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PSEUDOHERMITIAN BIMINIMAL LEGENDRE SURFACES IN THE 5-DIMENSIONAL SPHERE

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Abstract

In this paper, we determine nonminimal pseudohermitian biminimal Legendre surfaces in the unit 5-sphere S^5. In fact, the product of a circle and a helix of order 4 is realized as a nonminimal pseudohermitian biminimal Legendre immersion into S^5. In addition, we obtain that there exist no nonminimal pseudohermitian biminimal Legendre surfaces in a 5-dimensional Sasakian space form of non-positive constant holomorphic sectional curvature for the Tanaka–Webster connection.

1. Introduction

A Legendre submanifold in contact manifolds is one of important subjects in contact geometry and the geometry of submanifolds. In a Sasakian manifold there exist no Legendre submanifolds with parallel mean curvature vector other than the minimal ones (cf. [17]). By using variational view point, the biminimality was introduced by Loubeau and Montaldo [10] as an extension of minimality. Inoguchi [7] showed that a 3-dimensional Sasakian space form admits a proper biminimal Legendre curve if and only if its holomorphic sectional curvature is greater than 1. In a continuing paper [8] he classified nongeodesic biminimal Legendre curves in a 3-dimensional Sasakian space form. Recently, Sasahara [12] gave a classification of nonminimal biminimal Legendre surfaces in a 5-dimensional Sasakian space form and showed that there exist no nonminimal biminimal Legendre surfaces in a 5-dimensional Sasakian space form of constant holomorphic sectional curvature ≤ -3.

On the other hand, for a given contact form we have two compatible structures: one is a Riemannian structure (or metric) and the other is a pseudohermitian structure (or almost CR-structure). In pseudohermitian geometry (CR-geometry) we use the *Tanaka–Webster connection* as a canonical connection instead of the Levi-Civita connection. In our previous works [5], [6], we defined pseudohermitian harmonicity, minimality, biharmonicity, and biminimality, respectively by using the Tanaka–Webster connection. Particularly in [5], we have classified pseudohermitian biharmonic curves (of constant contact angle) in a 3-dimensional Sasakian space form with respect to the Tanaka–Webster connection. While for the Levi-Civita connection, the unit sphere S^3 does not contain proper biharmonic

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Legendre curves, S^3 does contain proper pseudohermitian biharmonic Legendre curves with respect to the Tanaka–Webster connection, which are called pseudohermitian circles. In these situations, it is natural and intriguing to study pseudohermitian biharmonic or biminimal Legendre surfaces in a 5-dimensional Sasakian space form.

The main purpose of this paper is to prove their classification theorems (Theorem 4.1 and Corollary 5.1). In particular we show that the product of a circle and a helix of order 4 is realized as a nonminimal pseudohermitian biminimal Legendre immersion into S^5. Such a Legendre surface has another remarkable geometric property, namely, it is *mass-symmetric* and of 2-type. In fact, in [1] it was proved that a mass-symmetric 2-type Legendre surface of S^5 is the product of a plane circle and a helix of order 4 or the product of two circles. The latter one is characterized as a nonminimal Legendre surface in S^5 with respect to the Levi-Civita connection by Sasahara [12]. Moreover, we obtain that there exist no nonminimal pseudohermitian biminimal Legendre surfaces in a 5-dimensional Sasakian space form of non-positive constant holomorphic sectional curvature for the Tanaka–Webster connection (Corollary 4.2).

2. Preliminaries

A $(2n+1)$-dimensional manifold M^{2n+1} is said to be a *contact manifold* if it admits a global 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, the *characteristic vector field* satisfying $\eta(\xi)=1$ and $d\eta(\xi, X)=0$ for any vector field X. It is well-known that there exists an *associated Riemannian metric* g and a $(1,1)$-type tensor field φ such that

$$\eta(X)=g(X,\xi), \quad d\eta(X,Y)=g(X,\varphi Y), \quad \varphi^2 X=-X+\eta(X)\xi, \tag{2.1}$$

where X and Y are vector fields on M. From (2.1), it follows that

$$\varphi\xi=0, \quad \eta\circ\varphi=0, \quad g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\eta(Y).$$

A Riemannian manifold M equipped with the structure tensors (η,ξ,φ,g) satisfying (2.1) is said to be a *contact Riemannian manifold*. We denote it by $M=(M;\eta,\xi,\varphi,g)$. Given a contact Riemannian manifold M, we define an operator h by $h=(1/2)\mathcal{L}_\xi\varphi$, where $\mathcal{L}_\xi$ denotes Lie differentiation in the characteristic direction ξ. Then we may observe that the *structural operator* h is self-adjoint and satisfies

$$\begin{aligned} &h\xi=0, \quad h\varphi=-\varphi h,\\ &\nabla_X\xi=-\varphi X-\varphi hX, \end{aligned} \tag{2.2}$$

where ∇ is the Levi-Civita connection. A contact Riemannian manifold for which ξ is a Killing vector field is called a *K-contact manifold*. It is at once shown that a contact Riemannian manifold is K-contact if and only if $h=0$.

