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$\mathbb{Q}$-TRIVIAL GENERALIZED BOTT MANIFOLDS

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Abstract

When the cohomology ring of a generalized Bott manifold with $\mathbb{Q}$-coefficient is isomorphic to that of a product of complex projective spaces $\mathbb{C}P^{n_i}$, the generalized Bott manifold is said to be $\mathbb{Q}$-trivial. We find a necessary and sufficient condition for a generalized Bott manifold to be $\mathbb{Q}$-trivial. In particular, every $\mathbb{Q}$-trivial generalized Bott manifold is diffeomorphic to a $\prod_{n_i>1}\mathbb{C}P^{n_i}$-bundle over a $\mathbb{Q}$-trivial Bott manifold.

1. Introduction

A *generalized Bott tower of height h* is a sequence of complex projective space bundles

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\}, \tag{1.1}$$

where $B_i = P(\underline{\mathbb{C}} \oplus \xi_i)$, $\underline{\mathbb{C}}$ is a trivial complex line bundle, ξ_i is a Whitney sum of n_i complex line bundles over B_{i-1}, and $P(\,\cdot\,)$ stands a projectivization. Each B_i is called an *i-stage generalized Bott manifold*. When all n_i's are 1 for $i = 1, \dots, h$, the sequence (1.1) is called a *Bott tower of height h* and B_i is called an *i-stage Bott manifold*.

A (h-stage) generalized Bott manifold is said to be $\mathbb{Q}$-*trivial* (respectively, $\mathbb{Z}$-*trivial*) if $H^*(B_h;\mathbb{Q}) \cong H^*\bigl(\prod_{i=1}^{h}\mathbb{C}P^{n_i};\mathbb{Q}\bigr)$ (respectively, $H^*(B_h;\mathbb{Z}) \cong H^*\bigl(\prod_{i=1}^{h}\mathbb{C}P^{n_i};\mathbb{Z}\bigr)$). It is shown in [4] that if B_h is $\mathbb{Z}$-trivial, then every fiber bundle in the tower (1.1) is trivial so that B_h is diffeomorphic to $\prod_{i=1}^{h}\mathbb{C}P^{n_i}$. Furthermore, Choi and Masuda show that every ring isomorphism between $\mathbb{Z}$-cohomology rings of two $\mathbb{Q}$-trivial Bott manifolds is induced by some diffeomorphism between them (see Theorem 3.1 and [2]).

We find a necessary and sufficient condition for a generalized Bott manifold to be $\mathbb{Q}$-trivial. Namely, we have the following proposition.

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Proposition 1.1. *An h-stage generalized Bott manifold B_h is $\mathbb{Q}$-trivial if and only if each vector bundle ξ_i, $i = 1, \dots, h$, satisfies*

$$(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k \tag{1.2}$$

for $k = 1, \dots, n_i + 1$, where $B_i = P(\underline{\mathbb{C}} \oplus \xi_i)$.

Moreover, the following theorem says that a $\mathbb{Q}$-trivial generalized Bott manifold without $\mathbb{C}P^1$-fibration is weakly equivariantly diffeomorphic to a trivial generalized Bott manifold.

Theorem 1.2. *Let B_h be a generalized Bott manifold such that all n_i's are greater than* 1. *Then the following are equivalent*
(1) *B_h is $\mathbb{Q}$-trivial,*
(2) *total Chern class $c(\xi_i)$ is trivial for each $i = 1, \dots, h$,*
(3) *B_h is $\mathbb{Z}$-trivial, and*
(4) *B_h is diffeomorphic to the product of projective spaces $\prod_{i=1}^{h} \mathbb{C}P^{n_i}$.*

In the light of Theorem 1.2, we have a natural question.

QUESTION 1.1. Let B_h and B'_h be generalized Bott manifolds with $n_i > 1$, $i = 1, \dots, h$. Is $H^*(B_h; \mathbb{Z})$ isomorphic to $H^*(B'_h; \mathbb{Z})$ if $H^*(B_h; \mathbb{Q}) \cong H^*(B'_h; \mathbb{Q})$?

Unfortunately, Example 3.1 shows that the answer to the question is negative.

From the proposition, we can deduce the following theorem.

