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THE CHARACTERISTIC RANK AND CUP-LENGTH IN ORIENTED GRASSMANN MANIFOLDS

Dedicated to Professor Mamoru Mimura on the occasion of his 77-th birthday

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Abstract

In the first part, this paper studies the characteristic rank of the canonical oriented k-plane bundle over the Grassmann manifold $\tilde{G}_{n,k}$ of oriented k-planes in Euclidean n-space. It presents infinitely many new exact values if $k = 3$ or $k = 4$, as well as new lower bounds for the number in question if $k \geq 5$. In the second part, these results enable us to improve on the general upper bounds for the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{n,k}$. In particular, for $\tilde{G}_{2^t,3}$ ($t \geq 3$) we prove that the cup-length is equal to $2^t - 3$, which verifies the corresponding claim of Tomohiro Fukaya's conjecture from 2008.

1. Introduction and some preliminaries

Given a real vector bundle α over a path-connected CW-complex X, the *characteristic rank of* α, denoted charrank(α), is defined to be ([6]) the greatest integer q, $0 \leq q \leq \dim(X)$, such that every cohomology class in $H^j(X)$, $0 \leq j \leq q$, is a polynomial in the Stiefel–Whitney classes $w_i(\alpha) \in H^i(X)$. Here and elsewhere in this paper, we write $H^i(X)$ instead of $H^i(X; \mathbb{Z}_2)$.

In particular, if TM is the tangent bundle of a smooth closed connected manifold M, then charrank(TM) is nothing but the *characteristic rank of* M, denoted charrank(M); this homotopy invariant of smooth closed connected manifolds was introduced, and in some cases also computed, in [3]. Results on the characteristic rank of vector bundles over the Stiefel manifolds can be found in [4]. The characteristic rank is useful, for instance, in studying the cup-length of a given space (see [3], [6], and also Section 3 of the present paper).

It is readily seen that the characteristic rank of the canonical k-plane bundle $\gamma_{n,k}$ (briefly γ) over the Grassmann manifold $G_{n,k}$ ($k \leq n-k$) of all k-dimensional vector subspaces in $\mathbb{R}^n$ is equal to $\dim(G_{n,k}) = k(n-k)$. Indeed, as is well known ([1]), for

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the $\mathbb{Z}_2$-cohomology algebra $H^*(G_{n,k})$ we can write

$$H^*(G_{n,k}) = \mathbb{Z}_2[w_1, \dots, w_k]/I_{n,k}, \tag{1.1}$$

where $\dim(w_i) = i$ and the ideal $I_{n,k}$ is generated by the k homogeneous components of $(1 + w_1 + \cdots + w_k)^{-1}$ in dimensions $n-k+1, \dots, n$; here the indeterminate w_i is a representative of the i-th Stiefel–Whitney class $w_i(\gamma)$ in the quotient algebra $H^*(G_{n,k})$. For the latter class $w_i(\gamma)$, we shall also use w_i as an abbreviation.

In contrast to the situation for $G_{n,k}$, the $\mathbb{Z}_2$-cohomology algebra $H^*(\tilde{G}_{n,k})$ ($k \le n-k$) of the "oriented" Grassmann manifold $\tilde{G}_{n,k}$ of all *oriented* k-dimensional vector subspaces in $\mathbb{R}^n$ is in general unknown. Since $\tilde{G}_{n,1}$ can be identified with the $(n-1)$-dimensional sphere, and the complex quadrics $\tilde{G}_{n,2}$ are also well understood special cases, we shall suppose that $k \ge 3$ throughout the paper.

In Section 2, we derive infinitely many new exact values if $k = 3$ or $k = 4$, as well as new lower bounds for the characteristic rank of the canonical oriented k-plane bundle $\tilde{\gamma}_{n,k}$ (briefly $\tilde{\gamma}$) over $\tilde{G}_{n,k}$ if $k \ge 5$. As a consequence, for odd n, we also obtain better bounds (as compared to those known from [3, p. 73]) on the invariant $\mathrm{charrank}(\tilde{G}_{n,k})$. Then, in Section 3, our results on the characteristic rank of $\tilde{\gamma}$ enable us to improve on the general upper bounds for the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{n,k}$. In particular, for $\tilde{G}_{2^t,3}$ ($t \ge 3$) we prove that the cup-length is equal to $2^t - 3$; this verifies the corresponding claim of Fukaya's conjecture [2, Conjecture 1.2].

2. On the characteristic rank of the canonical vector bundle over $\tilde{G}_{n,k}$

Using the notation introduced in Section 1, we now state our main result.