For a contact Riemannian manifold M one may define naturally an almost complex structure J on $M \times \mathbb{R}$:

$$J\left(X,\ f\frac{d}{dt}\right) = \left(\varphi X - f\xi,\ \eta(X)\frac{d}{dt}\right)$$

where X is a vector field tangent to M, t the coordinate on $\mathbb{R}$ and f a function on $M \times \mathbb{R}$. If the almost complex structure J is integrable, M is said to be *normal* or *Sasakian*. It is known that a contact Riemannian manifold M is normal if and only if M satisfies

$$[\varphi, \varphi] + 2\, d\eta \otimes \xi = 0,$$

where $[\varphi,\varphi]$ is the Nijenhuis torsion of φ. We note that 3-dimensional K-contact manifolds are *Sasakian* (cf. p. 76 in [2]).

We denote by R the Riemannian curvature tensor define by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

where X, Y, Z are vector fields on M. A Sasakian manifold is also characterized by the condition

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for all vector fields X and Y on the manifold and this is equivalent to

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

for all vector fields X and Y.

Let $(M; \eta, \xi, \varphi, g)$ be a Sasakian manifold. Then M is called a space of constant holomorphic sectional curvature ϵ if M satisfies

$$g(R(X, \varphi X)\varphi X, X) = \epsilon$$

for any unit vector field $X \perp \xi$. A complete and simply connected Sasakian space of constant holomorphic sectional curvature is called a Sasakian space form. We denote by $M^{2n+1}(\epsilon)$ a Sasakian space form of constant holomorphic sectional curvature ϵ. Tanno ([14]) classified Sasakian space forms. The curvature tensor R of $M^{2n+1}(\epsilon)$ is given by (see [2])

$$\begin{aligned} R(X, Y)Z = {} & \frac{\epsilon+3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & + \frac{\epsilon-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi \\ & + g(Z, \varphi Y)\varphi X - g(Z, \varphi X)\varphi Y + 2g(X, \varphi Y)\varphi Z\}. \end{aligned} \tag{2.3}$$

For more information on contact geometry, we refer to [2].

Now let (N, h) and (M, g) be Riemannian manifolds and $f: N \to M$ a smooth map. Then the section $\tau(f) := \operatorname{tr} \nabla^f df$ of the pull-back bundle f^*TM is called the *tension field* of f. Here ∇^f is the connection on f^*TM induced form the Levi-Civita connection ∇ of M and $\nabla^f df$ is the *second fundamental form* of f. A map f is said to be *harmonic* if its tension field vanishes identically.

DEFINITION 2.1. A smooth map $f: N \to M$ is said to be *biharmonic* if it is a critical point of the *bienergy functional*:

$$E_2(f) = \int_N \frac{1}{2}|\tau(f)|^2 \, dv_h.$$

The Euler–Lagrange equation of the bienergy is:

$$\tau_2(f) := -\mathcal{J}_f(\tau(f)) = 0 \tag{2.4}$$

(cf. Jiang [9]). The section $\tau_2(f)$ is called the *bitension field* of f. The Jacobi equation for the vector field V along f is given by

$$\mathcal{J}_f(V) := \triangle_f V + \operatorname{tr} R(df, V)\, df = 0,$$

where the operator $\triangle_f$ is the *rough Laplacian* acting on the space $\Gamma(f^*TM)$ of all smooth sections of f^*TM is defined by

$$\triangle_f := -\sum_{i=1}^{n} \left(\nabla^f_{e_i} \nabla^f_{e_i} - \nabla^f_{\nabla^N_{e_i} e_i}\right),$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N. Obviously, every harmonic map is biharmonic.

In case $f: (N, h) \to (M, g)$ is an isometric immersion, the *biharmonic equation* of f is given by

$$\mathcal{J}_f(\mathbb{H}) = \triangle_f \mathbb{H} + \operatorname{tr} R(df, \mathbb{H})\, df = 0,$$

where $\mathbb{H} = \tau(f)/\dim N$ is the *mean curvature vector field*. Loubeau and Montaldo introduced the notion of the biminimal immersions.

DEFINITION 2.2 ([10]). An isometric immersion $f: (N, h) \to (M, g)$ is said to be *biminimal* if it is a critical point of the bienergy functional $E_2(f)$ with respect to all normal variation with compact support. Here, a normal variation means a variation f_t off $f = f_0$ such that the variational vector field $V = df_t/dt|_{t=0}$ is normal to N.

We observe that f is biminimal if and only it it satisfies a *biminimal equation*:

$$\mathcal{J}_f(\mathbb{H})^\perp = \{\triangle_f \mathbb{H} + \operatorname{tr} R(df, \mathbb{H})\, df\}^\perp = 0. \tag{2.5}$$

Every biharmonic submanifold is biminimal. However, there are many nonbiharmonic biminimal submanifolds (cf. [10]).