Theorem 1.3. *Every $\mathbb{Q}$-trivial generalized Bott manifold is diffeomorphic to a $\prod_{n_i > 1} \mathbb{C}P^{n_i}$-bundle over a $\mathbb{Q}$-trivial Bott manifold.*

The remainder of this paper is organized as follows. In Section 2, we recall general facts on a generalized Bott manifold and deal with its cohomology ring. In Section 3, we prove Proposition 1.1, Theorems 1.2 and 1.3.

2. Cohomology ring of a generalized Bott manifold

Let B be a smooth manifold and let E be a complex vector bundle over B. Let $P(E)$ denote the projectivization of E. Let $y \in H^2(P(E))$ be the negative of the first Chern class of the tautological line bundle over $P(E)$. Then $H^*(P(E))$ can be viewed as an algebra over $H^*(B)$ via $\pi^*\colon H^*(B) \to H^*(P(E))$, where $\pi\colon P(E) \to B$ denotes the projection. When $H^*(B)$ is finitely generated and torsion free (this is the case when

B is a toric manifold), π^* is injective and $H^*(P(E))$ as an algebra over $H^*(B)$ is known to be described as

$$H^*(P(E)) = H^*(B)[y] \Big/ \Big\langle \sum_{k=0}^{n} c_k(E) y^{n-k} \Big\rangle, \tag{2.1}$$

where n denotes the complex dimension of the fiber of E (see [1]).

For a generalized Bott manifold B_h in (1.1), since $\pi_j^*\colon H^*(B_{j-1}) \to H^*(B_j)$ is injective, we regard $H^*(B_{j-1})$ as a subring of $H^*(B_j)$ for each j so that we have a filtration

$$H^*(B_h) \supset H^*(B_{h-1}) \supset \cdots \supset H^*(B_1).$$

Let $x_j \in H^2(B_j)$ denote minus the first Chern class of the tautological line bundle over $B_j = P(\underline{\mathbb{C}} \oplus \xi_j)$. We may think of x_j as an element of $H^2(B_i)$ for $i \geq j$. Then the repeated use of (2.1) shows that the ring structure of $H^*(B_h)$ can be described as

$$H^*(B_h) = \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h\rangle.$$

Let $\xi_{2,1}$ be the tautological line bundle over $B_1 = \mathbb{C}P^{n_1}$ and let $\xi_{3,1} = \pi_2^*(\xi_{2,1})$ the pull-back bundle of the tautological line bundle over B_1 to B_2 via the projection $\pi_2\colon B_2 \to B_1$. In general, let $\xi_{j,j-1}$ be the tautological line bundle over B_{j-1} and we define inductively

$$\xi_{j,j-k} = \pi_{j-1}^* \circ \cdots \circ \pi_{j-k+1}^*(\xi_{j-k+1,j-k})$$

for $k = 2, \ldots, j-1$. Then one can see that the Whitney sum of complex line bundles ξ_i over B_{i-1} in the sequence (1.1) can be written as

$$\xi_i := \left(\xi_{i,1}^{a_{11}^i} \otimes \cdots \otimes \xi_{i,i-1}^{a_{1,i-1}^i}\right) \oplus \cdots \oplus \left(\xi_{i,1}^{a_{n_i,1}^i} \otimes \cdots \otimes \xi_{i,i-1}^{a_{n_i,i-1}^i}\right)$$

for some integers $a_{11}^i, \ldots, a_{n_i,i-1}^i$. Note that $\xi_1 = (\underline{\mathbb{C}})^{n_1}$. Hence, the total Chern class of ξ_i is

$$c(\xi_i) = \prod_{j=1}^{n_i} \left(1 + \sum_{k=1}^{i-1} a_{jk}^i x_k\right). \tag{2.2}$$

Therefore, the cohomology ring of B_h is

$$\begin{aligned} &H^*(B_h;\mathbb{Z}) \\ &= \mathbb{Z}[x_1, \ldots, x_h]/\langle x_i^{n_i+1} + c_1(\xi_i)x_i^{n_i} + \cdots + c_{n_i}(\xi_i)x_i \mid i = 1, \ldots, h\rangle \\ &= \mathbb{Z}[x_1, \ldots, x_h] \Big/ \Big\langle x_i \prod_{j=1}^{n_i} \Big(\sum_{k=1}^{i-1} a_{jk}^i x_k + x_i\Big) \;\Big|\; i = 1, \ldots, h \Big\rangle. \end{aligned} \tag{2.3}$$

