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# $L^2$-ESTIMATES ON WEAKLY $q$-CONVEX DOMAINS

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**Abstract**

We establish an estimate on weakly $q$-convex domains in $\mathbb{C}^n$ which provides a unified approach to various existence results for the $\bar{\partial}$-problem. We also prove a Diederich–Fornaess type result for weakly $q$-convex domains.

## 1. Introduction

Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain and let $\phi \in C^2(\Omega)$ be a strictly plurisubharmonic function. A variant of Hörmander's theorem ([10]) states that for any $\bar{\partial}$-closed (0, 1)-form $f = f_{\bar{j}}\, d\overline{z_j} \in L^2_{0,1}(\Omega, \mathrm{loc})$ there exists a solution of $\bar{\partial} u = f$ satisfying

$$\int_\Omega |u|^2 e^{-\phi} \leq \int_\Omega |f|^2_{\sqrt{-1}\,\partial\bar{\partial}\phi} e^{-\phi}$$

where $|f|^2_{\sqrt{-1}\partial\bar{\partial}\phi} := \phi^{\bar{j}k} f_{\bar{j}} \overline{f_{\bar{k}}}$ and $(\phi^{\bar{j}k}) := (\phi_{j\bar{k}})^{-1}$. A geometric observation is that $\sqrt{-1}\partial\bar{\partial}\phi$ is the curvature form of the Hermitian metric $e^{-\phi}$ on the trivial line bundle. As proved in [9], the length of the (0, 1)-form could be calculated w.r.t. another curvature form. The pointwise norm $|f|^2_{\sqrt{-1}\partial\bar{\partial}\psi}$ is used in [9] instead of $|f|^2_{\sqrt{-1}\partial\bar{\partial}\varphi}$ where $\psi$ is any strictly plurisubharmonic function such that $-e^{-\psi}$ is plurisubharmonic. The latter result was then further generalized to non-plurisubharmonic weights ([7], [8], [2], [3]), i.e., the curvature of the Hermitian metric on trivial bundle is not necessarily positive. Berndtsson–Błocki–Donnelly–Fefferman type results are closely related to the Ohsawa–Takegoshi extension theorem and Bergman metric (see [4], [5], [2], [8]).

We will consider, in the present paper, the $\bar{\partial}$-problem on $q$-convex domains. We follow [11] in defining the notions of $q$-convexity and $q$-subharmonicity. We begin by recalling some basic notions and related preliminaries on exterior algebra. We prove a Diederich–Fornaess type result for weakly $q$-convex domains (Theorem 1). Let $\varphi \in C^\infty(\overline{\Omega})$ be a $q$-subharmonic function and let $\psi \in C^\infty(\overline{\Omega}))$ be a function such that the real (1,1)-form $\delta\sqrt{-1}\,\partial\bar{\partial}\varphi - \sqrt{-1}\,\partial\psi \wedge \bar{\partial}\psi$ is $q$-positive semi-definite (see Definition 3)

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for some constant $\delta \in [0, 4)$, we will establish the following a priori estimate

$$(*) \qquad \|\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_\varphi + \|\bar\partial g\|^2_\varphi \geq \frac{(2-\sqrt{\delta})^2}{4} \int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi},$$

for any $(p, q)$-form $g \in \mathrm{Dom}(\bar\partial^*) \cap C^\infty_{p,q}(\overline{\Omega})$ on weakly $q$-convex domains with smooth boundary. Here we have used the notation $F_\varphi = \varphi_{j\bar k}\, d\overline{z_k} \wedge \partial/\partial\overline{z_j} \lrcorner$. When $\psi = 0$ we can choose $\delta = 0$, so $(*)$ generalizes Hörmander's estimate to $q$-convex domains and $q$-subharmonic weight functions. Actually, $(*)$ also implies the following Donnelly–Fefferman type estimate.

$$(**) \qquad \|\bar\partial^*_{\varphi+\tau\psi} g\|^2_{\varphi+\psi} + \|\bar\partial g\|^2_{\varphi+\psi} \geq \tau^2 \int_\Omega \langle F_\psi g, g\rangle e^{-\varphi-\psi},$$

for any $g \in \mathrm{Dom}(\bar\partial^*) \cap C^\infty_{p,q}(\overline{\Omega})$ where $\varphi \in C^\infty(\overline{\Omega})$ is a $q$-subharmonic function, $\psi \in C^\infty(\overline{\Omega})$ with $-e^{-\psi}$ being $q$-subharmonic and $\tau \in (0, 1/2]$ is a constant. This estimate implies an existence theorem of Berndtsson–Błocki–Donnelly–Fefferman type (see Corollary 2 below). This kind of theorems may help produce a desired curvature term without the contribution of the metric which has important applications (e.g., Ohsawa–Takegoshi type extension theorems). The curvature operator $F_\varphi$ of a certain Hermitian metric will play an important role in our formulation of main results. Applications for $p$-convex Riemannian manifolds can be found in [12].

