Hai, D.D.
Osaka J. Math.
52 (2015), 393–408

A COMPARISON PRINCIPLE AND APPLICATIONS TO ASYMPTOTICALLY *p*-LINEAR BOUNDARY VALUE PROBLEMS

DANG DINH HAI

(Received January 16, 2013, revised November 18, 2013)

Abstract

Consider the problems

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega, \\ -\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$. We prove a strong comparison principle that allows $f - g$ to change sign. An application to singular asymptotically p-linear boundary problems is given.

1. Introduction

Consider the problems

$$\begin{cases} -\Delta_p u = f \text{ in } \Omega, & u = 0 \text{ on } \partial\Omega, \\ -\Delta_p v = g \text{ in } \Omega, & v = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0, 1)$, $\Delta_p z = \operatorname{div}(|\nabla z|^{p-2}\nabla z)$, $p > 1$, and $f, g\colon \Omega \to \mathbb{R}$.

In this paper, we shall establish a strong comparison principle

$$u > v \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial\Omega,$$

without requiring that $f \geq g$ a.e. in Ω. Here ν denotes the outer unit normal vector on $\partial\Omega$. It should be noted that the assumptions $f \geq g$ and $f \not\equiv g$ in Ω are needed in previous literature (see e.g. [9] and the references therein). We also provide an application to the existence of positive solutions for a class of singular p-Laplacian boundary value problems with asymptotically p-linear nonlinearity.

Let $d(x) = d(x, \partial\Omega)$ be the distance from x to $\partial\Omega$, we prove the following result:

2010 Mathematics Subject Classification. 35J95, 35J70.

Theorem 1.1. *Let $f, g, g_0 \in L^1(\Omega)$ with $g \geq g_0 \geq 0$, and $g_0 \not\equiv 0$. Suppose there exist constants $C > 0$ and $\gamma \in (0, 1)$ such that*

$$|f(x)|, g(x) \leq \frac{C}{d^\gamma(x)}$$

for a.e. $x \in \Omega$, and there exist a function $h \in C(\Omega)$, $h > 0$, and constants $\varepsilon \geq 0$, $m, M > 0$ with $m \leq M$ such that

$$f - g \geq m\left(h - \frac{\varepsilon}{d^\gamma}\right) \quad \text{in} \quad \Omega.$$

Let $u, v \in W_0^{1,p}(\Omega)$ be solutions of (1.1). Then there exists a positive constant ε_0 depending on n, Ω, p, γ, C, M, h, g_0 (but not on m), such that

$$u > v \quad \text{in} \quad \Omega \quad \text{and} \quad \frac{\partial u}{\partial \nu} < \frac{\partial v}{\partial \nu} \quad \text{on} \quad \partial\Omega$$

for $\varepsilon < \varepsilon_0$. If $\varepsilon = 0$, the result holds under the weaker condition that h is a nonnegative nontrivial measurable function in Ω.

REMARK 1.1. When $g \equiv 0$, the conclusion of Theorem 1.1 holds under the weaker assumption that h is a nonnegative nontrivial measurable function in Ω. In this case, ε_0 is independent of M. Indeed, let $\bar{u}$, $\bar{v}$ be the solutions of

$$-\Delta_p \bar{u} = \tilde{h} - \frac{\varepsilon}{d^\gamma} \text{ in } \Omega, \quad \bar{u} = 0 \text{ on } \partial\Omega,$$
$$-\Delta_p \bar{v} = \tilde{h} \text{ in } \Omega, \quad \bar{v} = 0 \text{ on } \partial\Omega,$$

respectively, where $\tilde{h} = \min(h, 1/d^\gamma)$. By the strong maximum principle [12, 14], there exists a constant $\delta > 0$ such that $\bar{v} \geq \delta d$ in Ω. Using Lemma 2.3 in Section 2, we deduce that

$$\bar{u} \geq \bar{v} - \frac{\delta}{2} d \geq \frac{\delta}{2} d$$

if ε is sufficiently small. This implies

$$u \geq m^{1/(p-1)} \bar{u} > m^{1/(p-1)} \frac{\delta}{2} d > 0 \quad \text{in} \quad \Omega$$

and $\partial u / \partial \nu < 0$ on $\partial\Omega$.

As an application of Theorem 1.1, consider the boundary value problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{u^\beta} + \lambda f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $\beta \in (0,1)$, q, f satisfy the following assumptions:
(A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and there exists a constant $k > 0$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{p-1}} = k.$$

(A2) There exists a constant $\delta \in (0,1)$ such that

$$\limsup_{u\to 0^+} u^{\delta}|f(u)| < \infty.$$

(A3) There exist constants $A, \varepsilon_0 > 0$ such that

$$f(u) \geq ku^{p-1} + \varepsilon_0 \quad \text{for} \quad u > A.$$

(A4) $q : \Omega \to \mathbb{R}$ is measurable and there exist constants η, $L > 0$ with $\beta + \eta < 1$, such that

$$|q(x)| \leq \frac{L}{d^{\eta}(x)}$$

for a.e. $x \in \Omega$.