REMARK 1. We can associate a generalized Bott manifold B_h with an $h \times h$ vector matrix A as follows:

$$A^T = \begin{pmatrix} \mathbf{1} & & & \\ \mathbf{a}_1^2 & \mathbf{1} & & \\ \vdots & \vdots & \ddots & \\ \mathbf{a}_1^h & \mathbf{a}_2^h & \cdots & \mathbf{1} \end{pmatrix}, \tag{2.4}$$

where

$$\mathbf{a}_k^i = \begin{pmatrix} a_{1k}^i \\ \vdots \\ a_{n_i k}^i \end{pmatrix}$$

and

$$\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Moreover we can consider B_h as a quasitoric manifold over the product of simplices $\prod_{i=1}^h \Delta^{n_i}$ with the reduced characteristic matrix $\Lambda_* = -A^T$.

3. $\mathbb{Q}$-trivial generalized Bott manifolds

As we mentioned in the introduction, Choi and Masuda classify $\mathbb{Q}$-trivial Bott manifolds as follows.

Theorem 3.1 ([2]). (1) *A Bott manifold B_h is $\mathbb{Q}$-trivial if and only if for each $i = 1, \dots, h$, each line bundle ξ_i satisfies $c_1(\xi_i)^2 = 0$ in $H^*(B_h; \mathbb{Z})$.*
(2) *Every ring isomorphism φ between two $\mathbb{Q}$-trivial Bott manifolds B_h and B'_h is induced by some diffeomorphism $B_h \to B'_h$.*

In this section we shall prove Proposition 1.1 and Theorem 1.2. To prove them, we need the following lemmas.

Lemma 3.2. *If a generalized Bott manifold B_h is $\mathbb{Q}$-trivial, then there exist linearly independent primitive elements $z_1, \dots, z_h$ in $H^2(B_h; \mathbb{Z})$ such that $z_i^{n_i}$ is not zero but $z_i^{n_i+1}$ is zero in $H^*(B_h; \mathbb{Z})$ for $i = 1, \dots, h$.*

Proof. Let $H^*(B_h; \mathbb{Z})$ be generated by $x_1, \dots, x_h$ as in (2.3) and let

$$H^*\left(\prod_{i=1}^h B_h; \mathbb{Q}\right) = \mathbb{Q}[y_1, \dots, y_h]/\langle y_i^{n_i+1} \mid i = 1, \dots, h\rangle.$$

Since both $\{x_1,\dots,x_h\}$ and $\{y_1,\dots,y_h\}$ are sets of generators of $H^2(B_h;\mathbb{Q})$, we can write

$$y_i = \sum_{j=1}^{h} c_{ij}x_j \quad \text{for} \quad i = 1, \dots, h \quad \text{and} \quad c_{ij} \in \mathbb{Q},$$

where the determinant of the matrix $C = (c_{ij})_{h\times h}$ is non-zero. We may assume that c_{ij}'s are irreducible fractions. Multiplying $(c_{i,1}, \dots, c_{i,h})$ by the least common denominator r_i of a set $\{c_{i,1},\dots,c_{i,h}\}$, we can get a primitive element $z_i = r_i y_i = r_i \sum_{j=1}^{h} c_{ij}x_j$ in $H^2(B_h;\mathbb{Z})$ such that $z_i^{n_i+1} = r_i^{n_i+1}y_i^{n_i+1}$ is zero in $H^*(B_h;\mathbb{Z})$ for each $i = 1, \dots, h$. Since the elements $y_1, \dots, y_h$ are linearly independent, the elements $z_1, \dots, z_h$ are also linearly independent. Since $y_i^{n_i}$ is not zero in $H^*(B_h;\mathbb{Q})$, $z_i^{n_i}$ cannot be zero in $H^*(B_h;\mathbb{Z})$. This proves the lemma. □

Lemma 3.3 ([4]). *Let B_m be an m-stage generalized Bott manifold. Then the set*

$$\{bx_m + w \in H^2(B_m) \mid 0 \neq b \in \mathbb{Z},\ w \in H^2(B_{m-1}),\ (bx_m + w)^{n_m+1} = 0\}$$

lies in a one-dimensional subspace of $H^2(B_m)$ if it is non-empty.