Here are the main results of the present paper:

**Theorem 1.** *Let $\Omega \Subset \mathbb{C}^n$ be a weakly $q$-convex domain with smooth boundary and let $r \in C^\infty(\overline{\Omega})$ be a defining function for $\Omega$. Then for any strictly $q$-subharmonic function $\phi \in C^\infty(\overline{\Omega})$, there exist constants $K > 0$, $\eta_0 \in (0, 1)$ such that for any $\eta \in (0, \eta_0)$ the function $\rho := -(-re^{-K\phi})^\eta$ is strictly $q$-subharmonic on $\Omega$.*

**Theorem 2.** *Let $\Omega$ be a weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \leq q \leq n)$ and let $\varphi \in C^2(\Omega)$ be a $q$-subharmonic function on $\Omega$ and $\psi \in C^1(\Omega)$. Assume that the real $(1,1)$-form $\delta\sqrt{-1}\,\partial\bar\partial\varphi - \sqrt{-1}\,\partial\psi \wedge \bar\partial\psi$ is $q$-positive semi-definite for some constant $\delta \in [0,4)$. Then for any $\bar\partial$-closed $(p, q)$-form $f \in L^2_{p,q}(\Omega, \mathrm{loc})$ $(0 \leq p \leq n)$, if*

$$\int_\Omega \langle F^{-1}_\varphi f, f\rangle e^{-\varphi+\psi} < \infty,$$

*there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi - \psi)$ such that*

$$\bar\partial u = f, \quad \|u\|^2_{\varphi-\psi} \leq \frac{4}{(2-\sqrt{\delta})^2} \int_\Omega \langle F^{-1}_\varphi f, f\rangle e^{-\varphi+\psi},$$

*where $F^{-1}_\varphi$ is defined by (8) and it is required implicitly that $F^{-1}_\varphi f$ is defined almost everywhere in $\Omega$.*

**Corollary 1.** *Let $\Omega$ be a weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \le q \le n)$ and let $\varphi$ be a $q$-subharmonic function on $\Omega$. Then for any $\bar\partial$-closed $(p,q)$-form $f \in L^2_{p,q}(\Omega, \mathrm{loc})$ $(0 \le p \le n)$, if*

$$\int_\Omega \langle F_\varphi^{-1} f,\ f\rangle e^{-\varphi} < \infty,$$

*there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi - \psi)$ such that*

$$\bar\partial u = f, \quad \|u\|^2_\varphi \le \int_\Omega \langle F_\varphi^{-1} f,\ f\rangle e^{-\varphi}.$$

**Corollary 2.** *Let $\Omega$ be a weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \le q \le n)$ and let $\varphi$ be a $q$-subharmonic function on $\Omega$, $\psi \in C^2(\Omega)$ be a function such that $-e^{-\psi}$ is $q$-subharmonic. For any constant $\delta \in [0, 1)$ and $\bar\partial$-closed $(p, q)$-form $f \in L^2_{p,q}(\Omega, \mathrm{loc})$ $(0 \le p \le n)$, if*

$$\int_\Omega \langle F_\psi^{-1} f,\ f\rangle e^{-\varphi+\delta\psi} < \infty$$

*then there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi - \delta\psi)$ such that*

$$\bar\partial u = f, \quad \|u\|^2_{\varphi-\delta\psi} \le \frac{4}{(1-\delta)^2} \int_\Omega \langle F_\psi^{-1} f,\ f\rangle e^{-\varphi+\delta\psi}.$$

**Corollary 3.** *Let $\Omega$ be a weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \le q \le n)$ and let $\varphi$ be a $q$-subharmonic function on $\Omega$, $\psi \in C^2(\Omega)$ be a strictly plurisubharmonic function such that $-e^{-\psi}$ is $q$-subharmonic. For any constant $\delta \in [0,1)$ and $\bar\partial$-closed $(p,q)$-form $f \in L^2_{p,q}(\Omega, \mathrm{loc})$ $(0 \le p \le n)$, if*

$$\int_\Omega \psi^{\bar jk} f_{I,\overline{jK}}\, \overline{f_{I,\overline{kK}}} e^{-\varphi+\delta\psi} < \infty$$

*then there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi - \delta\psi)$ such that*

$$\bar\partial u = f, \quad \|u\|^2_{\varphi-\delta\psi} \le \frac{4}{q^2(1-\delta)^2} \int_\Omega \psi^{\bar jk} f_{I,\overline{jK}}\, \overline{f_{I,\overline{kK}}} e^{-\varphi+\delta\psi}$$

*where $(\psi^{\bar jk}) := (\psi_{j\bar k})^{-1}$.*

**Corollary 4.** *Let $\Omega$ be a weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \le q \le n)$ and let $\varphi$ be a $q$-subharmonic function on $\Omega$, $\psi \in C^2(\Omega)$ be a $q$-subharmonic function such that $-e^{-\psi}$ is $q$-subharmonic. For any $\bar\partial$-closed $(p,q)$-form $f \in L^2_{p,q}(\Omega, \mathrm{loc})$ $(0 \le p \le n)$, if*

$$\int_\Omega \langle F_\psi^{-1} f,\ f\rangle e^{-\varphi} < \infty$$

*then there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi)$ such that*

$$\bar{\partial} u = f, \quad \|u\|^2_\varphi \le 4 \int_\Omega \langle F_\psi^{-1} f, f \rangle e^{-\varphi}.$$