Theorem 2.1. *For the canonical k-plane bundle $\tilde{\gamma}_{n,k}$ over the oriented Grassmann manifold $\tilde{G}_{n,k}$ ($3 \le k \le n-k$), with $2^{t-1} < n \le 2^t$, we have*

(1) $$\mathrm{charrank}(\tilde{\gamma}_{n,3}) \begin{cases} = n-2 & \textit{if} \quad n = 2^t, \\ = n-5+i & \textit{if} \quad n = 2^t - i,\ i \in \{1, 2, 3\}, \\ \ge n-2 & \textit{otherwise}; \end{cases}$$

(2) $$\mathrm{charrank}(\tilde{\gamma}_{n,4}) \begin{cases} = n-5+i & \textit{if} \quad n = 2^t - i,\ i \in \{0, 1, 2, 3\}, \\ \ge n-3 & \textit{otherwise}; \end{cases}$$

(3) *if* $k \ge 5$, *then* $\mathrm{charrank}(\tilde{\gamma}_{n,k}) \ge n-k+1$.

In addition, if n is odd, then the replacement of the canonical bundle $\tilde{\gamma}_{n,j}$ by the corresponding manifold $\tilde{G}_{n,j}$, in (1)–(3), *gives the corresponding result on* $\mathrm{charrank}(\tilde{G}_{n,j})$.

We shall pass to a proof of this theorem after some preparations.

For the universal 2-fold covering $p\colon \tilde{G}_{n,k} \to G_{n,k}$ ($k \ge 3$), the pullback $p^*(\gamma)$ is $\tilde{\gamma}$, and for the induced homomorphism in cohomology we have that $p^*(w_i) = \tilde{w}_i$ for all i, where $\tilde{w}_i$ is an abbreviated notation, used throughout the paper, for the Stiefel–Whitney class $w_i(\tilde{\gamma}_{n,k})$. Of course, now $\mathrm{charrank}(\tilde{\gamma}_{n,k})$ is, in other words, the greatest integer q, $0 \le q \le k(n-k)$, such that $p^*\colon H^j(G_{n,k}) \to H^j(\tilde{G}_{n,k})$ is surjective for all j, $0 \le j \le q$.

To the covering p there is associated a uniquely determined non-trivial line bundle ξ such that $w_1(\xi) = w_1(\gamma_{n,k})$. This yields ([5, Corollary 12.3]) an exact sequence of Gysin type,

$$\to H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k}) \xrightarrow{p^*} H^j(\tilde{G}_{n,k}) \to H^j(G_{n,k}) \xrightarrow{w_1} . \tag{2.1}$$

As is certainly clear from the context, we write here and elsewhere $H^{j-1}(G_{n,k}) \xrightarrow{w_1} H^j(G_{n,k})$ for the homomorphism given by the cup-product with the Stiefel–Whitney class w_1.

Thus $p^*\colon H^j(G_{n,k}) \to H^j(\tilde{G}_{n,k})$ is surjective if and only if the subgroup

$$\operatorname{Ker}(H^j(G_{n,k}) \xrightarrow{w_1} H^{j+1}(G_{n,k})) \tag{2.2}$$

vanishes.

By (1.1), a $\mathbb{Z}_2$-polynomial

$$p_j(w_1, \dots, w_k) = \sum_{i_1+2i_2+\cdots+ki_k=j} a_{i_1,i_2,\dots,i_k} w_1^{i_1} w_2^{i_2} \cdots w_k^{i_k}, \tag{2.3}$$

with at least one coefficient $a_{i_1,i_2,\dots,i_k} \in \mathbb{Z}_2$ nonzero, represents zero in $H^j(G_{n,k})$ precisely when there exist some polynomials $q_i(w_1, \dots, w_k)$ (briefly q_i) such that

$$p_j = q_{j-n+k-1}\bar{w}_{n-k+1} + \cdots + q_{j-n}\bar{w}_n,$$

where $\bar{w}_i(w_1, \dots, w_k)$ (briefly $\bar{w}_i$) is the homogeneous component of $(1 + w_1 + \cdots + w_k)^{-1} = 1 + w_1 + \cdots + w_k + (w_1 + \cdots + w_k)^2 + \cdots$ in dimension i. Of course, we have

$$\bar{w}_i = w_1\bar{w}_{i-1} + w_2\bar{w}_{i-2} + \cdots + w_k\bar{w}_{i-k}. \tag{2.4}$$

We note that $\bar{w}_i$ represents the i-th dual Stiefel–Whitney class of γ, that is, the Stiefel–Whitney class $w_i(\gamma_{n,k}^{\perp}) \in H^i(G_{n,k})$ of the complementary $(n-k)$-plane bundle $\gamma_{n,k}^{\perp}$ (briefly $\gamma^{\perp}$); we shall also use $\bar{w}_i$ as an abbreviation for $w_i(\gamma^{\perp})$.