Proof. To satisfy $(bx_m + w)^{n_m+1} = 0$, we need $bc_1(\xi_m) = (n_m + 1)w$. □

Lemma 3.4 ([4]). *For an element $z = \sum_{i=1}^{h} b_i x_i \in H^2(B_h)$, if b_i is non-zero, then z^{n_i} cannot be zero in $H^*(B_h)$.*

Proof. If we expand $\left(\sum_{i=1}^{h} b_i x_i\right)^{n_i}$, there appears a non-zero scalar multiple of $x_i^{n_i}$ because $b_i \neq 0$. Then, z^{n_i} cannot belong to the ideal generated by the polynomials $x_i \prod_{j=1}^{n_i}\left(\sum_{k=1}^{i-1} a^i_{jk}x_k + x_i\right)$, hence it is not zero in $H^*(B_h)$. □

Now we can prove Proposition 1.1.

Proof of Proposition 1.1. If each vector bundle ξ_i satisfies the conditions (1.2), then $(x_i + 1/(n_i + 1)c_1(\xi_i))^{n_i+1}$ is zero in $H^*(B_h;\mathbb{Q})$. Since the set

$$\left\{ x_i + \frac{1}{n_i+1}c_1(\xi_i) \,\middle|\, i = 1, \dots, h \right\}$$

generates $H^*(B_h;\mathbb{Q})$ as a graded ring, this shows that B_h is $\mathbb{Q}$-trivial.

Conversely, if a generalized Bott manifold is $\mathbb{Q}$-trivial, then there are linearly independent and primitive elements $z_1, \dots, z_h$ in $H^2(B_h;\mathbb{Z})$ such that $z_i^{n_i+1}$ is zero but $z_i^{n_i}$ is not zero in $H^*(B_h)$ by Lemma 3.2. We can put $z_i = \sum_{j=1}^{h} b_{ij}x_j$ with $b_{ij} \in \mathbb{Z}$ for each $i = 1, \dots, h$.

Now, consider a map $\mu\colon \{1,\dots,h\} \to \mathbb{N}$ given by $j \mapsto n_j$. Further assume that the image of μ is the set $\{N_1,\dots,N_m\}$ with $N_1 < \cdots < N_m$. We will show inductively that each z_i can be written as $r_i(x_i + 1/(\mu(i)+1)c_1(\xi_i))$ for some $r_i \in \mathbb{Z}\setminus\{0\}$.

CASE 1: Assume $i \in \mu^{-1}(N_1)$. Let $\mu^{-1}(N_1) := \{i_1,\dots,i_\alpha\}$ with $i_1 < \cdots < i_\alpha$. We have $z_i^{N_1+1} = 0$. Then, by Lemma 3.4, we can see that

$$z_i = \sum_{j\in\mu^{-1}(N_1)} b_{ij}x_j, \tag{3.1}$$

that is, $b_{ij'} = 0$ for $j' \notin \mu^{-1}(N_1)$. Note that for each $i \in \mu^{-1}(N_1)$, one of b_{ij}'s is nonzero for $j \in \mu^{-1}(N_1)$ because the set $\{z_i \mid i \in \mu^{-1}(N_1)\}$ is linearly independent. For some $i_p \in \mu^{-1}(N_1)$, if $b_{i_p i_\alpha}$ is nonzero, then $z_{i_p} \in H^2(B_{i_\alpha})$ and $b_{i i_\alpha} = 0$ for all $i \in \mu^{-1}(N_1)\setminus\{i_p\}$ by Lemma 3.3. Put $w_{i_\alpha} := z_{i_p}$. If $b_{i_q i_{\alpha-1}}$ is nonzero for some $i_q \in \mu^{-1}(N_1)\setminus\{i_p\}$, then $z_{i_q} \in H^2(B_{i_{\alpha-1}})$ and $b_{i i_{\alpha-1}} = 0$ for all $i \in \mu^{-1}(N_1)\setminus\{i_p, i_q\}$. Now, put $w_{i_{\alpha-1}} := z_{i_q}$. In this way, for each $i \in \mu^{-1}(N_1)$, we can obtain $w_i \in H^2(B_i)$ such that $w_i \notin H^2(B_{i-1})$ and $w_i^{N_1+1} = 0$ in $H^*(B_h)$. Moreover, from the proof of Lemma 3.3, we can write

$$w_i := r_i\left(x_i + \frac{1}{N_1+1}c_1(\xi_i)\right) \in H^2(B_i) \tag{3.2}$$

for each $i \in \mu^{-1}(N_1)$. In particular, if $N_1 = 1$, then w_i is of the form either $\pm x_i$ or $\pm(2x_i + c_1(\xi_i))$ for $i \in \mu^{-1}(N_1)$. Furthermore, without loss of generality, we may assume that $z_i = w_i$ for $i \in \mu^{-1}(N_1)$.