**Corollary 5.** *Let $\Omega$ be a bounded weakly $q$-convex domain in $\mathbb{C}^n$ $(1 \le q \le n)$ and let $\varphi$ be a $q$-subharmonic function on $\Omega$. For any $\bar{\partial}$-closed $(p, q)$-form $f \in L^2_{p,q}(\Omega, \varphi)$ $(0 \le p \le n)$, there exists a $(p, q-1)$-form $u \in L^2_{p,q-1}(\Omega, \varphi)$ such that*

$$\bar{\partial} u = f, \quad \|u\|_\varphi \le \frac{2d}{q} \|f\|_\varphi$$

*where $d$ is the diameter of $\Omega$.*

Since there are plenty of $q$-subharmonic functions which are not plurisubharmonic when $q \ge 2$, our results provide more flexibility in choosing weights for $L^2$-estimates. Such flexibility may help us make generalizations and improvements on existence results for the $\bar{\partial}$-problem. Let $\rho$ be the function in Theorem 1 above, then it is easy to see that $-e^{-\psi}$ is strictly $q$-subharmonic on $\Omega$ where $\psi := -\log(-\rho)$, as a consequence, we obtain Theorem 2.4 in [11]. Theorem 1 was originally proved by Diederich and Fornæss ([6]) for pseudoconvex domains, i.e. the case of $q = 1$. Theorem 2 was obtained by Błocki ([5]) for (0, 1)-forms on pseudoconvex domains. Corollary 1 is a strengthen version of Theorem 3.1 in [11]. In the case of $q = 1$, Corollary 2 recovers a result due to Błocki ([3]). The arguments used in [3] and [5] do not indicate the estimates $(*)$, $(**)$. Corollary 3 above improves the main result in [1] and our Corollary 5 improves slightly a result due to Hörmander (Theorem 2.2.3 in [10]) when $q \ge 2$.

## 2. Weakly $q$-convex domains

We begin by establishing the basic notation.

We will adhere to the summation convention that sum is performed over strictly increasing multi-indices. The coordinates of $\mathbb{C}^n$ are chosen such that the standard Kähler form of $\mathbb{C}^n$ is given by $\sqrt{-1}\, dz_j \wedge d\overline{z_j}$. Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $\phi \in C^\infty(\Omega)$, we denote by $\nabla^{0,1}\phi$ the (0, 1)-part of the gradient $\nabla\phi$ of $\phi$ w.r.t. the standard Kähler metric, i.e. $\nabla^{0,1}\phi = \phi_j\, \partial/\partial\overline{z_j}$. We use $\langle \cdot, \cdot \rangle$ to denote the induced (pointwise) Hermitian inner product of $(p, q)$-forms on $\Omega$. Following [10], the weighted $L^2$ Hermitian inner product of $(p, q)$-forms will be denoted by $(\cdot, \cdot)_\phi$ and the corresponding Hilbert space will be denoted by $L^2_{p,q}(\Omega, \phi)$.

Let $\Omega$ be a domain in $\mathbb{C}^n$, $1 \le q \le n$, we recall the notion of $q$-subharmonicity ([11], [14] and [13]).

DEFINITION 1. Let $\varphi$ be an upper semi-continuous function on $\Omega$, we say $\varphi$ is $q$-subharmonic on $\Omega$ if the restriction of $\varphi$ to any $q$ dimensional complex submanifold of $\Omega$ is subharmonic w.r.t. the induced metric.

REMARK 1. It is easy to show (see [1], [14]) that for any $\varphi \in C^2(\Omega)$, $\varphi$ is $q$-subharmonic if and only if any sum of $q$ eigenvalues of the complex Hessian

$$\varphi_{i\bar{j}}\, dz_i \otimes d\overline{z_j}$$

of $\varphi$ is nonnegative. If any sum of $q$ eigenvalues of the complex Hessian is positive, $\phi$ is then called strictly $q$-subharmonic. Moreover, every $q$-subharmonic function could be approximated by a decreasing sequence of smooth $q$-subharmonic functions. It is easy to see that 1-subharmonicity is equivalent to plurisubharmonicity.

Following [11] and [13], we introduce the notion of $q$-convexity.

DEFINITION 2. Assume $\Omega$ is a smooth domain, and $r$ a defining function for $\Omega$, then we say that $\Omega$ is weakly $q$-convex if at every point $b \in \partial\Omega$ we have

$$r_{i\bar{j}}(b) g_{\overline{iK}} \overline{g_{\overline{jK}}} \geq 0$$

for every $(0, q)$-form $g = g_{\bar{J}}\, d\overline{z_J}$ such that

$$r_i g_{\overline{iK}} = 0$$

for all multi-indices $K$ with $|K| = q - 1$. For a general domain $\Omega \subseteq \mathbb{C}^n$, we call it weakly $q$-convex if it could be exhausted by smooth weakly $q$-convex domains.

REMARK 2. It is easy to see that $q$-subharmonicity (convexity) implies $(q+1)$-subharmonicity (convexity). The notions of $q$-subharmonicity and $q$-convexity are both invariant under a unitary change of coordinates, but not preserved by biholomorphic transformations.

*Assume $\Omega \subseteq \mathbb{C}^n$ is a smooth domain, and $r \in C^\infty(\overline{\Omega})$ is a defining function for $\Omega$.* Let $\phi \in C^\infty(\overline{\Omega})$ and $g \in C^\infty_{p,q}(\overline{\Omega})$ satisfy

$$r_i g_{I,\overline{iK}} = 0$$

for all multi-indices $K$ with $|I| = p$, $|K| = q - 1$, then we have the standard Kohn–Morrey–Hörmander identity

$$\begin{aligned}\|\bar{\partial} g\|_\phi^2 + \|\bar{\partial}_\phi^* g\|_\phi^2 = &\int_\Omega \partial_j \partial_{\bar{k}} \phi\, g_{I,\overline{jK}} \overline{g_{I,\overline{kK}}} e^{-\phi} \\ &+ \int_{\partial\Omega} \partial_j \partial_{\bar{k}} r\, g_{I,\overline{jK}} \overline{g_{I,\overline{kK}}} \frac{1}{|\nabla r|} e^{-\phi} \\ &+ \int_\Omega |\partial_{\bar{j}} g_{I,\bar{J}}|^2 e^{-\phi}.\end{aligned}$$