Let λ_1 be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition, and let ϕ_1 be the corresponding positive eigenfunction with $\|\phi_1\|_\infty = 1$. Note that, since $\partial\phi_1/\partial\nu < 0$ on $\partial\Omega$, Theorem 1.1 holds if d is replaced by ϕ_1. Let $\lambda_\infty = \lambda_1/k$. Then we have

Theorem 1.2. *Let* (A1)–(A4) *hold. Then there exists a constant* $\tilde{\varepsilon} > 0$ *such that for* $\lambda_\infty - \tilde{\varepsilon} < \lambda < \lambda_\infty$, *problem* (1.2) *has a positive solution* $u_\lambda \in C^{1,\kappa}(\bar{\Omega})$ *for some* $\kappa \in (0,1)$ *with*

$$(*) \qquad u_\lambda \geq \left(\frac{\lambda_\infty \varepsilon_0}{4k(\lambda_\infty - \lambda)}\right)^{1/(p-1)} \phi_1 \quad \textit{in} \quad \Omega.$$

Theorem 1.3. *Let* $q \geq 0$, $q \not\equiv 0$. *Suppose* $f \geq 0$, (A2), (A4) *hold, and*

$$\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} = k$$

for some $k \in (0,\infty)$. *Then problem* (1.2) *has a positive solution* u_λ *for* $\lambda < \lambda_\infty$. *If, in addition,*

$$f(u) \geq ku^{p-1} \quad \textit{for all} \quad u > 0,$$

then (1.2) *has no positive solution for* $\lambda \geq \lambda_\infty$, *and*

$$\|u_\lambda\|_\infty \to \infty \quad \textit{as} \quad \lambda \to \lambda_\infty^-.$$

EXAMPLE 1.1. (i) Let $f(u) = -1/u^{\delta} + u^{p-1} + u^{q}$, where $\delta \in (0, 1)$ and $0 \leq q < p-1$. Then f satisfies (A1)–(A3) with $k = 1$, and so (1.2) has a positive solution when λ is sufficiently close to λ_1 and $\lambda < \lambda_1$, by Theorem 1.2.
(ii) Let $f(u) = 1/u^{\delta} + u^{p-1}(m|\sin u| + e^{1/(1+u)})$, where $\delta \in (0,1)$, $m \geq 0$. Then it follows from Theorem 1.3 that, if $m > 0$, (1.2) has a positive solution for $\lambda < \lambda_1/(m+1)$, and, if $m = 0$, (1.2) has a positive solution if and only if $\lambda < \lambda_1$.

REMARK 1.2. It should be noted that Theorem 1.2 may not be true when $\varepsilon_0 = 0$. Indeed, by multiplying the equation in (1.2) by u and integrating, we see that (1.2) has no positive solution for $\lambda < \lambda_\infty$ when $q \leq 0$ and $f(u) = ku^{p-1}$.

REMARK 1.3. In [15], assuming that f is continuous and nonnegative on $[0,\infty)$, $\lim_{u\to\infty} f(u)/u = k \in (0, \infty)$, and f satisfies some additional conditions at 0, Zhang showed via variational method that (1.2) with $p = 2$ has a positive solution for $\lambda \in (0, \lambda_1/k)$, provided that $q \geq 0$, $q \not\equiv 0$, $q\phi_1^{-\beta} \in L^r(\Omega)$, where $n/2 < r$. The result in [15] was improved by Hai in [4], using sub- and super solutions approach. The proof in [4] depends on the linearity of the Laplacian and can not be applied to the general case where $p > 1$, except for radial solutions in a ball [6]. Related results on the case where f is nonsingular can be found in Ambrosetti, Arcoya, and Buffoni [1], Ambrosetti and Hess [2], and Ambrosetti, Garcia Azorero, and Peral [3]. The approach in [1, 2, 3] was via bifurcation theory for $p = 2$ in [1, 2] and $p > 1$ in [3]. Thus, Theorems 1.2 and 1.3 provide extensions of corresponding results in [1, 2, 3, 4, 6, 15] to the singular p-Laplacian case. Note that the precise lower bound estimate $(*)$ has not been obtained in previous literature.

2. Preliminary results

Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary ∂D.

We shall denote the norm in $C^{k,\alpha}(\bar{D})$ and $L^k(D)$ by $|\cdot|_{k,\alpha}$ and $\|\cdot\|_k$ respectively. The distance from x to ∂D is denoted by $d(x, \partial D)$.

We first recall the following regularity result in [5, Lemma 3.1], which plays an important role in the proofs of our main results.

Lemma A. *Let $h \in L^\infty_{\rm loc}(\Omega)$ and suppose there exist numbers $\gamma \in (0,1)$ and $C > 0$ such that*

$$|h(x)| \leq \frac{C}{d^{\gamma}(x)} \tag{3.1}$$

for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the solution of

$$\begin{cases} -\Delta_p u = h & in \quad \Omega, \\ u = 0 & on \quad \partial\Omega. \end{cases} \tag{3.2}$$

Then there exist constants $\alpha \in (0, 1)$ *and* $\tilde{M} > 0$ *depending only on* C, γ, Ω *such that* $u \in C^{1,\alpha}(\bar{\Omega})$ *and* $|u|_{1,\alpha} < \tilde{M}$.