CASE 2: Assume that $z_k = r_k(x_k + 1/(\mu(k)+1)c_1(\xi_k))$ for $N_1 \le \mu(k) \le N_{n-1}$ and let $l \in \mu^{-1}(N_n)$. Then we have $z_l^{N_n+1} = 0$. Then by Lemma 3.4, we can easily see that

$$z_l = \sum_{k\in\mu^{-1}(N_{<n})} b_{lk}x_k + \sum_{j\in\mu^{-1}(N_n)} b_{lj}x_j,$$

where $N_{<n} = \{N_1,\dots,N_{n-1}\}$. That is, $b_{lj'} = 0$ for $j' \notin \mu^{-1}(N_{\le n})$. Since $z_l^{N_n+1}$ is zero in $H^*(B_h)$, we have

$$\begin{aligned} &\left(\sum_{k\in\mu^{-1}(N_{<n})} b_{lk}x_k + \sum_{j\in\mu^{-1}(N_n)} b_{lj}x_j\right)^{N_n+1} \\ &= \sum_{k\in\mu^{-1}(N_{<n})} f_k(x_1,\dots,x_h)(x_k^{\mu(k)+1} + c_1(\xi_k)x_k^{\mu(k)} + \cdots + c_{\mu(k)}(\xi_k)x_k) \\ &\quad + \sum_{j\in\mu^{-1}(N_n)} b_{lj}^{N_n+1}(x_j^{N_n+1} + c_1(\xi_j)x_j^{N_n} + \cdots + c_{N_n}(\xi_j)x_j) \end{aligned} \tag{3.3}$$

as polynomials, where $f_k(x_1,\dots,x_h)$ is a homogeneous polynomial of degree $N_n - \mu(k)$ for each $k \in \mu^{-1}(N_{<n})$. Note that for each $l \in \mu^{-1}(N_n)$, one of b_{lj}'s is non-zero

for $j \in \mu^{-1}(N_n)$ from the linearly independency of the set $\{z_i \mid i \in \mu^{-1}(N_{\leq n})\}$. Let $\mu^{-1}(N_n) := \{l_1, \dots, l_\beta\}$ with $l_1 < \cdots < l_\beta$. Assume $b_{l_p l_\beta}$ is nonzero for some $l_p \in \mu^{-1}(N_n)$. Substituting $l = l_p$ into (3.3) and comparing the monomials containing $x_{l_\beta}^{N_n}$ as a factor on both sides of (3.3), we have

$$(N_n + 1)\left(\sum_{k\in\mu^{-1}(N_{<n})} b_{l_p k}x_k + \sum_{\substack{j\in\mu^{-1}(N_n)\\ j\neq l_\beta}} b_{l_p j}x_j\right) = b_{l_p l_\beta}c_1(\xi_{l_\beta}).$$

Since $c_1(\xi_{l_\beta})$ belongs to $H^2(B_{l_\beta - 1})$, we can see that $b_{l_p k} = 0$ for $k > l_\beta$. That is,

$$z_{l_p} = \sum_{\substack{k\in\mu^{-1}(N_{<n})\\ k<l_\beta}} b_{l_p k}x_k + \sum_{j\in\mu^{-1}(N_n)} b_{l_p j}x_j.$$

Thus, we can see that $z_{l_p} \in H^2(B_{l_\beta})$ and $b_{l l_\beta} = 0$ for all $l \in \mu^{-1}(N_n) \setminus \{l_p\}$ by Lemma 3.3. Put $w_{l_\beta} := z_{l_p}$. Now assume that $b_{l_q l_{\beta-1}}$ is nonzero for some $l_q \in \mu^{-1}(N_n) \setminus \{l_p\}$. Substituting $l = l_q$ into (3.3) and comparing the monomials containing $x_{l_{\beta-1}}^{N_n}$ as a factor on both sides of (3.3), we have

$$(N_n + 1)\left(\sum_{k\in\mu^{-1}(N_{<n})} b_{l_q k}x_k + \sum_{\substack{j\in\mu^{-1}(N_n)\\ j<l_{\beta-1}}} b_{l_q j}x_j\right) = b_{l_q l_{\beta-1}}c_1(\xi_{l_{\beta-1}}).$$

