ON MULTIDIMENSIONAL
DIFFUSION PROCESSES WITH JUMPS

Dedicated to professor Richard F. Bass on his sixtieth birthday

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Abstract

Let G be an open set of $\mathbb{R}^d$ ($d \geq 2$) and dx denotes the Lebesgue measure on it. We construct a diffusion process with jumps associated with diffusion data (diffusion coefficients $\{a_{ij}(x)\}$, a drift coefficient $\{b_i(x)\}$ and a killing function $c(x)$) and a Lévy kernel $k(x, y)$ in terms of a lower bounded semi-Dirichlet form on $L^2(G; dx)$. When G is the whole space, we allow that the diffusion coefficients may degenerate. We also show some Sobolev inequalities for the Dirichlet form and then show the absolute continuity of its resolvent.

1. Introduction

Consider the following (formal) second order partial differential operator with a non-local part:

$$
\begin{aligned}
\mathcal{L}u(x) &:= \mathcal{L}_c u(x) + \mathcal{L}_j u(x) \\
&= \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)u(x) - \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}u(x) - c(x)u(x) \\
&\quad + \lim_{n\to\infty}\frac{1}{2}\int_{|x-y|>1/n}(u(y)-u(x))k(x,y)\,dy, \quad x \in G,
\end{aligned}
$$
(1.1)

where a_{ij}, b_i and c are measurable functions defined on an open set G of $\mathbb{R}^d$ for $i, j = 1, 2, \ldots, d$ and $k(x, y)$ is a measurable function defined on $G \times G \setminus \{(x, x): x \in G\}$.

A main purpose of the present paper is devoted to construct a diffusion process with jumps on G associated with the operator $\mathcal{L}$. To carry out this program, we adopt the lower bounded semi-Dirichlet form theory, which has been developed recently (see [8, 18]), to show the existence of a diffusion process with jumps on G associated with $\mathcal{L}$ under some assumptions on the diffusion data $\{a_{ij}(x), b_i(x), c(x)\}$ and the Lévy kernel $k(x, y)$.

A construction of diffusion processes with jumps have been made by many people including Komatsu [11], Stroock [23] and Lepeltier and Marchal [14] in the 1970s

already by making use of the theory of martingale problems or the theory of pseudo differential operators (see [10, 2]). Bensoussan and Lions [3] considered the elliptic differential operators with jumps to study the stochastic control and stopping problems of diffusion processes with jumps (see also [9]). In a symmetric process case, many examples are considered using the Dirichlet form theory ([7]). In [15], Ma and Röckner also gave some examples of diffusion processes with jumps via non-symmetric Dirichlet forms. In the papers/books mentioned above, the diffusion coefficients must not degenerate when the drift term does not vanish (including the case where the jump term vanishes).

In this paper, we will pay special attention to the following two types of conditions on the data in the subsequent sections. We emphasize that, taking the jump term into consideration, we can allow the diffusion coefficients may degenerate even when the drift term does not vanish (see Section 4).

To construct a diffusion process with jumps, we consider the following quadratic form: For each $n \in \mathbb{N}$,

(1.2)
$$\eta^n(u, v)$$

$$:= -\int_G \mathcal{L}^n u(x) v(x)\, dx = -\int_G (\mathcal{L}_c u(c) v(x) + \mathcal{L}_j^n u(x) v(x))\, dx$$

$$:= \eta^{(c)}(u, v) + \eta^{(j,n)}(u, v)$$

$$:= \frac{1}{2} \sum_{i,j=1}^d \int_G a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}\, dx + \sum_{i=1}^d \int_G b_i(x) u(x) \frac{\partial v(x)}{\partial x_i}\, dx + \int_G u(x) v(x) c(x)\, dx$$

$$- \frac{1}{2} \int_{|x-y|>1/n} (u(y) - u(x)) v(x) k(x, y)\, dx\, dy.$$

We will show the finite limit $\eta(u, v) = \lim_{n \to \infty} \eta^n(u, v)$ exists for appropriate functions u, v and then consider a question whether the limit produces a Hunt process by using the lower bounded semi-Dirichlet form. We will also see that the limit has the following expression:

(1.3)
$$\eta(u, v) = \frac{1}{2} \sum_{ij=1}^d \int_G a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j}\, dx + \sum_{i=1}^d \int_G b_i(x) u(x) \frac{\partial v(x)}{\partial x_i}\, dx$$

$$+ \int_G u(x) v(x) c(x)\, dx + \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) k_s(x, y)\, dx\, dy$$

$$+ \iint_{x \neq y} (u(x) - u(y)) v(x) k_a(x, y)\, dx\, dy,$$

where $k_s(x, y) = (1/2)(k(x, y) + k(y, x))$ and $k_a(x, y) = (1/2)(k(x, y) - k(y, x))$ for $x \neq y$ (see the condition (J.2) in Section 3).

The organization of this paper is as follows: In the next section, we introduce a notion of lower bounded semi-Dirichlet forms. In Sections 3 and 4, we construct a regular lower bounded semi-Dirichlet form under the two cases respectively. Note that, in Section 4, we will show that it is possible to construct a diffusion process with jumps in the case where the diffusion coefficients may degenerate and the drift coefficient does not vanish. In Section 5, after stating the association of the diffusion process with jumps, we give a martingale characterization of the process and we also give a conservativeness criteria for the process. We will discuss a simple example in the last section.

2. Preliminaries—lower bounded semi-Dirichlet form—

In this section, we give a definition of lower bounded semi-Dirichlet forms. To this end, let X be a locally compact separable metric space and m a positive Radon measure on X with full support. Let $\mathcal{F}$ be a dense subspace of $L^2(X; m)$ satisfying $f \wedge 1 \in \mathcal{F}$ whenever $f \in \mathcal{F}$. Denote by $(\cdot, \cdot)$ the inner product in L^2 and by $\|\cdot\|_{L^p}$ the L^p-norm in L^p for $1 \leq p < \infty$. A bilinear form η defined on $\mathcal{F} \times \mathcal{F}$ is called a lower bounded closed form on $L^2(X; m)$ if the following conditions are satisfied: there exists a $\beta \geq 0$ such that

(B.1) (lower boundedness): for any $u \in \mathcal{F}$, $\eta_\beta(u, u) \geq 0$, where

$$\eta_\beta(u, v) = \eta(u, v) + \beta(u, v), \quad u, v \in \mathcal{F}.$$

(B.2) (weak sector condition): there exists a constant $K \geq 1$ so that

$$|\eta(u, v)| \leq K \sqrt{\eta_\beta(u, u)} \cdot \sqrt{\eta_\beta(v, v)} \quad \text{for} \quad u, v \in \mathcal{F}.$$

(B.3) (closedness): the space $\mathcal{F}$ is closed with respect to the norm $\sqrt{\eta_\alpha(u, u)}$, $u \in \mathcal{F}$, for some, or equivalently, for all $\alpha > \beta$.