Let

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right),$$

where $a_{ij} \in C^{0,\alpha}(\bar{D})$, $1 \leq i, j \leq n$, for some $\alpha \in (0,1)$, and suppose there exist constants $m_0, m_1 > 0$ such that

$$|a_{ij}|_{0,\alpha} \leq m_1 \tag{2.1}$$

for $1 \leq i,\ j \leq n$, and

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq m_0|\xi|^2 \tag{2.2}$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

Lemma 2.1. *Let* $h \in L^1(D)$ *and suppose there exist constants* $C > 0$ *and* $\gamma \in (0, 1)$ *such that*

$$|h(x)| \leq \frac{C}{d^{\gamma}(x, \partial D)} \tag{2.3}$$

for a.e. $x \in D$. *Let* $w \in H_0^1(D)$ *be the solution of*

$$\begin{cases} Lw = h & \text{in} \quad D, \\ w = 0 & \text{on} \quad \partial D. \end{cases} \tag{2.4}$$

Then there exist constants $\beta \in (0, 1)$ *and* $\tilde{M} > 0$ *depending only on* m_0, m_1, C, γ, D, n, *such that* $w \in C^{1,\beta}(\bar{D})$ *and*

$$|w|_{1,\beta} \leq \tilde{M}.$$

Proof. Let $\phi \in C^1(\bar{D})$ be the solution of

$$L\phi = 1 \text{ in } D, \quad \phi = 0 \text{ on } \partial D.$$

Then there exists a constant $C_0 > 0$ independent of a_{ij} such that $\phi(x) \leq C_0 d(x, \partial D)$ for all $x \in D$. Let $a = 2^{1/(1-\gamma)}\|\phi\|_\infty$ and $h_0\colon [0, a] \to \mathbb{R}$ satisfy

$$\begin{cases} -h_0'' = \dfrac{1}{t^{\gamma}}, & 0 < t < a, \\ h_0(0) = 0, & h_0'(a) = 0. \end{cases}$$

Note that $h_0(t) = (t/(1-\gamma))(a^{1-\gamma} - t^{1-\gamma}/(2-\gamma))$. A calculation shows that

$$L(h_0(\phi)) = -h_0''(\phi) \sum_{i,j=1}^{n} a_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + h_0'(\phi)$$
$$\geq \frac{m_0}{\phi^\gamma} |\nabla \phi|^2 + \frac{a^{1-\gamma} - \phi^{1-\gamma}}{1-\gamma} \geq \frac{m_0}{\phi^\gamma} |\nabla \phi|^2 + \frac{a^{1-\gamma}}{2(1-\gamma)} \geq \frac{m_2}{d^\gamma(x, \partial D)},$$

where m_2 is independent of a_{ij}. By the weak comparison principle ([11, Lemma A.2], [13, Lemma 3.1]),

$$|w| \leq \frac{C}{m_2} h_0(\phi) \quad \text{in} \quad D,$$

i.e. w is bounded in D. By Lemma A, the problem

$$\begin{cases} -\Delta z = h & \text{in} \quad D, \\ z = 0 & \text{on} \quad \partial D, \end{cases}$$

has a solution $z \in C^{1,\alpha}(\bar{D})$ for some $\alpha \in (0, 1)$. Since w satisfies

$$-\mathrm{div}(A(x, \nabla w) - \nabla z) = 0 \quad \text{in} \quad D,$$

where $A = (A_1, \ldots, A_n)$, $A_i(x, \eta) = \sum_{j=1}^n a_{ij}(x)\eta_j$, $\eta = (\eta_1, \ldots, \eta_n)$, the result now follows from Lieberman [8, Theorem 1]. □

Lemma 2.2. *Let h satisfy* (2.3), $h \geq 0$, $h \not\equiv 0$, *and let $w \in H_0^1(D)$ be the solution of* (2.4). *Then there exists a constant $k_0 > 0$ depending only on h, m_0, m_1, C, γ, D, n such that*

$$w(x) \geq k_0 d(x, \partial D)$$

for all $x \in D$.

Proof. Let Λ be the set of all solutions w of (2.4) among the coefficients a_{ij} that satisfy (2.1) and (2.2). By the strong maximum principle, $w > 0$ in Ω and $\partial w/\partial \nu < 0$ on ∂D. By Lemma 2.1, $w \in C^{1,\beta}(\bar{D})$ and there exists a constant $\tilde{M} > 0$ such that $|w|_{1,\beta} \leq \tilde{M}$ for all $w \in \Lambda$. Since Λ is closed in $C^1(\bar{D})$, Λ is compact in $C^1(\bar{D})$. Define $G \colon \Lambda \to \mathbb{R}$ by

$$Gw = \inf_{x \in D} \frac{w(x)}{d(x, \partial D)}.$$

Then G is continuous and positive on Λ, and therefore has a positive minimum, which completes the proof. □

Lemma 2.3. *Let $f, g \in L^1(D)$ satisfy*

$$|f(x)|, |g(x)| \le \frac{C}{d^\gamma(x, \partial D)}$$

for a.e. $x \in \Omega$ for some constant $C > 0$. Let u, v be the solutions of (1.1). *Then* $|u - v|_{0,1} \to 0$ *as* $\|f - g\|_1 \to 0$.