Generalizing submanifolds with harmonic mean curvature ($\triangle \mathbb{H} = 0$) or normal harmonic mean curvature ($\triangle^\perp \mathbb{H} = 0$), submanifolds with property $\triangle \mathbb{H} = \lambda \mathbb{H}$ or $\triangle^\perp \mathbb{H} = \lambda \mathbb{H}$ have been extensively studied by many authors. We may find references in [7]. (Here, $\triangle^\perp$ is the Laplace–Beltrami operator of the normal bundle, which is called *normal Laplacian*.) More generally, the notion of λ-biminimal immersion was introduced by Loubeau and Montaldo:

DEFINITION 2.3. An isometric immersion $f : N \to M$ is called a *λ-biminimal* immersion if it is a critical point of the function:

$$E_{2,\lambda}(f) = E_2(f) + \lambda E(f), \quad \lambda \in \mathbb{R}.$$

The Euler–Lagrange equation for λ-biminimal immersions is

$$\tau_2(f)^\perp = \lambda \tau(f),$$

or equivalently,

$$\mathcal{J}_f(\mathbb{H})^\perp = -\lambda \mathbb{H}.$$

3. CR structures

For a contact Riemannian manifold $M = (M^{2n+1}; \eta, \xi, \varphi, g)$, the tangent space T_pM of M at a point $p \in M$ can be decomposed as the direct sum $T_pM = D_p \oplus \{\xi\}_p$, with $D_p = \{v \in T_pM \mid \eta(v) = 0\}$. Then $D : p \to D_p$ defines a $2n$-dimensional distribution orthogonal to ξ, called the *contact distribution*. We see that the restriction $J = \varphi|_D$ of φ to D defines an almost complex structure on D. Then the associated *almost CR-structure* of the contact Riemannian manifold M is given by the holomorphic subbundle

$$\mathcal{H} = \{X - iJX \mid X \in D\}$$

of the complexification $TM^{\mathbb{C}}$ of the tangent bundle TM. Then we see that each fiber $\mathcal{H}_p$ is of complex dimension n, $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$, and $\mathbb{C}D = \mathcal{H} \oplus \bar{\mathcal{H}}$. We say that the associated almost CR-structure is *integrable* if $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$. In such a case, $\mathcal{H}$ is called a CR-structure associated to the contact Riemannian structure (η, ξ, φ, g).

For an associated almost CR-structure $\mathcal{H}$ of a contact Riemannian manifold M, we define the Levi form L by

$$L : D \times D \to \mathcal{F}(M), \quad L(X, Y) = -d\eta(X, JY),$$

where $\mathcal{F}(M)$ denotes the algebra of differential functions on M. Then we see that the Levi form is Hermitian and positive definite. We call the pair (η, L) a *strongly pseudoconvex pseudohermitian structure* on M. Now, we review the *Tanaka–Webster connection* [13], [16] on a strongly pseudoconvex pseudohermitian manifold $M = (M; \eta, L)$ with the associated contact Riemannian structure (η, ξ, φ, g). The Tanaka–Webster connection $\hat{\nabla}$ is defined by

$$\hat{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi,$$

for all vector fields X, Y on M. Together with (2.2), $\hat{\nabla}$ may be rewritten as

$$\hat{\nabla}_X Y = \nabla_X Y + A(X, Y), \tag{3.1}$$

where we have put

$$A(X, Y) = \eta(X)\varphi Y + \eta(Y)(\varphi X + \varphi h X) - g(\varphi X + \varphi h X, Y)\xi. \tag{3.2}$$

We see that the Tanaka–Webster connection $\hat{\nabla}$ has the torsion

$$\hat{T}(X, Y) = 2g(X, \varphi Y)\xi + \eta(Y)\varphi h X - \eta(X)\varphi h Y. \tag{3.3}$$

In particular, for a K-contact manifold (3.2) and the above equation reduce as follows:

$$\begin{aligned} A(X, Y) &= \eta(X)\varphi Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi, \\ \hat{T}(X, Y) &= 2g(X, \varphi Y)\xi. \end{aligned} \tag{3.4}$$

Furthermore, it was proved in [15] that

Proposition 3.1. *The Tanaka–Webster connection $\hat{\nabla}$ on a contact Riemannian manifold $M = (M^{2n+1}; \eta, \xi, \varphi, g)$ with the associated (integrable) CR-structure is the unique linear connection satisfying the following conditions*:
(i) $\hat{\nabla}\eta = 0$, $\hat{\nabla}\xi = 0$;
(ii) $\hat{\nabla}g = 0$, $\hat{\nabla}\varphi = 0$;
(iii-1) $\hat{T}(X, Y) = -\eta([X, Y])\xi$, X, $Y \in D$;
(iii-2) $\hat{T}(\xi, \varphi Y) = -\varphi\hat{T}(\xi, Y)$, $Y \in D$.

