)$$

is an isomorphism. They have also given the following geometric description of $Z(U_\zeta)$ (see also De Concini–Procesi [8]). Denote by G the connected simply-connected simple algebraic group over $\mathbb{C}$ with root system Δ. Take Borel subgroups B^+ and B^- of G such that $B^+ \cap B^-$ is a maximal torus of G. We set $H = H(\Delta) = B^+ \cap B^-$. Denote by $N^\pm$ the unipotent radical of $B^\pm$. Define a subgroup $K = K(\Delta)$ of $B^+ \times B^-$ by

$$K = \{(tx, t^{-1}y) \in B^+ \times B^- \mid t \in H,\ x \in N^+,\ y \in N^-\}.$$

Then we have

$$Z_{\mathrm{Fr}}(U_\zeta) \cong U_1 \cong \mathbb{C}[K], \quad Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[H/W],$$
$$Z_{\mathrm{Fr}}(U_\zeta) \cap Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[H/W],$$

and the morphisms $K \to H/W$, $H/W \to H/W$ corresponding to the embeddings $Z_{\mathrm{Fr}}(U_\zeta) \cap Z_{\mathrm{Har}}(U_\zeta) \subset Z_{\mathrm{Fr}}(U_\zeta)$ and $Z_{\mathrm{Fr}}(U_\zeta) \cap Z_{\mathrm{Har}}(U_\zeta) \subset Z_{\mathrm{Har}}(U_\zeta)$ are given by $(g_1, g_2) \mapsto \mathrm{Ad}(G)((g_1 g_2^{-1})_s) \cap H$, and $[t] \mapsto [t^l]$, respectively. Here, g_s for $g \in G$ denotes the semisimple part of g in its Jordan decomposition. In conclusion, we obtain

$$Z(U_\zeta) \cong \mathbb{C}[K \times_{H/W} H/W].$$

Now assume that l is even, or Δ is of type G_2 and l is an odd multiple of 3. We can similarly define $Z_{\mathrm{Har}}(U_\zeta(\Delta))$ as a subalgebra of $Z(U_\zeta(\Delta))$ isomorphic to $\mathbb{C}[2P]^W \cong \mathbb{C}[H(\Delta)/W]$. However, it is a more delicate problem to define $Z_{\mathrm{Fr}}(U_\zeta(\Delta))$. We have an injective Hopf algebra homomorphism $F\colon U_\varepsilon(\Delta') \to U_\zeta(\Delta)$, where $\varepsilon \in \{\pm 1\}$, $\Delta' \in \{\Delta, \Delta^\vee\}$ are determined from Δ and l. Here, $\Delta^\vee$ denotes the set of coroots. This F is a dual version of the Frobenius homomorphism for the Lusztig forms defined in [17]. In the case Δ is of type G_2 and l is an odd multiple of 3 we have $\varepsilon = 1$, $\Delta' = \Delta^\vee$ and $\mathrm{Im}(F) \subset Z(U_\zeta(\Delta))$. In the case l is even and $\varepsilon = 1$, $U_1(\Delta')$ is commutative, but $\mathrm{Im}(F)$ is not a subalgebra of $Z(U_\zeta(\Delta))$. In the case $\varepsilon = -1$ $U_{-1}(\Delta')$ is non-commutative. We define $Z_{\mathrm{Fr}}(U_\zeta(\Delta))$ to be the intersection $\mathrm{Im}(F) \cap Z(U_\zeta(\Delta))$. Then the conclusion is similar to the odd order case. Namely, the canonical homomorphism

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \otimes_{Z_{\mathrm{Fr}}(U_\zeta(\Delta))\cap Z_{\mathrm{Har}}(U_\zeta(\Delta))} Z_{\mathrm{Har}}(U_\zeta(\Delta)) \to Z(U_\zeta(\Delta))$$

turns out to be an isomorphism. Moreover, we have

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cong \mathbb{C}[K(\Delta')/\Gamma], \quad Z_{\mathrm{Har}}(U_\zeta(\Delta)) \cong \mathbb{C}[H(\Delta)/W],$$
$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) \cong \mathbb{C}[H(\Delta')/W],$$

where, Γ is a certain finite group acting on the algebraic variety $K(\Delta')$, and the morphism $K(\Delta')/\Gamma \to H(\Delta')/W$ is induced by $K(\Delta') \to H(\Delta')/W$. The definition of $H(\Delta)/W \to H(\Delta')/W$ is more involved and omitted here. In conclusion, we obtain

$$Z(U_\zeta(\Delta)) \cong \mathbb{C}[(K(\Delta')/\Gamma) \times_{H(\Delta')/W} H(\Delta)/W].$$

The proof is partially similar to that for the odd order case in De Concini–Kac–Procesi [6]. However, some arguments are simplified using certain bilinear forms arising from the Drinfeld pairing. We also note that we have avoided the usage of quantum coadjoint orbits in this paper. We hope to investigate the quantum coadjoint orbits in the even order case in the near future since they should be indispensable in developing the representation theory.

In dealing with the case $\varepsilon = -1$ we use $U_{-1}(\Delta')^\Gamma \cong U_1(\Delta')^\Gamma$. We establish it using a result of [12] relating U_{-q} with U_q. I would like to thank Masaki Kashiwara for explaining it to me.

2. Quantized enveloping algebras

2.1. Let Δ be a (finite) reduced irreducible root system in a vector space $\mathfrak{h}^*_{\mathbb{Q}}$ over $\mathbb{Q}$ (we assume that $\mathfrak{h}^*_{\mathbb{Q}}$ is spanned by the elements of Δ). We denote by W the Weyl group. We fix a W-invariant positive definite symmetric bilinear form

$$(\ ,\ ) \colon \mathfrak{h}^*_{\mathbb{Q}} \times \mathfrak{h}^*_{\mathbb{Q}} \to \mathbb{Q}. \tag{2.1}$$

For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha,\alpha) \in \mathfrak{h}^*_{\mathbb{Q}}$. Then $\Delta^\vee = \{\alpha^\vee \mid \alpha \in \Delta\}$ is also an irreducible root system in a vector space $\mathfrak{h}^*_{\mathbb{Q}}$. Set

$$Q = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha, \quad Q^\vee = \sum_{\alpha\in\Delta} \mathbb{Z}\alpha^\vee,$$
$$P = \{\lambda \in \mathfrak{h}^*_{\mathbb{Q}} \mid (\lambda, \alpha^\vee) \in \mathbb{Z}\ (\alpha \in \Delta)\},$$
$$P^\vee = \{\lambda \in \mathfrak{h}^*_{\mathbb{Q}} \mid (\lambda, \alpha) \in \mathbb{Z}\ (\alpha \in \Delta)\}.$$

Take a set $\Pi = \{\alpha_i\}_{i\in I}$ of simple roots of Δ, and denote by Δ^+ the corresponding set of positive roots of Δ. Then $\Pi^\vee = \{\alpha_i^\vee\}_{i\in I}$ is a set of simple roots of $\Delta^\vee$, and $\Delta^{\vee+} = \{\alpha^\vee \mid \alpha \in \Delta^+\}$ is the corresponding set of positive roots of $\Delta^\vee$. We set

$$Q^+ = \sum_{\alpha\in\Delta^+} \mathbb{Z}_{\geqq 0}\alpha,$$
$$P^+ = \{\lambda \in \mathfrak{h}^*_{\mathbb{Q}} \mid (\lambda, \alpha^\vee) \in \mathbb{Z}_{\geqq 0}\ (\alpha \in \Delta^+)\}.$$

For $i \in I$ let $s_i \in W$ be the corresponding simple reflection. We denote the standard partial order on W by $\geqq$. We denote by Δ_{short} (resp. Δ_{long}) the set of short (resp. long) roots. In our convention we have $\Delta_{\mathrm{short}} = \Delta_{\mathrm{long}} = \Delta$ if Δ is of type A, D, E. We set

$$d = \frac{(\alpha, \alpha)}{(\beta, \beta)} \quad (\alpha \in \Delta_{\mathrm{long}},\ \beta \in \Delta_{\mathrm{short}}),$$
$$d_\alpha = \frac{(\alpha, \alpha)}{(\beta, \beta)} \quad (\alpha \in \Delta,\ \beta \in \Delta_{\mathrm{short}}), \quad d_i = d_{\alpha_i} \quad (i \in I).$$

Define $\rho \in P \cap (Q/2)$ by $(\rho, \alpha_i^\vee) = 1$ $(i \in I)$. Define $\tilde{\rho} \in Q^\vee/2$ by $\tilde{\rho} = (1/2)\sum_{\alpha \in \Delta^+} d_\alpha \alpha^\vee$. We have $\rho = ((\alpha, \alpha)/2)\tilde{\rho}$ for $\alpha \in \Delta_{\mathrm{short}}$.

For $n \in \mathbb{Z}_{\geqq 0}$ we set

$$[n]_t = \frac{t^n - t^{-n}}{t - t^{-1}} \in \mathbb{Z}[t, t^{-1}], \quad [n]_t! = [n]_t[n]_{t-1} \cdots [1]_t \in \mathbb{Z}[t, t^{-1}].$$

2.2. Let $\mathbb{F} = \mathbb{Q}(q)$ be the rational function field in the variable q, and set

$$q_\alpha = q^{d_\alpha} \quad (\alpha \in \Delta), \qquad q_i = q_{\alpha_i} \quad (i \in I).$$

We denote by $U = U(\Delta)$ the corresponding simply-connected quantized enveloping algebra over $\mathbb{F}$, i.e., U is an associative algebra over $\mathbb{F}$ generated by the elements k_λ $(\lambda \in P)$, e_i, f_i $(i \in I)$ satisfying the fundamental relations

$$k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu} \quad (\lambda, \mu \in P),$$
$$k_\lambda e_i k_\lambda^{-1} = q_i^{(\lambda, \alpha_i^\vee)} e_i \quad (\lambda \in P,\ i \in I),$$
$$k_\lambda f_i k_\lambda^{-1} = q_i^{-(\lambda, \alpha_i^\vee)} f_i \quad (\lambda \in P,\ i \in I),$$
$$e_i f_j - f_j e_i = \delta_{ij}(k_i - k_i^{-1})/(q_i - q_i^{-1}) \quad (i, j \in I),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n e_i^{(1-a_{ij}-n)} e_j e_i^{(n)} = 0 \quad (i, j \in I,\ i \neq j),$$
$$\sum_{n=0}^{1-a_{ij}} (-1)^n f_i^{(1-a_{ij}-n)} f_j f_i^{(n)} = 0 \quad (i, j \in I,\ i \neq j),$$

where $k_i = k_{\alpha_i}$ $(i \in I)$, $a_{ij} = (\alpha_i^\vee, \alpha_j)$ $(i, j \in I)$, $e_i^{(n)} = e_i^n/[n]_{q_i}!$, $f_i^{(n)} = f_i^n/[n]_{q_i}!$ $(i \in I,\ n \in \mathbb{Z}_{\geqq 0})$. Note that the above definition of $U(\Delta)$ does not depend on the choice of the symmetric bilinear form $(\ ,\ )$.

We regard U as a Hopf algebra by

$$\begin{aligned}
&\Delta(k_\lambda) = k_\lambda \otimes k_\lambda \quad (\lambda \in P),\\
&\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i \quad (i \in I),\\
&\varepsilon(k_\lambda) = 1 \quad (\lambda \in P),\\
&\varepsilon(e_i) = \varepsilon(f_i) = 0 \quad (i \in I),\\
&S(k_\lambda) = k_\lambda^{-1} \quad (\lambda \in P),\\
&S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i \quad (i \in I).
\end{aligned}$$

Define subalgebras U^0, U^+, U^-, $U^{\geqq 0}$, $U^{\leqq 0}$ of U by

$$\begin{aligned}
&U^0 = \langle k_\lambda \mid \lambda \in P \rangle, \quad U^+ = \langle e_i \mid i \in I \rangle, \quad U^- = \langle f_i \mid i \in I \rangle,\\
&U^{\geqq 0} = \langle k_\lambda,\ e_i \mid \lambda \in P,\ i \in I \rangle, \quad U^{\leqq 0} = \langle k_\lambda,\ f_i \mid \lambda \in P,\ i \in I \rangle.
\end{aligned}$$

We have $U^0 = \bigoplus_{\lambda \in P} \mathbb{F} k_\lambda$, and the multiplication of U induces isomorphisms

$$\begin{aligned}
&U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+ \cong U,\\
&U^+ \otimes U^0 \cong U^0 \otimes U^+ \cong U^{\geqq 0}, \quad U^- \otimes U^0 \cong U^0 \otimes U^- \cong U^{\leqq 0}
\end{aligned}$$

of vector spaces.

We denote by U_{ad} the $\mathbb{F}$-subalgebra of U generated by k_λ $(\lambda \in Q)$, e_i, f_i $(i \in I)$. We also set

$$\begin{aligned}
&U_{\mathrm{ad}}^0 = \langle k_\lambda \mid \lambda \in Q \rangle,\\
&U_{\mathrm{ad}}^{\geqq 0} = \langle k_\lambda,\ e_i \mid \lambda \in Q,\ i \in I \rangle, \quad U_{\mathrm{ad}}^{\leqq 0} = \langle k_\lambda,\ f_i \mid \lambda \in Q,\ i \in I \rangle.
\end{aligned}$$

Then we have

$$\begin{aligned}
&U^+ \otimes U_{\mathrm{ad}}^0 \otimes U^- \cong U^- \otimes U_{\mathrm{ad}}^0 \otimes U^+ \cong U_{\mathrm{ad}},\\
&U^+ \otimes U_{\mathrm{ad}}^0 \cong U_{\mathrm{ad}}^0 \otimes U^+ \cong U_{\mathrm{ad}}^{\geqq 0},\\
&U^- \otimes U_{\mathrm{ad}}^0 \cong U_{\mathrm{ad}}^0 \otimes U^- \cong U_{\mathrm{ad}}^{\leqq 0}.
\end{aligned}$$

We denote by $\mathrm{Mod}(U_{\mathrm{ad}})$ the category of finite-dimensional U_{ad}-modules M with weight space decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$, where

$$M_\lambda = \left\{ m \in M \;\middle|\; k_i m = q_i^{(\lambda, \alpha_i^\vee)} m \ (i \in I) \right\}.$$

2.3. The modified quantized enveloping algebra $\dot{U} = \dot{U}(\Delta)$ is defined as follows (see Lusztig [17]). For $\gamma \in Q$ set $U_{\mathrm{ad},\gamma} = \{u \in U_{\mathrm{ad}} \mid k_i u k_i^{-1} = q_i^{(\gamma, \alpha_i^\vee)} u \ (i \in I)\}$. For

$\lambda, \mu \in P$ we set

$$ {}_\lambda\bar{U}_\mu = U_{\mathrm{ad}} \Big/ \left(\sum_{i \in I} (k_i - q_i^{(\lambda, \alpha_i^\vee)}) U_{\mathrm{ad}} + \sum_{i \in I} U_{\mathrm{ad}} (k_i - q_i^{(\mu, \alpha_i^\vee)}) \right) $$

(note ${}_\lambda\bar{U}_\mu = 0$ unless $\lambda - \mu \in Q$), and let ${}_\lambda p_\mu \colon U_{\mathrm{ad}} \to {}_\lambda\bar{U}_\mu$ be the natural map. For $\lambda \in P$ set $1_\lambda = {}_\lambda p_\lambda(1)$. Set

$$ \dot{U} = \bigoplus_{\lambda, \mu \in P} {}_\lambda\bar{U}_\mu. $$

Then $\dot{U}$ is an associative algebra (without 1) by

$$ {}_\lambda p_\mu(x) {}_{\lambda'} p_{\mu'}(y) = \begin{cases} {}_\lambda p_{\mu'}(xy) & (\mu = \lambda'), \\ 0 & (\mu \neq \lambda') \end{cases} $$

for $x \in U_{\mathrm{ad},\lambda-\mu}$, $y \in U_{\mathrm{ad},\lambda'-\mu'}$. Moreover, $\dot{U}$ is a U_{ad}-bimodule by

$$ u \cdot {}_\lambda p_\mu(x) \cdot u' = {}_{\lambda+\gamma} p_{\mu-\gamma'}(uxu') \quad (x \in U_{\mathrm{ad},\lambda-\mu},\ u \in U_{\mathrm{ad},\gamma},\ u' \in U_{\mathrm{ad},\gamma'}). $$

Then we have an isomorphism

$$ \bigoplus_{\lambda \in P} (U^- \otimes U^+) \cong \dot{U} \quad ((u_\lambda \otimes u'_\lambda)_{\lambda \in P} \leftrightarrow \textstyle\sum_{\lambda \in P} u_\lambda 1_\lambda u'_\lambda). $$

We denote by $\mathrm{Mod}(\dot{U})$ the category of finite-dimensional $\dot{U}$-modules M with weight space decomposition $M = \bigoplus_{\lambda \in P} 1_\lambda M$. Then any $M \in \mathrm{Mod}(U_{\mathrm{ad}})$ is regarded as an object of $\mathrm{Mod}(\dot{U})$ via the action of $\dot{U}$ on M given by

$$ (u 1_\lambda u') m = u p_\lambda^M(u' m) \quad (u \in U^-,\ u' \in U^+), $$

where $p_\lambda^M \colon M \to 1_\lambda M$ is the projection with respect to the weight space decomposition of M. Moreover, this correspondence gives the equivalence of categories $\mathrm{Mod}(U_{\mathrm{ad}}) \cong \mathrm{Mod}(\dot{U})$ (see Lusztig [17]). It follows that for each $\lambda \in P^+$ there exists uniquely (up to isomorphism) a finite-dimensional irreducible $\dot{U}$-module $L(\lambda)$ such that

$$ L(\lambda) = \bigoplus_{\mu \in \lambda - Q^+} 1_\mu L(\lambda), \quad \dim 1_\lambda L(\lambda) = 1, $$

and that any $M \in \mathrm{Mod}(\dot{U})$ is isomorphic to a direct sum of $L(\lambda)$'s for $\lambda \in P^+$.