For a lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(X; m)$, there exist unique semigroups $\{T_t; t > 0\}$, $\{\hat{T}_t; t > 0\}$ of linear operators on $L^2(X; m)$ satisfying

(2.1) $(T_t f, g) = (f, \hat{T}_t g), \quad \|T_t f\|_{L^2} \leq e^{\beta t}, \quad \|\hat{T}_t f\|_{L^2} \leq e^{\beta t}, \quad f, g \in L^2(X; m), \ t > 0,$

such that their Laplace transforms G_α and $\hat{G}_\alpha$ are determined for $\alpha > \beta$ by

$$G_\alpha f, \hat{G}_\alpha f \in \mathcal{F}, \quad \eta_\alpha(G_\alpha f, u) = \eta_\alpha(u, \hat{G}_\alpha f) = (f, u), \quad f \in L^2(X; m), \ u \in \mathcal{F}.$$

$\{T_t; t > 0\}$ is said to be *Markovian* if $0 \leq T_t f \leq 1$, $t > 0$, whenever $f \in L^2(X; m)$, $0 \leq f \leq 1$. H. Kunita showed in [12] that the semi-group $\{T_t; t > 0\}$ is Markovian if and only if

(2.2) $\qquad Uu \in \mathcal{F} \quad \text{and} \quad \eta(Uu, u - Uu) \geq 0 \quad \text{for any} \quad u \in \mathcal{F},$

where Uu denotes the unit contraction of u: $Uu = (0 \vee u) \wedge 1$. A lower bounded closed form $(\eta, \mathcal{F})$ on $L^2(X; m)$ satisfying (2.2) is called a *lower bounded semi-Dirichlet form on $L^2(X; m)$*.

A lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$ is said to be regular if $\mathcal{F} \cap C_0(X)$ is uniformly dense in $C_0(X)$ and η_α-dense in $\mathcal{F}$ for $\alpha > \beta$, where $C_0(X)$ denotes the space of continuous functions on X with compact support. Carrillo-Menendez [4] constructed a Hunt process properly associated with any regular lower bounded semi-Dirichlet form on $L^2(X; m)$.

3. Diffusion process with jumps—uniformly elliptic case—

Let G be an open set of $\mathbb{R}^d$. Throughout this section, we make the following assumptions on a_{ij}, b_i, c and k:

(D.1) there exists $0 < \lambda \leq \Lambda$ such that

$$\lambda |\xi|^2 \leq \sum_{ij=1}^{d} a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \text{for} \quad x \in G, \ \xi \in \mathbb{R}^d.$$

(D.2) $b_i \in L^{p_0}(G)$ for some p_0 with $d \leq p_0 \leq \infty$ if G is bounded and $b_i \in L^d(G) \cup L^\infty(G)$ when G is unbounded for $i = 1, 2, \ldots, d$.

(D.3) $c \in L_+^{d/2}(G) \cup L^\infty(G)$.

(J.1) $M_s \in L^1_{\text{loc}}(G)$ for $M_s(x) = \int_{y \neq x}(1 \wedge |x - y|^2)k_s(x, y)\, dy$, $x \in G$.

(J.2) $C_1 := \sup_{x \in G} \int_{|x-y| \geq 1, y \in G} |k_a(x, y)|\, dy < \infty$ and there exists a constant $\gamma \in (0, 1]$ such that

$$C_2 := \sup_{x \in G} \int_{|x-y| < 1, y \in G} |k_a(x, y)|^\gamma \, dy < \infty$$

and, for some constant $C_3 > 0$,

$$|k_a(x, y)|^{2-\gamma} \leq C_3 k_s(x, y), \quad x, y \in G \quad \text{with} \quad 0 < |x - y| < 1.$$

Here k_s and k_a are defined by

$$k_s(x, y) = \frac{1}{2}(k(x, y) + k(y, x)), \quad k_a(x, y) = \frac{1}{2}(k(x, y) - k(y, x)), \quad x, y \in G, \ x \neq y,$$

respectively.

In [8, Proposition 1], we showed that for any $u, v \in C_0^{\text{lip}}(G)$, the limit

$$\eta^{(j)}(u, v) := \lim_{n \to \infty} \eta^{(j,n)}(u, v) = -\lim_{n \to \infty} \int_{|x-y| > 1/n} (u(y) - u(x))v(x)k(x, y)\, dx\, dy$$

exists under the assumptions (J.1) and (J.2). Moreover the limit has the following expression:

$$\eta^{(j)}(u, v) = \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(u(x) - u(y))k_s(x, y) \, dx \, dy$$

$$+ \iint_{x \neq y} (u(x) - u(y))v(y)k_a(x, y) \, dx \, dy.$$

REMARK 3.1. Quite recently, Schilling and Wang in [19] simplified the conditions (J.2) as follows:

$$(3.1) \qquad \sup_{x \in G} \int_{\{y \in G : k_s(x, y) \neq 0\}} \frac{k_a(x, y)^2}{k_s(x, y)} \, dy < \infty$$

and investigated the generator and the co-generator of the form. But in this paper, we keep the conditions as (J.1), (J.2). Note that under the condition (3.1), they showed that the quadratic form η becomes indeed a lower-bounded semi-Dirichlet form in the same way as ours [8].

Let us now define for $u, v \in C_0^1(G)$,

$$\mathcal{E}(u, v) := \mathcal{E}^{(c)}(u, v) + \mathcal{E}^{(j)}(u, v)$$

$$(3.2) \qquad = \frac{1}{2} \sum_{i=1}^{d} \int_G \frac{\partial u}{\partial x_i}(x) \frac{\partial v}{\partial x_i}(x) \, dx$$

$$+ \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(u(x) - u(y))k_s(x, y) \, dx \, dy.$$

Under the assumption (J.1), we easily see $(\mathcal{E}, C_0^1(G))$ is a closable symmetric form on $L^2(G)$ and denote by $\mathcal{F}$ the closure of $C_0^1(G)$ with respect to $\sqrt{\mathcal{E}_1(\cdot, \cdot)}$:

$$\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + \int_G u(x)v(x) \, dx, \quad u, v \in C_0^1(G).$$

We now show that the form η satisfies the weak sector condition and the lower boundedness condition: there exists a positive constant $K > 0$ and $\beta \geq 0$ so that

$$(3.3) \qquad \eta_\beta(u, u) \geq 0, \quad u \in \mathcal{F}$$

and

$$(3.4) \qquad |\eta(u, v)| \leq K \sqrt{\eta_\beta(u, u)} \sqrt{\eta_\beta(v, v)}, \quad \text{for} \quad u, v \in \mathcal{F}.$$