Since $c_1(\xi_{l_{\beta-1}})$ belongs to $H^2(B_{l_{\beta-1}-1})$, we can see that $b_{l_q k} = 0$ for $k > l_{\beta-1}$, and hence,

$$z_{l_q} = \sum_{\substack{k\in\mu^{-1}(N_{<n})\\ k<l_{\beta-1}}} b_{l_p k}x_k + \sum_{\substack{j\in\mu^{-1}(N_n)\\ j<l_\beta}} b_{l_p j}x_j.$$

Thus, we can see that $z_{l_q} \in H^2(B_{l_{\beta-1}})$ and $b_{l l_{\beta-1}} = 0$ for all $l \in \mu^{-1}(N_n) \setminus \{l_p, l_q\}$ by Lemma 3.3. Now, put $w_{l_{\beta-1}} := z_{l_q}$. In this way, for each $l \in \mu^{-1}(N_n)$, we can obtain $w_l \in H^2(B_l)$ such that $w_l \notin H^2(B_{l-1})$ and $w_l^{N_n+1} = 0$ in $H^2(B_h)$. Moreover, from the proof of Lemma 3.3, w_l can be written as $r_l(x_l + 1/(N_n + 1)c_1(\xi_l))$. Furthermore, without loss of generality, we may assume that $z_l = w_l$ for $l \in \mu^{-1}(N_n)$.

By Cases 1 and 2, we can see that, for each $i = 1, \dots, h$, we can write

$$z_i = r_i\left(x_i + \frac{1}{n_i + 1}c_1(\xi_i)\right)$$

for some $r_i \in \mathbb{Z} \setminus \{0\}$. Therefore, $\{(n_i + 1)x_i + c_1(\xi_i)\}^{n_i+1}$ is zero in $H^*(B_h)$. From this, we can see

$$(n_i + 1)^k c_k(\xi_i) = \binom{n_i + 1}{k} c_1(\xi_i)^k \quad \text{and} \quad c_1(\xi_i)^{n_i+1} = 0$$

$k = 1, \dots, n_i$. □

By using Proposition 1.1, we can prove Theorem 1.2.

Proof of Theorem 1.2. We first prove the implication (1) $\Rightarrow$ (2). By Proposition 1.1, we have the relation

$$(n_i + 1)^2 c_2(\xi_i) = \frac{n_i(n_i + 1)}{2} c_1(\xi_i)^2. \tag{3.4}$$

If $n_i = 2$, from (2.2) and (3.4), we have

$$\begin{aligned} &\{(a^i_{11}x_1 + \cdots + a^i_{1,i-1}x_{i-1}) + (a^i_{21}x_1 + \cdots + a^i_{2,i-1}x_{i-1})\}^2 \\ &= 3(a^i_{11}x_1 + \cdots + a^i_{1,i-1}x_{i-1})(a^i_{21}x_1 + \cdots + a^i_{2,i-1}x_{i-1}). \end{aligned} \tag{3.5}$$

For $j = 1, \dots, i-1$, since $x_j^2 \neq 0$ in $H^*(B_i)$, by comparing the coefficients of x_j^2 on both sides of (3.5), we have $(a^i_{1j} + a^i_{2j})^2 = 3a^i_{1j}a^i_{2j}$ whose integer solution is only $a^i_{1j} = a^i_{2j} = 0$. If $n_i = n > 2$, then we have

$$\begin{aligned} &n\{(a^i_{11}x_1 + \cdots + a^i_{1,i-1}x_{i-1}) + \cdots + (a^i_{21}x_1 + \cdots + a^i_{2,i-1}x_{i-1})\}^2 \\ &= 2(n+1)\{(a^i_{11}x_1 + \cdots + a^i_{1,i-1}x_{i-1})(a^i_{21}x_1 + \cdots + a^i_{2,i-1}x_{i-1}) + \cdots \\ &\quad + (a^i_{n-1,1}x_1 + \cdots + a^i_{n-1,i-1}x_{i-1})(a^i_{n,1}x_1 + \cdots + a^i_{n,i-1}x_{i-1})\}. \end{aligned} \tag{3.6}$$