2.4. We denote by $V = V(\Delta)$ the associative algebra over $\mathbb{F}$ generated by the elements t_λ $(\lambda \in P)$, x_i, y_i $(i \in I)$ satisfying the fundamental relations

$$
\begin{aligned}
&t_0 = 1, \qquad t_\lambda t_\mu = t_{\lambda+\mu} \quad (\lambda, \mu \in P),\\
&t_\lambda x_i t_\lambda^{-1} = q_i^{(\lambda, \alpha_i^\vee)} x_i \quad (\lambda \in P,\ i \in I),\\
&t_\lambda y_i t_\lambda^{-1} = q_i^{(\lambda, \alpha_i^\vee)} y_i \quad (\lambda \in P,\ i \in I),\\
&x_i y_j - y_j x_i = 0 \quad (i, j \in I),\\
&\sum_{n=0}^{1-a_{ij}} (-1)^n x_i^{(1-a_{ij}-n)} x_j x_i^{(n)} = 0 \quad (i, j \in I,\ i \neq j),\\
&\sum_{n=0}^{1-a_{ij}} (-1)^n y_i^{(1-a_{ij}-n)} y_j y_i^{(n)} = 0 \quad (i, j \in I,\ i \neq j),
\end{aligned}
$$

where $x_i^{(n)} = x_i^n/[n]_{q_i}!$, $y_i^{(n)} = y_i^n/[n]_{q_i}!$ $(i \in I,\ n \in \mathbb{Z}_{\geqq 0})$. We set $t_i = t_{\alpha_i}$ for $i \in I$.

Define subalgebras V^0, V^+, V^-, $V^{\geqq 0}$, $V^{\leqq 0}$ of V by

$$
\begin{aligned}
&V^0 = \langle t_\lambda \mid \lambda \in P \rangle, \quad V^+ = \langle x_i \mid i \in I \rangle, \quad V^- = \langle y_i \mid i \in I \rangle,\\
&V^{\geqq 0} = \langle t_\lambda, x_i \mid \lambda \in P,\ i \in I \rangle, \quad V^{\leqq 0} = \langle t_\lambda, y_i \mid \lambda \in P,\ i \in I \rangle.
\end{aligned}
$$

We have $V^0 = \bigoplus_{\lambda \in P} \mathbb{F} t_\lambda$, and the multiplication of V induces isomorphisms

$$
\begin{aligned}
&V^+ \otimes V^0 \otimes V^- \cong V^- \otimes V^0 \otimes V^+ \cong V,\\
&V^+ \otimes V^0 \cong V^0 \otimes V^+ \cong V^{\geqq 0}, \quad V^- \otimes V^0 \cong V^0 \otimes V^- \cong V^{\leqq 0}
\end{aligned}
$$

of vector spaces. Moreover, we have algebra isomorphisms

$$
\begin{aligned}
&{}_J{}^+ \colon V^+ \to U^+ \quad (x_i \mapsto e_i),\\
&{}_J{}^- \colon V^- \to U^- \quad (y_i \mapsto f_i).
\end{aligned}
$$

REMARK 2.1. V is a q-analogue of the enveloping algebra of a certain solvable Lie subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$, where $\mathfrak{g}$ is a simple Lie algebra with root system Δ (see Subsection 2.14 below).

2.5. The modified version $\dot{V} = \dot{V}(\Delta)$ is defined similarly to $\dot{U}$ as follows. Denote by V_{ad} the $\mathbb{F}$-subalgebra of V generated by t_λ $(\lambda \in Q)$, x_i, y_i $(i \in I)$. For $\gamma \in Q^+$ set $V_{\mathrm{ad},\gamma} = \{v \in V_{\mathrm{ad}} \mid t_i v t_i^{-1} = q_i^{(\gamma, \alpha_i^\vee)} v \ (i \in I)\}$. For $\lambda, \mu \in P$ we set

$${}_\lambda \overline{V}_\mu = V_{\mathrm{ad}} \Big/ \left(\sum_{i \in I} (t_i - q_i^{(\lambda, \alpha_i^\vee)}) V_{\mathrm{ad}} + \sum_{i \in I} V_{\mathrm{ad}} (t_i - q_i^{(\mu, \alpha_i^\vee)}) \right)$$

(note ${}_\lambda\overline{V}_\mu = 0$ unless $\lambda - \mu \in Q^+$), and let ${}_\lambda\pi_\mu : V_{\mathrm{ad}} \to {}_\lambda\overline{V}_\mu$ be the natural map. For $\lambda \in P$ set $1_\lambda = {}_\lambda\pi_\lambda(1)$. Set

$$\dot{V} = \bigoplus_{\lambda,\mu\in P} {}_\lambda\overline{V}_\mu.$$

Then $\dot{V}$ is an associative algebra (without 1) by

$${}_\lambda\pi_\mu(x){}_{\lambda'}\pi_{\mu'}(y) = \begin{cases} {}_\lambda\pi_{\mu'}(xy) & (\mu = \lambda'), \\ 0 & (\mu \neq \lambda') \end{cases}$$

for $x \in V_{\mathrm{ad},\lambda-\mu}$, $y \in V_{\mathrm{ad},\lambda'-\mu'}$. Moreover, $\dot{V}$ is a V_{ad}-bimodule by

$$v \cdot {}_\lambda\pi_\mu(x) \cdot v' = {}_{\lambda+\gamma}\pi_{\mu-\gamma'}(vxv') \quad (x \in V_{\mathrm{ad},\lambda-\mu},\ u \in V_{\mathrm{ad},\gamma},\ u' \in V_{\mathrm{ad},\gamma'}).$$

Then we have an isomorphism

$$\bigoplus_{\lambda\in P}(V^- \otimes V^+) \cong \dot{V} \quad ((v_\lambda \otimes v'_\lambda)_{\lambda\in P} \leftrightarrow \textstyle\sum_{\lambda\in P} v_\lambda v'_\lambda 1_\lambda).$$

Denote by $\mathrm{Mod}(\dot{V})$ (resp. $\mathrm{Mod}(V_{\mathrm{ad}})$) the category of finite-dimensional $\dot{V}$-module (resp. V_{ad}-module) with weight space decomposition. Then we have a natural equivalence $\mathrm{Mod}(\dot{V}) \cong \mathrm{Mod}(V_{\mathrm{ad}})$ of categories.

2.6. We denote by

$$\tau : U_{\mathrm{ad}}^{\geqq 0} \times U_{\mathrm{ad}}^{\leqq 0} \to \mathbb{F}$$

the Drinfeld pairing. It is a bilinear form uniquely determined by the properties

$$\tau(1, 1) = 1, \tag{2.2}$$

$$\tau(x, y_1y_2) = (\tau \otimes \tau)(\Delta(x), y_1 \otimes y_2) \quad (x \in U_{\mathrm{ad}}^{\geqq 0},\ y_1, y_2 \in U_{\mathrm{ad}}^{\leqq 0}), \tag{2.3}$$

$$\tau(x_1x_2, y) = (\tau \otimes \tau)(x_2 \otimes x_1, \Delta(y)) \quad (x_1, x_2 \in U_{\mathrm{ad}}^{\geqq 0},\ y \in U_{\mathrm{ad}}^{\leqq 0}), \tag{2.4}$$

$$\tau(k_i, k_j) = q_i^{-(\alpha_i^\vee,\alpha_j)} \quad (i, j \in I), \tag{2.5}$$

$$\tau(k_\lambda, f_i) = \tau(e_i, k_\lambda) = 0 \quad (\lambda \in Q,\ i \in I), \tag{2.6}$$

$$\tau(e_i, f_j) = \delta_{ij}/(q_i^{-1} - q_i) \quad (i, j \in I). \tag{2.7}$$

We define a bilinear form

$$\sigma : U \times \dot{V} \to \mathbb{F}$$

by

$$\sigma(u_+k_\mu(Su_-), v_-v_+1_\lambda) = \tau(u_+, \jmath^-(v_-))\delta_{\lambda,\mu}\tau(\jmath^+(v_+), u_-)$$
$$(u_\pm \in U^\pm,\ v_\pm \in V^\pm,\ \lambda, \mu \in P).$$

The following result is a consequence of Gavarini [9, Theorem 6.2] (see also [20, Proposition 3.4]).

Proposition 2.2. *We have*

$$\sigma(u, vv') = (\sigma \otimes \sigma)(\Delta(u), v \otimes v') \quad (u \in U,\ v, v' \in \dot{V}).$$

2.7. For a Hopf algebra H we define a left action of H on H by

$$\mathrm{ad}(h)(h') = \sum_j h_{0j} h' (S h_{1j}) \quad (h, h' \in H,\ \Delta(h) = \textstyle\sum_j h_{0j} \otimes h_{1j}).$$

We define a right action of U_{ad} on $\dot{U}$ by

$$x \cdot \widetilde{\mathrm{ad}}(u) = \sum_j (S u_{0j}) x u_{1j} \quad (x \in \dot{U},\ u \in U_{\mathrm{ad}},\ \Delta(u) = \textstyle\sum_j u_{0j} \otimes u_{1j}).$$

We set

$${}^eU = \sum_{\lambda \in P} U^+ k_{2\lambda} (SU^-) \subset U.$$

Then ${}^eU$ is a subalgebra of U satisfying $\mathrm{ad}(U)({}^eU) \subset {}^eU$. Define a bilinear form

$$\omega : {}^eU \times \dot{U} \to \mathbb{F}$$

by

$$\omega(u_+ k_{2\mu}(Su_-), w_- 1_\lambda (S w_+)) = \tau(u_+, w_-) \delta_{\lambda,-\mu} \tau(w_+, u_-)$$
$$(u_\pm, w_\pm \in U^\pm,\ \lambda, \mu \in P).$$

The following result is a consequence of [19, Proposition 2.2.1].

Proposition 2.3. *We have*

$$\omega(\mathrm{ad}(u')(u), x) = \omega(u, x \cdot \widetilde{\mathrm{ad}}(u')) \quad (u \in {}^eU,\ u' \in U,\ x \in \dot{U}).$$

Set

$${}^fU = \{u \in U \mid \dim \mathrm{ad}(U)(u) < \infty\}.$$

Then ${}^fU$ is a subalgebra of ${}^eU$ and we have

$${}^fU = \sum_{\lambda \in P^+} \mathrm{ad}(U)(k_{-2\lambda})$$

(see [11]).

2.8. We denote Lusztig's braid group action on U by T_i $(i \in I)$. Namely, $T_i\colon U \to U$ is the algebra automorphism given by

$$T_i(k_\lambda) = k_{s_i(\lambda)} \quad (\lambda \in P),$$

$$T_i(e_j) = \begin{cases} -f_i k_i & (i = j), \\ \sum_{r=0}^{-a_{ij}} (-1)^{-a_{ij}-r} q_i^{-r} e_i^{(-a_{ij}-r)} e_j e_i^{(r)} & (i \neq j), \end{cases}$$

$$T_i(f_j) = \begin{cases} -k_i^{-1} e_i & (i = j), \\ \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-a_{ij}-r} f_i^{(-a_{ij}-r)} f_j f_i^{(r)} & (i \neq j). \end{cases}$$

We denote by w_0 the longest element of W. We fix a reduced expression $w_0 = s_{i_1} \cdots s_{i_N}$ $(i_1, \dots, i_N \in I)$ in the following. For $j = 1, \dots, N$ set $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$, and

$$e_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(e_{i_j}), \quad f_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}),$$
$$e_{\beta_j}^{(n)} = T_{i_1} \cdots T_{i_{j-1}}(e_{i_j}^{(n)}), \quad f_{\beta_j}^{(n)} = T_{i_1} \cdots T_{i_{j-1}}(f_{i_j}^{(n)}) \quad (n \in \mathbb{Z}_{\geqq 0}).$$

Then we have $\Delta^+ = \{\beta_j \mid j = 1, \dots, N\}$, and $e_\beta \in U^+$, $f_\beta \in U^-$ $(\beta \in \Delta^+)$. Moreover, the set $\{e_{\beta_N}^{m_N} \cdots e_{\beta_1}^{m_1} \mid m_j \in \mathbb{Z}_{\geqq 0}\}$ (resp. $\{f_{\beta_N}^{m_N} \cdots f_{\beta_1}^{m_1} \mid m_j \in \mathbb{Z}_{\geqq 0}\}$) is known to be a basis of U^+ (resp. U^-).

2.9. We set $\mathcal{G} = \mathcal{G}(\Delta) = P/P_0$, where

$$P_0 = \{\lambda \in P \mid d_i(\lambda, \alpha_i^\vee) \in 2\mathbb{Z} \ (i \in I)\}.$$

Note that $\mathcal{G}$ is a 2-elementary finite group. For $\lambda \in P$ we denote by $\delta_\lambda \in \mathcal{G}$ the element represented by λ. We define an action of $\mathcal{G}$ on the algebra U by

$$\delta_\lambda(k_\mu) = k_\mu, \quad \delta_\lambda(e_i) = (-1)^{d_i(\lambda, \alpha_i^\vee)} e_i, \quad \delta_\lambda(f_i) = (-1)^{d_i(\lambda, \alpha_i^\vee)} f_i$$

for $\lambda, \mu \in P$, $i \in I$. We define an $\mathbb{F}$-algebra structure of $\tilde{U} = \tilde{U}(\Delta) = U \otimes \mathbb{F}[\mathcal{G}]$ by

$$(u \otimes \delta)(v \otimes \delta') = u\delta(v) \otimes \delta\delta' \quad (u, v \in U, \ \delta, \delta' \in \mathcal{G}).$$

We will identify U and $\mathbb{F}[\mathcal{G}]$ with the subalgebras $U \otimes 1$ and $1 \otimes \mathbb{F}[\mathcal{G}]$ of $\tilde{U}$ respectively. We extend the $\mathcal{G}$-action on U to that on $\tilde{U}$ by $\delta(x) = \delta x \delta^{-1}$ $(\delta \in \mathcal{G}, \ x \in \tilde{U})$. Set

$$U^{\mathcal{G}} = \{u \in U \mid \delta(u) = u \ (\delta \in \mathcal{G})\}, \quad \tilde{U}^{\mathcal{G}} = \{x \in \tilde{U} \mid \delta(x) = x \ (\delta \in \mathcal{G})\}.$$

Then we see easily that $\tilde{U}^{\mathcal{G}} = U^{\mathcal{G}}\mathbb{F}[\mathcal{G}]$.