By what we have said, no nonzero homogeneous polynomials in $w_1, \dots, w_k$ in dimensions $\le n-k$ represent 0 in cohomology; therefore the kernel (2.2) is the zero-subgroup for all $j \le n-k-1$, and we always have

$$\operatorname{charrank}(\tilde{\gamma}_{n,k}) \ge n-k-1. \tag{2.5}$$

For the Grassmann manifold $G_{n,k}$ $(3 \le k \le n-k)$, let $g_i(w_2, \dots, w_k)$ (briefly just g_i) denote the reduction of $\bar{w}_i(w_1, \dots, w_k)$ modulo w_1.

The following fact is obvious.

Fact 2.2. *Let $r < k$. If $\bar{w}_i(w_1, \dots, w_k) = 0$, then also $\bar{w}_i(w_1, \dots, w_r) = 0$ and, similarly, if $g_i(w_2, \dots, w_k) = 0$, then also $g_i(w_2, \dots, w_r) = 0$.*

For $G_{n,k}$, the formula (2.4) implies that $g_i = w_2 g_{i-2} + w_3 g_{i-3} + \cdots + w_k g_{i-k}$, and an obvious induction proves that

$$g_i = w_2^{2^s} g_{i-2\cdot 2^s} + w_3^{2^s} g_{i-3\cdot 2^s} + \cdots + w_k^{2^s} g_{i-k\cdot 2^s} \tag{2.6}$$

for all s such that $i \geq 1 + k \cdot 2^s$.

In our proof of Theorem 2.1, we shall use the following.

Lemma 2.3. *For the Grassmann manifold $G_{n,k}$ $(3 \leq k \leq n-k)$,*
(i) *$g_i(w_2, w_3) = 0$ if and only if $i = 2^t - 3$ for some $t \geq 2$;*
(ii) *$g_i(w_2, w_3, w_4) = 0$ if and only if $i = 2^t - 3$ for some $t \geq 2$;*
(iii) *if $k \geq 5$ then, for $i \geq 2$, we never have $g_i(w_2, \dots, w_k) = 0$.*

Proof of Lemma 2.3. PART (i). In view of Fact 2.2, the equality

$$g_{2^t-3}(w_2, w_3) = 0$$

for $t \geq 2$ (already proved, in a different way, in [3]) is a direct consequence of the equality $g_{2^t-3}(w_2, w_3, w_4) = 0$; the latter will be verified in the proof of Part (ii).

Now we prove that $g_i(w_2, w_3) \neq 0$ for $i \neq 2^t - 3$. For $i < 14$, this is readily verified by a direct calculation. Let us suppose that $i \geq 14$. Then, for each i, there exists a uniquely determined integer λ ($\lambda \geq 2$) such that $2^\lambda < i/3 \leq 2^{\lambda+1}$. For proving the claim, it suffices to verify it in each of the following three situations:
(a) $3 \cdot 2^\lambda + 1 \leq i < 5 \cdot 2^\lambda$;
(b) $i = 5 \cdot 2^\lambda$;
(c) $5 \cdot 2^\lambda + 1 \leq i \leq 6 \cdot 2^\lambda$.

CASE (a). By (2.6), we have

$$g_i = w_2^{2^\lambda} g_{i-2\cdot 2^\lambda} + w_3^{2^\lambda} g_{i-3\cdot 2^\lambda}.$$

By our assumption, i is not of the form $2^j - 3$, and one sees that $i - 2\cdot 2^\lambda$ or $i - 3\cdot 2^\lambda$ is not of the form $2^j - 3$. If just one of the numbers $i - 2\cdot 2^\lambda$, $i - 3\cdot 2^\lambda$ is not of the form $2^j - 3$, then it suffices to apply the inductive hypothesis (and the proved fact that $g_{2^t-3} = 0$ for $t \geq 2$). If none of the numbers $i - 2\cdot 2^\lambda$ and $i - 3\cdot 2^\lambda$ have the form $2^j - 3$ then, by the inductive hypothesis, both $g_{i-2\cdot 2^\lambda}$ and $g_{i-3\cdot 2^\lambda}$ are nonzero and, as a consequence, also $g_i \neq 0$. Indeed, now a necessary condition for $g_i = 0$ is that $g_{i-2\cdot 2^\lambda}$ should contain the term $w_3^{2^\lambda}$; but the latter implies that $i - 2\cdot 2^\lambda \geq 3\cdot 2^\lambda$, thus $i \geq 5\cdot 2^\lambda$, which is not fulfilled.