For the non-local part $\eta^{(j)}$, we have already shown in [8, Theorem 1] that

$$(3.5) \qquad |\eta^{(j)}(u, v)| \leq 2\sqrt{2} \sqrt{\mathcal{E}_{\beta_0}^{(j)}(u, u)} \sqrt{\mathcal{E}_{\beta_0}^{(j)}(v, v)}$$

and

(3.6) $\eta_{\beta_0}^{(j)}(u, u) \geq \frac{1}{4}\mathcal{E}_{\beta_0}^{(j)}(u, u), \quad u, v \in C_0^1(G)$

for $\beta_0 = 8(C_1 \vee C_2 C_3)$ under the assumption (J.1) and (J.2). As for the local part $\eta^{(c)}$:

$$\eta^{(c)}(u, v) := \sum_{ij=1}^{d} \int_G a_{ij}(x)\frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx + \sum_{i=1}^{d}\int_G b_i(x)u(x)\frac{\partial v(x)}{\partial x_i}\,dx$$
$$+ \int_G u(x)v(x)c(x)\,dx,$$

Stampacchia showed in [22] (see also [13]) the weak sector condition with respect to the Sobolev norm for the form $\eta^{(c)}$. We give the proof for reader's convenience. In showing these properties, the following Gagliardo–Nirenberg–Sobolev inequality plays an important role:

Lemma 3.1 (see [5, p. 138] and [16]). *For* $1 \leq p < d$, *there exists a positive constant* $C > 0$ *depending only on* p *and* d *such that*

(3.7) $\|u\|_{L^{pd/(d-p)}} \leq C \sum_{i=1}^{d}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^p}, \quad u \in C_0^1(G).$

Proposition 3.1. *Let* G *be an open set of* $\mathbb{R}^d$. *Assume* (D.1)–(D.3) *hold. Then it follows that, for some constant* $K_1 > 0$,

$|\eta^{(c)}(u, v)|$

$\leq K_1\left(\sum_{i=1}^{d}\int_G\left|\frac{\partial u}{\partial x_i}\right|^2 dx + \int_G u^2\,dx\right)^{1/2}\cdot\left(\sum_{i=1}^{d}\int_G\left|\frac{\partial v}{\partial x_i}\right|^2 dx + \int_G v^2\,dx\right)^{1/2}$

$= K_1\sqrt{\mathcal{E}_1^{(c)}(u, u)}\cdot\sqrt{\mathcal{E}_1^{(c)}(v, v)}$

for any $u, v \in C_0^1(G)$.

Proof.   According to Assumptions (D.1) and (D.2), we find that for $u, v \in C_0^1(G)$,

$$\left|\sum_{i,j=1}^{d}\int_G a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx\right| \leq \Lambda\sqrt{\sum_{i=1}^{d}\int_G\left|\frac{\partial u}{\partial x_i}\right|^2 dx}\cdot\sqrt{\sum_{i=1}^{d}\int_G\left|\frac{\partial v}{\partial x_i}\right|^2 dx}$$

and

$$\left| \sum_{i=1}^{d} \int_{G} b_i u \frac{\partial v}{\partial x_i} \, dx \right|$$

$$\leq \sqrt{\sum_{i=1}^{d} \int_{G} b_i^2 u^2 \, dx} \cdot \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx}$$

$$\leq \sqrt{\sum_{i=1}^{d} \left(\int_{G} |b_i|^d \, dx \right)^{2/d} \left(\int_{G} |u|^{2d/(d-2)} \, dx \right)^{(d-2)/d}} \cdot \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx}$$

$$\leq \left(\sum_{i=1}^{d} \int_{G} |b_i|^d \, dx \right)^{1/d} \cdot \| u \|_{L^{2d/(d-2)}} \cdot \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx}.$$

Here we used the Hölder inequality in the last inequality to the pair (p, q) with $1/p = (d-2)/d$ and $1/q = 1 - 1/p = 2/d$. We now estimate the term $\int uvc \, dx$ in $\eta^{(c)}$. First we assume that $c \in L^{\infty}(G)$. Then we see that

$$\left| \int_{G} u(x)v(x)c(x) \, dx \right| \leq \| c \|_{\infty} \| u \|_{L^2} \| v \|_{L^2}.$$

When $c \in L^{d/2}(G)$, using the Hölder inequality and then the Schwarz inequality, we find that

$$\left| \int_{G} u(x)v(x)c(x) \, dx \right|$$

$$\leq \left(\int_{G} |u(x)v(x)|^{d/(d-2)} \, dx \right)^{(d-2)/d} \cdot \| c \|_{L^{d/2}}$$

$$\leq \left(\int_{G} |u(x)|^{2d/(d-2)} \, dx \right)^{(d-2)/(2d)} \left(\int_{G} |v(x)|^{2d/(d-2)} \, dx \right)^{(d-2)/(2d)} \cdot \| c \|_{L^{d/2}}$$

$$= \| u \|_{L^{2d/(d-2)}} \cdot \| v \|_{L^{2d/(d-2)}} \cdot \| c \|_{L^{d/2}}.$$

Then using the previous lemma (in the case $p = 2$), it follows that

$$|\eta^{(c)}(u, v)|$$

$$\leq \Lambda \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial u}{\partial x_i} \right|^2 \, dx} \cdot \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx}$$

$$+ C \left(\sum_{i=1}^{d} \int_{G} |b_i|^d \, dx \right)^{1/d} \cdot \left(\| u \|_{L^2} + \sum_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} \right) \cdot \sqrt{\sum_{i=1}^{d} \int_{G} \left| \frac{\partial v}{\partial x_i} \right|^2 \, dx}$$

$$+ C^2 \| c \|_{L^2} \cdot \left(\| u \|_{L^2} + \sum_{i=1}^{d} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2} \right) \cdot \left(\| v \|_{L^2} + \sum_{i=1}^{d} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2} \right)$$

$$\leq \Lambda \sqrt{\int_G u^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial u}{\partial x_i}\right|^2 dx} \cdot \sqrt{\int_G v^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial v}{\partial x_i}\right|^2 dx}$$

$$+ (C+1)\sqrt{d+1}\left(\sum_{i=1}^{d} \int_G |b_i|^d\, dx\right)^{1/d} \cdot \sqrt{\int_G u^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial u}{\partial x_i}\right|^2 dx}$$

$$\cdot \sqrt{\int_G v^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial v}{\partial x_i}\right|^2 dx}$$

$$+ C^2(d+1)\|c\|_{L^{d/2}} \sqrt{\int_G u^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial u}{\partial x_i}\right|^2 dx} \cdot \sqrt{\int_G v^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial v}{\partial x_i}\right|^2 dx}$$

$$\leq K_1 \sqrt{\int_G u^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial u}{\partial x_i}\right|^2 dx} \cdot \sqrt{\int_G v^2\, dx + \sum_{i=1}^{d} \int_G \left|\frac{\partial v}{\partial x_i}\right|^2 dx}$$

$$= K_1 \sqrt{\mathcal{E}_1^{(c)}(u, u)} \cdot \sqrt{\mathcal{E}_1^{(c)}(v, v)},$$

where

$$K_1 := \Lambda + (C+1)\sqrt{d+1}\left(\sum_{i=1}^{d} \int_G |b_i|^d\, dx\right)^{1/d} + C^2(d+1)\|c\|_{L^{d/2}}. \qquad \square$$