2.10. Let θ be the automorphism of the field $\mathbb{F}$ sending q to $-q$. For an $\mathbb{F}$-algebra R we denote by ${}^\theta R$ the $\mathbb{F}$-algebra obtained by twisting the $\mathbb{F}$-module structure

of R via θ. Namely, ${}^{\theta}R$ is isomorphic to R as a ring via the correspondence $R \ni x \leftrightarrow {}^{\theta}x \in {}^{\theta}R$, and the $\mathbb{F}$-module structure is given by $c^{\theta}x = {}^{\theta}(\theta(c)x)$ $(c \in \mathbb{F},\ x \in R)$.

Now we are going to define an embedding of ${}^{\theta}U$ into $\tilde{U}$ following [12]. We can take a subset J of I such that for $i, j \in I$ with $i \neq j$ we have

$$d_i(\alpha_i^{\vee}, \alpha_j) \notin 2\mathbb{Z} \Longrightarrow |\{i, j\} \cap J| = 1.$$

For $i \in I$ set

$$\varphi_i = \begin{cases} \delta_{\alpha_i} & (i \in J), \\ 1 & (i \notin J), \end{cases} \quad \psi_i = (-1)^{d_i}\varphi_i\delta_{\alpha_i}.$$

For $\gamma = \sum_{i \in I} m_i\alpha_i \in Q$ we further set

$$\varphi_\gamma = \prod_{i \in I} \varphi_i^{m_i}, \quad \psi_\gamma = \prod_{i \in I} \psi_i^{m_i}.$$

Proposition 2.4 ([12])**.** *An embedding ${}^{\theta}U \to \tilde{U}$ of $\mathbb{F}$-algebras is given by*

$${}^{\theta}k_\lambda \mapsto k_\lambda\delta_\lambda, \quad {}^{\theta}e_i \mapsto e_i\varphi_i, \quad {}^{\theta}f_i \mapsto f_i\psi_i.$$

REMARK 2.5. In [12] Kashiwara–Kang–Oh established using Proposition 2.4 the equivalence $\mathrm{Mod}(U) \cong \mathrm{Mod}({}^{\theta}U)$, where $\mathrm{Mod}(U)$ (resp. $\mathrm{Mod}({}^{\theta}U)$) denotes the category of U-modules (resp. ${}^{\theta}U$-modules) with weight space decompositions (see also Andersen [1]).

We will identify ${}^{\theta}U$ with a subalgebra of $\tilde{U}$. We can easily check the following.

Lemma 2.6. (i) *The multiplication of $\tilde{U}$ gives an isomorphism ${}^{\theta}U \otimes \mathbb{F}[\mathcal{G}] \cong \tilde{U}$ of $\mathbb{F}$-modules.*
(ii) *For any $\delta \in \mathcal{G}$ and ${}^{\theta}u \in {}^{\theta}U$ we have $\delta^{\theta}u\delta^{-1} = {}^{\theta}(\delta(u))$.*

Proposition 2.7. *For any $\beta \in \Delta^+$ we have*

$${}^{\theta}e_\beta = \pm e_\beta\varphi_\beta, \quad {}^{\theta}(Sf_\beta) = \pm(Sf_\beta)\varphi_\beta.$$

Proof. For $i \in I$ define ${}^{\theta}T_i : {}^{\theta}U \to {}^{\theta}U$ by ${}^{\theta}T_i({}^{\theta}u) = {}^{\theta}(T_i(u))$ $(u \in U)$. For $\gamma \in Q^+$ set

$$U^+_\gamma = \{x \in U^+ \mid k_i x k_i^{-1} = q_i^{(\alpha_i^{\vee}, \gamma)}x \ (i \in I)\}.$$

For $i \in I$ we set

$$\varpi_i = \begin{cases} -1 & (i \in J), \\ 1 & (i \notin J). \end{cases}$$

In order to show the statement for e_β, it is sufficient to show that for $\gamma \in Q^+$ and $i \in I$ there exists $c_{i,\gamma} \in \{\pm 1\}$ satisfying

$$ {}^\theta T_i(x\varphi_\gamma) = c_{i,\gamma} T_i(x)\varphi_{s_i\gamma} \quad (x \in U_\gamma^+). \tag{2.8} $$

We first note that for $i, j \in I$ we have

$$ {}^\theta T_i(e_j\varphi_j) = c_{i,j} T_i(e_j)\varphi_{s_i\alpha_j}, \tag{2.9} $$

where

$$ c_{ij} = (-1)^{d_i a_{ij}(a_{ij}+1)/2} \varpi_i^{d_i a_{ij}} \in \{\pm 1\}. $$

The verification of (2.9) in the case $i = j$ is easy. In the case $i \neq j$ one needs some case by case calculation according to the relative position of α_i and α_j. Details are omitted. Now let us show (2.8) using (2.9). For $j_1, \dots, j_r \in I$ with $\sum_p \alpha_{j_p} = \gamma$ we have

$$ (e_{j_1}\varphi_{j_1}) \cdots (e_{j_r}\varphi_{j_r}) = A e_{j_1} \cdots e_{j_r}\varphi_\gamma $$

where $A = \prod_{p=1}^{r-1} \varpi_{j_p}^{d_{j_p}(\alpha_{j_p}^\vee, \alpha_{j_{p+1}}+\cdots+\alpha_{j_r})}$. Hence we have

$$ \begin{aligned} {}^\theta T_i((e_{j_1} \cdots e_{j_r})\varphi_\gamma) &= A\, {}^\theta T_i((e_{j_1}\varphi_{j_1}) \cdots (e_{j_r}\varphi_{j_r})) \\ &= A\left(\prod_p c_{ij_p}\right)((T_i e_{j_1})\varphi_{\alpha_{j_1}}\varphi_{\alpha_i}^{-a_{ij_1}}) \cdots ((T_i e_{j_r})\varphi_{\alpha_{j_r}}\varphi_{\alpha_i}^{-a_{ij_r}}) \\ &= AA'\left(\prod_p c_{ij_p}\right) T_i(e_{j_1} \cdots e_{j_r})\varphi_{s_i\gamma}, \end{aligned} $$

with

$$ \begin{aligned} A' &= \left(\prod_{p=1}^{r-1} \varpi_{j_p}^{d_{j_p}(\alpha_{j_p}^\vee, s_i(\alpha_{j_{p+1}}+\cdots+\alpha_{j_r}))}\right)\left(\prod_{p=1}^{r-1} \varpi_i^{-d_i a_{ij_p}(\alpha_i^\vee, s_i(\alpha_{j_{p+1}}+\cdots+\alpha_{j_r}))}\right) \\ &= A\left(\prod_{p=1}^{r-1} \varpi_{j_p}^{d_{j_p}(\alpha_{j_p}^\vee, \alpha_i)(\alpha_i^\vee, \alpha_{j_{p+1}}+\cdots+\alpha_{j_r})}\right)\left(\prod_{p=1}^{r-1} \varpi_i^{d_{j_p}(\alpha_{j_p}^\vee, \alpha_i)(\alpha_i^\vee, \alpha_{j_{p+1}}+\cdots+\alpha_{j_r})}\right) \\ &= A\prod_{p=1}^{r-1} (\varpi_i\varpi_{j_p})^{d_{j_p}(\alpha_{j_p}^\vee, \alpha_i)(\alpha_i^\vee, \alpha_{j_{p+1}}+\cdots+\alpha_{j_r})} \\ &= A\prod_{p=1}^{r-1} (-1)^{d_i(\alpha_i^\vee, \alpha_{j_p})(\alpha_i^\vee, \alpha_{j_{p+1}}+\cdots+\alpha_{j_r})} \\ &= A\prod_{1 \leqq p < p' \leqq r} (-1)^{d_i(\alpha_i^\vee, \alpha_{j_p})(\alpha_i^\vee, \alpha_{j_{p'}})}. \end{aligned} $$

We thus obtain (2.8), where

$$c_{i,\gamma} = \left(\prod_p c_{ij_p}\right)\left(\prod_{p<p'}(-1)^{d_i(\alpha_i^\vee,\alpha_{j_p})(\alpha_i^\vee,\alpha_{j_{p'}})}\right).$$

The proof of the assertion for f_β is similar. □

2.11. Set $\mathcal{H} = \mathcal{H}(\Delta) = Q^\vee/2Q^\vee$. For $\nu \in Q^\vee$ we denote by γ_ν the element of $\mathcal{H}$ represented by ν. Define an action of $\mathcal{H}$ on the $\mathbb{F}$-algebra $U^0 = \bigoplus_{\lambda\in P}\mathbb{F}k_\lambda \cong \mathbb{F}[P]$ by

$$\gamma_\nu \cdot k_\lambda = (-1)^{(\nu,\lambda)}k_\lambda \quad (\nu \in Q^\vee,\ \lambda \in P).$$

We can extend this $\mathcal{H}$-action on U^0 to that on the algebra $U \cong U^+ \otimes SU^- \otimes U^0$ by

$$\gamma \cdot (ut) = u(\gamma \cdot t) \quad (\gamma \in \mathcal{H},\ u \in U^+(SU^-),\ t \in U^0).$$

Since this action commutes with that of $\mathcal{G}$, we get an action of $\mathcal{G} \times \mathcal{H}$ on U.

2.12. Set $\mathbb{A} = \mathbb{Q}[q^{\pm 1}]$. Following De Concini–Procesi [8] we define $U_\mathbb{A}$ to be the smallest $\mathbb{A}$-subalgebra of U that contains k_λ $(\lambda \in P)$, $(q_i - q_i^{-1})e_i$, $(q_i - q_i^{-1})f_i$ $(i \in I)$ and is stable under the action of T_i $(i \in I)$. It is a Hopf algebra over $\mathbb{A}$. Set

$$U_\mathbb{A}^0 = U_\mathbb{A} \cap U^0, \quad U_\mathbb{A}^\pm = U_\mathbb{A} \cap U^\pm, \quad U_\mathbb{A}^{\geqq 0} = U_\mathbb{A} \cap U^{\geqq 0}, \quad U_\mathbb{A}^{\leqq 0} = U_\mathbb{A} \cap U^{\leqq 0}.$$

Then we have $U_\mathbb{A}^0 = \bigoplus_{\lambda\in P}\mathbb{A}k_\lambda$, and the multiplication of $U_\mathbb{A}$ induces isomorphisms

$$\begin{aligned}
&U_\mathbb{A}^+ \otimes U_\mathbb{A}^0 \otimes U_\mathbb{A}^- \cong U_\mathbb{A}^- \otimes U_\mathbb{A}^0 \otimes U_\mathbb{A}^+ \cong U_\mathbb{A},\\
&U_\mathbb{A}^+ \otimes U_\mathbb{A}^0 \cong U_\mathbb{A}^0 \otimes U_\mathbb{A}^+ \cong U_\mathbb{A}^{\geqq 0},\\
&U_\mathbb{A}^- \otimes U_\mathbb{A}^0 \cong U_\mathbb{A}^0 \otimes U_\mathbb{A}^- \cong U_\mathbb{A}^{\leqq 0}
\end{aligned}$$

of $\mathbb{A}$-modules. For $\beta \in \Delta^+$ we define $a_\beta \in U_\mathbb{A}^+$, $b_\beta \in U_\mathbb{A}^-$ by

$$a_\beta = (q_\beta - q_\beta^{-1})e_\beta, \quad b_\beta = (q_\beta - q_\beta^{-1})f_\beta.$$

Then $\{a_{\beta_N}^{m_N}\cdots a_{\beta_1}^{m_1} \mid m_j \in \mathbb{Z}_{\geqq 0}\}$ (resp. $\{b_{\beta_N}^{m_N}\cdots b_{\beta_1}^{m_1} \mid m_j \in \mathbb{Z}_{\geqq 0}\}$) is a free $\mathbb{A}$-basis of $U_\mathbb{A}^+$ (resp. $U_\mathbb{A}^-$).

Set

$$U_{\mathrm{ad},\mathbb{A}} = U_\mathbb{A} \cap U_{\mathrm{ad}}, \quad U_{\mathrm{ad},\mathbb{A}}^\flat = U_\mathbb{A} \cap U_{\mathrm{ad}}^\flat \quad (\flat = 0,\ \geqq 0,\ \leqq 0).$$

Then we have $U_{\mathrm{ad},\mathbb{A}}^0 = \bigoplus_{\lambda\in Q}\mathbb{A}k_\lambda$, and

$$\begin{aligned}
&U_\mathbb{A}^+ \otimes U_{\mathrm{ad},\mathbb{A}}^0 \otimes U_\mathbb{A}^- \cong U_\mathbb{A}^- \otimes U_{\mathrm{ad},\mathbb{A}}^0 \otimes U_\mathbb{A}^+ \cong U_{\mathrm{ad},\mathbb{A}},\\
&U_\mathbb{A}^+ \otimes U_{\mathrm{ad},\mathbb{A}}^0 \cong U_{\mathrm{ad},\mathbb{A}}^0 \otimes U_\mathbb{A}^+ \cong U_{\mathrm{ad},\mathbb{A}}^{\geqq 0},\\
&U_\mathbb{A}^- \otimes U_{\mathrm{ad},\mathbb{A}}^0 \cong U_{\mathrm{ad},\mathbb{A}}^0 \otimes U_\mathbb{A}^- \cong U_{\mathrm{ad},\mathbb{A}}^{\leqq 0}.
\end{aligned}$$

Denote by $U^L_{\mathrm{ad},\mathbb{A}}$ the $\mathbb{A}$-subalgebra of U generated by the elements $\{e_i^{(n)},\ f_i^{(n)},\ k_\lambda \mid i \in I,\ n \in \mathbb{Z}_{\geqq 0},\ \lambda \in Q\}$, and set

$$U^{L,\flat}_{\mathrm{ad},\mathbb{A}} = U^L_{\mathrm{ad},\mathbb{A}} \cap U^\flat_{\mathrm{ad}} \quad (\flat = 0,\ \geqq 0,\ \leqq 0),$$
$$U^{L,\pm}_{\mathbb{A}} = U^L_{\mathrm{ad},\mathbb{A}} \cap U^\pm.$$

Then we have

$$U^{L,+}_{\mathbb{A}} \otimes U^{L,0}_{\mathrm{ad},\mathbb{A}} \otimes U^{L,-}_{\mathbb{A}} \cong U^{L,-}_{\mathbb{A}} \otimes U^{L,0}_{\mathrm{ad},\mathbb{A}} \otimes U^{L,+}_{\mathbb{A}} \cong U^L_{\mathrm{ad},\mathbb{A}},$$
$$U^{L,+}_{\mathbb{A}} \otimes U^{L,0}_{\mathrm{ad},\mathbb{A}} \cong U^{L,0}_{\mathrm{ad},\mathbb{A}} \otimes U^{L,+}_{\mathbb{A}} \cong U^{L,\geqq 0}_{\mathrm{ad},\mathbb{A}},$$
$$U^{L,-}_{\mathbb{A}} \otimes U^{L,0}_{\mathrm{ad},\mathbb{A}} \cong U^{L,0}_{\mathrm{ad},\mathbb{A}} \otimes U^{L,-}_{\mathbb{A}} \cong U^{L,\leqq 0}_{\mathrm{ad},\mathbb{A}}.$$

Moreover, $U^{L,0}_{\mathrm{ad},\mathbb{A}}$ is generated by the elements of the form k_λ $(\lambda \in Q)$,

$$\left[\begin{array}{c} k_i \\ m \end{array}\right] = \prod_{s=0}^{m-1} \frac{q_i^{-s}k_i - q_i^s k_i^{-1}}{q_i^{s+1} - q_i^{-s-1}} \quad (i \in I,\ m \geqq 0),$$

and $\left\{e_{\beta_N}^{(m_N)} \cdots e_{\beta_1}^{(m_1)} \;\middle|\; m_j \in \mathbb{Z}_{\geqq 0}\right\}$ (resp. $\left\{f_{\beta_N}^{(m_N)} \cdots f_{\beta_1}^{(m_1)} \;\middle|\; m_j \in \mathbb{Z}_{\geqq 0}\right\}$) is a free $\mathbb{A}$-basis of $U^{L,+}_{\mathbb{A}}$ (resp. $U^{L,-}_{\mathbb{A}}$).