When $\Omega$ is $q$-convex, we obtain the following inequality

$$\|\bar\partial g\|^2_\phi + \|\bar\partial^*_\phi g\|^2_\phi \geq \int_\Omega \partial_j\partial_{\bar k}\phi g_{I,\overline{jK}}\overline{g_{I,\overline{kK}}}e^{-\phi}. \tag{1}$$

We denote by $\bigwedge^{p,q}$ the linear space of $(p,q)$-forms, i.e. $\bigwedge^{p,q} = \mathrm{span}_{\mathbb{C}}\{dz_I \wedge d\overline{z_J} \mid |I| = p,\ |J| = q\}$. For any real (1, 1)-form $\theta = \sqrt{-1}\theta_{j\bar k}\, dz_j \wedge d\overline{z_k}$, we introduce a self-adjoint linear operator on $\bigwedge^{p,q}$ by setting

$$F_\theta = \theta_{j\bar k}\, d\overline{z_k} \wedge \frac{\partial}{\partial \overline{z_j}}\lrcorner \tag{2}$$

where $\lrcorner$ means the interior product. We also set the notation $F_\phi := F_{\sqrt{-1}\partial\bar\partial\phi}$ for a smooth function $\phi$.

With the linear operator $F_\phi$, we can rewrite the integrand on the right hand side of (1) as follows

$$\begin{aligned} \partial_j\partial_{\bar k}\phi g_{I,\overline{jK}}\overline{g_{I,\overline{kK}}} &= \left(\phi_{j\bar k}\frac{\partial}{\partial\overline{z_j}}\lrcorner g\right)_{I,\bar K} \cdot \overline{\left(\frac{\partial}{\partial\overline{z_k}}\lrcorner g\right)_{I,\bar K}} \\ &= \left\langle \phi_{j\bar k}\frac{\partial}{\partial\overline{z_j}}\lrcorner g,\ \frac{\partial}{\partial\overline{z_k}}\lrcorner g\right\rangle \\ &= \langle F_\phi g, g\rangle. \end{aligned} \tag{3}$$

Consequently, we obtain by the Kohn–Morrey–Hörmander identity and (3)

$$\|\bar\partial g\|^2_\phi + \|\bar\partial^*_\phi g\|^2_\phi \geq \int_\Omega \langle F_\phi g, g\rangle e^{-\phi} := (F_\phi g, g)_\phi. \tag{4}$$

Denote the eigenvalues of the matrix $(\theta_{j\bar k})$ by

$$\lambda_1 \leq \cdots \leq \lambda_n,$$

after a unitary change of coordinates, we have $F_\theta = \lambda_j\, d\overline{z_j} \wedge \partial/\partial\overline{z_j}\lrcorner$. For any multi-indices $I, J$ with $|I| = p$, $|J| = q$, set

$$\lambda_{I,J} := \sum_{j\in J}\lambda_j, \tag{5}$$

it holds that

$$\begin{aligned} F_\theta\, dz_I \wedge d\overline{z_J} &= \lambda_j\, dz_I \wedge d\overline{z_j} \wedge \frac{\partial}{\partial\overline{z_j}}\lrcorner\, d\overline{z_J} \\ &= \lambda_j\, dz_I \wedge d\overline{z_j} \wedge \sum_{a=1}^{q}(-1)^{a-1}\delta_{jj_a}\, d\overline{z_{j_1}} \wedge \cdots \wedge \widehat{d\overline{z_{j_a}}} \wedge \cdots \wedge d\overline{z_{j_q}} \\ &= \sum_{j\in J}\lambda_j\, dz_I \wedge d\overline{z_J} = \lambda_{I,J}\, dz_I \wedge d\overline{z_J} \end{aligned}$$

where the circumflex over a term means that it is to be omitted. Hence eigenvalues of the map $F_\theta$ are given by

$$\lambda_{I,J}, \quad |I| = p,\ |J| = q. \tag{6}$$

DEFINITION 3. Let $\theta = \sqrt{-1}\theta_{j\bar{k}}\,dz_j \wedge d\overline{z_k}$ be a real (1,1)-form on $\mathbb{C}^n$, $1 \le q \le n$. $\theta$ is said to be $q$-positive semi-definite ($q$-positive) if $\lambda_1 + \cdots + \lambda_q \ge 0$ ($> 0$) where $\lambda_1 \le \cdots \le \lambda_n$ are the eigenvalues of the matrix $(\theta_{j\bar{k}})$.

REMARK 3. By formula (6), $\theta$ is $q$-positive semi-definite if and only if the operator $F_\theta\colon \bigwedge^{p,q} \to \bigwedge^{p,q}$ is a positive semi-definite for any $0 \le p \le n$. We have the following criterion for $q$-subharmonicity of a smooth function $\phi$.

$\phi$ is $q$-subharmonic (strictly $q$-subharmonic) on $\Omega$ if and only if $F_\phi$ is $q$-positive semi-definite (definite) at each point of $\Omega$.

Since $F_\theta\colon \bigwedge^{p,q} \to \bigwedge^{p,q}$ is self-adjoint, we have the following orthogonal decomposition

$$\bigwedge^{p,q} = \mathrm{Ker}\, F_\theta \oplus \mathrm{Im}\, F_\theta, \tag{7}$$

which implies that $F_\theta$ induces an isomorphism $F_\theta|_{\mathrm{Im}\, F_\theta}\colon \mathrm{Im}\, F_\theta \to \mathrm{Im}\, F_\theta$. We can therefore define

$$F_\theta^{-1} := (F_\theta|_{\mathrm{Im}\, F_\theta})^{-1}\colon \mathrm{Im}\, F_\theta \to \mathrm{Im}\, F_\theta \tag{8}$$

for any real (1, 1)-form $\theta$. Notice that $F_\theta$ itself is not required to be invertible in the above definition.