Combining the proposition with (3.5), we have the following:

Proposition 3.2. *Assume that* (D.1)–(D.3) *and* (J.1)–(J.2) *hold for some large* $\lambda > 0$. *Then there exists a positive constant* $K > 0$ *and* $\beta \geq 0$ *such that*

$$\eta_\beta(u, u) \geq 0, \quad \forall u \in C_0^1(G)$$

and

$$|\eta(u, v)| \leq K \sqrt{\eta_\beta(u, u)} \cdot \sqrt{\eta_\beta(v, v)}, \quad \forall u, v \in C_0^1(G).$$

Proof. Since the lower boundedness and the weak sector condition of the jump part are known by (3.5) and (3.6), we only consider the diffusion part $\eta^{(c)}$. In fact, suppose that

$$|\eta^{(j)}(u, v)| \leq C_1 \sqrt{\eta_{\beta_1}^{(j)}(u, u)} \cdot \sqrt{\eta_{\beta_1}^{(j)}(v, v)}$$

and

$$|\eta^{(c)}(u, v)| \leq C_2 \sqrt{\eta_{\beta_2}^{(c)}(u, u)} \cdot \sqrt{\eta_{\beta_2}^{(c)}(v, v)}$$

for $u, v \in C_0^1(G)$. Then by using an elementary inequality:

$$\sqrt{A} \cdot \sqrt{B} + \sqrt{C} \cdot \sqrt{D} \leq \sqrt{2} \cdot \sqrt{A + C} \cdot \sqrt{B + D}$$

for nonnegative numbers A, B, C and D, the weak sector condition of the form η holds for putting $\beta = \beta_1 + \beta_2$:

$$|\eta(u,v)| \leq |\eta^{(j)}(u,v)| + |\eta^{(c)}(u,v)| \leq \sqrt{2} \cdot (C_1 \vee C_2) \sqrt{\eta_\beta(u,u)} \cdot \sqrt{\eta_\beta(v,v)}, \quad u, v \in C_0^1(G).$$

We adopt an argument developed in [22] to estimate the diffusion part $\eta^{(c)}$ as follows. First we assume $c \in L^{d/2}(G; m)$ in (D.3). By using the uniformly ellipticity (D.1) and Proposition 3.1, we find that

$$
\begin{aligned}
\eta^{(c)}(u,u) &\geq \lambda \sum_{i=1}^d \int_G \left|\frac{\partial u}{\partial x_i}\right|^2 dx - \sum_{i=1}^d \|b_i\|_{L^d} \cdot \|u\|_{L^{2d/(d-2)}} \cdot \sqrt{\sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2} \\
&\quad - \|c\|_{L^{d/2}} \cdot \|u\|_{L^{2d/(d-2)}}^2 \\
&\geq \lambda \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2 - \sum_{i=1}^d \|b_i\|_{L^d} \cdot C \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2 - \|c\|_{L^{d/2}} \cdot C \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2 \\
&\geq \left(\lambda - C \sum_{i=1}^d \|b_i\|_{L^d} - C \|c\|_{L^{d/2}}\right) \sum_{i=1}^d \left\|\frac{\partial u}{\partial x_i}\right\|_{L^2}^2 \\
&= 2\left(\lambda - C \sum_{i=1}^d \|b_i\|_{L^d} - C \|c\|_{L^{d/2}}\right) \mathcal{E}^{(c)}(u,u).
\end{aligned}
$$

Hence, if we assume that, for example,

$$(3.8) \qquad\qquad C \sum_{i=1}^d \|b_i\|_{L^d} + C \|c\|_{L^{d/2}} < \frac{\lambda}{2},$$

then we see for $u \in C_0^1(G)$, $\eta^{(c)}(u,u) \geq \lambda \mathcal{E}^{(c)}(u,u)$ and this gives us the lower boundedness of $\eta^{(c)}$. When $c \in L^\infty(G; m)$, the elliptic constant λ can be taken a bit smaller:

$$C \sum_{i=1}^d \|b_i\|_{L^d} < \frac{\lambda}{2},$$

but β then should be chosen as $\lambda + \|c\|_{L^\infty}$ in this case.

On the other hand, according to Proposition 3.1, we have for some constant $K_1 > 0$,

$$
\begin{aligned}
|\eta^{(c)}(u,v)| &\leq K_1 \sqrt{\mathcal{E}^{(c)}(u,u)} \cdot \sqrt{\mathcal{E}^{(c)}(v,v)} \\
&\leq K_1 \sqrt{\mathcal{E}_1^{(c)}(u,u)} \cdot \sqrt{\mathcal{E}_1^{(c)}(v,v)}, \quad u, v \in C_0^1(G).
\end{aligned}
$$

Then it follows that

$$|\eta^{(c)}(u, v)| \leq \frac{K_1}{\lambda} \sqrt{\eta^{(c)}_{\beta_2}(u, u)} \cdot \sqrt{\eta^{(c)}_{\beta_2}(u, u)}, \quad u, v \in C_0^1(G)$$

for putting $\beta_2 \geq \lambda$ if $c \in L^{d/2}(G; dx)$ (resp. $\beta_2 \geq \lambda + \|c\|_\infty$ if $c \in L^\infty(G; dx)$). Hence, combining the calculus done above with the result for the jump part, we see that the lower boundedness of $\eta^{(c)}$ is satisfied. $\qquad\square$

We now state a main theorem in this section:

Theorem 3.1. *Assume* (D.1)–(D.3), (J.1) *and* (J.2). *Assume also that the elliptic constant* $\lambda > 0$ *satisfies* (3.8). *Then the form* η *defined as*

$$\eta(u, v) = \eta^{(c)}(u, v) + \eta^{(j)}(u, v), \quad u, v \in C_0^1(G)$$

extends from $C_0^1(G) \times C_0^1(G)$ *to* $\mathcal{F} \times \mathcal{F}$ *to be a lower bounded closed form on* $L^2(G)$. *Moreover the pair* $(\eta, \mathcal{F})$ *is a regular lower bounded semi-Dirichlet form on* $L^2(G)$.

Proof. We only need to show the Markov property (2.2). Since $(\mathcal{E}, \mathcal{F})$ defined in (3.2) is a Dirichlet form on $L^2(G)$ and satisfies that, for each $\alpha > \beta$, there exist $c, c' > 0$ so that

$$c\mathcal{E}_1(u, u) \leq \eta_\alpha(u, u) \leq c'\mathcal{E}_1(u, u), \quad u \in \mathcal{F}.$$

Then it follows that $Uu := u \wedge 1 \in \mathcal{F}$ whenever $u \in \mathcal{F}$. We have shown in [8] that $\eta^{(j)}(Uu, u - Uu) \geq 0$ for any $u \in C_0^{\mathrm{lip}}(G)$. It is extended to the inequality for $u \in \mathcal{F}$ (see e.g. [17]). The Markov property for the form $\eta^{(c)}$ is shown in Section II.2 in [15]. $\quad\square$

4. Diffusion process with jumps—degenerate case—

In this section we assume the following conditions instead of (D.1) on the whole space $G = \mathbb{R}^d$.
(D.1)' $\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq 0$ for any $\xi \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ and, the functions a_{ij}, $(\partial/\partial x_i)a_{ij}$ belong to $L_{\mathrm{loc}}^2(\mathbb{R}^d)$ for each $i, j = 1, 2, \ldots, d$.