CASE (b). One directly sees, from $(1 + w_2 + w_3)^{-1} = 1 + w_2 + w_3 + (w_2 + w_3)^2 + \cdots$, that

$$g_{5\cdot 2^\lambda} = w_2^{5\cdot 2^{\lambda-1}} + \text{different terms} \neq 0.$$

CASE (c). By a repeated use of (2.6), we now have that

$$\begin{aligned} g_i &= w_2^{2^\lambda}(w_2^{2^\lambda} g_{i-4\cdot 2^\lambda} + w_3^{2^\lambda} g_{i-5\cdot 2^\lambda}) \\ &\quad + w_3^{2^\lambda}(w_2^{2^{\lambda-1}} g_{i-4\cdot 2^\lambda} + w_3^{2^{\lambda-1}} g_{i-9\cdot 2^{\lambda-1}}) \\ &= (w_2^{2^{\lambda+1}} + w_2^{2^{\lambda-1}} w_3^{2^\lambda}) g_{i-4\cdot 2^\lambda} \\ &\quad + w_2^{2^\lambda} w_3^{2^\lambda} g_{i-5\cdot 2^\lambda} + w_3^{3\cdot 2^{\lambda-1}} g_{i-9\cdot 2^{\lambda-1}}. \end{aligned} \tag{2.7}$$

If $i-4\cdot 2^\lambda$ is of the form 2^j-3, then one verifies that $i-5\cdot 2^\lambda$ or $i-9\cdot 2^{\lambda-1}$ is not of the form 2^j-3. If just one of the numbers $i-5\cdot 2^\lambda$, $i-9\cdot 2^{\lambda-1}$ is not of the form 2^j-3, then it suffices to apply the inductive hypothesis (and the proved fact that $g_{2^t-3}=0$ for $t\geq 2$). If none of the numbers $i-5\cdot 2^\lambda$ and $i-9\cdot 2^{\lambda-1}$ have the form 2^j-3 then, by the inductive hypothesis, both $g_{i-5\cdot 2^\lambda}$ and $g_{i-9\cdot 2^{\lambda-1}}$ are nonzero and, as a consequence, also $g_i\neq 0$. Indeed, now a necessary condition for $g_i=0$ is that $g_{i-5\cdot 2^\lambda}$ should contain the term $w_3^{2^{\lambda-1}}$; but the latter implies that $i-5\cdot 2^\lambda\geq 3\cdot 2^{\lambda-1}$, thus $i>6\cdot 2^\lambda$, which is not fulfilled.

Finally, let us suppose that $i-4\cdot 2^\lambda$ is not of the form 2^j-3 (thus, by the inductive hypothesis, $g_{i-4\cdot 2^\lambda}\neq 0$). Then, in order to have $g_i=0$, it would be necessary to "eliminate" $w_2^{2^{\lambda+1}} g_{i-4\cdot 2^\lambda}$. This would only be possible if $g_{i-5\cdot 2^\lambda}$ contains $w_2^{2^\lambda}$, thus if $i-5\cdot 2^\lambda\geq 2\cdot 2^\lambda$, hence $i\geq 7\cdot 2^\lambda$, which is not fulfilled, or if $g_{i-9\cdot 2^{\lambda-1}}$ contains $w_2^{2^{\lambda+1}}$, thus if $i-9\cdot 2^{\lambda-1}\geq 2\cdot 2^{\lambda+1}$, hence $i\geq 17\cdot 2^{\lambda-1}\geq 8\cdot 2^\lambda$, which is not fulfilled.

PART (ii). We first prove that $g_{2^t-3}(w_2,w_3,w_4)=0$ for $t\geq 2$. We directly see that $g_1=0$ and $g_5=0$. For $t\geq 3$ we have, by (2.6) and the inductive hypothesis, that

$$g_{2^t-3} = w_2^{2^{t-3}} g_{3\cdot 2^{t-2}-3} + w_3^{2^{t-3}} g_{5\cdot 2^{t-3}-3}. \tag{2.8}$$

Thus, again by (2.6) and the inductive hypothesis, we obtain

$$\begin{aligned} g_{2^t-3} &= w_2^{2^{t-3}}(w_2^{2^{t-3}} g_{2^{t-1}-3} + w_3^{2^{t-3}} g_{3\cdot 2^{t-3}-3} + w_4^{2^{t-3}} g_{2^{t-2}-3}) \\ &\quad + w_3^{2^{t-3}}(w_2^{2^{t-3}} g_{3\cdot 2^{t-3}-3} + w_3^{2^{t-3}} g_{2^{t-2}-3} + w_4^{2^{t-3}} g_{2^{t-3}-3}) \\ &= 0. \end{aligned} \tag{2.9}$$