Proof. Note that $f, g \in L^1(\Omega)$ (see [7, p. 6]). By Lemma A, $u, v \in C^{1,\alpha}(\bar{D})$ for some $\alpha \in (0,1)$, and there exists a constant $\tilde{M} > 0$ independent of u, v, such that $|u|_{1,\alpha}$, $|v|_{1,\alpha} \le \tilde{M}$. Multiplying the equation

$$-(\Delta_p u - \Delta_p v) = f - g \quad \text{in} \quad \Omega$$

by $u - v$ and integrating, we obtain

$$\int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla u|^{p-2}\nabla u) \cdot \nabla(u - v)\, dx = \int_\Omega (f - g)(u - v)\, dx.$$

Using the inequality [10, Lemma 30.1],

$$(|x| + |y|)^{2-\min(p,2)}(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \ge c|x - y|^{\max(p,2)}$$

for $x, y \in \mathbb{R}^n$, where c is a positive constant depending only on p, we obtain

$$\int_\Omega |\nabla(u - v)|^r\, dx \le c_1 \|f - g\|_{L^1} \|u - v\|_\infty \le c_2 \|f - g\|_{L^1},$$

where $r = \max(p, 2)$ and c_1, c_2 are constants depending only on p, $\tilde{M}$.

Hence

$$\|u - v\|_2 \to 0$$

as $\|f - g\|_1 \to 0$, and since $C^{1,\alpha}(\bar{D})$ is compactly imbedded in $C^1(\bar{D})$, Lemma 2.3 follows. □

3. Proofs of the main results

Proof of Theorem 1.1. By the strong maximum principle, there exists a constant $\delta > 0$ such that $v \ge \delta d$ in Ω. Let $\varepsilon \in [0, 1)$, $m_\varepsilon = \min(m, \varepsilon)$ and $\tilde{h} = \min(h, 1/d^\gamma)$. Then

$$f \ge g + m_\varepsilon \tilde{h} - \frac{M\varepsilon}{d^\gamma} \equiv \tilde{f} \quad \text{in} \quad \Omega.$$

Let $\tilde{u}$ satisfy

$$-\Delta_p \tilde{u} = \tilde{f} \text{ in } \Omega, \quad \tilde{u} = 0 \text{ on } \partial\Omega.$$

Then $u \geq \tilde{u}$ in Ω, by the weak comparison principle. Since

$$|\tilde{f}| \leq \frac{C + 1 + M}{d^\gamma} \equiv \frac{\tilde{C}}{d^\gamma}$$

and

$$\int_\Omega |\tilde{f} - g|\, dx \leq \varepsilon(1 + M) \int_\Omega \frac{1}{d^\gamma}\, dx,$$

it follows that $\tilde{f} \to g$ in $L^1(\Omega)$ as $\varepsilon \to 0$. Here we have used the fact that $1/d^\gamma \in L^1(\Omega)$ (see e.g. [7, p. 6]). Using Lemma 2.3, we see that $\tilde{u} \to v$ in $C^1(\bar{\Omega})$ as $\varepsilon \to 0$. Hence

$$u \geq \tilde{u} \geq \frac{\delta d}{2} \quad \text{in} \quad \Omega$$

if ε is sufficiently small, which we shall assume.

By Lemma A, there exist constants $\tilde{M} > 0$ and $\alpha \in (0, 1)$ independent of u, v such that $|u|_{1,\alpha}, |v|_{1,\alpha} \leq \tilde{M}$. Thus

$$\frac{\delta d}{2} \leq u, \quad v \leq \tilde{M} d \quad \text{in} \quad \Omega, \tag{3.1}$$

and therefore

$$u \geq c_0 v \quad \text{in} \quad \Omega,$$

where $c_0 = \delta/(2\tilde{M})$.

Let c be the largest number such that $u \geq cv$ in Ω, and suppose that $c \leq 1$. From (3.1), it follows that

$$\frac{\partial u}{\partial \nu}, \frac{\partial v}{\partial \nu} \leq -\frac{\delta}{2} \quad \text{on} \quad \partial\Omega.$$

For $t \in [0, 1]$, let $w_t = t\nabla u + (1 - t)c\nabla v$. Then

$$\begin{aligned} w_t \cdot \nu = t\frac{\partial u}{\partial \nu} + (1 - t)c\frac{\partial v}{\partial \nu} &\leq -\frac{tc\delta}{2} - \frac{(1 - t)c\delta}{2} \\ = -\frac{c\delta}{2} &\leq -\frac{c_0\delta}{2} \quad \text{on} \quad \partial\Omega, \end{aligned}$$

which implies

$$|w_t| \geq \frac{c_0\delta}{2} \equiv c_1 \quad \text{on} \quad \partial\Omega \tag{3.2}$$

for all $t \in [0, 1]$.

Let $x \in \Omega$ and $x_0 \in \partial\Omega$ be such that $d(x) = |x - x_0|$. Since $|w_t|_{0,\alpha} \leq \tilde{M}$, it follows that

$$|w_t(x) - w_t(x_0)| \leq \tilde{M} d^\alpha(x),$$

which, together with (3.2), implies

$$|w_t(x)| \geq c_1 - \tilde{M} d^{\alpha}(x) \geq \frac{c_1}{2} \equiv c_2 \tag{3.3}$$

for $x \in \Omega_\eta \equiv \{x \in \Omega\colon d(x) < \eta\}$, where $\eta = (c_1/2\tilde{M})^{1/\alpha}$.