Consider a quadratic form $\tilde{\mathcal{E}}(u, v)$ for $u, v \in C_0^1(\mathbb{R}^d)$, a similar one as in the previous section (3.2): for $u, v \in C_0^1(\mathbb{R}^d)$,

$$\tilde{\mathcal{E}}(u, v) := \tilde{\mathcal{E}}^{(c)}(u, v) + \mathcal{E}^{(j)}(u, v)$$

$$= \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} \widetilde{a_{ij}}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \, dx$$

$$+ \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(u(x) - u(y))k_s(x, y) \, dx \, dy,$$

where $\widetilde{a_{ij}}(x) = (1/2)(a_{ij}(x) + a_{ij}(x))$, $x \in \mathbb{R}^d$. Then we easily see the following lemma (see e.g. Section 3.1 in [7]):

Lemma 4.1. *Assume* (D.1)$'$ *and* (J.1) *hold by* $G = \mathbb{R}^d$. *Then the pair* $(\tilde{\mathcal{E}}, C_0^1(\mathbb{R}^d))$ *is a symmetric closable form on* $L^2(\mathbb{R}^d)$ *and, denoting* $\mathcal{F}$ *the closure of* $C_0^1(\mathbb{R}^d)$ *with respect to the norm* $\sqrt{\tilde{\mathcal{E}}_1(\,\cdot\,,\,\cdot\,)}$, $(\tilde{\mathcal{E}}, \mathcal{F})$ *is a regular symmetric Dirichlet form on* $L^2(\mathbb{R}^d)$.

We now consider a bilinear form η^n on $C_0^1(\mathbb{R}^d) \times C_0^1(\mathbb{R}^d)$ in (1.2) for each $n \in \mathbb{N}$. As stated in the previous section, the forms $\eta^{(j,n)}(u, v)$ converges to $\eta^{(j)}(u, v)$ as $n \to \infty$ for $u, v \in C_0^1(\mathbb{R}^d)$ under the assumptions (J.1) and (J.2). So in order to show that the limit $\eta = \eta^{(c)} + \eta^{(j)}$ becomes a lower bounded semi-Dirichlet form under the assumption (D.1)$'$ imposed on the diffusion coefficients $\{a_{ij}\}$, we make the following assumptions on the functions c, b_i for $i = 1, 2, \ldots, d$ and the kernel $k(x, y)$ as well:
(D.2)$'$ there exists a vector ${}^t(b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$, so that $b_i(x) = b_i$ for $x \in \mathbb{R}^d$ and $i = 1, 2, \ldots, d$. (Namely, the function b is a constant drift.)
(D.3)$'$ $c \in L_+^\infty(\mathbb{R}^d)$.
(J.3) there exists a $\kappa > 0$ such that

$$k(x, y) \geq \kappa |x - y|^{-d-1}, \quad x, y \in \mathbb{R}^d, \ 0 < |x - y| < 1.$$

We show a simple lemma:

Lemma 4.2. *Assume* (J.3). *Then for any* $u \in C_0^1(\mathbb{R}^d)$ *and each* $i = 1, 2, \ldots, d$, *it follows that*

$$\frac{\kappa \pi^{(d+2)/2}}{\Gamma((1+d)/2)} \left| \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x) u(x)\, dx \right|$$
$$\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} (u(x) - u(y))^2 k(x, y)\, dx\, dy + 4\kappa c_d \int_{\mathbb{R}^d} u(x)^2\, dx,$$

where κ *is the constant in* (J.3), Γ *is the Gamma function and* c_d *is the surface measure of the unit ball in* $\mathbb{R}^d$.

Proof.   For any $u \in C_0^1(\mathbb{R}^d)$ and any $i = 1, 2, \ldots, d$, we see

$$\iint_{x \neq y} (u(x) - u(y))^2 k(x, y)\, dx\, dy$$
$$\geq \iint_{0 < |x-y| < 1} (u(x) - u(y))^2 k(x, y)\, dx\, dy$$
$$\geq \kappa \iint_{0 < |x-y| < 1} \frac{(u(x) - u(y))^2}{|x - y|^{d+1}}\, dx\, dy$$

$$= \kappa \iint_{x \neq y} \frac{(u(x) - u(y))^2}{|x - y|^{d+1}} \, dx \, dy - \kappa \iint_{|x-y| \geq 1} \frac{(u(x) - u(y))^2}{|x - y|^{d+1}} \, dx \, dy$$

$$\geq \frac{\kappa \pi^{(d+2)/2}}{\Gamma((1+d)/2)} \int_{\mathbb{R}^d} |\xi| \cdot |\hat{u}(\xi)|^2 \, d\xi - 4\kappa \int_{\mathbb{R}^d} u(x)^2 \left(\int_{|x-y| \geq 1} |x - y|^{-d-1} \, dy \right) dx$$

$$\geq \frac{\kappa \pi^{(d+2)/2}}{\Gamma((1+d)/2)} \left| \int_{\mathbb{R}^d} \widehat{\frac{\partial u}{\partial x_i}}(\xi) \cdot \hat{u}(\xi) \, d\xi \right| - 4\kappa c_d \int_{\mathbb{R}^d} u(x)^2 \, dx$$

$$= \frac{\kappa \pi^{(d+2)/2}}{\Gamma((1+d)/2)} \left| \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x) \cdot u(x) \, dx \right| - 4\kappa c_d \int_{\mathbb{R}^d} u(x)^2 \, dx,$$

where we used (J.3) in the second inequality, the Planchrel theorem in the second equality (see e.g. [1]) and Parseval's identity in the last equality. Thus the desired inequality holds. $\qquad \square$

Lemma 4.3. *Assume* (D.1)′–(D.3)′, (J.1), (J.2) *and* (J.3) *hold. Then there exists a constant* $K > 0$ *so that*

$$|\eta(u, v)| \leq K \sqrt{\tilde{\mathcal{E}}_1(u, u)} \cdot \sqrt{\tilde{\mathcal{E}}_1(v, v)}, \quad u, v \in C_0^1(\mathbb{R}^d).$$