PART (iii). First, one readily calculates that $g_5(w_2,w_3,w_4,w_5)=w_5\neq 0$. Then for completing the proof of Part (iii), in view of what we have proved up to now and Fact 2.2, it suffices to verify that $g_{2^t-3}(w_2,w_3,w_4,w_5)\neq 0$ for $t\geq 4$. For this, we show that $h_{2^t-3}(w_4,w_5)$ is nonzero for $t\geq 4$, where $h_{2^t-3}(w_4,w_5)$ (briefly h_{2^t-3}) is obtained by reducing $g_{2^t-3}(w_2,w_3,w_4,w_5)$ modulo w_2 and w_3. Indeed, by (2.6), we see that

$$h_{2^t-3} = w_4^{2^{t-3}} h_{2^{t-1}-3} + w_5^{2^{t-3}} h_{3\cdot 2^{t-3}-3}. \tag{2.10}$$

By the inductive hypothesis, $h_{2^{t-1}-3}\neq 0$; thus a necessary condition for $h_{2^t-3}=0$ is that the term $w_5^{2^{t-3}}$ should be contained in $h_{2^{t-1}-3}$. But this would require that $2^{t-1}-3 \geq 5 \cdot 2^{t-3}$, which is not fulfilled. This finishes the proof of Lemma 2.3. $\square$

The announced preparations are finished, and we can prove Theorem 2.1.

Proof of Theorem 2.1. Recall that, for $G_{n,k}$ ($k \leq n-k$) there are no polynomial relations among $w_1, w_2, \ldots, w_k$ in dimensions $\leq n-k$, and a nonzero polynomial $p_{n-k+1} \in \mathbb{Z}_2[w_1, w_2, \ldots, w_k]$ represents $0 \in H^{n-k+1}(G_{n,k})$ if and only if $p_{n-k+1} = \bar{w}_{n-k+1}$. From the Gysin sequence (2.1) we see that

$$\begin{aligned} &p^*\colon H^{n-k}(G_{n,k}) \to H^{n-k}(\tilde{G}_{n,k}) \quad \text{is surjective} \\ &\text{and, equivalently, } \operatorname{charrank}(\tilde{\gamma}_{n,k}) \geq n-k, \\ &\text{precisely when } g_{n-k+1}(w_2, \ldots, w_k) \neq 0. \end{aligned} \tag{2.11}$$

We still observe that, for $3 \leq k \leq n-k$,

$$\text{if} \quad g_{n-k+1} \neq 0 \quad \text{and} \quad g_{n-k+2} \neq 0, \quad \text{then} \quad \operatorname{charrank}(\tilde{\gamma}_{n,k}) \geq n-k+1. \tag{2.12}$$

Indeed, by the criterion (2.11), we have $\operatorname{charrank}(\tilde{\gamma}_{n,k}) \geq n-k$. To show that this inequality can be improved as claimed in (2.12), let us suppose that a nonzero polynomial $p_{n-k+1} \in \mathbb{Z}_2[w_1, \ldots, w_k]$ represents an element in $\operatorname{Ker}(H^{n-k+1}(G_{n,k}) \xrightarrow{w_1} H^{n-k+2}(G_{n,k}))$. Thus $w_1 p_{n-k+1}$ represents $0 \in H^{n-k+2}(G_{n,k})$. This means that, in $\mathbb{Z}_2[w_1, \ldots, w_k]$, $w_1 p_{n-k+1} = a w_1 \bar{w}_{n-k+1} + b \bar{w}_{n-k+2}$, where $a = 1$ or $b = 1$. Of course, since $g_{n-k+2} \neq 0$, necessarily $b = 0$, $a = 1$. But the polynomial equality $w_1 p_{n-k+1} = w_1 \bar{w}_{n-k+1}$ implies that $p_{n-k+1} = \bar{w}_{n-k+1}$, thus p_{n-k+1} represents $0 \in H^{n-k+1}(G_{n,k})$. So we see that $\operatorname{Ker}(H^{n-k+1}(G_{n,k}) \xrightarrow{w_1} H^{n-k+2}(G_{n,k})) = 0$ and $\operatorname{charrank}(\tilde{\gamma}_{n,k}) \geq n-k+1$.

Proof of Parts (1) and (2). By Lemma 2.3(i), (ii), $g_{n-k+1}(w_2, \ldots, w_k)$ vanishes if $(n, k) \in \{(2^t - 1, 3), (2^t, 4)\}$. By the criterion (2.11), for these pairs (n, k), the homomorphism $p^*\colon H^{n-k}(G_{n,k}) \to H^{n-k}(\tilde{G}_{n,k})$ is not surjective; thus, there is a non-Stiefel–Whitney generator in $H^{n-k}(\tilde{G}_{n,k})$ if $(n,k) \in \{(2^t-1,3),(2^t,4)\}$, and we conclude that $\operatorname{charrank}(\tilde{\gamma}_{2^t-1,3}) = 2^t - 5 = \operatorname{charrank}(\tilde{\gamma}_{2^t,4})$.