We define $\dot{U}_{\mathbb{A}}$ to be the $\mathbb{A}$-subalgebra of $\dot{U}$ consisting of elements of the form

$$\sum_{\lambda \in P} u_\lambda 1_\lambda u'_\lambda \quad (u_\lambda \in U^{L,-}_{\mathbb{A}},\ u'_\lambda \in U^{L,+}_{\mathbb{A}}).$$

For $\lambda \in P^+$ we define an $\mathbb{A}$-form $L_{\mathbb{A}}(\lambda)$ of $L(\lambda)$ by

$$L_{\mathbb{A}}(\lambda) = \dot{U}_{\mathbb{A}} v \quad (1_\lambda L(\lambda) = \mathbb{F}v).$$

We define $\dot{V}_{\mathbb{A}}$ to be the $\mathbb{A}$-subalgebra of $\dot{V}$ consisting of elements of the form

$$\sum_{\lambda \in P} v_\lambda v'_\lambda 1_\lambda \quad (v_\lambda \in (J^-)^{-1}(U^{L,-}_{\mathbb{A}}),\ v'_\lambda \in (J^+)^{-1}(U^{L,+}_{\mathbb{A}})).$$

We set

$${}^eU_{\mathbb{A}} = {}^eU \cap U_{\mathbb{A}}, \quad {}^fU_{\mathbb{A}} = {}^fU \cap U_{\mathbb{A}}.$$

By [13], [14], [15] we have

$$\begin{aligned} \tau(e_{\beta_N}^{(m_N)} \cdots e_{\beta_1}^{(m_1)},\ b_{\beta_N}^{n_N} \cdots b_{\beta_1}^{n_1}) &= \tau(a_{\beta_N}^{m_N} \cdots a_{\beta_1}^{m_1},\ f_{\beta_N}^{(n_N)} \cdots f_{\beta_1}^{(n_1)}) \\ &= \prod_{s=1}^{N} \delta_{m_s,n_s}(-1)^{m_s} q_{\beta_s}^{m_s(m_s-1)/2}, \end{aligned} \tag{2.10}$$

and hence τ induces bilinear forms

$$\tau_{\mathbb{A}}^{\emptyset,L}\colon U_{\mathrm{ad},\mathbb{A}}^{\geqq 0}\times U_{\mathrm{ad},\mathbb{A}}^{L,\leqq 0}\to\mathbb{A},\quad \tau_{\mathbb{A}}^{L,\emptyset}\colon U_{\mathrm{ad},\mathbb{A}}^{L,\geqq 0}\times U_{\mathrm{ad},\mathbb{A}}^{\leqq 0}\to\mathbb{A}.$$

It follows that σ and ω also induce perfect bilinear forms

$$\sigma_{\mathbb{A}}\colon U_{\mathbb{A}}\times\dot{V}_{\mathbb{A}}\to\mathbb{A},\quad \omega_{\mathbb{A}}\colon {}^{e}U_{\mathbb{A}}\times\dot{U}_{\mathbb{A}}\to\mathbb{A}.$$

Set $\tilde{U}_{\mathbb{A}}=U_{\mathbb{A}}\otimes\mathbb{A}[\mathcal{G}]$. It is an $\mathbb{A}$-subalgebra of $\tilde{U}$. We also have an obvious $\mathbb{A}$-form ${}^{\theta}U_{\mathbb{A}}$ of ${}^{\theta}U$. By Proposition 2.7 the embedding ${}^{\theta}U\subset\tilde{U}$ induces ${}^{\theta}U_{\mathbb{A}}\to\tilde{U}_{\mathbb{A}}$.

2.13. Let $z\in\mathbb{C}^{\times}$, and set

$$z_\beta=z^{d_\beta}\quad(\beta\in\Delta),\quad z_i=z_{\alpha_i}\quad(i\in I). \tag{2.11}$$

Set

$$U_z=U_z(\Delta)=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}},$$

where $\mathbb{A}\to\mathbb{C}$ is given by $q\mapsto z$. We also set

$$\begin{aligned}
&U_z^{\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{\flat}\quad(\flat=\emptyset,+,-,0,\geqq 0,\leqq 0),\\
&U_{\mathrm{ad},z}^{\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathrm{ad},\mathbb{A}}^{\flat},\quad U_{\mathrm{ad},z}^{L,\flat}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathrm{ad},\mathbb{A}}^{\flat}\quad(\flat=\emptyset,0,\geqq 0,\leqq 0),\\
&U_z^{L,\pm}=\mathbb{C}\otimes_{\mathbb{A}}U_{\mathbb{A}}^{L,\pm},\\
&\dot{U}_z=\mathbb{C}\otimes_{\mathbb{A}}\dot{U}_{\mathbb{A}},\quad \dot{V}_z=\mathbb{C}\otimes_{\mathbb{A}}\dot{V}_{\mathbb{A}},\\
&{}^{e}U_z=\mathbb{C}\otimes_{\mathbb{A}}{}^{e}U_{\mathbb{A}},\quad {}^{f}U_z=\mathbb{C}\otimes_{\mathbb{A}}{}^{f}U_{\mathbb{A}}.
\end{aligned}$$

Then we have

$$U_z\cong U_z^{-}\otimes U_z^{0}\otimes U_z^{+},\quad \dot{U}_z\cong\bigoplus_{\lambda\in P}U_z^{L,-}1_\lambda U_z^{L,+}.$$

Since $U_{\mathbb{A}}$ is a free $\mathbb{A}$-module, we have ${}^{f}U_z\subset{}^{e}U_z\subset U_z$.

We denote by $\mathrm{Mod}(\dot{U}_z)$ the category of finite-dimensional $\dot{U}_z$-modules M with weight space decomposition $M=\bigoplus_{\lambda\in P}1_\lambda M$. For $\lambda\in P^{+}$ we define $L_z(\lambda)\in\mathrm{Mod}(\dot{U}_z)$ by

$$L_z(\lambda)=\mathbb{C}\otimes_{\mathbb{A}}L_{\mathbb{A}}(\lambda).$$

Note that $\tau_{\mathbb{A}}^{\emptyset,L}$, $\tau_{\mathbb{A}}^{L,\emptyset}$, $\sigma_{\mathbb{A}}$ and $\omega_{\mathbb{A}}$ induce bilinear forms

$$\begin{aligned}
&\tau_z^{\emptyset,L}\colon U_{\mathrm{ad},z}^{\geqq 0}\times U_{\mathrm{ad},z}^{L,\leqq 0}\to\mathbb{C},\quad \tau_z^{L,\emptyset}\colon U_{\mathrm{ad},z}^{L,\geqq 0}\times U_{\mathrm{ad},z}^{\leqq 0}\to\mathbb{C},\\
&\sigma_z\colon U_z\times\dot{V}_z\to\mathbb{C},\quad \omega_z\colon {}^{e}U_z\times\dot{U}_z\to\mathbb{C}.
\end{aligned}$$

By (2.10) $\tau_z^{\emptyset,L}|_{U_z^+ \times U_z^{L,-}}$, $\tau_z^{L,\emptyset}|_{U_z^{L,+} \times U_z^-}$, σ_z, ω_z are perfect.

Set $\tilde{U}_z = \mathbb{C} \otimes_{\mathbb{A}} \tilde{U}_{\mathbb{A}} = U_z \otimes \mathbb{C}[\mathcal{G}]$. Then we have a natural embedding $U_z \subset \tilde{U}_z$, which is compatible with the $\mathcal{G}$-actions. Note that the embedding ${}^{\theta}U_{\mathbb{A}} \to \tilde{U}_{\mathbb{A}}$ also induces an embedding $U_{-z} \subset \tilde{U}_z$, which is compatible with $\mathcal{G}$-actions. Hence setting

$$U_z^{\mathcal{G}} = \{u \in U_z \mid \delta(u) = u \ (\delta \in \mathcal{G})\},$$
$$\tilde{U}_z^{\mathcal{G}} = \{x \in \tilde{U}_z \mid \delta(x) = x \ (\delta \in \mathcal{G})\},$$

we obtain embeddings

$$U_{-z} \subset \tilde{U}_z \supset U_z, \quad U_{-z}^{\mathcal{G}} \subset \tilde{U}_z^{\mathcal{G}} \supset U_z^{\mathcal{G}}.$$

We denote by $\tilde{\Xi}_z \colon U_{-z} \to U_z$ the restriction of the linear map $\tilde{U}_z \to U_z$, which sends $u\delta_\lambda$ for $u \in U_z$, $\lambda \in P$ to u.

Proposition 2.8. *The linear map $\tilde{\Xi}_z$ induces an isomorphism*

$$\Xi_z \colon U_{-z}^{\mathcal{G}} \to U_z^{\mathcal{G}} \tag{2.12}$$

of $\mathbb{C}$-algebras, which is compatible with the $\mathcal{H}$-actions.

Proof. Since $\tilde{\Xi}_z$ is a linear isomorphism compatible with $\mathcal{G}$-actions, it induces a linear isomorphism $\Xi_z \colon U_{-z}^{\mathcal{G}} \to U_z^{\mathcal{G}}$. Note that $U_{-z}^{\mathcal{G}} \subset \tilde{U}_z^{\mathcal{G}} = U_z^{\mathcal{G}}\mathbb{C}[\mathcal{G}]$. For $u, u' \in U_z^{\mathcal{G}}$, $\delta, \delta' \in \mathcal{G}$ we have

$$\tilde{\Xi}_z((u\delta)(u'\delta')) = \tilde{\Xi}_z(uu'\delta\delta') = uu' = \tilde{\Xi}_z(u\delta)\tilde{\Xi}_z(u'\delta').$$

Hence $\tilde{\Xi}_z|U_z^{\mathcal{G}}\mathbb{C}[\mathcal{G}] \colon U_z^{\mathcal{G}}\mathbb{C}[\mathcal{G}] \to U_z^{\mathcal{G}}$ is an algebra homomorphism. It follows that its restriction $\Xi_z \colon U_{-z}^{\mathcal{G}} \to U_z^{\mathcal{G}}$ is also an algebra homomorphism. The remaining statement about the action of $\mathcal{H}$ is obvious. □

2.14. Let $G = G(\Delta)$ be a connected, simply-connected semisimple algebraic group over $\mathbb{C}$ with root system Δ. Take a maximal torus $H = H(\Delta)$ of G and Borel subgroups B^+, B^- of G such that $B^+ \cap B^- = H$. Set $N^{\pm} = [B^{\pm}, B^{\pm}]$, and define a closed subgroup $K = K(\Delta)$ of $B^+ \times B^-$ by

$$K = \{(gh, g'h^{-1}) \mid h \in H, \ g \in N^+, \ g' \in N^-\}.$$

Setting

$$K^0 = \{(h, h^{-1}) \mid h \in H\} \cong H, \tag{2.13}$$

$$K^+ = \{(g, 1) \mid g \in N^+\}, \quad K^- = \{(1, g') \mid g' \in N^-\} \cong N^-, \tag{2.14}$$

we obtain an isomorphism

$$K^+ \times K^0 \times K^- \to K \quad ((a, b, c) \mapsto abc)$$

of algebraic varieties. Denote by $\mathfrak{g}$ the Lie algebra of G. It is generated by the elements $\bar{h}_i$, $\bar{e}_i$, $\bar{f}_i$ $(i \in I)$ satisfying the fundamental relations

$$\begin{aligned}
&[\bar{h}_i, \bar{h}_j] = 0 \quad (i, j \in I),\\
&[\bar{h}_i, \bar{e}_j] = a_{ij}\bar{e}_j, \quad [\bar{h}_i, \bar{f}_j] = -a_{ij}\bar{f}_j \quad (i, j \in I),\\
&[\bar{e}_i, \bar{f}_j] = \delta_{ij}\bar{h}_i \quad (i, j \in I),\\
&\mathrm{ad}(\bar{e}_i)^{1-a_{ij}}(\bar{e}_j) = \mathrm{ad}(\bar{f}_i)^{1-a_{ij}}(\bar{f}_j) = 0 \quad (i, j \in I,\ i \neq j).
\end{aligned}$$

Then the Lie algebra $\mathfrak{k}$ of K is the subalgebra of $\mathfrak{g} \oplus \mathfrak{g}$ generated by elements $\bar{t}_i = (\bar{h}_i, -\bar{h}_i)$, $\bar{x}_i = (\bar{e}_i, 0)$, $\bar{y}_i = (0, \overline{f_i})$ $(i \in I)$. Those generators satisfy the fundamental relations

$$\begin{aligned}
&[\bar{t}_i, \bar{t}_j] = 0 \quad (i, j \in I),\\
&[\bar{t}_i, \bar{x}_j] = a_{ij}\bar{x}_j, \quad [\bar{t}_i, \bar{y}_j] = a_{ij}\bar{y}_j \quad (i, j \in I),\\
&[\bar{x}_i, \bar{y}_j] = 0 \quad (i, j \in I),\\
&\mathrm{ad}(\bar{x}_i)^{1-a_{ij}}(\bar{x}_j) = \mathrm{ad}(\bar{y}_i)^{1-a_{ij}}(\bar{y}_j) = 0 \quad (i, j \in I,\ i \neq j).
\end{aligned}$$

Let $U(\mathfrak{k})$ be the enveloping algebra of $\mathfrak{k}$. We can define the modified version $\dot{U}(\mathfrak{k})$ of $U(\mathfrak{k})$ similarly to $\dot{V}$ as follows. For $\gamma \in Q^+$ set $U(\mathfrak{k})_\gamma = \{u \in U(\mathfrak{k}) \mid [\bar{t}_i, u] = (\gamma, \alpha_i^\vee)u \ (i \in I)\}$. For $\lambda, \mu \in P$ we set

$${}_\lambda\bar{U}(\mathfrak{k})_\mu = U(\mathfrak{k}) \Big/ \left(\sum_{i \in I} (\bar{t}_i - (\lambda, \alpha_i^\vee))U(\mathfrak{k}) + \sum_{i \in I} U(\mathfrak{k})(\bar{t}_i - (\mu, \alpha_i^\vee)) \right),$$

and let ${}_\lambda\pi^1_\mu \colon U(\mathfrak{k}) \to {}_\lambda\bar{U}(\mathfrak{k})_\mu$ be the natural map. Set

$$\dot{U}(\mathfrak{k}) = \bigoplus_{\lambda, \mu \in P} {}_\lambda\bar{U}(\mathfrak{k})_\mu.$$

Then $\dot{U}(\mathfrak{k})$ is an associative algebra (without 1) by

$${}_\lambda\pi^1_\mu(x)\,{}_{\lambda'}\pi^1_{\mu'}(y) = \begin{cases} {}_\lambda\pi^1_{\mu'}(xy) & (\mu = \lambda'),\\ 0 & (\mu \neq \lambda') \end{cases}$$

for $x \in U(\mathfrak{k})_{\lambda - \mu}$, $y \in U(\mathfrak{k})_{\lambda' - \mu'}$. It is easily seen that we have $\dot{V}_1 \cong \dot{U}(\mathfrak{k})$.