We define the *pseudohermitian curvature tensor* (or *Tanaka–Webster curvature tensor*) $\hat{R}$ on a contact Riemannian manifold equipped with the associated CR-structure and Tanaka–Webster connection $\hat{\nabla}$ by

$$\hat{R}(X, Y)Z = \hat{\nabla}_X(\hat{\nabla}_Y Z) - \hat{\nabla}_Y(\hat{\nabla}_X Z) - \hat{\nabla}_{[X,Y]}Z \tag{3.5}$$

for all vector fields X, Y, Z in M. Then from the definition of $\hat{R}$, we have

$$
\begin{aligned}
\hat{R}(X,Y)Z = R(X,Y)Z& \\
&+ \eta(Y)((\nabla_X\varphi)Z - g(X + hX, Z)\xi) \\
&- \eta(X)((\nabla_Y\varphi)Z - g(Y + hY, Z)\xi) \\
&+ \eta(Z)((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X \\
&+ \eta(Y)(X + hX) - \eta(X)(Y + hY)) - 2g(\varphi X, Y)\varphi Z \\
&- g(\varphi X + \varphi hX, Z)(\varphi Y + \varphi hY) + g(\varphi Y + \varphi hY, Z)(\varphi X + \varphi hX) \\
&- g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X + (\nabla_X\varphi h)Y - (\nabla_Y\varphi h)X, Z)\xi
\end{aligned}
\tag{3.6}
$$

for all vector fields X, Y, Z in M. In [4] the first author studied the relation between pseudohermitian geometry and Riemannian geometry. Indeed, for Sasakian space forms $M^{2n+1}(\epsilon)$ the holomorphic sectional curvature for $\hat{\nabla}$ is $\hat{\epsilon} = \epsilon + 3$.

4. Pseudohermitian biminimal submanifolds

Let M^{2n+1} be a contact Riemannian manifold and $f\colon N^m \to M^{2n+1}$ be an isometric immersion of a Riemannian manifold (N, h). Then we have the basic formulas for $\hat{\nabla}$:

$$
\hat{\nabla}^f_X Y = f_*\hat{\nabla}^h_X Y + \hat{\sigma}(X,Y) \quad \text{and} \quad \hat{\nabla}^f_X V = -f_*\hat{S}_V X + \hat{D}_X V, \tag{4.1}
$$

where $X, Y \in TN^m$, $V \in T^\perp N^m$, $\hat{\sigma}$, $\hat{S}$ and $\hat{D}$ are the *second fundamental form*, the *shape operator* and the *normal connection* with respect to $\hat{\nabla}$. The connection $\hat{\nabla}^h$ is the connection on N induced from $\hat{\nabla}$. The first formula is called the *Gauss formula* and the second formula is called the *Weingarten formula* with respect to Tanaka–Webster connection. Then we can find the relation:

$$
g(\hat{\sigma}(X,Y), V) = h(\hat{S}_V X, Y).
$$

If η restricted to N^m vanishes, then N^m is called an *integral submanifold*, in particular if $m = n$, it is called a *Legendre submanifold*.

Let N^n be a Legendre submanifold of a Sasakian manifold M^{2n+1} and let e_i $(i = 1, \dots, n)$ be an orthonormal frame along N^n such that $\{e_i\}$ are tangent to N^n, $\varphi e_1 = e_{n+1}, \dots, \varphi e_n = e_{2n}$, $\xi = e_{2n+1}$. It follows from (3.4), we can see that

$$
A(X,Y) = 0, \tag{4.2}
$$

for $X, Y \in TN$, and then we find that $\hat{\sigma} = \sigma$. This implies that $\hat{\nabla}^h$ coincides with the Levi-Civita connection ∇^h of (N, h). Moreover, we have

$$
f_* S_{\varphi Y} X = -\varphi\sigma(X,Y) = f_* S_{\varphi X} Y, \quad S_\xi = 0. \tag{4.3}
$$

Differentiating $g(\sigma(X, Y), Z) = 0$, we get

$$\begin{aligned}0 &= g(\hat{\nabla}_W \sigma(X, Y), Z) + g(\sigma(X, Y), \hat{\nabla}_W Z)\\ &= g((\hat{\nabla}_W \sigma)(X, Y), Z) + g(\sigma(X, Y), \sigma(W, Z))\\ &= g((\hat{\nabla}_W \sigma)(X, Y), Z) + h(S_{\sigma(X,Y)} W, Z),\end{aligned}$$

for $W, X, Y, Z \in TN$.

Proposition 4.1. *Let N be an n-dimensional Legendre submanifold of a $(2n+1)$-dimensional Sasakian manifold M. If the second fundamental form of N is is parallel with respect to Tanaka–Webster connection, then N is totally geodesic in M.*

Now we suppose that the ambient space $M = M^{2n+1}(\hat{\epsilon})$ be a Sasakian space form. Since φ is parallel for Tanaka–Webster connection $\hat{\nabla}$, we get

$$\hat{D}_X \varphi Y = \varphi f_* \hat{\nabla}^h_X Y, \quad f_* S_{\varphi Y} X = -\varphi \sigma(X, Y).$$

Then by using a straightforward computation the equations of Gauss and Codazzi of Legendre submanifolds for Tanaka–Webster connection are given respectively by:

$$h(R^h(X, Y)Z, W) = g(\hat{R}(f_* X, f_* Y) f_* Z, f_* W) + h([S_{\varphi Z}, S_{\varphi W}]X, Y), \tag{4.4}$$

$$(\hat{\nabla}_X \sigma)(Y, Z) = (\hat{\nabla}_Y \sigma)(X, Z). \tag{4.5}$$

We prepare some more notions which will be needed. (cf. [6]).