Of course, again by Lemma 2.3 (i), (ii), we have $g_{n-k+1}(w_2,\ldots,w_k) \neq 0$ if $(n,k) \notin \{(2^t - 1, 3), (2^t, 4)\}$ and $k \in \{3, 4\}$. By the criterion (2.11), for these pairs (n, k), the homomorphism $p^*\colon H^{n-k}(G_{n,k}) \to H^{n-k}(\tilde{G}_{n,k})$ is surjective; so we have that $\operatorname{charrank}(\tilde{\gamma}_{n,3}) \geq n-3$ if $n \neq 2^t - 1$ and $\operatorname{charrank}(\tilde{\gamma}_{n,4}) \geq n-4$ if $n \neq 2^t$.

To prove the result for $\tilde{G}_{2^t-2,3}$, we first recall (Lemma 2.3 (i)) that $g_{2^t-4} \neq 0$, $g_{2^t-3} = 0$, and $g_{2^t-2} \neq 0$. Thus $\bar{w}_{2^t-3} = w_1 p_{2^t-4}$ for some polynomial p_{2^t-4}. The latter cannot represent 0 in the cohomology group $H^{2^t-4}(G_{2^t-2,3})$; indeed, if p_{2^t-4} represents zero, then necessarily $p_{2^t-4} = \bar{w}_{2^t-4}$ (as polynomials), thus we have a relation $\bar{w}_{2^t-3} = w_1 \bar{w}_{2^t-4}$, which is impossible. This implies (see (2.1)) that $p^*\colon H^{2^t-4}(G_{2^t-2,3}) \to$

$H^{2^t-4}(\tilde{G}_{2^t-2,3})$ is not an epimorphism, thus $\operatorname{charrank}(\tilde{\gamma}_{2^t-2,3}) \le 2^t - 5$. By (2.11), since $g_{2^t-4} \ne 0$, we have $\operatorname{charrank}(\tilde{\gamma}_{2^t-2,3}) \ge 2^t - 5$, which proves the claim for $\tilde{G}_{2^t-2,3}$. The result for $\tilde{G}_{2^t-1,4}$ can be derived in an analogous way.

Now we prove the claim for $\tilde{G}_{2^t-3,3}$. We have $g_{2^t-5} \ne 0$, $g_{2^t-4} \ne 0$, and $g_{2^t-3} = 0$. Thus $\bar{w}_{2^t-3} = w_1 p_{2^t-4}$ for some polynomial p_{2^t-4}. The latter cannot represent 0 in $H^{2^t-4}(G_{2^t-3,3})$. Indeed, if p_{2^t-4} represents zero, then $p_{2^t-4} = a w_1 \bar{w}_{2^t-5} + b\bar{w}_{2^t-4}$ in $\mathbb{Z}_2[w_1, w_2, w_3]$, with $a = 1$ or $b = 1$; as a consequence, we would have $\bar{w}_{2^t-3} = a w_1^2 \bar{w}_{2^t-5} + b w_1 \bar{w}_{2^t-4}$, which is impossible. From the Gysin sequence (2.1), we see that $p^*\colon H^{2^t-4}(G_{2^t-3,3}) \to H^{2^t-4}(\tilde{G}_{2^t-3,3})$ is not an epimorphism. Thus $\operatorname{charrank}(\tilde{\gamma}_{2^t-3,3}) \le 2^t - 5$. At the same time, by the observation (2.12), we have $\operatorname{charrank}(\tilde{\gamma}_{2^t-3,3}) \ge 2^t - 5$. This proves the claim for $\tilde{G}_{2^t-3,3}$; again, the result for $\tilde{G}_{2^t-2,4}$ can be proved analogously.

We pass to proving the result for $\tilde{G}_{2^t,3}$. We know that none of g_{2^t-2}, g_{2^t-1}, g_{2^t} vanishes. By (2.12), we see that $\operatorname{charrank}(\tilde{\gamma}_{2^t,3}) \ge 2^t - 2$. At the same time, since $w_2 g_{2^t-2} + g_{2^t} = w_3 g_{2^t-3} = 0$, we have (as for $\mathbb{Z}_2$-polynomials) $w_2 \bar{w}_{2^t-2} + \bar{w}_{2^t} = w_1 p_{2^t-1}$, for some polynomial p_{2^t-1}. The latter cannot represent $0 \in H^{2^t-1}(G_{2^t,3})$. Indeed, p_{2^t-1} representing 0 would mean that $p_{2^t-1} = a w_1 \bar{w}_{2^t-2} + b\bar{w}_{2^t-1}$ (where $a = 1$ or $b = 1$), which implies an impossible relation $\bar{w}_{2^t} = (a w_1^2 + w_2)\bar{w}_{2^t-2} + b w_1 \bar{w}_{2^t-1}$. Thus p_{2^t-1} represents a nonzero element in

$$\operatorname{Ker}(H^{2^t-1}(G_{2^t,3}) \xrightarrow{w_1} H^{2^t}(G_{2^t,3})),$$

and we have that $\operatorname{charrank}(\tilde{\gamma}_{2^t,3}) \le 2^t - 2$, which proves the claim for $\tilde{G}_{2^t,3}$.