Next, we have

$$-(\Delta_p u - \Delta_p(cv)) = f - c^{p-1} g \quad \text{in} \quad \Omega, \tag{3.4}$$

and the left hand side of (3.4) can be linearized as $L(u - cv)$, where

$$Lw = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial w}{\partial x_j}\right)$$

and $a_{ij}(x) = \int_0^1 (\partial a^i/\partial z_j)(t\nabla u + (1-t)c\nabla v)\, dt$, $a^i(z) = |z|^{p-2} z$.

Note that, in view of (3.3) and the fact that $|w_t|_{0,\alpha} \leq \tilde{M}$ for $t \in [0, 1]$, the coefficients a_{ij} satisfy (2.1) and (2.2) in Ω_η with $m_0 = (p-1)\min(\tilde{M}^{p-2}, c_2^{p-2})$.

Let w_0, w_1 be the solutions of

$$Lw_0 = \tilde{h} \text{ in } \Omega_\eta, \quad w_0 = 0 \text{ on } \partial\Omega_\eta,$$

and

$$Lw_1 = \frac{1}{d^{\gamma}(x, \partial\Omega_\eta)} \text{ in } \Omega_\eta, \quad w_1 = 0 \text{ on } \partial\Omega_\eta$$

respectively. By Lemmas 2.1 and 2.2, there exist positive constants M_0 and k_0 such that

$$w_0 \geq k_0 d(x, \partial\Omega_\eta), \quad w_1 \leq M_0 d(x, \partial\Omega_\eta) \quad \text{in} \quad \Omega_\eta. \tag{3.5}$$

Since $c \leq 1$ and $d(x) \geq d(x, \partial\Omega_\eta)$ for $x \in \Omega_\eta$,

$$L(u - cv) \geq f - g \geq m\left(\tilde{h} - \frac{\varepsilon}{d^{\gamma}(x, \partial\Omega_\eta)}\right) \quad \text{in} \quad \Omega_\eta,$$

and since $u \geq cv$ on $\partial\Omega_\eta$, it follows from the weak comparison principle and (3.5) that, for $x \in \Omega_{\eta/2}$,

$$\begin{aligned} u - cv &\geq m(w_0 - \varepsilon w_1) \geq m(k_0 - \varepsilon M_0) d(x, \partial\Omega_\eta) = m(k_0 - \varepsilon M_0) d(x) \\ &\geq m\left(\frac{k_0}{2}\right) d(x) \end{aligned} \tag{3.6}$$

if $\varepsilon < k_0/2M_0$. In particular,

$$u - cv \geq m\left(\frac{k_0}{2}\right)\frac{\eta}{2} \equiv mk_1 \quad \text{when} \quad d(x) = \frac{\eta}{2}.$$

If $\varepsilon = 0$ then it follows from

$$-\Delta_p u = f \geq c^{p-1} g = -\Delta_p(cv + mk_1) \quad \text{in} \quad \Omega$$

and $u \geq cv + mk_1$ on $\partial(\Omega \setminus \Omega_{\eta/2})$ that

$$u \geq cv + mk_1 \quad \text{in} \quad \Omega \setminus \Omega_{\eta/2}. \tag{3.7}$$

Suppose $\varepsilon > 0$ and $h \geq a > 0$ in $\Omega \setminus \Omega_{\eta/2}$. Then we have

$$-\Delta_p u = f \geq g + m\left(a - \frac{\varepsilon}{d^\gamma(x)}\right) \geq g + m\left(a - \frac{\varepsilon}{(\eta/2)^\gamma}\right) \geq g \quad \text{in} \quad \Omega \setminus \Omega_{\eta/2},$$

if ε is sufficiently small. Hence (3.7) holds by the weak comparison principle. This, together with (3.6), gives the existence of a constant $\tilde{c} > c$ such that $u \geq \tilde{c}v$ in Ω, a contradiction. Hence $c > 1$ and therefore $u > v$ in Ω and

$$\frac{\partial(u - v)}{\partial \nu} \leq (c-1)\frac{\partial v}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega,$$

which completes the proof. □

Proof of Theorem 1.2. Let $\lambda > 0$ be such that $\lambda_\infty/2 < \lambda < \lambda_\infty$. Let $c = (\lambda_\infty \varepsilon_0/(4k(\lambda_\infty - \lambda)))^{1/(p-1)}$ and M be a constant such that $M > c$. Define

$$\mathbf{K} = v \in C(\bar{\Omega}) \colon c\phi_1 \leq v \leq M\phi_1 \text{ in } \Omega\}.$$

For each $v \in \mathbf{K}$, it follows from Lemma A that the problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{v^\beta} + \lambda f(v) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \equiv Tv \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ such that $|u|_{1,\alpha} < \tilde{M}$, where α, $\tilde{M}$ are independent of $v \in \mathbf{K}$. We shall show that $T \colon \mathbf{K} \to C(\bar{\Omega})$ is a compact operator. In view of the compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$, we need only to show that T is continuous. Let (v_n) be a sequence in $\mathbf{K}$ such that $v_n \to v$ in $C(\bar{\Omega})$, and let $u_n = Tv_n$, $u = Tv$. Let $G(w) = q(x)/w^\beta + \lambda f(w)$ for $w \in \mathbf{K}$. Then

$$G(v_n) \to G(v) \quad \text{pointwise in} \quad \Omega,$$

and it follows from (A2) and (A4) that there exist constants $K, C_0 > 0$ such that

$$|G(v_n)| \leq \frac{q(x)}{(c\phi_1)^\beta} + \frac{\lambda_\infty K}{(c\phi_1)^\delta} \leq \frac{C_0}{d^\gamma}$$

for all n, where $\gamma = \max(\beta + \eta, \delta)$. Hence $G(v_n) \to G(v)$ in $L^1(\Omega)$, and Lemma 2.3 implies $u_n \to u$ in $C^1(\bar{\Omega})$, i.e., T is continuous on $\mathbf{K}$.