DEFINITION 4.1. Let (N, h) be a Riemannian manifold and $f\colon N \to (M, \eta, g, \hat{\nabla})$ a smooth map into a strongly pseudoconvex pseudohermitian manifold equipped with Tanaka–Webster connection. Then f is said to be *pseudohermitian harmonic* if it is harmonic with respect to the metric h and the Tanaka–Webster connection $\hat{\nabla}$ of M. The tension field $\hat{\tau}(f) = \mathrm{tr}_h(\hat{\nabla}\, \mathrm{d} f)$ is called the *pseudohermitian tension field.*

DEFINITION 4.2 ([6]). Let (N, h) be a Riemannian m-manifold and $f\colon N \to (M, \eta, g, \hat{\nabla})$ an isometric immersion into a strongly pseudoconvex pseudohermitian manifold equipped with Tanaka–Webster connection. Then (N, f) is said to be *pseudohermitian minimal* if its *pseudohermitian mean curvature vector field* $\hat{\mathbb{H}}$ vanishes. Here the pseudohermitian mean curvature vector field is defined by

$$\hat{\mathbb{H}} = \frac{1}{m} \hat{\tau}(f),$$

where $\hat{\tau}(f)$ is the pseudohermitian tension field.

Now let $f\colon N \to M^{2n+1}$ be a Legendre submanifold in a Sasakian manifold. Then as we have seen before, $\hat{\nabla}^h = \nabla^h$ and $\hat{\sigma} = \sigma$, so the pseudohermitain mean curvature vector vector field $\hat{\mathbb{H}}$ is nothing but the mean curvature vector field $\mathbb{H}$. Thus minimality and pseudohermitian minimality is equivalent for Legendre submanifolds in Sasakian manifolds. From Proposition 4.1 we get at once

Corollary 4.1. *In a Sasakian manifold, there exist no Legendre submanifolds with pseudohermitian parallel mean curvature vector, which means* $\hat{\nabla}\mathbb{H} = 0$, *other than minimal ones.*

We consider some extensions of pseudohermitian minimal Legendre submanifolds.

DEFINITION 4.3. A smooth map $f\colon (N, h) \to (M, \eta, g, \hat{\nabla})$ is said to be *pseudohermitian biharmonic* if it satisfies the Jacobi equation for the $\hat{\nabla}$-tension field $\hat{\tau}(f)$ of f:

$$\hat{\mathcal{J}}_f(\hat{\tau}(f)) = \hat{\Delta}_f \hat{\tau}(f) + \operatorname{tr}_h \hat{T}(\mathrm{d}f, \hat{\nabla}^f \hat{\tau}(f)) + \operatorname{tr}_h \hat{R}(\mathrm{d}f, \hat{\tau}(f))\,\mathrm{d}f = 0. \tag{4.6}$$

f is pseudohermitian biminimal immersion if and only if

$$\{\hat{\Delta}_f \hat{\mathbb{H}} + \operatorname{tr}_h \hat{T}(\mathrm{d}f, \hat{\nabla}^f \hat{\mathbb{H}}) + \operatorname{tr}_h \hat{R}(\mathrm{d}f, \hat{\mathbb{H}})\,\mathrm{d}f\}^{\perp} = 0. \tag{4.7}$$

We call (4.7) a pseudohermitian biminimal equation. Clearly, pseudohermitian biharmonic submanifolds are pseudohermitian biminimal.

Analogously to λ-biminimal immersion, we may define the following

DEFINITION 4.4. An isometric immersion $f\colon N \to M$ is called a *pseudohermitian λ-biminimal* immersion if it satisfies:

$$\hat{\tau}_2(f)^{\perp} = \lambda\hat{\tau}(f).$$

More explicitly,

$$\{\hat{\Delta}_f \hat{\mathbb{H}} + \operatorname{tr}_h \hat{T}(\mathrm{d}f, \hat{\nabla}^f \hat{\mathbb{H}}) + \operatorname{tr}_h \hat{R}(\mathrm{d}f, \hat{\mathbb{H}})\,\mathrm{d}f\}^{\perp} = -\lambda\hat{\mathbb{H}}.$$

The main purpose of the present paper is to prove

Theorem 4.1. *Let N^2 be a nonminimal pseudohermitian biminimal Legendre surface in a 5-dimensional Sasakian space form $M^5(\hat{\epsilon})$ of constant holomorphic sectional curvature $\hat{\epsilon}$ for $\hat{\nabla}$. Then $\hat{\epsilon} > 0$ and at each point $p \in N^2$ we have a local coordinate system $\{\mathrm{U}; x, y\}$ on a neighborhood $\mathrm{U}(p)$ such that the metric tensor*
(1) $g = dx^2 + dy^2$
and the second fundamental form σ takes the form

(2)

$$\begin{cases} \sigma\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial x}\right) = \dfrac{\hat{\epsilon}}{\alpha}\varphi\dfrac{\partial}{\partial x}, \\ \sigma\left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right) = \left(\alpha - \dfrac{\hat{\epsilon}}{\alpha}\right)\varphi\dfrac{\partial}{\partial y}, \\ \sigma\left(\dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial y}\right) = \left(\alpha - \dfrac{\hat{\epsilon}}{\alpha}\right)\varphi\dfrac{\partial}{\partial x}, \end{cases}$$

where $\alpha = \sqrt{(\hat{\epsilon}/8)(13 \pm \sqrt{41})}$.