Now we shall pass to $\tilde{G}_{2^t-3,4}$. Then we have $g_{2^t-6} \ne 0$, $g_{2^t-5} \ne 0$, $g_{2^t-4} \ne 0$, $g_{2^t-3} = 0$. By (2.12), we know that $\operatorname{charrank}(\tilde{\gamma}_{2^t-3,4}) \ge 2^t - 6$. To improve this inequality, we now show that

$$\operatorname{Ker}(H^{2^t-5}(G_{2^t-3,4}) \xrightarrow{w_1} H^{2^t-4}(G_{2^t-3,4})) = 0. \tag{2.13}$$

Let a nonzero polynomial p_{2^t-5} represent an element in the kernel under question. This means that the polynomial $w_1 p_{2^t-5}$ represents $0 \in H^{2^t-4}(G_{2^t-3,4})$. Consequently, $w_1 p_{2^t-5} = a w_1^2 \bar{w}_{2^t-6} + b w_2 \bar{w}_{2^t-6} + c w_1 \bar{w}_{2^t-5} + d\bar{w}_{2^t-4}$ in $\mathbb{Z}_2[w_1, w_2, w_3, w_4]$, where at least one of the coefficients a, b, c, d is equal to 1. We cannot have $b = d = 1$, because $w_2 \bar{w}_{2^t-6} + \bar{w}_{2^t-4}$ reduced mod w_1 is $w_2 g_{2^t-6} + g_{2^t-4}$ and, as we shall see in the next step, the latter is not zero. Indeed, let z_i denote the reduction of g_i modulo w_2 and w_3. Then $w_2 g_{2^t-6} + g_{2^t-4}$ reduced modulo w_2 and w_3 is equal to z_{2^t-4}. A direct calculation gives that $z_{12} = w_4^3$ and, by induction, we obtain that $z_{2^t-4} = w_4^{2^{t-3}} z_{2^{t-1}-4} = w_4^{2^{t-3}} w_4^{2^{t-3}-1} = w_4^{2^{t-2}-1} \ne 0$. So we have shown that $w_2 g_{2^t-6} + g_{2^t-4} \ne 0$. One also readily sees that it is impossible to have $(b, d) = (1, 0)$ as well as $(b, d) = (0, 1)$. Thus the only remaining possibility is $(b,d) = (0,0)$. So we obtain $w_1 p_{2^t-5} = w_1(a w_1 \bar{w}_{2^t-6} + c\bar{w}_{2^t-5})$, thus $p_{2^t-5} = a w_1 \bar{w}_{2^t-6} + c\bar{w}_{2^t-5}$. This means that p_{2^t-5} represents $0 \in H^{2^t-5}(G_{2^t-3,4})$, and we have proved the equality (2.13).

As a consequence, we have $\operatorname{charrank}(\tilde{\gamma}_{2^t-3,4}) \geq 2^t - 5$. Since $g_{2^t-3} = 0$, we have that $\bar{w}_{2^t-3} = w_1 p_{2^t-4}$ for some polynomial p_{2^t-4}, about which one can show (similarly to situations of this type dealt with above) that it cannot represent zero in cohomology. Thus we also have $\operatorname{charrank}(\widetilde{\gamma}_{2^t-3,4}) \leq 2^t - 5$, and finally $\operatorname{charrank}(\tilde{\gamma}_{2^t-3,4}) = 2^t - 5$.

In view of Lemma 2.3 (i), (ii), for all the manifolds $\tilde{G}_{n,3}$ and $\tilde{G}_{n,4}$ that remain, the observation (2.12) implies the lower bounds stated in Theorem 2.1 (1), (2).

Proof of Part (3). For $k \geq 5$, Lemma 2.3 (iii) says that $g_{n-k+1} \neq 0$ and $g_{n-k+2} \neq 0$; thus the observation (2.12) applies, giving that $\operatorname{charrank}(\tilde{\gamma}_{n,k}) \geq n - k + 1$ in all these cases.