Proof.   First note that the limit $\eta(u, v)$ has the following expression for $u, v \in C_0^1(\mathbb{R}^d)$:

$$\eta(u, v) = \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx + \sum_{i=1}^d b_i \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} v(x) \, dx + \int_{\mathbb{R}^d} c(x) u(x) v(x) \, dx$$

$$+ \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))(u(x) - u(y)) k_s(x, y) \, dx \, dy$$

$$= \tilde{\mathcal{E}}(u, v) + \sum_{i=1}^d b_i \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} v(x) \, dx + \int_{\mathbb{R}^d} c(x) u(x) v(x) \, dx.$$

So we see that

$$|\eta(u, v)| \leq \sqrt{\tilde{\mathcal{E}}(u, u)} \cdot \sqrt{\tilde{\mathcal{E}}(v, v)} + \sum_{i=1}^d |b_i| \cdot \left| \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x) v(x) \, dx \right| + \|c\|_\infty \|u\|_{L^2} \cdot \|v\|_{L^2}$$

We need to estimate the second term of the right hand side. To this end, by making use of the Plancherel theorem, the Schwarz inequality and Lemma 4.2, we find that

$$\left| \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}(x) v(x) \, dx \right| = \left| \int_{\mathbb{R}^d} \widehat{\frac{\partial u}{\partial x_i}}(\xi) \cdot \hat{v}(\xi) \, d\xi \right| \leq \int_{\mathbb{R}^d} |\xi| \cdot |\hat{u}(\xi)| \cdot |\hat{v}(\xi)| d\xi$$

$$\leq \sqrt{\int_{\mathbb{R}^d} |\xi| \cdot |\hat{u}(\xi)|^2 \, d\xi} \cdot \sqrt{\int_{\mathbb{R}^d} |\xi| \cdot |\hat{v}(\xi)|^2 \, d\xi}$$

$$= c_1 \sqrt{\iint_{x \neq y} \frac{(u(x)-u(y))^2}{|x-y|^{d+1}} \, dx \, dy} \cdot \sqrt{\iint_{x \neq y} \frac{(v(x)-v(y))^2}{|x-y|^{d+1}} \, dx \, dy}$$

$$\leq c_2 \sqrt{\int_{|x-y| \leq 1} (u(x)-u(y))^2 k(x, y) \, dx \, dy + \int_{\mathbb{R}^d} u(x)^2 \, dx}$$

$$\cdot \sqrt{\int_{|x-y| \leq 1} (v(x)-v(y))^2 k(x, y) \, dx \, dy + \int_{\mathbb{R}^d} v(x)^2 \, dx}$$

$$\leq c_2 \sqrt{\tilde{\mathcal{E}}_1(u, u)} \cdot \sqrt{\tilde{\mathcal{E}}_1(v, v)}.$$

Hence it follows that

$$|\eta(u, v)| \leq (1 + c_3) \sqrt{\tilde{\mathcal{E}}(u, u)} \cdot \sqrt{\tilde{\mathcal{E}}(v, v)} + \|c\|_\infty \|u\|_{L^2} \cdot \|v\|_{L^2}$$

$$\leq \{(1 + c_3) \vee \|c\|_\infty\} \sqrt{\tilde{\mathcal{E}}_1(u, u)} \cdot \sqrt{\tilde{\mathcal{E}}_1(v, v)},$$

where $c_3 := c_2 \cdot d \cdot \sup_{1 \leq i \leq d} |b_i|$. $\qquad\square$

From this lemma, we have the main theorem in this section:

Theorem 4.1. *Assume* (D.1)′–(D.3)′, (J.1), (J.2) *and* (J.3) *and the constant* $\kappa > 0$ *satisfies*

(4.1)
$$\frac{\sum_{i=1}^d |b_i|}{C(d, 1)} \leq \frac{\kappa}{8}.$$

Then the form η *defined as*

$$\eta(u, v) = \eta^{(c)}(u, v) + \eta^{(j)}(u, v), \quad u, v \in C_0^1(\mathbb{R}^d)$$

extends from $C_0^1(\mathbb{R}^d) \times C_0^1(\mathbb{R}^d)$ *to* $\mathcal{F} \times \mathcal{F}$ *to be a lower bounded closed form on* $L^2(\mathbb{R}^d)$. *Moreover the pair* $(\eta, \mathcal{F})$ *is a regular lower bounded semi-Dirichlet form on* $L^2(\mathbb{R}^d)$.

Proof. We only show the lower boundedness and the weak sector condition of the form. According to Lemma 4.2 and the assumption on the kernel k, we find

$$\eta^{(c)}(u, u)$$

$$= \tilde{\mathcal{E}}^{(c)}(u, u) + \sum_{i=1}^d b_i \int \frac{\partial u}{\partial x_i} \cdot u(x) \, dx + \int c(x) u^2(x) \, dx$$

$$\geq \tilde{\mathcal{E}}^{(c)}(u, u) - \sum_{i=1}^{d} |b_i| \left(2 \frac{\Gamma((1+d)/2)}{\kappa \pi^{(d+2)/2}} \iint_{x \neq y} (u(x) - u(y))^2 k_s(x, y) \, dx \, dy \right.$$

$$\left. + \frac{c_d \Gamma((1+d)/2)}{\kappa \pi^{(d+2)/2}} \|u\|_{L^2}^2 \right)$$

$$- \|c\|_\infty \cdot \|u\|_{L^2}^2$$

$$\geq \tilde{\mathcal{E}}^{(c)}(u, u) - \frac{1}{8} \mathcal{E}^{(j)}(u, u) - \left(\|c\|_\infty + \frac{c_d}{8} \right) \cdot \|u\|_{L^2}^2.$$

Therefore

$$\eta_{\beta_0}(u, u) = \eta^{(c)}(u, u) + \eta_{\beta_0}^{(j)}(u, u)$$

$$\geq \tilde{\mathcal{E}}^{(c)}(u, u) - \frac{1}{8} \mathcal{E}^{(j)}(u, u) - \left(\|c\|_\infty + \frac{c_d}{8} \right) \cdot \|u\|^2 + \frac{1}{4} \mathcal{E}^{(j)}(u, u) + \frac{\beta_0}{4} \|u\|_{L^2}^2$$

$$\geq \frac{1}{8} \tilde{\mathcal{E}}(u, u) + \left(\frac{\beta_0}{4} - \|c\|_\infty - \frac{c_d}{8} \right) \|u\|_{L^2}^2.$$

Hence if we take β as $\beta_0 + \|c\|_\infty + c_d/8$, then we see $\eta_\beta(u, u) \geq 0$ for $u \in C_0^1(\mathbb{R}^d)$ and the weak sector condition from the preceding lemma. The Markov property also holds as in the uniformly elliptic case. □

REMARK 4.1.   Note that, when the drift term does not appear in the form (that is, $b = 0$), only the condition (J.1) on the kernel k (not necessarily to assume neither (J.2) nor (J.3)) guarantees that the form $(\tilde{\mathcal{E}}, \mathcal{F})$ becomes a regular symmetric Dirichlet form.