Regard $\mathbb{C}[K]$ as a $U(\mathfrak{k})$-module by differentiating the K-action

$$(kf)(k') = f(k'k) \quad (k, k' \in \mathbb{C}[K],\ f \in \mathbb{C}[K])$$

on $\mathbb{C}[K]$. Since $\mathbb{C}[K]$ is a sum of finite dimensional $U(\mathfrak{k})$-submodules with weight space decomposition, we obtain a natural action of $\dot{U}(\mathfrak{k})$ on $\mathbb{C}[K]$. Cosider the bilinear form

$$\bar{\sigma} : \mathbb{C}[K] \times \dot{U}(\mathfrak{k}) \to \mathbb{C} \quad (\bar{\sigma}(f, x) = (xf)(1)),$$

By Proposition 2.2 and $K \cong K^+ \times K^0 \times K^-$, we see easily that an isomorphism

$$\Upsilon : U_1 \to \mathbb{C}[K] \tag{2.15}$$

of coalgebras is given by

$$\bar{\sigma}(\Upsilon(u), x) = \sigma_1(u, x) \quad (u \in U_1,\ x \in \dot{V}_1 = \dot{U}(\mathfrak{k})).$$

Since U_1 and $\mathbb{C}[K]$ are commutative, it is easily seen that (2.15) is an isomorphism of Hopf algebras (see [8], [9], [20]).

3. Harish-Chandra center

3.1. For a ring R we denote its center by $Z(R)$.
Consider the composite of

$$Z(U) \hookrightarrow U \cong U^- \otimes U^0 \otimes U^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U^0 \cong \mathbb{F}[P],$$

where $\mathbb{F}[P] = \bigoplus_{\lambda\in P} \mathbb{F}e(\lambda)$ is the group algebra of P, and the isomorphism $U^0 \cong \mathbb{F}[P]$ is given by $k_\lambda \leftrightarrow e(\lambda)$. By [5], [11], [19] this linear map $Z(U) \to \mathbb{F}[P]$ is an injective algebra homomorphism whose image coincides with

$$\mathbb{F}[2P]^{W\circ} = \{x \in \mathbb{F}[2P] \mid w \circ x = x \ (w \in W)\},$$

where the action of W on $\mathbb{F}[2P]$ is given by

$$w \circ e(2\lambda) = q^{(w\lambda-\lambda,2\tilde{\rho})} e(2w\lambda) \quad (w \in W,\ \lambda \in P).$$

Hence we have an isomorphism

$$\iota : Z(U) \to \mathbb{F}[2P]^{W\circ}. \tag{3.1}$$

We recall here a description of $Z(U)$ in terms of the characters of finite-dimensional U-modules. For $M \in \mathrm{Mod}(\dot{U})$ we define $\tilde{t}_M \in \dot{U}^*$ by

$$\langle \tilde{t}_M, x \rangle = \mathrm{Tr}(xk_{2\rho}, M) \quad (x \in \dot{U}).$$

Then there exists uniquely an element $t_M \in {}^eU$ satisfying

$$\omega(t_M, x) = \langle \tilde{t}_M, x \rangle \quad (x \in \dot{U}).$$

More explicitly, we have

$$t_M = \sum_{\lambda\in 2P,\ \sum_{j=1}^N (m_j-m_j')\beta_j=0} c_{\lambda,\{m_j\}_{j=1}^N,\{m_j'\}_{j=1}^N} a_{\beta_1}^{m_1}\cdots a_{\beta_N}^{m_N} k_\lambda S(b_{\beta_N}^{m_N'}\cdots b_{\beta_1}^{m_1'}),$$

where

$$\begin{aligned} &c_{\lambda,\{m_j\}_{j=1}^N,\{m_j'\}_{j=1}^N} \\ &= \prod_{j=1}^N (-1)^{m_j+m_j'} q_{\beta_j}^{-m_j(m_j-1)/2-m_j'(m_j'-1)/2} \\ &\qquad \times \operatorname{Tr}\bigl(\pi\bigl\{f_{\beta_1}^{(m_1)}\cdots f_{\beta_N}^{(m_N)} 1_{-\lambda/2} S(e_{\beta_N}^{(m_N')}\cdots e_{\beta_1}^{(m_1')}) k_{2\rho}\bigr\},\ 1_{-\lambda/2-\sum_j m_j'\beta_j} M\bigr). \end{aligned}$$

We can show $t_M \in Z(U)$ using $k_{2\rho}^{-1} u k_{2\rho} = S^2 u$ $(u \in U)$, $Z(U) = \{v \in U \mid \operatorname{ad}(u)(v) = \varepsilon(u)v\ (u \in U)\}$, and Proposition 2.3 (see [19]). We have

$$\iota(t_M) = \sum_{\lambda\in P} (\dim 1_\lambda M) q^{(\lambda,2\tilde{\rho})} e(-2\lambda).$$

Proposition 3.1. (i) $Z(U) \subset U^{\mathcal{G}}$.

(ii) *We have*

$$Z({}^\theta U) = Z(\tilde{U}) = Z(U)$$

as subalgebras of $\tilde{U}$. *Moreover, the composite of*

$$\mathbb{F}[2P]^{W\circ} \cong Z(U) = Z({}^\theta U) \cong {}^\theta Z(U) \cong {}^\theta(\mathbb{F}[2P]^{W\circ})$$

is induced by the $\mathbb{F}$*-linear isomorphism*

$$\mathbb{F}[2P] \ni e(2\lambda) \mapsto {}^\theta e(2\lambda) \in {}^\theta\mathbb{F}[2P].$$

Proof. (i) Let $\delta \in \mathcal{G}$. Since δ acts on U as an algebra automorphism, we have $\delta(Z(U)) = Z(U)$. It is easily seen from the definition of δ that $\iota(\delta(z)) = \iota(z)$ for any $z \in Z(U)$. Hence δ acts as identity on $Z(U)$.

(ii) By (i) we have $Z(U) \subset Z(\tilde{U})$. Let us show $Z(U) \supset Z(\tilde{U})$. Let $z = \sum_{\delta\in\mathcal{G}} u_\delta \delta \in Z(\tilde{U})$, where $u_\delta \in U$. By $uz = zu$ for $u \in U$ we have $uu_\delta = u_\delta \delta(u)$. By considering the corresponding identity in the associated graded algebra $\operatorname{Gr} U$ introduced in [7] we see easily that $u_\delta = 0$ for $\delta \neq 1$. Hence $z \in Z(U)$. The proof of $Z({}^\theta U) = Z(\tilde{U})$ is similar. The remaining statement is a consequence of ${}^\theta k_{2\lambda} = k_{2\lambda}$ for $\lambda \in P$. □

3.2. By $Z(U_{\mathbb{A}}) = U_{\mathbb{A}} \cap Z(U)$, ι induces an injective algebra homomorphism

$$\iota_{\mathbb{A}} : Z(U_{\mathbb{A}}) \to \mathbb{A}[2P]^{W\circ}.$$

Proposition 3.2. *$\iota_{\mathbb{A}}$ is an isomorphism of $\mathbb{A}$-algebras.*

Proof. For $\lambda \in P^+$ we have $t_{L(\lambda)} \in U_{\mathbb{A}}$, and $\mathbb{A}[2P]^{W\circ}$ is spanned over $\mathbb{A}$ by $\iota(t_{L(\lambda)})$ for $\lambda \in P^+$. □

3.3. Let $z \in \mathbb{C}^\times$. We denote by $Z_{\mathrm{Har}}(U_z)$ the image of $Z(U_{\mathbb{A}}) \to Z(U_z)$, and call it the Harish-Chandra center of U_z. We can similarly consider the composite of

$$Z_{\mathrm{Har}}(U_z) \hookrightarrow U_z \cong U_z^- \otimes U_z^0 \otimes U_z^+ \xrightarrow{\varepsilon\otimes 1\otimes\varepsilon} U_z^0 \cong \mathbb{C}[P].$$

We define an action $\circ_z$ of W on $\mathbb{C}[2P]$ by

$$w \circ_z e(2\lambda) = z^{(w\lambda-\lambda,2\tilde{\rho})} e(2w\lambda) \quad (w \in W,\ \lambda \in P).$$

Proposition 3.3. *The above linear map $Z_{\mathrm{Har}}(U_z) \to \mathbb{C}[P]$ induces an isomorphism*

$$\iota_z \colon Z_{\mathrm{Har}}(U_z) \to \mathbb{C}[2P]^{W\circ_z}$$

of $\mathbb{C}$-algebras.

Proof. By $Z(U_{\mathbb{A}}) = U_{\mathbb{A}} \cap Z(U)$ the canonical map $\mathbb{C} \otimes_{\mathbb{A}} Z(U_{\mathbb{A}}) \to U_z$ is injective. Hence $Z_{\mathrm{Har}}(U_z) \cong \mathbb{C} \otimes_{\mathbb{A}} Z(U_{\mathbb{A}}) \cong \mathbb{C}[2P]^{W\circ_z}$. □

For $M \in \mathrm{Mod}(\dot{U}_z)$ we can similarly define $t_M \in {}^eU_z$ by

$$\omega_z(t_M, x) = \mathrm{Tr}(xk_{2\rho}, M) \quad (x \in \dot{U}_z).$$

By our construction $\{t_{L_z(\lambda)} \mid \lambda \in P^+\}$ is a basis of $Z_{\mathrm{Har}}(U_z)$. Indeed for $M \in \mathrm{Mod}(\dot{U}_z)$ we can write

$$[M] = \sum_{\lambda\in P^+} m_\lambda [L_z(\lambda)] \quad (m_\lambda \in \mathbb{Z})$$

in an appropriate Grothendieck group, and in this case we have

$$t_M = \sum_{\lambda\in P^+} m_\lambda t_{L_z(\lambda)} \in Z_{\mathrm{Har}}(U_z).$$

Note that for $z \in \mathbb{C}^\times$ the two actions $\circ_z$ and $\circ_{-z}$ of W on $\mathbb{C}[2P]$ are the same. By Proposition 3.1 we have the following.

Proposition 3.4. *For $z \in \mathbb{C}^\times$ we have $U_z^{\mathcal{G}} \supset Z_{\mathrm{Har}}(U_z)$, and the isomorphism $\Xi_z \colon U_{-z}^{\mathcal{G}} \to U_z^{\mathcal{G}}$ induces the isomorphism $Z_{\mathrm{Har}}(U_{-z}) \cong Z_{\mathrm{Har}}(U_z)$ given by*

$$Z_{\mathrm{Har}}(U_{-z}) \xrightarrow{\iota_{-z}} \mathbb{C}[2P]^{W\circ_{-z}} = \mathbb{C}[2P]^{W\circ_z} \xleftarrow{\iota_z} Z_{\mathrm{Har}}(U_z).$$

3.4. We consider the case where $z = 1$. Since the action $\circ_1$ of W on $\mathbb{C}[2P]$ is nothing but the ordinary one, we have

$$Z_{\mathrm{Har}}(U_1) \cong \mathbb{C}[2P]^W \cong \mathbb{C}[P]^W \cong \mathbb{C}[H]^W \cong \mathbb{C}[H/W].$$

Here the second isomorphism is induced by $\mathbb{C}[2P] \ni e(2\lambda) \leftrightarrow e(\lambda) \in \mathbb{C}[P]$. Recall also that we have an isomorphism

$$U_1 \cong \mathbb{C}[K].$$

Hence the inclusion $Z_{\mathrm{Har}}(U_1) \to U_1$ induces a morphism $f\colon K \to H/W$ of algebraic varieties. Let us give an explicit description of this morphism. Define a morphism $\kappa\colon K \to G$ of algebraic varieties by $\kappa((g_1, g_2) = g_1 g_2^{-1}$. We also define $\upsilon\colon G \to H/W$ as follows. Let $g \in G$. Let g_s be the semisimple part of g with respect to the Jordan decomposition. Then $\mathrm{Ad}(G)(g_s) \cap H$ consists of a single W-orbit. We define $\upsilon(g)$ to be this W-orbit.

Proposition 3.5 ([8]). *The morphism $f\colon K \to H/W$ is the composite of $\kappa\colon K \to G$ and $\upsilon\colon G \to H/W$.*

Proof. For the convenience of the readers we give a sketch of the proof using the bilinear forms ω_1 ant θ_1. First note that

$$Z_{\mathrm{Har}}(U_1) \subset {}^f U_1 \subset {}^e U_1 \subset U_1.$$

Via $\omega_1\colon {}^e U_1 \times \dot{U}_1 \to \mathbb{C}$ we obtain embeddings ${}^f U_1 \subset {}^e U_1 \subset (\dot{U}_1)^*$. Identifying $\dot{U}_1$ with the modified enveloping algebra of Lie(G) we have ${}^f U_1 \cong \mathbb{C}[G]$ (see [3]). On the other hand we see from $\dot{U}_1 \cong \bigoplus_{\lambda \in P} U_1^{L,-} 1_\lambda U_1^{L,+}$ that ${}^e U_1$ is identified with $\mathbb{C}[N^- \times H \times N^+]$. Consequently we obtain a sequence

$$\mathbb{C}[H/W] \to \mathbb{C}[G] \to \mathbb{C}[N^- \times H \times N^+] \to \mathbb{C}[K]$$

of algebra embeddings. We can easily check that the corresponding morphisms of algebraic varieties are given by

$$K \ni (g_+ g_0, g_- g_0^{-1}) \mapsto (g_-, g_0^{-2}, g_+^{-1}) \in N^- \times H \times N^+ \quad (g_\pm \in N^\pm,\ g_0 \in H),$$
$$N^- \times H \times N^+ \ni (x_-, x_0, x_+) \mapsto x_- x_0 x_+ \in G,$$
$$G \ni g \mapsto \upsilon(g)^{-1} \in H/W. \qquad \square$$

4. Frobenius center

4.1. Fix a positive integer l. If l is odd (resp. even), then we set $r = l$ (resp. $r = l/2$). Note that r is the order of ζ^2. We assume

$$r > d \tag{4.1}$$

in the following. We take $\zeta \in \mathbb{C}$ to be a primitive l-th root of 1. Define ζ_β ($\beta \in \Delta$), ζ_i ($i \in I$) as in (2.11) for $z = \zeta$. For $\beta \in \Delta$ we denote the orders of ζ_β, ζ_β^2 by l_β, r_β respectively. For $i \in I$ we set $l_i = l_{\alpha_i}$, $r_i = r_{\alpha_i}$.

4.2. For $\alpha \in \Delta$ set $\alpha' = r_\alpha \alpha \in \mathfrak{h}_{\mathbb{Q}}^*$. Then $\Delta' = \{r_\alpha \alpha \mid \alpha \in \Delta\}$ is a root system with $\{\alpha_i' \mid i \in I\}$ a set of simple roots. Note that as an abstract root system (disregarding the inner product) we have $\Delta' \cong \Delta$ or $\Delta' \cong \Delta^\vee$. Set

$$P' = \{\mu \in \mathfrak{h}_{\mathbb{Q}}^* \mid (\mu, \alpha^\vee) \in r_\alpha \mathbb{Z} \quad (\forall \alpha \in \Delta)\}.$$

Then P' is the weight lattice for Δ', and we have $P' \subset P$.

Set

$$\varepsilon = \zeta_\alpha^{r_\alpha^2} \quad (\alpha \in \Delta,\ \alpha' \in (\Delta')_{\mathrm{short}}). \tag{4.2}$$

Then we have $\varepsilon = \pm 1$. Furthermore, $\varepsilon = -1$ if and only if we have either
(a) r is odd and $l = 2r$,
or
(b) $d = 2$, r is even with $r/2$ odd.
Set

$$\varepsilon_{\alpha'} = \varepsilon^{(\alpha',\alpha')/(\beta',\beta')} \quad (\alpha' \in \Delta',\ \beta' \in (\Delta')_{\mathrm{short}}).$$

Then we have

$$\varepsilon_{\alpha'} = \zeta_\alpha^{r_\alpha^2} \quad (\alpha \in \Delta). \tag{4.3}$$

An explicit description of (Δ', ε) in each case is given in Table 1.