Next, we shall show that if λ is sufficiently close to λ_∞ and M is large enough then T maps $\mathbf{K}$ into $\mathbf{K}$. By (A2) and (A3), there exists a constant $M_0 > 0$ such that

$$f(z) \geq kz^{p-1} + \varepsilon_0 - \frac{M_0}{z^\delta} \tag{3.8}$$

for all $z > 0$. Let $v \in \mathbf{K}$ and $u = Tv$. Then (3.8) and (A4) imply

$$-\Delta_p u \geq -\frac{L}{(c\phi_1)^\beta d^\eta} + \lambda\left(k(c\phi_1)^{p-1} + \varepsilon_0 - \frac{M_0}{(c\phi_1)^\delta}\right) \quad \text{in} \quad \Omega.$$

Consequently,

$$-\Delta_p\left(\frac{u}{c}\right) \geq -\frac{L_1}{c^{p-1+\beta}\phi_1^\gamma} - \frac{\lambda_\infty M_0}{c^{p-1+\delta}\phi_1^\gamma} + \lambda\left(k\phi_1^{p-1} + \frac{\varepsilon_0}{c^{p-1}}\right) \equiv f_{c,\lambda} \quad \text{in} \quad \Omega,$$

where L_1 is a positive constant such that $d/\phi_1 \geq (L/L_1)^{1/\eta}$.

Let $\bar{u}_c$, $\bar{z}_c$ be the solutions of

$$-\Delta_p \bar{u}_c = f_{c,\lambda} \text{ in } \Omega, \quad \bar{u}_c = 0 \text{ on } \partial\Omega,$$

and

$$-\Delta_p \bar{z}_c = \lambda\left(k\phi_1^{p-1} + \frac{\varepsilon_0}{2c^{p-1}}\right) \equiv g_{c,\lambda} \text{ in } \Omega, \quad \bar{z}_c = 0 \text{ on } \partial\Omega,$$

respectively. Then $u \geq c\bar{u}_c$ in Ω. Note that

$$|f_{c,\lambda}|, |g_{c,\lambda}| \leq \frac{\tilde{C}}{\phi_1^\gamma},$$

where $\tilde{C} > 0$ depends only on ε_0, k, p, L_1, λ_∞, M_0. Since

$$f_{c,\lambda} - g_{c,\lambda} \geq \frac{1}{c^{p-1}}\left[\frac{\lambda_\infty\varepsilon_0}{4} - \left(\frac{L_1}{c^\beta} + \frac{\lambda_\infty M_0}{c^\delta}\right)\frac{1}{\phi_1^\gamma}\right] \quad \text{in} \quad \Omega,$$

and

$$c^{1-p} \leq \frac{2k}{\varepsilon_0},$$

it follows from Theorem 1.1 with $m = c^{1-p}$, $M = 2k/\varepsilon_0$, $h = \lambda_\infty\varepsilon_0/4$, $g_0 = (\lambda_\infty/2)k\phi_1^{p-1}$, that $\bar{u}_c > \bar{z}_c$ in Ω for $c \gg 1$, which implies

$$u \geq c\bar{z}_c \equiv \tilde{z}_c \quad \text{in} \quad \Omega. \tag{3.9}$$

By the choice of c,

$$(\lambda_1 - \lambda k)c^{p-1} = \frac{\lambda_\infty \varepsilon_0}{4} \le \frac{\lambda \varepsilon_0}{2}.$$

Hence

$$-\Delta_p \tilde{z}_c = \lambda k\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{2k}\right) \ge \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

and since

$$-\Delta_p(c\phi_1) = \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

it follows that

$$\tilde{z}_c \ge c\phi_1 \quad \text{in} \quad \Omega. \tag{3.10}$$

Hence, if λ is sufficiently close to λ_∞, it follows from (3.9) and (3.10) that $u \ge c\phi_1$ in Ω.

Next, let $\tilde{\lambda}_\infty > 0$ and $b > 1$ be such that $\lambda b < \tilde{\lambda}_\infty < \lambda_\infty$. In view of (A1) and (A2), there exists a constant $D > 0$ such that

$$f(z) \le kbz^{p-1} + \frac{D}{z^\delta}$$

for all $z > 0$. Hence

$$-\Delta_p u \le \lambda kb(M\phi_1)^{p-1} + \frac{\lambda_\infty D + L_1}{\phi_1^\gamma} \quad \text{in} \quad \Omega,$$

for $c > 1$, which implies

$$-\Delta_p\left(\frac{u}{M}\right) \le \lambda kb\phi_1^{p-1} + \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma} \equiv f_M \quad \text{in} \quad \Omega.$$