When $\theta$ is $q$-positive, we know by (6)

$$(F_\theta^{-1} g)_{I,\bar{J}} = \lambda_{I,J}^{-1} g_{I,\bar{J}} \tag{9}$$

holds for any $g = g_{I,\bar{J}}\, dz_I \wedge d\overline{z_J} \in \bigwedge^{p,q}$ and any given multi-indices $I$, $J$ satisfying $|I| = p$, $|J| = q$.

If the function $\phi$ is further assumed to be strictly plurisubharmonic, we denote by $(\phi^{\bar{j}k})$ the inverse matrix of the complex Hessian matrix $(\phi_{j\bar{k}})$, then we have

$$\begin{aligned}
\langle F_\phi^{-1} g, g\rangle &= \lambda_{I,J}^{-1} |g_{I,\bar{J}}|^2 \\
&= \left(\sum_{j\in J} \lambda_j\right)^{-1} |g_{I,\bar{J}}|^2 \\
&\le \frac{1}{q^2} \sum_{j\in J} \lambda_j^{-1} |g_{I,\bar{J}}|^2 \\
&\overset{(3)\,(5)}{=} \frac{1}{q^2} \phi^{\bar{j}k} g_{I,\overline{jK}} \overline{g_{I,\overline{kK}}}
\end{aligned} \tag{10}$$

for arbitrary $g = g_{I,\bar{J}}\, dz_I \wedge d\overline{z_J} \in \bigwedge^{p,q}$.

We conclude this section by proving a Diederich–Fornaess type result for smooth bounded weakly $q$-convex domains.

Proof of Theorem 1. By Remark 3, it suffices to show that $\langle F_\rho g, g\rangle > 0$ for any $0 \neq g \in \bigwedge^{0,q}$. A direct computation gives

$$\begin{aligned}\langle F_\rho g, g\rangle = \eta(-r)^{\eta-2}e^{-\eta K\phi}\cdot [Kr^2(\langle F_\phi g, g\rangle - \eta K|\nabla^{0,1}\phi \lrcorner g|^2)\\ - r(\langle F_r g, g\rangle - 2\eta K \Re\langle \nabla^{0,1}\phi\lrcorner g, \nabla^{0,1} r\lrcorner g\rangle)\\ + (1-\eta)|\nabla^{0,1} r\lrcorner g|^2].\end{aligned} \tag{11}$$

Throughout the proof, we denote by $A_1, A_2, \ldots$ various constants which are independent of $\eta$, $K$.

Since the boundary of $\Omega$ is assumed to be smooth, for any sufficiently small $\varepsilon > 0$ there is a smooth map $\pi : N_\varepsilon \to \partial\Omega$ such that

$$\operatorname{dist}(z, \partial\Omega) = |z - \pi(z)|, \quad z \in N_\varepsilon \tag{12}$$

where $N_\varepsilon := \{z \in \Omega \mid r(z) > -\varepsilon\}$. As the function $r \in C^\infty(\overline{\Omega})$ is a defining function for $\Omega$, there exists a constant $A_1 > 0$ which only depends on $\varepsilon$ such that

$$\operatorname{dist}(z, \partial\Omega) \le -A_1 r(z), \quad A_1 \le |\nabla r(z)|,\ z \in N_\varepsilon. \tag{13}$$

For any $g \in \bigwedge^{0,q}$, $z \in N_\varepsilon$, set

$$g_1(z) = \frac{1}{|\nabla^{0,1} r(z)|^2}\nabla^{0,1} r(z)\lrcorner\bar\partial r(z)\wedge g, \quad g_2(z) = \frac{1}{|\nabla^{0,1} r(z)|^2}\bar\partial r(z)\wedge \nabla^{0,1} r(z)\lrcorner g,$$

then we have $g = g_1(z) + g_2(z)$, $|g|^2 = |g_1(z)|^2 + |g_2(z)|^2$ and

$$\nabla^{0,1} r(z)\lrcorner g_1(z) = 0, \quad |g_2(z)| \le \frac{1}{|\nabla^{0,1} r(z)|}|\nabla^{0,1} r(z)\lrcorner g| \tag{14}$$

for every $z \in N_\varepsilon$. From (12) and the first inequality in (13), there is a constant $A_2 > 0$ such that

$$\begin{aligned}|\langle F_r g_1, g_1\rangle(z) - \langle F_r g_1, g_1\rangle(\pi(z))| &= \left|\int_0^1 \frac{d}{dt}\langle F_r g_1, g_1\rangle(tz + (1-t)\pi(z))\, dt\right|\\ &\le -A_2 r(z)|g|^2\end{aligned} \tag{15}$$

holds for any $z \in N_\varepsilon$. By (3), the identity in (14) and Definition 2, we get

$$\langle F_r g_1, g_1\rangle(\pi(z)) \ge 0, \quad z \in N_\varepsilon.$$

Therefore, for any $z \in N_\varepsilon$, the following estimate follows from (15)

$$\langle F_r g_1, g_1\rangle(z) \geq A_2 r(z)|g|^2.$$

Taking into account of the inequality in (14) and $|g_1(z)| \leq |g|$, the above estimate implies that

$$\langle F_r g, g\rangle(z) \geq A_2 r(z)|g|^2 - \frac{A_3}{|\nabla^{0,1} r(z)|}|\nabla^{0,1} r(z) \lrcorner g||g| \tag{16}$$

holds for any $z \in N_\varepsilon$ where $A_3 > 0$ is another constant.