4.3. Similarly to the Frobenius homomorphism

$$\mathrm{Fr}\colon \dot{U}_\zeta(\Delta) \to \dot{U}_\varepsilon(\Delta') \tag{4.4}$$

given in [17, Theorem 35.1.9] we can define an algebra homomorphism

$$\xi\colon \dot{V}_\zeta(\Delta) \to \dot{V}_\varepsilon(\Delta') \tag{4.5}$$

such that

- for $\lambda \notin P'$ we have $\xi(x_i^{(n)} 1_\lambda) = \xi(y_i^{(n)} 1_\lambda) = 0$ ($i \in I$, $n \in \mathbb{Z}_{\geqq 0}$),
- for $\lambda \in P'$ we have

$$\xi(x_i^{(n)} 1_\lambda) = \begin{cases} x_i^{(n/r_i)} 1_\lambda & (r_i \mid n), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\xi(y_i^{(n)} 1_\lambda) = \begin{cases} y_i^{(n/r_i)} 1_\lambda & (r_i \mid n), \\ 0 & (\text{otherwise}). \end{cases}$$

Table 1.

type of Δ	l	r	Δ'	ε
A_n, D_n, E_6, E_7, E_8	$l \in 2\mathbb{Z}+1$	l	$r\Delta$	1
	$l \in 4\mathbb{Z}$	$l/2$	$r\Delta$	1
	$l \in 4\mathbb{Z}+2$	$l/2$	$r\Delta$	-1
B_n, C_n, F_4	$l \in 2\mathbb{Z}+1$	l	$r\Delta$	1
	$l \in 4\mathbb{Z}+2$	$l/2$	$r\Delta$	-1
	$l \in 8\mathbb{Z}$	$l/2$	$\frac{r}{2}(2\Delta_{\text{short}} \sqcup \Delta_{\text{long}})$	1
	$l \in 8\mathbb{Z}+4$	$l/2$	$\frac{r}{2}(2\Delta_{\text{short}} \sqcup \Delta_{\text{long}})$	-1
G_2	$l \in 6\mathbb{Z} \pm 1$	l	$r\Delta$	1
	$l \in 6\mathbb{Z}+3$	l	$\frac{r}{3}(3\Delta_{\text{short}} \sqcup \Delta_{\text{long}})$	1
	$l \in 12\mathbb{Z}$	$l/2$	$\frac{r}{3}(3\Delta_{\text{short}} \sqcup \Delta_{\text{long}})$	1
	$l \in 12\mathbb{Z} \pm 4$	$l/2$	$r\Delta$	1
	$l \in 12\mathbb{Z}+6$	$l/2$	$\frac{r}{3}(3\Delta_{\text{short}} \sqcup \Delta_{\text{long}})$	-1
	$l \in 12\mathbb{Z} \pm 2$	$l/2$	$r\Delta$	-1

The fact that ξ is well-defined follows easily from the corresponding fact for Fr. Moreover, for $\lambda \in P'$ and $\beta \in \Delta^+$

$$\xi(x_\beta^{(n)}1_\lambda) = \begin{cases} x_{\beta'}^{(n/r_\beta)}1_\lambda & (r_\beta \mid n), \\ 0 & (\text{otherwise}), \end{cases}$$

$$\xi(y_\beta^{(n)}1_\lambda) = \begin{cases} y_{\beta'}^{(n/r_\beta)}1_\lambda & (r_\beta \mid n), \\ 0 & (\text{otherwise}) \end{cases}$$

by [17, 41.1.9].

Proposition 4.1. *There exists uniquely an injective homomorphism*

$$^t\xi : U_\varepsilon(\Delta') \to U_\zeta(\Delta)$$

of coalgebras satisfying

$$\sigma_\zeta(^t\xi(u), v) = \sigma_\varepsilon(u, \xi(v)) \quad (u \in U_\varepsilon(\Delta'),\ v \in \dot{V}_\zeta(\Delta)). \tag{4.6}$$

Moreover, we have

$$
\begin{aligned}
&{}^t\xi(a_{\beta'_N}^{n_N}\cdots a_{\beta'_1}^{n_1}k_\mu S(b_{\beta'_N}^{n'_N}\cdots b_{\beta'_1}^{n'_1}))\\
&=c_{\beta_1}^{n_1+n'_1}\cdots c_{\beta_N}^{n_N+n'_N}a_{\beta_N}^{r_{\beta_N}n_N}\cdots a_{\beta_1}^{r_{\beta_1}n_1}k_\mu S(b_{\beta_N}^{r_{\beta_N}n'_N}\cdots b_{\beta_1}^{r_{\beta_1}n'_1})\\
&(\mu\in P',\ n_1,\dots,n_N,n'_1,\dots,n'_N\in\mathbb{Z}_{\geqq 0}),
\end{aligned}
\tag{4.7}
$$

where

$$c_\beta=(-1)^{r_\beta+1}\zeta_\beta^{-r_\beta(r_\beta-1)/2}\quad(\beta\in\Delta^+).$$

Proof. It is easily seen from (2.10) that there exists uniquely a linear map ${}^t\xi: U_\varepsilon(\Delta')\to U_\zeta(\Delta)$ satisfying (4.6), and it is given by (4.7). Then we conclude from Proposition 2.2 that ${}^t\xi$ is a homomorphism of coalgebras. □

Similarly we have the following.

Proposition 4.2. *We have* ${}^t\xi({}^eU_\varepsilon(\Delta'))\subset{}^eU_\zeta(\Delta)$, *and*

$$\omega_\zeta({}^t\xi(u),x)=\omega_\varepsilon(u,\mathrm{Fr}(x))\quad(u\in{}^eU_\varepsilon(\Delta'),\ x\in\dot{U}_\zeta(\Delta)).$$

4.4. For $\beta\in\Delta$ we set $\eta_\beta=\zeta_\beta^{r_\beta}$. We have $\eta_\beta=\pm1$, and $\eta_\beta=-1$ if and only l_β is even.

Proposition 4.3 (De Concini–Kac [5])**.** *For* $\alpha,\beta\in\Delta^+$, $\lambda\in P$, $\mu\in P'$ *we have*

$$
\begin{aligned}
&a_\alpha^{r_\alpha}a_\beta=\eta_\alpha^{(\alpha^\vee,\beta)}a_\beta a_\alpha^{r_\alpha},\quad (Sb_\alpha^{r_\alpha})(Sb_\beta)=\eta_\alpha^{(\alpha^\vee,\beta)}(Sb_\beta)(Sb_\alpha^{r_\alpha}),\\
&a_\alpha^{r_\alpha}(Sb_\beta)=\eta_\alpha^{(\alpha^\vee,\beta)}(Sb_\beta)a_\alpha^{r_\alpha},\quad (Sb_\alpha^{r_\alpha})a_\beta=\eta_\alpha^{(\alpha^\vee,\beta)}a_\beta(Sb_\alpha^{r_\alpha}),\\
&k_\lambda a_\alpha^{r_\alpha}=\eta_\alpha^{(\lambda,\alpha^\vee)}a_\alpha^{r_\alpha}k_\lambda,\quad k_\lambda(Sb_\alpha^{r_\alpha})=\eta_\alpha^{(\lambda,\alpha^\vee)}(Sb_\alpha^{r_\alpha})k_\lambda,\\
&k_\mu a_\alpha=\eta_\alpha^{(\mu,\alpha^\vee)/r_\alpha}a_\alpha k_\mu,\quad k_\mu(Sb_\alpha)=\eta_\alpha^{(\mu,\alpha^\vee)/r_\alpha}(Sb_\alpha)k_\mu
\end{aligned}
$$

in $U_\zeta(\Delta)$.

Proposition 4.4. *For* $\alpha',\beta'\in(\Delta')^+$, $\mu\in P'$ *we have*

$$
\begin{aligned}
&a_{\alpha'}a_{\beta'}=\varepsilon_{\alpha'}^{((\alpha')^\vee,\beta')}a_{\beta'}a_{\alpha'},\quad (Sb_{\alpha'})(Sb_{\beta'})=\varepsilon_{\alpha'}^{((\alpha')^\vee,\beta')}(Sb_{\beta'})(Sb_{\alpha'}),\\
&a_{\alpha'}(Sb_{\beta'})=\varepsilon_{\alpha'}^{((\alpha')^\vee,\beta')}(Sb_{\beta'})a_{\alpha'},\quad (Sb_{\alpha'})a_{\beta'}=\varepsilon_{\alpha'}^{((\alpha')^\vee,\beta')}a_{\beta'}(Sb_{\alpha'}),\\
&k_\mu a_{\alpha'}=\varepsilon_{\alpha'}^{(\mu,(\alpha')^\vee)}a_{\alpha'}k_\mu,\quad k_\mu(Sb_{\alpha'})=\varepsilon_{\alpha'}^{(\mu,(\alpha')^\vee)}(Sb_{\alpha'})k_\mu
\end{aligned}
$$

in $U_\varepsilon(\Delta')$.

Proof. Let

$$(\ |\ )' : \mathfrak{h}_{\mathbb{Q}} \times \mathfrak{h}_{\mathbb{Q}} \to \mathbb{Q}$$

be the W-invariant non-degenerate symmetric bilinear form such that $(\alpha' \mid \alpha')' = 2$ for $\alpha' \in (\Delta')_{\text{short}}$. Then we have $\varepsilon_{\alpha'}^{((\alpha')^\vee, \beta')} = \varepsilon^{(\alpha'|\beta')'}$ for $\alpha', \beta' \in (\Delta')^+$.

In order to show the first formula $a_{\alpha'} a_{\beta'} = \varepsilon^{(\alpha'|\beta')'} a_{\beta'} a_{\alpha'}$ for $\alpha', \beta' \in (\Delta')^+$, it is sufficient to show

$$\tau_\epsilon^{\emptyset, L}(a_{\alpha'} a_{\beta'}, y) = \varepsilon^{(\alpha'|\beta')'} \tau_\epsilon^{\emptyset, L}(a_{\beta'} a_{\alpha'}, y)$$

for any $y \in U_\varepsilon^{L,-} = U_\varepsilon^{L,-}(\Delta')$, where $\tau_\epsilon^{\emptyset, L}$ is defined for Δ'. Write

$$\Delta(y) = \sum_{\gamma, \delta \in (Q')^+} u_{\gamma,\delta}^y (k_\delta^{-1} \otimes 1) \quad (u_{\gamma,\delta}^y \in U_{\varepsilon,-\gamma}^{L,-} \otimes U_{\varepsilon,-\delta}^{L,-}),$$

where for $\gamma = \sum_{i \in I} m_i \alpha_i' \in (Q')^+$ we set

$$U_{\varepsilon,-\gamma}^{L,-} = \sum_{\sum_{k_j = i} n_j = m_i} \mathbb{C} f_{k_1}^{(n_1)} \cdots f_{k_s}^{(n_s)} \subset U_\varepsilon^{L,-}.$$

Then we have

$$\begin{aligned} \tau_\epsilon^{\emptyset, L}(a_{\alpha'} a_{\beta'}, y) &= (\tau_\epsilon^{\emptyset, L} \otimes \tau_\epsilon^{\emptyset, L})(a_{\beta'} \otimes a_{\alpha'}, \Delta(y)) \\ &= (\tau_\epsilon^{\emptyset, L} \otimes \tau_\epsilon^{\emptyset, L})(a_{\alpha'} \otimes a_{\beta'}, \mathcal{P}(u_{\beta', \alpha'}^y)), \end{aligned}$$

where $\mathcal{P}(y_1 \otimes y_2) = y_2 \otimes y_1$. Similarly, we have

$$\tau_\epsilon^{\emptyset, L}(a_{\beta'} a_{\alpha'}, y) = (\tau_\epsilon^{\emptyset, L} \otimes \tau_\epsilon^{\emptyset, L})(a_{\alpha'} \otimes a_{\beta'}, u_{\alpha', \beta'}^y).$$

Hence it is sufficient to show

$$\mathcal{P}(u_{\gamma,\delta}^y) = \varepsilon^{(\gamma|\delta)'} u_{\delta,\gamma}^y \quad (y \in U_\varepsilon^{L,-},\ \gamma, \delta \in (Q')^+). \tag{4.8}$$

We can easily check that if (4.8) holds for $y = y_1, y_2$, then it also holds for $y = y_1 y_2$. Hence the assertion follows from (4.8) for $y = f_i^{(n)}$, which is easily checked.

The second formula is equivalent to $b_{\alpha'} b_{\beta'} = \varepsilon^{(\alpha'|\beta')'} b_{\beta'} b_{\alpha'}$ for $\alpha', \beta' \in (\Delta')^+$, and is proved similarly to the first formula.

Let us show the third and the fourth formula. They are equivalent to $a_{\alpha'} b_{\beta'} = b_{\beta'} a_{\alpha'}$ for $\alpha', \beta' \in (\Delta')^+$. Take $1 \leqq j, k \leqq N$ such that $\alpha' = \beta_j', \beta' = \beta_k'$. If $j = k$, then the assertion is a consequence of $a_{\alpha_i'} b_{\alpha_i'} = b_{\alpha_i'} a_{\alpha_i'}$ in $U_\varepsilon(\Delta')$ for $i \in I$. Assume $j > k$. Setting

$$w = s_{i_1} \cdots s_{i_{k-1}}, \quad y = s_{i_k} \cdots s_{i_{j-1}}, \quad i_j = m, \quad i_k = n$$

we have

$$a_{\alpha'} = T_w T_y(a_{\alpha'_m}), \quad b_{\beta'} = T_w(b_{\alpha'_n}),$$

and hence it is sufficient to show

$$b_{\alpha'_n} T_y(a_{\alpha'_m}) = T_y(a_{\alpha'_m}) b_{\alpha'_n}.$$

By $s_n y < y$ this is equivalent to

$$T_n^{-1}(b_{\alpha'_n}) T_{s_n y}(a_{\alpha'_m}) = T_{s_n y}(a_{\alpha'_m}) T_n^{-1}(b_{\alpha'_n}).$$

By $T_n^{-1}(b_{\alpha'_n}) = -a_{\alpha'_n} k_n$ this is again equivalent to

$$a_{\alpha'_n} T_{s_n y}(a_{\alpha'_m}) = \varepsilon^{(\alpha'_n | s_n y(\alpha'_m))'} T_{s_n y}(a_{\alpha'_m}) a_{\alpha'_n}.$$

By $s_n y < s_n y s_m$ we have $s_n y(\alpha'_m) \in (\Delta')^+$ and $T_{s_n y}(a_{\alpha'_m})$ is a linear combination of the elements of the form $a_{\beta'_N}^{m_N} \cdots a_{\beta'_1}^{m_1}$ with $\sum_j m_j \beta'_j = s_n y(\alpha'_m)$. Hence the assertion follows from the first formula. The case $j < k$ can be handled in a similar way.

The remaining formulas are obvious. □

We see easily from Proposition 4.3, Proposition 4.4 the following.

Proposition 4.5. *${}^t\xi$ is a homomorphism of Hopf algebras.*

4.5. We define the Frobenius center $Z_{\mathrm{Fr}}(U_\zeta)$ of U_ζ by $Z_{\mathrm{Fr}}(U_\zeta) = \mathrm{Im}({}^t\xi) \cap Z(U_\zeta)$. Note

$$\mathrm{Im}({}^t\xi) \cong \left(\bigotimes_{\alpha \in \Delta^+} \mathbb{C}[a_\alpha^{r_\alpha}, Sb_\alpha^{r_\alpha}] \right) \otimes \mathbb{C}[P'].$$

Namely, the image of ${}^t\xi$ consists of the linear combinations of the monomials of the form

$$z = a_{\beta_1}^{r_{\beta_1} m_{\beta_1}} \cdots a_{\beta_N}^{r_{\beta_N} m_{\beta_N}} k_\mu (Sb_{\beta_1}^{r_{\beta_1} m'_{\beta_1}}) \cdots (Sb_{\beta_N}^{r_{\beta_1} m'_{\beta_N}}) \quad (\mu \in P'). \tag{4.9}$$

If l is odd, then we have $\eta_\alpha = 1$ for any $\alpha \in \Delta^+$, and hence $Z_{\mathrm{Fr}}(U_\zeta) = \mathrm{Im}({}^t\xi)$ by Proposition 4.3.