Since $x_j^2 \neq 0$ in $H^*(B_i)$ for $j = 1, \dots, i-1$, by comparing the coefficients of x_j^2 on both sides of (3.6) we have

$$n(a^i_{1,j} + \cdots + a^i_{nj})^2 = 2(n+1) \sum_{1 \le k < l \le n} a^i_{kj}a^i_{lj}. \tag{3.7}$$

The equation (3.7) is equivalent to

$$\sum_{m=1}^{n} (a^i_{m,j})^2 + \sum_{1 \le k < l \le n} (a^i_{kj} - a^i_{lj})^2 = 0.$$

Therefore, $a^i_{1j} = \cdots = a^i_{nj} = 0$ for each $j = 1, \dots, i-1$, and hence, in any case, $c(\xi_i)$ is trivial for all $i = 1, \dots, h$.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear.
The implication (3) $\Leftrightarrow$ (4) is proved by Choi-Masuda-Suh [4].
Therefore, all four conditions are equivalent. $\square$

From Theorem 1.2, we have the following corollary.

Corollary 3.5. *Let M be a quasitoric manifold. If $H^*(M;\mathbb{Q})$ is isomorphic to $H^*\big(\prod_{i=1}^h \mathbb{C}P^{n_i};\mathbb{Q}\big)$, then M is homeomorphic to $\prod_{i=1}^h \mathbb{C}P^{n_i}$ provided $n_i > 1$ for all i.*

Proof. By [3], if $H^*(M;\mathbb{Q})$ is isomorphic to $H^*\big(\prod_{i=1}^h \mathbb{C}P^{n_i};\mathbb{Q}\big)$, then M is homeomorphic to a generalized Bott manifold. But a $\mathbb{Q}$-trivial generalized Bott manifolds with $n_i > 1$ is diffeomorphic to $\prod_{i=1}^h \mathbb{C}P^{n_i}$. Hence, M is homeomorphic to $\prod_{i=1}^h \mathbb{C}P^{n_i}$. $\square$

The following is the counter-example of Question 1.1.

EXAMPLE 3.1. Let B be a fiber bundle $P(\underline{\mathbb{C}}^3 \oplus \xi)$ over $\mathbb{C}P^2$ and let B' be a fiber bundle $P(\underline{\mathbb{C}}^3 \oplus \xi^{\otimes 2})$ over $\mathbb{C}P^2$, where ξ is the tautological line bundle over $\mathbb{C}P^2$. Let y (respectively, Y) denote the negative of the first Chern class of the tautological line bundle over B_2 (respectively, B'_2). Then their cohomology rings are

$$H^*(B) = \mathbb{Z}[x, y]/\langle x^3, y(y^3 + xy^2)\rangle$$

and

$$H^*(B') = \mathbb{Z}[X, Y]/\langle X^3, Y(Y^3 + 2XY^2)\rangle.$$

Then the map ϕ defined by $\phi(x) = 2X$ and $\phi(y) = Y$ is an isomorphism from $H^*(B;\mathbb{Q}) \to H^*(B';\mathbb{Q})$. But this ϕ is not a $\mathbb{Z}$-isomorphism. Suppose that ψ is an isomorphism $H^*(B;\mathbb{Z}) \to H^*(B';\mathbb{Z})$. Then there exist $\alpha, \beta, \gamma, \delta$ in $\mathbb{Z}$ such that

$$\begin{pmatrix} \psi(x) \\ \psi(y) \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

and $\alpha\delta - \beta\gamma = \pm 1$. Since $\psi(x^3) = 0$ in $H^*(B';\mathbb{Z})$, we have

$$(\alpha X + \beta Y)^3 = \alpha^3 X^3$$

as polynomials. So, we can see that β is zero and $\alpha = \pm 1$, and hence $\delta = \pm 1$. Since $\psi(y(y^3 + xy^2))$ is zero in $H^*(B';\mathbb{Z})$, we have

$$(3.8) \qquad (\gamma X + \delta Y)^3((\alpha + \gamma)X + \delta Y) = (aX + bY)X^3 + cY(Y^3 + 2XY^2)$$

as polynomials in $\mathbb{Z}[X, Y]$. By comparing the coefficients of XY^3 on both sides of (3.8), we can see that

$$(3.9) \qquad 2c = 3\gamma\delta^3 + (\alpha + \gamma)\delta^3 = \delta(\alpha + 4\gamma).$$

Since the right hand side of (3.9) is odd, there is no such an integer c. Hence, there is no such $\mathbb{Z}$-isomorphism ψ.