To prove the final statement of the theorem, it suffices to recall that, if n is odd, then (see [3, p. 72]) we have $w_i(\tilde{G}_{n,k}) = \tilde{w}_i + Q_i(\tilde{w}_2, \dots, \tilde{w}_{i-1})$ $(i \leq k)$, where Q_i is a $\mathbb{Z}_2$-polynomial, and $\tilde{w}_j = w_j(\tilde{G}_{n,k}) + P_j(w_2(\tilde{G}_{n,k}), \dots, w_{j-1}(\tilde{G}_{n,k}))$ $(j \geq 2)$ for some $\mathbb{Z}_2$-polynomial P_j.

The proof of Theorem 2.1 is finished. □

3. On the cup-length of the Grassmann manifold $\tilde{G}_{n,k}$

Recall that the $\mathbb{Z}_2$-cup-length, $\operatorname{cup}(X)$, of a compact path connected topological space X is defined to be the maximum of all numbers c such that there exist, in positive degrees, cohomology classes $a_1, \dots, a_c \in H^*(X)$ such that their cup product $a_1 \cdots a_c$ is nonzero. In [3] and, independently, in [2], it was proved that for $t \geq 3$ we have

$$\operatorname{cup}(\tilde{G}_{2^t-1,3}) = 2^t - 3;$$

in addition, [3, Theorem 1.3] gave certain upper bounds for $\operatorname{cup}(\tilde{G}_{n,k})$.

Now Theorem 2.1 implies the following exact result for $\tilde{G}_{2^t,3}$, confirming the corresponding claim in Fukaya's conjecture [2, Conjecture 1.2], or improvements on the results of [3, Theorem 1.3] in the other cases.

Theorem 3.1. *For the oriented Grassmann manifold* $\tilde{G}_{n,k}$ $(3 \leq k \leq n-k)$, *with* $2^{t-1} < n \leq 2^t$, *we have*

(1) $$\operatorname{cup}(\tilde{G}_{n,3}) \begin{cases} = n-3 & \text{if} \quad n = 2^t, \\ \leq (2n-3-i)/2 & \text{if} \quad n = 2^t - i,\ i \in \{2,3\}, \\ \leq n-3 & \text{otherwise, for } n \neq 2^t - 1; \end{cases}$$

(2) $$\operatorname{cup}(\tilde{G}_{n,4}) \begin{cases} \leq (3n-10-i)/2 & \text{if} \quad n = 2^t - i,\ i \in \{0,1,2,3\}, \\ \leq (3n-12)/2 & \text{otherwise}; \end{cases}$$

(3) *if* $k \geq 5$, *then* $\operatorname{cup}(\tilde{G}_{n,k}) \leq (k-1)(n-k)/2$.

Proof. For a connected finite CW-complex X, let r_X denote the smallest positive integer such that $\tilde{H}^{r_X}(X) \neq 0$. In the case that such an integer does not exist, that is, all the reduced cohomology groups $\tilde{H}^i(X)$ $(1 \leq i \leq \dim(X))$ vanish, we set $r_X =$

$\dim(X)+1$; thus always $r_X \geq 1$. To obtain the upper bounds stated in the theorem, we use the following generalization of [3, Theorem 1.1].

Theorem 3.2 (A. Naolekar–A. Thakur [6]). *Let X be a connected closed smooth d-dimensional manifold. Let ξ be a vector bundle over X satisfying the following: there exists j, $j \leq \mathrm{charrank}_X(\xi)$, such that every monomial $w_{i_1}(\xi)\cdots w_{i_r}(\xi)$, $0 \leq i_t \leq j$, in dimension d vanishes. Then*

$$\mathrm{cup}(X) \leq 1 + \frac{d-j-1}{r_X}.$$

For the manifold $\tilde{G}_{n,k}$, every top-dimensional monomial in the Stiefel–Whitney classes of the canonical bundle $\tilde{\gamma}_{n,k}$ vanishes (indeed, if a top-dimensional monomial in the Stiefel–Whitney classes of $\tilde{\gamma}_{n,k}$ does not vanish, then it is a p^*-image of the corresponding non-vanishing top-dimensional monomial in the Stiefel–Whitney classes of $\gamma_{n,k}$; due to Poincaré duality, the latter monomial can be replaced with a monomial divisible by $w_1(\gamma_{n,k})$; but p^* maps this monomial to zero). Now the upper bounds stated in Theorem 3.1 are obtained by taking $X = \tilde{G}_{n,k}$ $(3 \leq k \leq n-k)$, $\xi = \tilde{\gamma}_{n,k}$, and j equal to the right-hand side of the corresponding (in)equality given in Theorem 2.1.

For $\tilde{G}_{2^t,3}$, it was proved in [3, p. 77] that $w_2(\widetilde{\gamma})^{2^t-4}$ does not vanish. This implies that $\mathrm{cup}(\tilde{G}_{2^t,3}) \geq 2^t-3$; this lower bound coincides with the upper bound proved above. The proof is finished. $\square$

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