Conversely, suppose that g *is the metric tensor on a* (*simply connected*) *domain* $\mathrm{V} \subset R^2$ *defined by* (1). *Then there exists a unique Legendre immersion of* (V, g) *into* $M^5(\hat{\epsilon})$ *whose second fundamental form is given by* (2) (*up to rigid motions of* $M^5(\hat{\epsilon})$). *In addition, such an immersion is nonminimal pseudohermitian biminimal.*

Corollary 4.2. *There exist no nonminimal pseudohermitian biminimal Legendre surfaces in a* 5*-dimensional Sasakian space form* $M^5(\hat{\epsilon})$ *for* $\hat{\epsilon} \leq 0$.

Here we recall some fundamental results on submanifolds in the unit sphere. A compact submanifold M^n of the unit hypersphere S^m of E^{m+1} is said to be *mass-symmetric* in S^m if the center of mass of M^n in E^{m+1} is exactly the center of S^m in E^{m+1}. Mass symmetric 2-type submanifolds of a hypersphere can be regarded as the "simplest" submanifolds of E^{m+1} next to minimal submanifolds (for the definition of 2-type submanifold, we refer to Chen's book [3]).

Lemma 4.1 ([1])**.** *Let* M *be a mass-symmetric* 2*-type Legendre surface in* S^5 *in* E^6. *Then* M *is locally isometric to the Riemannian product of a circle and a helix of order* 4 *or the product of two circles.*

Now we put $c = \alpha - \hat{\epsilon}/\alpha$. Then for the unit 5-sphere S^5, $\hat{\epsilon} = 4$, and we can see that $c^2 \neq 1$ in S^5 and by the similar arguments in [1] we can see that N^2 in Theorem 4.1 is locally isometric to the Riemannian product of a circle and a helix of order 4. Namely, we have

Corollary 4.3. *Let* $f\colon N^2 \to S^5 \subset C^3$ *be a nonminimal pseudohermitian biminimal Legendre immersion into the unit* 5*-sphere. Then the position vector* $f(x, y)$ *of* N^2 *in* C^3 *is given by*

$$f(x, y) = \frac{1}{\sqrt{c^2+1}}(ce^{i(x/c)}, ie^{-icx}\sin\sqrt{c^2+1}y, ie^{-icx}\cos\sqrt{c^2+1}y).$$

REMARK 4.1. The above corollary says that the product of a circle and a helix of order 4 is characterized by a nonminimal pseudohermitian biminimal Legendre immersion into the unit 5-sphere. On the other hand, Sasahara [12] showed that the product of two circles is realized as a nonminimal biminimal (with respect to ∇) Legendre immersion into the unit 5-sphere.

From Definition 2.2 and Definition 4.4, we can see that a nonminimal biminimal Legendre surface M in a 5-dimensional Sasakian space form corresponds to pseudohermitian 4-biminimal (for $\hat{\nabla}$) Legendre surface. Thus Corollary 2 in [12] can be restated as:

Corollary 4.4. *Let $f : N^2 \to S^5 \subset C^3$ be a nonminimal pseudohermitian 4-biminimal Legendre immersion into the unit 5-sphere. Then the position vector $f(x,y)$ of N^2 in C^3 is given by*

$$f(x, y) = \frac{1}{\sqrt{2}}(e^{ix}, ie^{-ix} \sin \sqrt{2}y, ie^{-ix} \cos \sqrt{2}y).$$

5. Proof of Theorem 4.1

Let $f : N^2 \to M^5(\hat{\epsilon})$ be a Legendre surface. Then from (4.2) and (4.3) we have

$$A(X, Y) = 0, \tag{5.1}$$

$$S_{\varphi Y} X = -\varphi\sigma(X, Y) = S_{\varphi X} Y, \quad S_\xi = 0. \tag{5.2}$$

for $X, Y \in TN$. Assume that the mean curvature vanishes nowhere. Let e_i $(i = 1, \dots, 5)$ be an orthonormal frame field along N^2 such that e_1, e_2 are tangent to N^2, $\varphi e_1 = e_3$, $\varphi e_2 = e_4$, $\xi = e_5$ and $\hat{\mathbb{H}} = \mathbb{H} = (\alpha/2)\varphi e_1$, with $\alpha > 0$. Using (5.2), we have $g(\sigma(e_1, e_1), \varphi e_2) = g(\sigma(e_1, e_2), \varphi e_1)$ and $g(\sigma(e_2, e_2), \varphi e_1) = g(\sigma(e_1, e_2), \varphi e_2)$. Then we may write the second fundamental form σ as follows:

$$\begin{aligned} \sigma(e_1, e_1) &= (\alpha - c)\varphi e_1 + b\varphi e_2, \\ \sigma(e_1, e_2) &= b\varphi e_1 + c\varphi e_2, \\ \sigma(e_2, e_2) &= c\varphi e_1 - b\varphi e_2, \end{aligned} \tag{5.3}$$