Now consider $\mathbb{Q}$-trivial generalized Bott manifolds B_h which have $\mathbb{C}P^1$-fibers, that is, $n_k = 1$ for some $k \in [h]$.

Lemma 3.6. *Let B_h and B'_h be two h-stage generalized Bott towers. If the associated vector matrices to them are*

$$A = \begin{pmatrix} \mathbf{1} & & & & \\ * & \ddots & & & \\ * & * & \mathbf{1} & & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{h-2} & \mathbf{1} & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_{h-2} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

and

$$A' = \begin{pmatrix} \mathbf{1} & & & & \\ * & \ddots & & & \\ * & * & \mathbf{1} & & \\ \mathbf{b}_1 & \cdots & \mathbf{b}_{h-2} & \mathbf{1} & \\ \mathbf{a}_1 & \cdots & \mathbf{a}_{h-2} & \mathbf{0} & \mathbf{1} \end{pmatrix},$$

respectively, then B_h and B'_h are equivariantly diffeomorphic.

Proof. Note that this lemma can be seen by the fact that B_h and B'_h are equivariantly diffeomorphic if two associated vector matrices are conjugated by a permutation matrix, see the paper [3]. It is obvious that

$$A' = E_\sigma A E_\sigma^{-1},$$

where $\sigma := (1, \ldots, h-2, h, h-1)$ is the permutation on $[h]$ which permutes only $h-1$ and h. □

Now, we can prove Theorem 1.3.

Proof of Theorem 1.3. Let B_h be a $\mathbb{Q}$-trivial generalized Bott manifold whose associated matrix is of the form (2.4).

Consider a map $\mu\colon \{1, \ldots, h\} \to \mathbb{N}$ given by $j \mapsto n_j$ and assume that the image of μ is the set $\{N_1, \ldots, N_m\}$ with $1 = N_1 < N_2 < \cdots < N_m$.

For each $i \in \mu^{-1}(1)$, by Proposition 1.1, we have $c_1(\xi_i)^2 = 0$ in $H^*(B_h)$. Since $x_k^2 \neq 0$ in $H^*(B_h)$ for $k \notin \mu^{-1}(1)$, we can see that $a_{1k}^i = 0$ for $k \in [i-1]$ with $n_k > 1$.

Now suppose that $n_j > 1$. Then by Proposition 1.1, we have the relation

$$(n_j + 1)^2 c_2(\xi_j) = \frac{n_j(n_j + 1)}{2} c_1(\xi_j)^2.$$

Since $x_k^2 \neq 0$ in $H^*(B_h)$ for $n_k > 1$, we can show that $\mathbf{a}_k^j = \mathbf{0}$ by using the same argument to the proof of Theorem 1.2.

Since $\mathbf{a}_k^j = \mathbf{0}$ for all $n_k > 1$, by Lemma 3.6, B_h is diffeomorphic to the $\mathbb{Q}$-trivial generalized Bott manifold B' whose associated matrix is of the form

$$(A')^T = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ a_{11}^2 & 1 & & & & & & \\ \vdots & \vdots & \ddots & & & & & \\ a_{11}^r & a_{1,2}^r & \cdots & 1 & & & & \\ \hline \mathbf{a}_1^{r+1} & \mathbf{a}_2^{r+1} & \cdots & \mathbf{a}_r^{r+1} & \mathbf{1} & & & \\ \mathbf{a}_1^{r+2} & \mathbf{a}_2^{r+2} & \cdots & \mathbf{a}_r^{r+2} & \mathbf{0} & \mathbf{1} & & \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & & \\ \mathbf{a}_1^h & \mathbf{a}_2^h & \cdots & \mathbf{a}_r^h & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{1} \end{array}\right), \tag{3.10}$$

where r is the cardinality of the set $\mu^{-1}(1)$, that is, $r = |\mu^{-1}(1)|$. This proves the theorem. □

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