5.   Associated diffusion process with jumps

Let $(\eta, \mathcal{F})$ be a regular lower bounded semi-Dirichlet form on $L^2(X; m)$ as defined in Section 2. For the symmetrization $\tilde{\eta}$, the pair $(\tilde{\eta}, \mathcal{F})$ is then a closed symmetric form on $L^2(X; m)$ but not necessarily a symmetric Dirichlet form. A symmetric Dirichlet form $\mathcal{E}$ on $L^2(X; m)$ with domain $\mathcal{F}$ is called a *reference* (*symmetric Dirichlet*) *form* of η as in [8] if, for each fixed $\alpha > \beta$,

$$(5.1) \qquad\qquad c_1 \mathcal{E}_1(u, u) \leq \eta_\alpha(u, u) \leq c_2 \mathcal{E}_1(u, u), \quad u \in \mathcal{F}$$

for some positive constants c_1, c_2 independent of $u \in \mathcal{F}$. The form $(\mathcal{E}, \mathcal{F})$ is then a regular Dirichlet form. In what follows, we assume that η admits a reference form $\mathcal{E}$.

In considering an association of a Hunt process with η, we need some potential theory attached to the form η. In order to formulate our assertion, denote by $\mathcal{O}$ the family of all open sets $O \subset G$ so that $\mathcal{L}_O := \{u \in \mathcal{F}: u \geq 1 \text{ a.e. on } O\} \neq \varnothing$. Fix $\alpha > \beta$ and for $O \in \mathcal{O}$, let e_O be the η_α-projection of 0 on $\mathcal{L}_O$ in Stampaccia's sense [21]:

$$(5.2) \qquad\qquad e_O \in \mathcal{L}_O, \quad \eta_\alpha(e_O, w) \geq \eta_\alpha(e_O, e_O), \quad \text{for any} \quad w \in \mathcal{L}_O.$$

A set $N \subset G$ is called η-polar if there exists decreasing $O_n \in \mathcal{O}$ containing N such that e_{O_n} is η_α-convergent to 0 as $n \to \infty$. A numerical function u on G is said to be η-*quasi-continuous* if there exists decreasing $O_n \in \mathcal{O}$ such that e_{O_n} is η_α-convergent to 0 as $n \to \infty$ and $u|_{G \setminus O_n}$ is continuous for each n.

The capacity Cap for the reference form $\mathcal{E}$ is defined by

$$(5.3) \qquad \mathrm{Cap}(O) := \inf\{\mathcal{E}_1(u, u) \colon u \in \mathcal{L}_O\}, \quad O \in \mathcal{O}.$$

Then it follows that

$$c_1 \, \mathrm{Cap}(O) \le \eta_\alpha(e_O, e_O) \le c_2 K_\alpha^2 \, \mathrm{Cap}(O), \quad O \in \mathcal{O}, \quad K_\alpha = K + \frac{\alpha}{\alpha - \beta},$$

since (5.1) and (B.2) imply that $\eta_\alpha(e_O, e_O) \le K_\alpha^2 \eta_\alpha(w, w)$, $w \in \mathcal{L}_O$. (5.3) means that a set N is η-polar if and only if it is $\mathcal{E}$-polar in the sense that $\mathrm{Cap}(N) = 0$, and a function u is η-quasi-continuous if and only if it is $\mathcal{E}$-quasi-continuous in the sense that there exist decreasing $O_n \in \mathcal{O}$ with $\mathrm{Cap}(O_n) \downarrow 0$ as $n \to \infty$ and $X \setminus O_n$ is continuous for each n. Every element of F admits its η-quasi-continuous m-version. If $\{u_n\} \subset \mathcal{F}$ is η_α-convergent to $u \in \mathcal{F}$ and if each u_n is η-quasi-continuous, then (5.1) implies that a subsequence of $\{u_n\}$ converges η-q.e., namely, outside some η-polar set, to an η-quasi-continuous version of u. We shall occasionally drop η from the terms η-polar, η-q.e. and η-quasi-continuity for simplicity. Then the following theorem is shown in [8, Theorem 4.1] by making use of the result of Carrillo-Menendez [4].

Theorem 5.1. *There exist a Borel η-polar set $N_0 \subset X$ and a Hunt process $\mathsf{M} = (X_t, P_x)$ on $X \setminus N_0$ which is properly associated with $(\eta, \mathcal{F})$ in the sense that $R_\alpha f$ is a quasi continuous version of $G_\alpha f$ for any $\alpha > 0$ and any bounded Borel $f \in L^2(X; m)$. Here R_α is the resolvent of M and G_α is the resolvent associated with η.*

In the following, we will assert that the resolvent of a Hunt process associated to our Dirichlet form is absolutely continuous with respect to Lebesgue measure using a Sobolev inequality.

Theorem 5.2. *Let $(\eta, \mathcal{F})$ be the lower bounded semi-Dirichlet form on $L^2(G)$ defined in* Section 3 *for an open set $G \subset \mathbb{R}^d$ (resp. in* Section 4 *for $G = \mathbb{R}^d$). We state the results separately*:
(i) *Assume* (D.1), (D.2), (J.1) *and* (J.2) *and the elliptic constant $\lambda > 0$ satisfies* (3.8). *Moreover we assume $d \ge 3$.*
(ii) *Set $G = \mathbb{R}^d$. Assume* (D.1)′–(D.3)′, (J.1)–(J.3), *$d \ge 2$ and the constant $\kappa > 0$ satisfies* (4.1).
In each case, there exist $\alpha > \beta$ and $q > 2$ such that

$$(5.4) \qquad \|u\|_{L^q}^2 \le C \eta_\alpha(u, u), \quad u \in \mathcal{F}$$

for some constant $C > 0$. So, we then see that there exists a Borel η-polar set N_0 such that $G \setminus N_0$ is M-invariant and $R_\alpha(x, \cdot)$ is absolute continuous with respect to Lebesgue measure on G for each $\alpha > 0$ and $x \in G \setminus N_0$.

Proof.   CASE (i):   By the proof of Proposition 3.2, we find that the inequality

$$\eta_\alpha(u, u) \geq C\mathcal{E}_1(u, u), \quad u \in \mathcal{F}$$

holds for some $C > 0$ and $\alpha > \beta$. Here the form $\mathcal{E}_1$ is defined as

$$\mathcal{E}_1(u, u) = \frac{1}{2} \sum_{i,j=1}^{d} \int_G \left(\frac{\partial u}{\partial x_i}\right)^2 dx + \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))^2 k(x, y) \, dx \, dy + \int_G u^2 \, dx$$

for $u \in \mathcal{F}$. So $\mathcal{E}$ and, hence η_α satisfies the Sobolev inequality (5.4) with $1/2 > 1/q = 1/2 - 2/d$, since

$$\eta_\alpha(u, u) \geq C\mathcal{E}_1(u, u) \geq \frac{C}{2} \sum_{i=1}^{d} \int_G \left(\frac{\partial u}{\partial x_i}\right)^2 dx \geq C' \|u\|_{L^q}^2, \quad u \in \mathcal{F}$$

by Gagliardo–Nirenberg–Sobolev inequality (the case $p = 2$ in (3.7)).