for some functions b, c. We put $\omega_i^j(e_k) = g(\hat{\nabla}^h_{e_k} e_i, e_j)$. Then we compute

$$\begin{aligned} &\hat{\nabla}_{e_1} e_1 = \omega_1^2(e_1)e_2 + (\alpha - c)\varphi e_1 + b\varphi e_2, \quad \hat{\nabla}_{e_1} e_2 = -\omega_1^2(e_1)e_1 + b\varphi e_1 + c\varphi e_2, \\ &\hat{\nabla}_{e_2} e_1 = \omega_1^2(e_2)e_2 + b\varphi e_1 + c\varphi e_2, \quad \hat{\nabla}_{e_2} e_2 = -\omega_1^2(e_2)e_1 + c\varphi e_1 - b\varphi e_2, \\ &\hat{\nabla}_{e_1} \varphi e_1 = -(\alpha - c)e_1 - be_2 + \omega_1^2(e_1)\varphi e_2, \quad \hat{\nabla}_{e_1} \varphi e_2 = -be_1 - ce_2 - \omega_1^2(e_1)\varphi e_1, \\ &\hat{\nabla}_{e_2} \varphi e_1 = -be_1 - ce_2 + \omega_1^2(e_2)\varphi e_2, \quad \hat{\nabla}_{e_2} \varphi e_2 = -ce_1 + be_2 - \omega_1^2(e_2)\varphi e_1, \\ &\hat{\nabla}_{e_1} \xi = \hat{\nabla}_{e_2} \xi = 0. \end{aligned} \tag{5.4}$$

Since $\hat{\nabla}$ parallelize φ, from (5.3) we have

$$\begin{aligned}
(\hat{\nabla}_{e_1}\sigma)(e_2, e_2) &= \{e_1 c + 3b\omega_1^2(e_1)\}\varphi e_1 - \{e_1 b - 3c\omega_1^2(e_1)\}\varphi e_2,\\
(\hat{\nabla}_{e_2}\sigma)(e_1, e_2) &= \{e_2 b + (\alpha - 3c)\omega_1^2(e_2)\}\varphi e_1 + \{e_2 c + 3b\omega_1^2(e_2)\}\varphi e_2,\\
(\hat{\nabla}_{e_1}\sigma)(e_1, e_2) &= \{e_1 b + (\alpha - 3c)\omega_1^2(e_1)\}\varphi e_1 + \{e_1 c + 3b\omega_1^2(e_1)\}\varphi e_2,\\
(\hat{\nabla}_{e_2}\sigma)(e_1, e_1) &= \{e_2(\alpha - c) - 3b\omega_1^2(e_2)\}\varphi e_1 + \{e_2 b + (\alpha - 3c)\omega_1^2(e_2)\}\varphi e_2.
\end{aligned}$$

From the Codazzi equation (4.5) we get

$$e_1 c + 3b\omega_1^2(e_1) = e_2 b + (\alpha - 3c)\omega_1^2(e_2), \tag{5.5}$$

$$-e_1 b + 3c\omega_1^2(e_1) = e_2 c + 3b\omega_1^2(e_2), \tag{5.6}$$

$$e_2(\alpha - c) - 3b\omega_1^2(e_2) = e_1 b + (\alpha - 3c)\omega_1^2(e_1). \tag{5.7}$$

Use (5.6) and (5.7) together to obtain

$$e_2\alpha = \alpha\omega_1^2(e_1). \tag{5.8}$$

Now we compute the pseudohermitian biminimal equation (4.7). First by using (5.4) we compute

$$\begin{aligned}
2\hat{\Delta}^h\mathbb{H} = {}&[\hat{\Delta}^h\alpha + \alpha\{(\alpha - c)^2 + c^2 + 2b^2 + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\}]\varphi e_1\\
&- [2(e_1\alpha)\omega_1^2(e_1) + 2(e_2\alpha)\omega_1^2(e_2) + \alpha\{e_1\omega_1^2(e_1) + e_2\omega_1^2(e_2)\} - \alpha^2 b]\varphi e_2.
\end{aligned} \tag{5.9}$$

Here we should remark that the Laplacian $\hat{\Delta}^h$ acting on the algebra $C^\infty(N)$ of smooth functions on M is defined by

$$\hat{\Delta}^h = -\sum_{i=1}^{2}\big(\hat{\nabla}^h_{e_i}\hat{\nabla}^h_{e_i} - \hat{\nabla}^h_{\hat{\nabla}^h_{e_i}e_i}\big),$$

where $\{e_1, e_2\}$ is a local orthonormal frame field on N. Since N is Legendre, $\hat{\nabla}^h = \nabla^h$, so we get $\hat{\Delta}^h$ is the Laplacian Δ of (N, h) with respect to the Riemannian metric h. From Proposition 3.1 and (5.4), we have

$$\mathrm{tr}_h\,\hat{T}(\mathrm{d}f, \hat{\nabla}^f\mathbb{H}) = -\{e_1(\alpha) + \alpha\omega_1^2(e_2)\}\xi. \tag{5.10}$$