Let $\bar{u}_M$ be the solution of

$$-\Delta_p(\bar{u}_M) = f_M \text{ in } \Omega, \quad \bar{u}_M = 0 \text{ on } \partial\Omega.$$

Then $u \le M\bar{u}_M$ in Ω. Since

$$-\Delta_p\phi_1 = \lambda_1\phi_1^{p-1} \quad \text{in} \quad \Omega,$$

and

$$\begin{aligned}\lambda_1\phi_1^{p-1} - f_M &= (\lambda_1 - \lambda kb)\phi_1^{p-1} - \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma} \\ &\ge k(\lambda_\infty - \tilde{\lambda}_\infty)\phi_1^{p-1} - \frac{\lambda_\infty D + L_1}{M^{p-1}\phi_1^\gamma},\end{aligned}$$

it follows from Theorem 1.1 with $u = \phi_1$, $v = \bar{u}_M$, $m = 1$, $h = k(\lambda_\infty - \tilde{\lambda}_\infty)\phi_1^{p-1}$ that $\bar{u}_M \le \phi_1$ in Ω for $M \gg 1$. Hence $u \le M\phi_1$ in Ω for $M \gg 1$. Thus $T : \mathbf{K} \to \mathbf{K}$ and the result now follows from the Schauder fixed point theorem. $\square$

Proof of Theorem 1.3. Let $z \in C^1(\bar{\Omega})$ be the solution of

$$-\Delta_p z = \frac{cq(x)}{\phi_1^\beta} \text{ in } \Omega, \quad z = 0 \text{ on } \partial\Omega,$$

where $c \in (0, 1)$. By Lemma 2.3, $z \le \phi_1$ in Ω if c is sufficiently small, which we assume. Let $M > 1$ be a large constant to be determined later and define

$$\mathbf{C} = \{v \in C(\bar{\Omega}) \colon v \le M\phi_1 \text{ in } \Omega\}.$$

Fix $\lambda \in (0, \lambda_\infty)$ and choose $b > 1$ so that $\lambda b < \lambda_\infty$. For each $v \in \mathbf{C}$, the problem

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{\max^\beta(v, z)} + \lambda f(\max(v, z)) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

has a unique solution $u \equiv Sv \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$ such that $|u|_{1,\alpha} < \tilde{M}$, where α, $\tilde{M}$ are independent of $v \in \mathbf{C}$. Since $z \ge \varepsilon_1 d$ in Ω for some $\varepsilon_1 > 0$, it follows as in the proof of Theorem 1.2 that $S \colon \mathbf{C} \to C(\bar{\Omega})$ is a compact operator. We shall show that $S \colon \mathbf{C} \to \mathbf{C}$ if M is large enough. Note that any fixed point of S is positive in Ω, by the strong maximum principle. Let $v \in \mathbf{C}$ and $u = Sv$. Since there exists a constant $D > 0$ such that

$$f(t) \le kbt^{p-1} + \frac{D}{t^\delta}$$

for $t > 0$, it follows that

$$-\Delta_p u \le \frac{L_1}{z^{\beta+\eta}} + \lambda\left(kb(M\phi_1)^{p-1} + \frac{D}{z^\delta}\right) \quad \text{in} \quad \Omega,$$

where L_1 is defined in the proof of Theorem 1.2. This implies

$$-\Delta_p\left(\frac{u}{M}\right) \le \lambda kb\phi_1^{p-1} + \left(\frac{L_1 + \lambda_\infty D}{M^{p-1}}\right)\frac{1}{z^\gamma} \equiv g_M,$$

where $\gamma = \max(\beta + \eta, \delta)$. Let u_M be the solution of

$$-\Delta_p(u_M) = g_M \text{ in } \Omega, \quad u_M = 0 \text{ on } \partial\Omega.$$

Then $u \le Mu_M$ in Ω. Since

$$\lambda_1\phi_1^{p-1} - g_M \ge k(\lambda_\infty - \lambda b)\phi_1^{p-1} - \left(\frac{L_1 + \lambda_\infty D}{M^{p-1}}\right)\frac{1}{z^\gamma},$$

it follows from Theorem 1.1 that $u_M \le \phi_1$ in Ω for $M \gg 1$, which implies

$$u \le M u_M \le M\phi_1 \quad \text{in} \quad \Omega$$

i.e. $u \in \mathbf{C}$ for $M \gg 1$. By the Schauder fixed point theorem, S has a fixed point u_λ in $\mathbf{C}$. We claim that $u_\lambda \ge z$ in Ω. Let $D = \{x \in \Omega\colon u_\lambda(x) < z(x)\}$ and suppose that $D \neq \varnothing$. Then, since $f \ge 0$,

$$-\Delta_p u_\lambda \ge \frac{q(x)}{u_\lambda^\beta} \ge \frac{q(x)}{z^\beta} \ge \frac{q(x)}{\phi_1^\beta} \ge -\Delta_p z \quad \text{in} \quad D.$$

Since $u_\lambda = z$ on ∂D, this implies $u_\lambda \ge z$ in D, a contradiction. Hence $D = \varnothing$ and therefore $u_\lambda \ge z$ in Ω as claimed. Thus u_λ is a positive solution of (1.2).