Since $\phi$ is strictly $q$-subharmonic on $\overline{\Omega}$, there is a constant $\sigma > 0$ such that

$$\langle F_\phi g, g\rangle(z) - \eta K|\nabla^{0,1}\phi(z)\lrcorner g|^2 \geq (\sigma - A_4\eta K)|g|^2 \tag{17}$$

holds for any $z \in \Omega$ where $A_4 := \sup_\Omega|\nabla^{0,1}\phi|^2$. From (11) and (17), there exists a constant $A_5 > 0$ such that

$$\langle F_\rho g, g\rangle(z) \geq \eta(-r)^{\eta-2}e^{-\eta K\phi}\left[Kr^2(z)\left(\sigma - \frac{\eta}{1-\eta}A_4 K\right) - A_5\right]|g|^2 \tag{18}$$

holds for any $z \in \Omega$.

When $K > 4A_5/(\sigma\varepsilon^2)$ and $\eta \in (0, \sigma/(2A_4K + \sigma))$, (18) implies that

$$\langle F_\rho g, g\rangle \geq \frac{1}{4}\eta(-r)^{\eta-2}e^{-\eta K\phi}K\varepsilon^2\sigma|g|^2 \tag{19}$$

holds on $\Omega \setminus N_\varepsilon$.

Similarly, for any constants $\eta \in (0, \sigma/(2A_4K))$ and $K > (4/\sigma)(A_2 + (\sigma^2/(4A_4)) + 2A_6^2 + \sigma^2)$, $A_6 := A_3/(2A_1)$, from (11), (16) and (17) it follows that the following inequality holds on $N_\varepsilon$

$$\begin{aligned}
\langle F_\rho g, g\rangle &\geq \eta(-r)^{\eta-2}e^{-\eta K\phi}[K(\sigma - A_4\eta K) - A_2]r^2|g|^2 \\
&\quad + 2(A_6 + A_4\eta K)|\nabla^{0,1} r\lrcorner g|r|g| \\
&\quad + (1-\eta)|\nabla^{0,1} r\lrcorner g|^2 \\
&\geq \eta(-r)^{\eta-2}e^{-\eta K\phi}\left[K(\sigma - A_4\eta K) - A_2 - \frac{2A_6^2 + 2A_4^2\eta^2K^2}{1-\eta}\right]r^2|g|^2 \\
&\geq \eta(-r)^{\eta-2}e^{-\eta K\phi}\left(\frac{K\sigma}{2} - A_2 - 4A_6^2 - \sigma^2\right)r^2|g|^2 \\
&\geq \frac{1}{4}\eta(-r)^{\eta-2}e^{-\eta K\phi}Kr^2\sigma|g|^2.
\end{aligned} \tag{20}$$

By combining (19) and (20), we know Theorem 1 is true for any constant $K > (4/\sigma)(A_2 + \sigma^2/(4A_4) + A_5/\varepsilon^2 + 2A_6^2 + \sigma^2)$ and $\eta_0 := \sigma/(2A_4K + \sigma)$. □

## 3. Donnelly–Fefferman type estimate

We will prove, in this section, the existence results in the present paper. The key for our proofs is to establish an a priori estimate of Donnelly–Fefferman type from which we get the existence theorem 1. Since the constant $\delta$ involved in this estimate would be allowed to have value zero, we also obtain an existence result of Donnelly–Fefferman type and Hömander type (with one weight function). We first recall a basic lemma from functional analysis which is due to Hörmander (see [10]).

**Lemma.** *Let* $H_1 \xrightarrow{T} H_2 \xrightarrow{S} H_3$ *be a complex of closed and densely defined operators between Hilbert spaces. For any* $f \in \operatorname{Ker} S$ *and any constant* $C > 0$, *the following conditions are equivalent.*
1. *There exists some* $u \in H_1$ *such that* $Tu = f$ *and* $\|u\|_{H_1} \leq C$.
2. $|(f, g)_{H_2}|^2 \leq C^2(\|T^*g\|^2_{H_1} + \|Sg\|^2_{H_3})$ *holds for each* $g \in \operatorname{Dom}(T^*) \cap \operatorname{Dom}(S)$.

Proof of Theorem 2. We consider first the case where $\Omega$ is a bounded domain in $\mathbb{C}^n$ with smooth boundary and $\varphi, \psi \in C^\infty(\overline{\Omega})$

We will apply the above lemma to following weighted $L^2$-spaces of differential forms

$$H_1 = L^2_{p,q-1}\left(\Omega, \varphi - \frac{1}{2}\psi\right), \quad H_2 = L^2_{p,q}\left(\Omega, \varphi - \frac{1}{2}\psi\right), \quad H_3 = L^2_{p,q+1}\left(\Omega, \varphi - \frac{1}{2}\psi\right)$$

and the operators

$$T = \bar{\partial} \circ e^{-(1/4)\psi}, \quad S = e^{-(1/4)\psi} \circ \bar{\partial}.$$

In order to use the above lemma, we need to show that the following estimate

$$\begin{aligned} &|(f, g)_{\varphi-(1/2)\psi}|^2 \\ &\leq \frac{4(F_\varphi^{-1} f,\ f)_{\varphi-\psi}}{(2-\sqrt{\delta})^2}(\|e^{(-1/4)\psi}\bar{\partial}^*_{\varphi-(1/2)\psi} g\|^2_{\varphi-(1/2)\psi} + \|e^{-(1/4)\psi}\bar{\partial} g\|^2_{\varphi-(1/2)\psi}) \end{aligned} \tag{21}$$

holds for arbitrary $g \in \operatorname{Dom}(\bar{\partial}^*) \cap C^\infty_{p,q}(\overline{\Omega})$.