Using (2.3) and (3.6), we get

$$\mathrm{tr}_h\,\hat{R}(\mathrm{d}f, \mathbb{H})\,\mathrm{d}f = -\frac{5}{4}\hat{\epsilon}\mathbb{H}. \tag{5.11}$$

Combining (5.9), (5.10) and (5.11), then the pseudohermitian biminimal equation yields:

$$\Delta\alpha + \alpha\left\{-\frac{5}{4}\hat{\epsilon} + (\alpha - c)^2 + c^2 + 2b^2 + (\omega_1^2(e_1))^2 + (\omega_1^2(e_2))^2\right\} = 0, \tag{5.12}$$

$$2(e_1\alpha)\omega_1^2(e_1) + 2(e_2\alpha)\omega_1^2(e_2) + \alpha\{e_1\omega_1^2(e_1) + e_2\omega_1^2(e_2)\} - \alpha^2 b = 0, \tag{5.13}$$

$$e_1\alpha + \alpha\omega_1^2(e_2) = 0. \tag{5.14}$$

Use (5.8) and (5.14) to get

$$\left[\frac{1}{\alpha}e_1, \frac{1}{\alpha}e_2\right] = 0.$$

From this observation, we may take a suitable local coordinate system $\{x, y\}$ such that

$$e_1 = \alpha\frac{\partial}{\partial x}, \quad e_2 = \alpha\frac{\partial}{\partial y}. \tag{5.15}$$

We adapt similar arguments in the proof of Theorem 1 in [12]. Then it follows from (5.15) that the metric tensor is given by

$$g = \frac{1}{\alpha^2}(dx^2 + dy^2).$$

Hence we have

$$\omega_1^2(e_1) = \alpha_y, \quad \omega_1^2(e_2) = -\alpha_x, \tag{5.16}$$

where $\alpha_x = \partial\alpha/\partial x$ and $\alpha_y = \partial\alpha/\partial y$. By substituting (5.15) and (5.16) into (5.13), we get $b = 0$. Hence, from (5.5), (5.6) and (5.12) we have

$$\alpha c_x = -(\alpha - 3c)\alpha_x, \tag{5.17}$$

$$3c\alpha_y = \alpha c_y, \tag{5.18}$$

$$\alpha\alpha_{yy} + \alpha\alpha_{xx} + \frac{5}{4}\hat{\epsilon} - \alpha^2 - 2c^2 + 2\alpha c - (\alpha_x)^2 - (\alpha_y)^2 = 0, \tag{5.19}$$

respectively. On the other hand, from the Gauss equation (4.4) we have

$$\begin{aligned} \alpha c - 2c^2 + \frac{1}{4}\hat{\epsilon} &= -(\omega_1^2(e_1))^2 - (\omega_1^2(e_2))^2 + e_2(\omega_1^2(e_1)) - e_1(\omega_1^2(e_2)) \\ &= -(\alpha_y)^2 - (\alpha_x)^2 + \alpha\alpha_{yy} + \alpha\alpha_{xx}. \end{aligned} \tag{5.20}$$

Combining (5.19) and (5.20) together, we obtain

$$\alpha^2 - 3\alpha c + 4c^2 - \frac{3}{2}\hat{\epsilon} = 0. \tag{5.21}$$

Differentiating (5.21) for x and y, respectively, then we have

$$(2\alpha - 3c)\alpha_i + (8c - 3\alpha)c_i = 0, \tag{5.22}$$

where $i = x, y$. Since $\alpha \neq 0$, from the system: (5.17), (5.18) and (5.22) for x and y, we find that α is a (positive) constant. Thus we have $\omega_1^2 = 0$ by (5.16) and have $\hat{\epsilon} > 0$ in (5.12). Consequently, the equation (5.19) is reduced to

$$\frac{5}{4}\hat{\epsilon} - \alpha^2 - 2c^2 + 2\alpha c = 0. \tag{5.23}$$

Solve (5.21) and (5.23) to get $c = \alpha - (1/\alpha)\hat{\epsilon}$. Then we get $\alpha = \sqrt{(\hat{\epsilon}/8)(13 \pm \sqrt{41})}$ again from (5.23). After all, we have $g = (1/\alpha^2)(dx^2 + dy^2)$ and

$$\begin{aligned}
\sigma(e_1, e_1) &= \frac{\hat{\epsilon}}{\alpha}\varphi e_1, \\
\sigma(e_1, e_2) &= \left(\alpha - \frac{\hat{\epsilon}}{\alpha}\right)\varphi e_2, \\
\sigma(e_2, e_2) &= \left(\alpha - \frac{\hat{\epsilon}}{\alpha}\right)\varphi e_1.
\end{aligned}$$

By virtue of the existence and uniqueness theorem (cf. Theorem 1 and Theorem 2 in [11]) we can prove the converse. Thus, we have proved Theorem 4.1. □

Corollary 5.1. *Let N^2 be a nonminimal pseudohermitian biharmonic Legendre surface in a* 5*-dimensional Sasakian space form $M^5(\hat{\epsilon})$ of constant holomorphic sectional curvature $\hat{\epsilon}$ for $\hat{\nabla}$. Then we have the same result as* Theorem 4.1.

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