Next, suppose $f(u) \ge k u^{p-1}$ for $u > 0$. Let $\lambda \ge \lambda_\infty$ and let u be a positive solution of (1.2). Then $u > 0$ in Ω and $\partial u/\partial \nu < 0$ on $\partial\Omega$ by the strong maximum principle. Let $c > 0$ be the largest number so that $u \ge c\phi_1$ in Ω. Then

$$-\Delta_p u \ge \frac{q(x)}{\|u\|_\infty^\beta} + \lambda k (c\phi_1)^{p-1} \ge \frac{q(x)}{\|u\|_\infty^\beta} + \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

and since

$$-\Delta_p (c\phi_1) = \lambda_1 (c\phi_1)^{p-1} \quad \text{in} \quad \Omega,$$

it follows from Theorem 1.1 with $\varepsilon = 0$ that $u > c\phi_1$ in Ω and

$$\frac{\partial u}{\partial \nu} < \frac{\partial (c\phi_1)}{\partial \nu} < 0 \quad \text{on} \quad \partial\Omega.$$

Hence there exists a constant $\tilde{c} > c$ such that $u \ge \tilde{c}\phi_1$ in Ω, a contradiction. Thus (1.2) has no positive solution for $\lambda \ge \lambda_\infty$. We shall verify next that $\lim\limits_{\lambda \to \lambda_\infty^-} \|u_\lambda\|_\infty = \infty$. Suppose otherwise, then there exist a sequence $(\lambda_n) \subset (0, \lambda_\infty)$ and a constant $C > 0$ such that $\lambda_n \to \lambda_\infty^-$ and $\|u_n\|_\infty < C$ for all n, where $u_n \equiv u_{\lambda_n}$. Since

$$-\Delta_p u_n \ge \frac{q(x)}{u_n^\beta} \ge \frac{q(x)}{C^\beta} \quad \text{in} \quad \Omega,$$

it follows that there exists a constant $\tilde{k} > 0$ such that $u_n \ge \tilde{k}\phi_1$ in Ω for all n. Hence there exists a constant $\tilde{C} > 0$ such that

$$\frac{q(x)}{u_n^\beta} + \lambda f(u_n) \le \frac{\tilde{C}}{\phi_1^\gamma} \quad \text{in} \quad \Omega$$

for all n. By Lemma A, there exist constants $\alpha \in (0, 1)$ and $\tilde{M} > 0$ such that $u_n \in C^{1,\alpha}(\bar{\Omega})$ and $|u_n|_{1,\alpha} < \tilde{M}$ for all n. By going to a subsequence, we assume that there exists $u \in C^1(\bar{\Omega})$ such that $u_n \to u$ in $C^1(\bar{\Omega})$. Let $\psi \in W_0^{1,p}(\Omega)$. Then

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi \, dx = \int_\Omega \left(\frac{q(x)}{u_n^\beta} + \lambda_n f(u_n)\right)\psi \, dx \tag{3.11}$$

for all n. Let $n \to \infty$ in (3.11) and using the Lebesgue dominated convergence theorem, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\psi \, dx = \int_\Omega \left(\frac{q(x)}{u^\beta} + \lambda_\infty f(u)\right)\psi \, dx$$

i.e. u is a positive solution of

$$\begin{cases} -\Delta_p u = \dfrac{q(x)}{u^\beta} + \lambda_\infty f(u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

a contradiction. This completes the proof of Theorem 1.3. $\square$

ACKNOWLEDGEMENT. The author thanks the referee for pointing out some errors in the original manuscript along with suggestions for improvement.

References

[1] A. Ambrosetti, D. Arcoya and B. Buffoni: *Positive solutions for some semi-positone problems via bifurcation theory*, Differential Integral Equations **7** (1994), 655–663.

[2] A. Ambrosetti and P. Hess: *Positive solutions of asymptotically linear elliptic eigenvalue problems*, J. Math. Anal. Appl. **73** (1980), 411–422.

[3] A. Ambrosetti, J. Garcia Azorero and I. Peral: *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. **137** (1996), 219–242.

[4] D.D. Hai: *On an asymptotically linear singular boundary value problems*, Topol. Methods Nonlinear Anal. **39** (2012), 83–92.

[5] D.D. Hai: *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. **383** (2011), 619–626.

[6] D.D. Hai and J.L. Williams: *Positive radial solutions for a class of quasilinear boundary value problems in a ball*, Nonlinear Anal. **75** (2012), 1744–1750.

[7] A.C. Lazer and P.J. McKenna: *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **111** (1991), 721–730.

[8] G.M. Lieberman: *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[9] M. Lucia and S. Prashanth: *Strong comparison principle for solutions of quasilinear equations*, Proc. Amer. Math. Soc. **132** (2004), 1005–1011.

[10] T. Oden: Qualitative Methods in Nonlinear Mechanics, Prentice-Hall, Inc, Englewood Cliffs, NJ, 1986.

[11] S. Sakaguchi: *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), 403–421.

[12] P. Tolksdorf: *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.
[13] P. Tolksdorf: *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), 773–817.
[14] J.L. Vázquez: *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202.
[15] Z. Zhang: *Critical points and positive solutions of singular elliptic boundary value problems*, J. Math. Anal. Appl. **302** (2005), 476–483.

Department of Mathematics and Statistics
Mississippi State University
Mississippi State
MS 39762
U.S.A.