Let $g \in \operatorname{Dom}\bar{\partial}^* \cap C^\infty_{p,q}(\overline{\Omega})$, from

$$\bar{\partial}^*_\varphi g = \bar{\partial}^*_{\varphi-(1/2)\psi} g + \frac{1}{2}\nabla^{0,1}\psi \lrcorner g,$$

by using Cauchy's inequality with $\varepsilon$, it follows that

$$\|\bar{\partial}^*_\varphi g\|^2_\varphi \leq \frac{1+\epsilon}{\epsilon}\|\bar{\partial}^*_{\varphi-(1/2)\psi} g\|^2_\varphi + \frac{1+\epsilon}{4}\|\nabla^{0,1}\psi \lrcorner g\|^2_\varphi$$

for any positive constant $\epsilon$. For any $\varepsilon \in [0, 4)$, let

$$\epsilon = \frac{2}{\sqrt{\delta}} - 1,$$

then the above inequality becomes

$$\|\bar\partial^*_\varphi g\|^2_\varphi \le \frac{2}{2-\sqrt{\delta}}\|\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_\varphi + \frac{1}{2\sqrt{\delta}}\|\nabla^{0,1}\psi \lrcorner g\|^2_\varphi. \tag{22}$$

Since $\delta\sqrt{-1}\partial\bar\partial\varphi - \sqrt{-1}\partial\psi \wedge \bar\partial\psi$ is $q$-positive semi-definite, we get the following inequality

$$\delta\langle F_\varphi g, g\rangle \ge |\nabla^{0,1}\psi \lrcorner g|^2. \tag{23}$$

Substituting (22) and (23) into Hörmander's estimate (4), the $q$-subharmonicity of $\varphi$ gives

$$\begin{aligned}\frac{2}{2-\sqrt{\delta}}\|\bar\partial^*_{\varphi-\frac{1}{2}\psi} g\|^2_\varphi + \|\bar\partial g\|^2_\varphi &\ge \|\bar\partial^*_\varphi g\|^2_\varphi + \|\bar\partial g\|^2_\varphi - \frac{1}{2\sqrt{\delta}}\|\nabla^{0,1}\psi \lrcorner g\|^2_\varphi \\ &\ge \frac{2-\sqrt{\delta}}{2}\int_\Omega \langle F_\psi g, g\rangle e^{-\varphi}\end{aligned}$$

which further implies the desired estimate $(*)$ as follows

$$\begin{aligned}\|e^{-(1/4)\psi}\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_{\varphi-(1/2)\psi} + \|e^{-(1/4)\psi}\bar\partial g\|^2_{\varphi-(1/2)\psi} &= \|\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_\varphi + \|\bar\partial g\|^2_\varphi \\ &\ge \|\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_\varphi + \frac{2-\sqrt{\delta}}{2}\|\bar\partial g\|^2_\varphi \\ &\ge \frac{(2-\sqrt{\delta})^2}{4}\int_\Omega \langle F_\varphi g, g\rangle e^{-\varphi}.\end{aligned}$$

Since $\varphi$ is $q$-subharmonic, the Cauchy–Schwarz inequality applied to the positive semi-definite Hermitian form $(F_\varphi\,\cdot\,,\,\cdot\,)_\varphi$ gives

$$\begin{aligned}|(f, g)_{\varphi-\frac{1}{2}\psi}|^2 &= |(F_\varphi \circ F_\varphi^{-1} e^{(1/2)\psi} f, g)_\varphi|^2 \\ &\le (e^{(1/2)\psi} f, e^{(1/2)\psi} F_\varphi^{-1} f)_\varphi (F_\varphi g, g)_\varphi \\ &\le \frac{4(F_\varphi^{-1} f, f)_{\varphi-\psi}}{(2-\sqrt{\delta})^2}(\|e^{-(1/4)\psi}\bar\partial^*_{\varphi-(1/2)\psi} g\|^2_{\varphi-\frac{1}{2}\psi} + \|e^{-(1/4)\psi}\bar\partial g\|^2_{\varphi-(1/2)\psi})\end{aligned}$$

where $F_\varphi^{-1}$ is defined by (8). Thus the estimate (21) has been proved for $g \in \mathrm{Dom}\bar\partial^* \cap C^\infty_{p,q}(\overline{\Omega})$. By using the density lemma (Proposition 1.2.4 in [10]), we know that (22) holds for any $g \in \mathrm{Dom}(T^*) \cap \mathrm{Dom}(S)$. Consequently, by the lemma we mentioned at the beginning of this section, there exists some $v \in L^2_{p,q-1}(\Omega, \varphi - (1/2)\psi)$ such that

$$Tv = f, \quad \|v\|^2_{\varphi-(1/2)\psi} \le \frac{4}{(2-\sqrt{\delta})^2}(F_\varphi^{-1} f, f)_{\varphi-\psi}.$$

Set $u = e^{-(1/4)\psi} v$, then we get $u \in L^2_{p,q-1}(\Omega, \varphi - \psi)$ and

$$\bar{\partial} u = f, \quad \|u\|^2_{\varphi-\psi} = \|v\|^2_{\varphi-(1/2)\psi} \le \frac{4}{(2-\sqrt{\delta})^2}(F_\varphi^{-1} f, f)_{\varphi-\psi}. \tag{24}$$