Assume l is even. By Proposition 4.3 we see easily that $Z_{\mathrm{Fr}}(U_\zeta)$ consists of the linear combinations of the monomials of the form (4.9) satisfying

$$\sum_{\alpha \in \Delta_1^+} (m_\alpha + m'_\alpha) \alpha^\vee \in 2Q^\vee, \tag{4.10}$$

$$(\mu, \gamma^\vee)/r_\gamma \in 2\mathbb{Z} \quad (\forall \gamma \in \Delta_1^+), \tag{4.11}$$

where

$$\Delta_1^+ = \{\alpha \in \Delta^+ \mid \eta_\alpha = -1\} = \begin{cases} \Delta_{\mathrm{short}} \cap \Delta^+ & (r \notin 2\mathbb{Z},\ l = 2r,\ d = 2), \\ \Delta^+ & (\text{otherwise}). \end{cases}$$

Note that (4.11) is equivalent to $\mu \in P''$, where

$$P'' = \begin{cases} 2P_0' & (r \notin 2\mathbb{Z},\ l = 2r,\ d = 2), \\ 2P' & (\text{otherwise}). \end{cases} \tag{4.12}$$

Here

$$P_0' = \{\lambda \in \mathfrak{h}_{\mathbb{Q}}^* \mid d_\alpha(\lambda, \alpha^\vee) \in r\mathbb{Z}\ (\alpha \in \Delta)\}.$$

Define subgroups Γ_1 and Γ_2 of $\mathcal{G}(\Delta')$ and $\mathcal{H}(\Delta')$ respectively by

$$\Gamma_1 = \begin{cases} \{1\} & (l \notin 2\mathbb{Z}), \\ \mathcal{G}(\Delta') & (l \in 2\mathbb{Z}), \end{cases}$$

$$\Gamma_2 = \begin{cases} \{1\} & (l \notin 2\mathbb{Z}), \\ (Q')^\vee_{\mathrm{short}}/2(Q')^\vee_{\mathrm{short}} & (l \in 2\mathbb{Z},\ r \notin 2\mathbb{Z},\ d = 2), \\ \mathcal{H}(\Delta') & (\text{otherwise}), \end{cases}$$

where

$$(Q')^\vee_{\mathrm{short}} = \sum_{\alpha' \in (\Delta')_{\mathrm{short}}} \mathbb{Z}(\alpha')^\vee.$$

Set

$$\Gamma = \Gamma_1 \times \Gamma_2.$$

By the above argument we have the following.

Proposition 4.6. *Under the identification* $\mathrm{Im}({}^t\xi) \cong U_\varepsilon(\Delta')$ *we have*

$$\begin{aligned} Z_{\mathrm{Fr}}(U_\zeta(\Delta)) &\cong (U_\varepsilon^+(\Delta') \otimes SU_\varepsilon^-(\Delta'))^{\Gamma_1} \otimes \mathbb{C}[P''] \\ &= (U_\varepsilon^+(\Delta') \otimes SU_\varepsilon^-(\Delta'))^{\Gamma_1} \otimes U_\varepsilon^0(\Delta')^{\Gamma_2} \\ &= U_\varepsilon(\Delta')^\Gamma. \end{aligned}$$

Proposition 4.7. *We have an isomorphism*

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cong \mathbb{C}[K(\Delta')]^\Gamma \quad (= \mathbb{C}[K(\Delta')/\Gamma]) \tag{4.13}$$

of algebras.

Proof. Assume $\varepsilon = 1$. Then the action of the group Γ on the algebra $U_1(\Delta')$ induces the action of Γ on the algebraic variety $K(\Delta')$ via the algebra isomorphism (2.15) for Δ', and hence we have $U_1(\Delta')^\Gamma \cong \mathbb{C}[K(\Delta')]^\Gamma \cong \mathbb{C}[K(\Delta')/\Gamma]$. Assume $\varepsilon = -1$. In this case we have $U_{-1}(\Delta')^\Gamma \cong U_1(\Delta')^\Gamma$ by Proposition 2.8. Hence we have also $U_{-1}(\Delta')^\Gamma \cong \mathbb{C}[K(\Delta')]^\Gamma \cong \mathbb{C}[K(\Delta')/\Gamma]$. □

By Proposition 4.6 and [10] we obtain the following.

Corollary 4.8. *$Z_{\mathrm{Fr}}(U_\zeta)$ is Cohen–Macaulay.*

5. Main result

Since the action $\circ_\varepsilon$ of W on $\mathbb{C}[2P']$ is the ordinary one, we have

$$Z_{\mathrm{Har}}(U_\varepsilon(\Delta')) \cong \mathbb{C}[2P']^W \cong \mathbb{C}[P']^W \cong \mathbb{C}[H(\Delta')/W], \tag{5.1}$$

where the second isomorphism is induced by $\mathbb{C}[P'] \cong \mathbb{C}[2P']$ ($e(\lambda) \leftrightarrow e(2\lambda)$). Similarly, we have

$$Z_{\mathrm{Har}}(U_\zeta(\Delta)) \cong \mathbb{C}[H(\Delta)/W]. \tag{5.2}$$

Note that the action of W on $H(\Delta')$ in (5.1) is the ordinary one, while that on $H(\Delta)$ in (5.2) is the twisted one given by

$$w\colon h \mapsto w(h_1 h)h_1^{-1} \quad (w \in W, h \in H(\Delta)),$$

where $h_1 \in H(\Delta)$ is given by $\lambda(h_1) = \zeta^{2(\lambda,\tilde{\rho})}$ ($\lambda \in P = \mathrm{Hom}(H(\Delta), \mathbb{C}^\times)$).

Proposition 5.1. *We have*

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) = {}^t\xi(Z_{\mathrm{Har}}(U_\varepsilon(\Delta'))),$$

and hence

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) \cong \mathbb{C}[H(\Delta')/W]. \tag{5.3}$$

Proof. Note that $Z_{\mathrm{Har}}(U_\varepsilon(\Delta'))$ is spanned by $\{t_{L_\varepsilon(\lambda)}\}_{\lambda \in (P')^+}$. By Proposition 4.2 we have ${}^t\xi(t_{L_\varepsilon(\lambda)}) = t_{\mathrm{Fr}^* L_\varepsilon(\lambda)}$, where $\mathrm{Fr}^* L_\varepsilon(\lambda)$ is the $\dot{U}_\zeta(\Delta)$-module induced via $\mathrm{Fr}\colon \dot{U}_\zeta(\Delta) \to \dot{U}_\varepsilon(\Delta')$. Hence we have

$$Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) \supset {}^t\xi(Z_{\mathrm{Har}}(U_\varepsilon(\Delta'))),$$

and $\iota_\zeta({}^t\xi(Z_{\mathrm{Har}}(U_\varepsilon(\Delta')))) = \mathbb{C}[2P']^W$. On the other hand by Proposition 4.6 we have $\iota_\zeta(Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta))) \subset \mathbb{C}[2P]^{W\circ_\zeta} \cap \mathbb{C}[P''] = \mathbb{C}[2P']^W$. □

By the definition of the Harish-Chandra isomorphism we have the following.

Proposition 5.2. *The morphism $H(\Delta)/W \to H(\Delta')/W$, which is associated to the inclusion $Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) \subset Z_{\mathrm{Har}}(U_\zeta(\Delta))$ together with the isomorphisms* (5.2) *and* (5.3)*, is the natural one induced from the canonical morphism $H(\Delta) \to H(\Delta')$ associated to the embedding $P' \subset P$.*

Note that we have the following commutative diagram

$$\begin{array}{ccccc}
Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) & \longrightarrow & Z_{\mathrm{Fr}}(U_\zeta(\Delta)) & & \\
\uparrow & & \uparrow & & \\
Z_{\mathrm{Har}}(U_\varepsilon(\Delta')) & \longrightarrow & U_\varepsilon(\Delta')^\Gamma & & \\
\uparrow & & \uparrow & & \\
Z_{\mathrm{Har}}(U_1(\Delta')) & \longrightarrow & U_1(\Delta')^\Gamma & \longrightarrow & U_1(\Delta') \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{C}[H(\Delta')/W] & \longrightarrow & \mathbb{C}[K(\Delta')/\Gamma] & \longrightarrow & \mathbb{C}[K(\Delta')]
\end{array}$$

where horizontal arrows are inclusions, and vertical arrows are isomorphisms. Note also that the inclusion $\mathbb{C}[H(\Delta')/W] \to \mathbb{C}[K(\Delta')]$ is induced by $\upsilon \circ \kappa$, where $\kappa\colon K(\Delta') \to G(\Delta')$ and $\upsilon\colon G(\Delta') \to H(\Delta')/W$ are morphisms of algebraic varieties we have already defined. Hence we have the following.

Proposition 5.3. *The morphism $K(\Delta')/\Gamma \to H(\Delta')/W$, which is associated to the inclusion $Z_{\mathrm{Fr}}(U_\zeta(\Delta)) \cap Z_{\mathrm{Har}}(U_\zeta(\Delta)) \subset Z_{\mathrm{Fr}}(U_\zeta(\Delta))$ together with the isomorphisms* (4.13) *and* (5.3)*, is induced by $\upsilon \circ \kappa\colon K(\Delta') \to H(\Delta')/W$.*

The main result of this paper is the following.

Theorem 5.4. *The natural homomorphism*

$$Z_{\mathrm{Fr}}(U_\zeta) \otimes_{Z_{\mathrm{Fr}}(U_\zeta) \cap Z_{\mathrm{Har}}(U_\zeta)} Z_{\mathrm{Har}}(U_\zeta) \to Z(U_\zeta)$$

is an isomorphism. In particular, we have

$$Z(U_\zeta) \cong \mathbb{C}[(K(\Delta')/\Gamma) \times_{H(\Delta')/W} (H(\Delta)/W)].$$

The rest of the paper is devoted to the proof of Theorem 5.4. The arguments below mostly follow that in De Concini–Kac–Procesi [6] (see also De Concini–Procesi [8]).

We set for simplicity

$$Z = Z(U_\zeta),$$
$$Z_{\mathrm{Fr}} = Z_{\mathrm{Fr}}(U_\zeta) \cong \mathbb{C}[K(\Delta')/\Gamma],$$
$$Z_{\mathrm{Har}} = Z_{\mathrm{Har}}(U_\zeta) \cong \mathbb{C}[H(\Delta)/W],$$

so that

$$Z_{\mathrm{Fr}} \cap Z_{\mathrm{Har}} \cong \mathbb{C}[H(\Delta')/W].$$

We are going to show that the canonical homomorphism

$$j : Z_{\mathrm{Fr}} \otimes_{Z_{\mathrm{Fr}} \cap Z_{\mathrm{Har}}} Z_{\mathrm{Har}} \to Z$$

is an isomorphism.

Proposition 5.5. *$Z_{\mathrm{Fr}} \otimes_{Z_{\mathrm{Fr}} \cap Z_{\mathrm{Har}}} Z_{\mathrm{Har}}$ is a normal domain.*

Proof. By Serre's criterion it is sufficient to show that the scheme $(K(\Delta')/\Gamma) \times_{H(\Delta')/W} (H(\Delta)/W)$ is smooth in codimension one and Cohen–Macaulay.

We first show that $(K(\Delta')/\Gamma) \times_{H(\Delta')/W} (H(\Delta)/W)$ is smooth in codimension one. Since $H(\Delta)/W$ is smooth and $H(\Delta)/W \to H(\Delta')/W$ is a finite morphism, it is sufficient to show that there exists a subvariety X of $K(\Delta')/\Gamma$ with codimension greater than one such that $(K(\Delta')/\Gamma) \setminus X \to H(\Delta')/W$ is smooth. Consider first $K(\Delta') \to H(\Delta')/W$. Then there exists a subvariety X_1 of $K(\Delta')$ with codimension greater than one such that $K(\Delta') \setminus X_1 \to H(\Delta')/W$ is smooth since a similar result is known to hold for $G(\Delta') \to H(\Delta')/W$ and $K(\Delta') \to G(\Delta')$ is smooth. Hence it is sufficient to show that there exists a subvariety X_2 of $K(\Delta')$ with codimension greater than one such that $K(\Delta') \setminus X_2 \to (K(\Delta')/\Gamma$ is smooth since $K(\Delta') \to (K(\Delta')/\Gamma$ is a finite morphism. We may assume $\Gamma \neq \{1\}$. In this case we have

$$K(\Delta') = Y \times \operatorname{Spec} \mathbb{C}[P'], \quad K(\Delta')/\Gamma = Y/P' \times \operatorname{Spec} \mathbb{C}[P''],$$

where $Y = \prod_{\alpha \in \Delta^+} \mathbb{C}^2$ and the action of P' on Y is given by

$$\lambda : (x_\alpha)_{\alpha \in \Delta^+} \mapsto ((-1)^{d_{\alpha'}(\lambda, (\alpha')^\vee)} x_\alpha)_{\alpha \in \Delta^+} \quad (\lambda \in P',\ x_\alpha \in \mathbb{C}^2).$$

Since $\operatorname{Spec} \mathbb{C}[P'] \to \operatorname{Spec} \mathbb{C}[P'']$ is smooth, it is sufficient to show that there exists a subvariety Z of Y with codimension greater than one such that $Y \setminus Z \to Y/P'$ is smooth. Note that the obvious action of $\prod_{\alpha \in \Delta^+} GL_2(\mathbb{C})$ on Y commutes with the action of P'. Hence $Y \to Y/P'$ is smooth on the open orbit $Y_0 = \prod_{\alpha \in \Delta^+} (\mathbb{C}^2 \setminus \{0\})$. Our claim is a consequence of $\dim(Y \setminus Y_0) \leqq \dim Y - 2$.