Theorem 2 now follows, in its full generality, from (24), the standard argument of smooth approximation and taking weak limit (see e.g. [10]). □

Proof of Corollary 1. Corollary 1 follows from Theorem 2 by choosing $\delta = 0$ and $\psi = 0$. □

Proof of Corollary 2. Let $\varphi_1 = \varphi + \psi$ and $\psi_1 = (1+\delta)\psi$, then $\varphi_1$ is $q$-subharmonic. Since

$$(1+\delta)^2\sqrt{-1}\,\partial\bar{\partial}\varphi_1 - \sqrt{-1}\,\partial\psi_1 \wedge \bar{\partial}\psi_1 = (1+\delta)^2[\sqrt{-1}\,\partial\bar{\partial}\varphi + \sqrt{-1}e^{\psi}\,\partial\bar{\partial}(-e^{-\psi})],$$

the assumption that $\varphi$ and $-e^{-\psi}$ are both $q$-subharmonic functions implies that $(1+\delta)^2\sqrt{-1}\,\partial\bar{\partial}\varphi_1 - \sqrt{-1}\,\partial\psi_1 \wedge \bar{\partial}\psi_1$ is $q$-positive semi-definite. Applying Theorem 2 to the weights $\varphi_1$ and $\psi_1$, we obtain Corollary 2. □

Proof of Corollary 3. Corollary 3 follows directly from Corollary 2 and the pointwise inequality (10). □

Proof of Corollary 4. Corollary 4 follows directly from Corollary 2 by choosing the constant $\delta$ to be 0. □

Proof of Corollary 5. Without loss of generality, we assume that $\Omega$ contains the origin of $\mathbb{C}^n$. Let $\psi = q|z|^2/d^2$, then (9) implies that $F_\psi^{-1} = (d^2/q^2)\mathrm{Id}$ on $(p,q)$-forms. Since the complex Hessian of $-e^{-\psi}$ is given by

$$\frac{q}{d^2}e^{-\psi}\left(dz_i \otimes d\overline{z_i} - \frac{q}{d^2}z_i\,dz_i \otimes z_j\,d\overline{z_j}\right),$$

we know that any sum of $q$ eigenvalues of the complex Hessian of $-e^{-\psi}$ is no less than

$$\frac{q}{d^2}e^{-\psi}\left[\left(1 - \frac{q}{d^2}|z|^2\right) + q - 1\right] = \frac{q^2}{d^2}e^{-\psi}\left(1 - \frac{|z|^2}{d^2}\right) \ge 0.$$

So $-e^{-\psi}$ is, by definition, a $q$-subharmonic function on $\Omega$ (but not plurisubharmonic). Applying Corollary 4 with the weight $\psi = q|z|^2/d^2$, we obtain the following estimate for the solution $u$

$$\|u\|_{\varphi}^2 \leq \frac{4d^2}{q^2}\|f\|_{\varphi}^2.$$

This completes the proof of Corollary 5. □

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## References


[1] H. Ahn and N.Q. Dieu: *The Donnelly–Fefferman theorem on $q$-pseudoconvex domains*, Osaka J. Math. **46** (2009), 599–610.

[2] B. Berndtsson: *The extension theorem of Ohsawa–Takegoshi and the theorem of Donnelly–Fefferman*, Ann. Inst. Fourier (Grenoble) **46** (1996), 1083–1094.

[3] Z. Błocki: *A note on the Hörmander, Donnelly–Fefferman, and Berndtsson $L^2$-estimates for the $\bar{\partial}$-operator*, Ann. Polon. Math. **84** (2004), 87–91.

[4] Z. Błocki: *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. **357** (2005), 2613–2625.

[5] Z. Błocki: *Suita conjecture and the Ohsawa–Takegoshi extension theorem*, to appear in Invent. Math.

[6] K. Diederich and J.E. Fornaess: *Pseudoconvex domains*: *bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), 129–141.

[7] K. Diederich and G. Herbort: *Extension of holomorphic $L^2$-functions with weighted growth conditions*, Nagoya Math. J. **126** (1992), 141–157.

[8] K. Diederich and T. Ohsawa: *An estimate for the Bergman distance on pseudoconvex domains*, Ann. of Math. (2) **141** (1995), 181–190.

[9] H. Donnelly and C. Fefferman: *$L^2$-cohomology and index theorem for the Bergman metric*, Ann. of Math. (2) **118** (1983), 593–618.

[10] L. Hörmander: *$L^2$ estimates and existence theorems for the $\bar{\partial}$ operator*, Acta Math. **113** (1965), 89–152.

[11] L.-H. Ho: *$\bar{\partial}$-problem on weakly $q$-convex domains*, Math. Ann. **290** (1991), 3–18.

[12] Q. Ji, X. Liu and G. Yu: *$L^2$-estimates on $p$-convex Riemannian manifolds*, Adv. Math. **253** (2014), 234–280.

[13] T. Napier and M. Ramachandran: *The Bochner–Hartogs dichotomy for weakly* 1*-complete Kähler manifolds*, Ann. Inst. Fourier (Grenoble) **47** (1997), 1345–1365.

[14] H. Wu: *On certain Kähler manifolds which are $q$-complete*; in Complex Analysis of Several Variables (Madison, Wis., 1982), Proc. Sympos. Pure Math. **41**, Amer. Math. Soc., Providence, RI, 1984, 253–276.

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