Let us show that $Z_{\mathrm{Fr}}(U_\zeta)\otimes_{\mathbb{C}[2P']^W}\mathbb{C}[2P]^{W\circ\zeta}$ is Cohen–Macaulay. By [18] $\mathbb{C}[2P']^W$ and $\mathbb{C}[2P]^{W\circ\zeta}$ are both isomorphic to the polynomial ring in $|I|$-variables. Hence we have

$$Z_{\mathrm{Fr}}(U_\zeta)\otimes_{\mathbb{C}[2P']^W}\mathbb{C}[2P]^{W\circ\zeta}\cong Z_{\mathrm{Fr}}(U_\zeta)[X_1,\dots,X_{|I|}]/(f_1,\dots,f_{|I|})$$

for some $f_1,\dots,f_{|I|}\in Z_{\mathrm{Fr}}(U_\zeta)[X_1,\dots,X_{|I|}]$. Moreover, we have obviously $\dim Z_{\mathrm{Fr}}(U_\zeta)\otimes_{\mathbb{C}[2P']^W}\mathbb{C}[2P]^{W\circ\zeta}=\dim Z_{\mathrm{Fr}}(U_\zeta)$. Hence our claim is a consequence of Corollary 4.8 and well-known results on Cohen–Macaulay rings. □

Lemma 5.6. *$Z_{\mathrm{Fr}}\otimes_{Z_{\mathrm{Fr}}\cap Z_{\mathrm{Har}}}Z_{\mathrm{Har}}$ is a free Z_{Fr}-module of rank P/P'.*

Proof. It is sufficient to show that Z_{Har} is a free $Z_{\mathrm{Fr}}\cap Z_{\mathrm{Har}}$-module of rank P/P'. Namely, we have only to show that $\mathbb{C}[2P]^{W\circ\zeta}$ is a free $\mathbb{C}[2P']^W$-module of rank P/P'. We may replace $\mathbb{C}[2P]^{W\circ\zeta}$ with $\mathbb{C}[2P]^W$ by applying an automorphism of $\mathbb{C}[P]$ which sends $\mathbb{C}[2P]^{W\circ\zeta}$ and $\mathbb{C}[2P']^W$ to $\mathbb{C}[2P]^W$ and $\mathbb{C}[2P']^W$ respectively. By Steinberg [18] $\mathbb{C}[2P]$ (resp. $\mathbb{C}[2P']$) is a free $\mathbb{C}[2P]^W$-module (resp. $\mathbb{C}[2P']^W$-module) of rank $|W|$. Since $\mathbb{C}[2P]$ is a free $\mathbb{C}[2P']$-module of rank $|P/P'|$, $\mathbb{C}[2P]$ is a free $\mathbb{C}[2P']^W$-module of rank $|W|\times|P/P'|$. Note that $\mathbb{C}[2P]^W$ is a direct summand of the free $\mathbb{C}[2P]^W$-module $\mathbb{C}[2P]$ by [18]. Hence $\mathbb{C}[2P]^W$ is also a direct summand of the free $\mathbb{C}[2P']^W$-module $\mathbb{C}[2P]$. It follows that $\mathbb{C}[2P]^W$ is a projective $\mathbb{C}[2P']^W$-module of rank $|P/P'|$. Since $\mathbb{C}[2P']^W$ is isomorphic to a polynomial ring by [18], we conclude from the Serre conjecture that $\mathbb{C}[2P]^W$ is a free $\mathbb{C}[2P']^W$-module of rank P/P'. □

Set

$$m=\begin{cases}1 & (l\notin 2\mathbb{Z}),\\ 2^{|\Delta_{\mathrm{short}}\cap\Pi|} & (l\in 2\mathbb{Z},\ r\notin 2\mathbb{Z},\ d=2),\\ 2^{|\Pi|} & (\text{otherwise}).\end{cases}\tag{5.4}$$

For a commutative domain S we denote by $Q(S)$ the quotient field.

Lemma 5.7. *U_ζ is a finitely generated Z_{Fr}-module, and we have*

$$\dim_{Q(Z_{\mathrm{Fr}})}Q(Z_{\mathrm{Fr}})\otimes_{Z_{\mathrm{Fr}}}U_\zeta=\left(m\prod_{\alpha\in\Delta^+}r_\alpha\right)^2\times|P/P'|.$$

Proof. Denote by C the image of ${}^t\xi\colon U_\varepsilon(\Delta')\to U_\zeta(\Delta)$. Then we have

$$Z_{\mathrm{Fr}}\subset C\subset U_\zeta.$$

Since U_ζ is a free C-module of rank $\left(\prod_{\alpha\in\Delta^+}r_\alpha\right)^2\times|P/P'|$, it is sufficient to show that C is a finitely generated Z_{Fr}-module and

$$\dim_{Q(Z_{\mathrm{Fr}})}Q(Z_{\mathrm{Fr}})\otimes_{Z_{\mathrm{Fr}}}C=m^2.$$

If l is odd, we have $C = Z_{\mathrm{Fr}}$, and hence we may assume that l is even. By the explicit description of Z_{Fr} given by (4.10), (4.11) we have

$$C \cong \mathbb{C}[P'] \otimes \mathbb{C}[(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}], \quad Z_{\mathrm{Fr}} \cong \mathbb{C}[P''] \otimes \mathbb{C}[L],$$

where

$$L = \left\{ (m_\beta, m'_\beta)_{\beta\in\Delta^+} \in (\mathbb{Z}^2_{\geqq 0})^{\Delta^+} \;\middle|\; \sum_{\beta\in\Delta_1^+} (m_\beta + m'_\beta)\beta^\vee \in 2Q^\vee \right\},$$

and $\mathbb{C}[(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}]$ and $\mathbb{C}[L]$ are the semigroup algebras of the semigroups $(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}$ and L respectively. Note that $\mathbb{C}[P']$ is a free $\mathbb{C}[P'']$-module of rank $|P'/P''|$. Since $\mathbb{C}[(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}]$ is a finitely generated $\mathbb{C}[2(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}]$-module, it is also a finitely generated $\mathbb{C}[L]$-module by $2(\mathbb{Z}^2_{\geqq 0})^{\Delta^+} \subset L$. Hence C is a finitely generated Z_{Fr}-module.

Set

$$\tilde{L} = \{(m_\beta, m'_\beta)_{\beta\in\Delta^+} \in (\mathbb{Z}^2)^{\Delta^+} \mid \sum_{\beta\in\Delta_1^+} (m_\beta + m'_\beta)\beta^\vee \in 2Q^\vee\}.$$

Then we have $(\mathbb{Z}^2)^{\Delta^+}/\tilde{L} \cong Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)$, where $Q_1^\vee = \sum_{\beta\in\Delta_1^+} \mathbb{Z}\beta^\vee$. Hence $\mathbb{C}[(\mathbb{Z}^2)^{\Delta^+}]$ is a free $\mathbb{C}[\tilde{L}]$-module of rank $|Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)|$. Since $\mathbb{C}[(\mathbb{Z}^2)^{\Delta^+}]$ and $\mathbb{C}[\tilde{L}]$ are localizations of $\mathbb{C}[(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}]$ and $\mathbb{C}[L]$ respectively with respect to the multiplicative set $S = 2(\mathbb{Z}^2_{\geqq 0})^{\Delta^+}$ of $\mathbb{C}[L]$, we obtain that $S^{-1}C$ is a free $S^{-1}Z_{\mathrm{Fr}}$-module of rank $|P'/P''| \times |Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)|$. Therefore, $Q(Z_{\mathrm{Fr}}) \otimes_{Z_{\mathrm{Fr}}} C$ is a free $Q(Z_{\mathrm{Fr}})$-module of rank $|P'/P''| \times |Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)|$. It remains to show $m^2 = |P'/P''| \times |Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)|$. In the case $\Delta_1^+ = \Delta^+$ we have $P'' = 2P'$, $Q_1^\vee = Q^\vee$, and hence the assertion is obvious. In the case $\Delta_1^+ = \Delta^+ \cap \Delta_{\mathrm{short}}$ we have $P' = rP$, $P'' = r(P \cap 2P_1)$, where

$$P_1 = \{\mu \in \mathfrak{h}^*_{\mathbb{Q}} \mid (\mu, \alpha^\vee) \in \mathbb{Z} \ (\alpha \in \Delta_{\mathrm{short}})\},$$

and hence $P'/P'' \cong P/(P \cap 2P_1)$. On the other hand we have

$$Q_1^\vee/(Q_1^\vee \cap 2Q^\vee) \cong (Q_1^\vee + 2Q^\vee)/2Q^\vee \cong \left(\frac{1}{2}Q_1^\vee + Q^\vee\right) \Big/ Q^\vee.$$

Since P and $P \cap 2P_1$ are lattices in $\mathfrak{h}^*_{\mathbb{Q}}$ dual to $Q^\vee$ and $(1/2)Q_1^\vee + Q$ respectively, we obtain $|P'/P''| = |Q_1^\vee/(Q_1^\vee \cap 2Q^\vee)|$. It remains to check $m = |(Q_1^\vee + 2Q^\vee)/2Q^\vee|$. For that it is sufficient to show

$$Q_1^\vee + 2Q^\vee = \sum_{\alpha\in\Delta_{\mathrm{short}}\cap\Pi} \mathbb{Z}\alpha^\vee + 2Q^\vee.$$

In order to prove this we have only to show that the right-hand side is stable under the action of the Weyl group. Hence it is sufficient to show $s_j(\alpha_i^\vee) \in \sum_{\alpha\in\Delta_{\mathrm{short}}\cap\Pi} \mathbb{Z}\alpha^\vee +$

$2Q^\vee$ for any $i, j \in I$ satisfying $\alpha_i \in \Delta_{\text{short}}$. This is obvious if $\alpha_j \in \Delta_{\text{short}}$. In the case where $\alpha_j \in \Delta_{\text{long}}$ we have $s_j(\alpha_i^\vee) = \alpha_i^\vee - (\alpha_i^\vee, \alpha_j)\alpha_j^\vee$ with $(\alpha_i^\vee, \alpha_j) \in \{0, -2\}$. We are done. □

In general let R be a $\mathbb{C}$-algebra. Assume that R is prime (i.e. $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$), and is finitely generated as a $Z(R)$-module. Then $Q(Z(R)) \otimes_{Z(R)} R$ is a finite-dimensional central simple algebra over the field $Q(Z(R))$. Hence $\overline{Q(Z(R))} \otimes_{Z(R)} R$ is isomorphic to the matrix algebra $M_n(\overline{Q(Z(R))})$ for some n, where $\overline{Q(Z(R))}$ denotes the algebraic closure of $Q(Z(R))$. Then this n is called the degree of R. Namely, the degree n of R is given by

$$\dim_{Q(Z(R))} Q(Z(R)) \otimes_{Z(R)} R = n^2.$$

Note that U_ζ is a finitely generated $Z(U_\zeta)$-module by Lemma 5.7. In [5] De Concini–Kac have shown that U_ζ has no zero divisors using a certain degeneration $\operatorname{Gr} U_\zeta$ of U_ζ. In particular, it is a prime algebra. Hence we have the notion of the degree of U_ζ. In [7] De Concini–Kac–Procesi proved that the degree of U_ζ is less than or equal to that of $\operatorname{Gr} U_\zeta$. They have also shown that the degree of $\operatorname{Gr} U_\zeta$ can be computed from the elementary divisors of a certain matrix with integral coefficients. The actual computation of the elementary divisors was done in [7] when l is odd, and in Beck [2] in the remaining cases. From these results we have the following.

Proposition 5.8. *We have*

$$\dim_{Q(Z)} Q(Z) \otimes_Z U_\zeta \leqq \left(m \prod_{\alpha \in \Delta^+} r_\alpha \right)^2.$$

Let us show that j is injective. By Proposition 5.5 $Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{Har}}} Z_{\text{Har}}$ is a domain. Note also that Z is a domain since U_ζ has no zero divisors. Hence we have only to show that

$$j^* \colon \operatorname{Spec} Z \to \operatorname{Spec} Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{Har}}} Z_{\text{Har}}$$

has a dense image. Consider the embedding $j' \colon Z_{\text{Fr}} \to Z$. Since j' is injective, $(j')^* \colon \operatorname{Spec} Z \to \operatorname{Spec} Z_{\text{Fr}}$ has a dense image. Note that $(j')^*$ is the composite of j^* with the natural morphism

$$\varphi \colon \operatorname{Spec} Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{Har}}} Z_{\text{Har}} \to \operatorname{Spec} Z_{\text{Fr}}.$$

Since $\operatorname{Spec} Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{Har}}} Z_{\text{Har}}$ is irreducible and φ is a finite morphism by Lemma 5.6, we conclude that j^* must have a dense image. The injectivity of j is verified.

Set for simplicity

$$Z' = Z_{\text{Fr}} \otimes_{Z_{\text{Fr}} \cap Z_{\text{Har}}} Z_{\text{Har}}.$$

Then we have

$$Z_{\mathrm{Fr}} \subset Z' \subset Z \subset U_\zeta.$$

We need to show $Z' = Z$.

Assume that

$$Q(Z) = Q(Z') \tag{5.5}$$

holds. Since U_ζ is a finitely generated Z_{Fr}-module, Z is a finitely generated Z'-module. It follows that $Z = Z'$ by Proposition 5.5. Hence it is sufficient to show (5.5).

Since Z' is a free Z_{Fr}-module of rank $|P/P'|$, we have $[Q(Z') : Q(Z_{Fr})] \geqq |P/P'|$. Hence it is sufficient to show

$$[Q(Z) : Q(Z_{Fr})] \leqq |P/P'|. \tag{5.6}$$

Note that we have $Q(Z_{\mathrm{Fr}}) \otimes_{Z_{\mathrm{Fr}}} Z \cong Q(Z)$ since Z is a finitely generated Z_{Fr}-module. Hence

$$Q(Z_{Fr}) \otimes_{Z_{Fr}} U_\zeta \cong Q(Z_{Fr}) \otimes_{Z_{Fr}} Z \otimes_Z U_\zeta \cong Q(Z) \otimes_Z U_\zeta.$$

Hence we obtain (5.6) by Lemma 5.7, Proposition 5.8. The proof of Theorem 5.4 is complete.

Corollary 5.9. *The degree of U_ζ is equal to $m \prod_{\alpha \in \Delta^+} r_\alpha$, where m is as in* (5.4).

REMARK 5.10. Corollary 5.9 was proved by De Concini–Kac–Procesi [7] in the case l is odd and by Beck [2] in the case l is divided by $4d$.

References

[1] H.H. Andersen: *Quantum groups at roots of* ± 1, Comm. Algebra **24** (1996), 3269–3282.

[2] J. Beck: *Representations of quantum groups at even roots of unity*, J. Algebra **167** (1994), 29–56.

[3] P. Caldero: *Éléments ad-finis de certains groupes quantiques*, C.R. Acad. Sci. Paris Sér. I Math. **316** (1993), 327–329.

[4] C. De Concini: *Poisson algebraic groups and representations of quantum groups at roots of* 1; in First European Congress of Mathematics **1** (Paris, 1992), Progr. Math. **119**, Birkhäuser, Basel, 1994, 93–119.

[5] C. De Concini and V.G. Kac: *Representations of quantum groups at roots of* 1; in Operator Algebras, Unitary Representations, Enveloping Algebras, and Invariant Theory (Paris, 1989), Progr. Math. **92**, Birkhäuser Boston, Boston, MA, 1990, 471–506.

[6] C. De Concini, V.G. Kac and C. Procesi: *Quantum coadjoint action*, J. Amer. Math. Soc. **5** (1992), 151–189.

[7] C. De Concini, V.G. Kac and C. Procesi: *Some remarkable degenerations of quantum groups*, Comm. Math. Phys. **157** (1993), 405–427.
[8] C. De Concini and C. Procesi: *Quantum groups*, in *D*-Modules, Representation Theory, and Quantum Groups (Venice, 1992), Lecture Notes in Math. **1565**, Springer, Berlin, 1993, 31–140.
[9] F. Gavarini: *Quantization of Poisson groups*, Pacific J. Math. **186** (1998), 217–266.
[10] M. Hochster and J.A. Eagon: *Cohen–Macaulay rings, invariant theory, and the generic perfection of determinantal loci*, Amer. J. Math. **93** (1971), 1020–1058.
[11] A. Joseph and G. Letzter: *Local finiteness of the adjoint action for quantized enveloping algebras*, J. Algebra **153** (1992), 289–318.
[12] S.-J. Kang, M. Kashiwara and S.-j. Oh: *Supercategorification of quantum Kac–Moody algebras* II, arXiv:1303.1916.
[13] A.N. Kirillov and N. Reshetikhin: *q-Weyl group and a multiplicative formula for universal R-matrices*, Comm. Math. Phys. **134** (1990), 421–431.
[14] S.M. Khoroshkin and V.N. Tolstoy: *Universal R-matrix for quantized* (*super*)*algebras*, Comm. Math. Phys. **141** (1991), 599–617.
[15] S.Z. Levendorskiĭ and Ya. S. Soĭbel'man: *Quantum Weyl group and multiplicative formula for the R-matrix of a simple Lie algebra*, Funct. Anal. Appl. **25** (1991), 143–145.
[16] G. Lusztig: *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), 89–113.
[17] G. Lusztig: Introduction to Quantum Groups, Progress in Mathematics **110**, Birkhäuser Boston, Boston, MA, 1993.
[18] R. Steinberg: *On a theorem of Pittie*, Topology **14** (1975), 173–177.
[19] T. Tanisaki: *Killing forms, Harish-Chandra isomorphisms, and universal R-matrices for quantum algebras*, Inter. J. Mod. Phys. **A7** (1992), 941–961.
[20] T. Tanisaki: *Manin triples and differential operators on quantum groups*, Tokyo J. Math. **36** (2013), 49–83.

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