CASE (ii):   By the proof of Theorem 4.1, for some constant $C > 0$ and any $\alpha > \beta$, it follows that

$$\eta_\alpha(u, u) \geq \frac{1}{8}\tilde{\mathcal{E}}(u, u) + C\|u\|_{L^2}^2, \quad u \in \mathcal{F},$$

where

$$\tilde{\mathcal{E}}(u, u) = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dx$$
$$+ \frac{1}{2} \iint_{x \neq y} (u(x) - u(y))^2 k(x, y) \, dx \, dy, \quad u \in \mathcal{F}.$$

From the assumptions (J.1) and (J.3), we see for $u \in C_0^1(\mathbb{R}^d)$,

$$\iint_{x \neq y} (u(x) - u(y))^2 k(x, y) \, dx \, dy$$
$$\geq \iint_{0 < |x-y| < 1} (u(x) - u(y))^2 k(x, y) \, dx \, dy$$
$$\geq \kappa \iint_{0 < |x-y| < 1} (u(x) - u(y))^2 |x - y|^{-d-1} \, dx \, dy$$
$$= \kappa \iint_{x \neq y} (u(x) - u(y))^2 |x - y|^{-d-1} \, dx \, dy$$
$$- \kappa \iint_{|x-y| \geq 1} (u(x) - u(y))^2 |x - y|^{-d-1} \, dx \, dy$$

$$\geq \kappa \iint_{x \neq y} (u(x) - u(y))^2 |x - y|^{-d-1} \, dx \, dy$$

$$- 4\kappa \int_{\mathbb{R}^d} u(x)^2 \int_{|x-y| \geq 1} |x - y|^{-d-1} \, dy \, dx$$

$$\geq M \|u\|_{L^q}^2 - 4\kappa c_d \|u\|_{L^2}^2,$$

where M is a positive constant and q satisfies $1/2 > 1/q > 1/2 - 1/(2d)$ (see e.g. [7, (1.4.32)]). This implies that, for some $\alpha_0 > \beta$, $q > 2$ and $C' > 0$,

$$\|u\|_{L^q} \leq C' \eta_{\alpha_0}(u, u), \quad u \in \mathcal{F}.$$

By making use of Theorem 1 and 2 in [6], the latter statement in the theorem follows in each case. $\qquad\square$

We now consider a conservativeness problem of a jump-diffusion associated with a regular lower bounded semi-Dirichlet form $(\eta, \mathcal{F})$. We assume that η admits an operator $(L, \mathcal{D}(L))$ satisfying the following:

$$(5.5) \qquad \eta(f, g) = -(Lf, g), \quad f \in D(L), \ g \in \mathcal{F},$$

where $D(L)$ is a dense subset of $\mathcal{F}$ with respect to the norm $\sqrt{\eta_\alpha(\cdot, \cdot)}$ for $\alpha > \beta$ (see cf. [15, Section I.2]). We further assume that
(L.1) $D(L)$ is a linear subspace of $\mathcal{F} \cap C_0(G)$,
(L.2) L is a linear operator sending $D(L)$ into $L^2(G) \cap C_b(G)$,
(L.3) there exists a countable subfamily D_0 of $D(L)$ such that each $f \in D(L)$ admits $f_n \in D_0$ satisfying that f_n, Lf_n are uniformly bounded and converge pointwise to f, Lf, respectively, as $n \to \infty$.
We also consider an additional condition that
(L.4) there exists $f_n \in D(L)$ such that f_n and Lf_n are uniformly bounded and converge to 1 and 0, respectively, as $n \to \infty$.
As in Theorem 4 in [8], we then see the following theorem:

Theorem 5.3. *Assume that the operator $(L, D(L))$ satisfies the conditions* (L.1)–(L.3).
(i) *There exists then a Borel properly exceptional set N containing N_0 such that, for every $f \in D(L)$,*

$$(5.6) \qquad M_t^{[f]} := f(X_t) - f(X_0) - \int_0^t (Lf)(X_s) \, ds, \quad t \geq 0$$

is a P_x-martingale for each $x \in G \setminus N$.
(ii) *If the additional condition* (L.4) *is satisfied, then the Hunt process $X_{G \setminus N}$ is conservative.*

The proof of this theorem is done quite the same way as that of Theorem 4 in [8]. So we omit it.

6. Example

In this section, we give an example which is related to a second order (degenerate) elliptic differential operator with stable-type generator. To this end, we assume $1 \le \alpha < 2$, $\alpha/2 < \delta \le 1$ and set $a_{ij}(x) = x_i \cdot x_j$, $i, j = 1, 2, \ldots, d$, $b(x) = (1, 1, \ldots, 1)$ for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. Put

$$k(x, y) = C(|x|^\delta + 1) \cdot |x - y|^{-d-\alpha}, \quad x, y \in \mathbb{R}^d, \ x \ne y$$

for some positive constant $C > 0$. According to the previous section, we find that, for $u, v \in C_0^2(\mathbb{R}^d)$, a quadratic form defined by

$$
\eta(u, v) = \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d} x_i x_j \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \, dx + \sum_{i=1}^d \int_{\mathbb{R}^d} \frac{\partial u(x)}{\partial x_i} v(x) \, dx
$$
$$
+ \frac{C}{2} \iint_{x \ne y} (u(x) - u(y))(v(x) - v(y)) \frac{|x|^\delta + 1}{|x - y|^{d+\alpha}} \, dx \, dy
$$
$$
+ \frac{C}{2} \iint_{x \ne y} (u(x) - u(y)) v(x)(|x|^\delta - |y|^\delta)|x - y|^{-d-\alpha} \, dx \, dy,
$$

produces a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$. In fact, since we easily see the functions $\{a_{ij}\}$ satisfy the condition (D.1)$'$, we only check the conditions (J.1) and (J,2).

<u>(J.1)</u>: Since $k(x, y) = C(|x|^\delta + 1) \cdot |x - y|^{-d-\alpha}$ for $x \ne y$,

$$
M_s(x) = C \int_{y \ne x} (1 \wedge |x - y|^2)(|x|^\delta + |y|^\delta + 2)|x - y|^{-d-\alpha} \, dy
$$
$$
= C \left(\int_{0 < |x-y| < 1} + \int_{|x-y| \ge 1} \right)(1 \wedge |x - y|^2)(|x|^\delta + |y|^\delta + 2)|x - y|^{-d-\alpha} \, dy
$$
$$
=: \text{(I)} + \text{(II)},
$$

$$
\text{(I)} = C \int_{0 < |x-y| < 1} (|x|^\delta + |y|^\delta + 2)|x - y|^{2-d-\alpha} \, dy
$$
$$
= C(|x|^\delta + 2) \int_{0 < |h| < 1} |h|^{2-d-\alpha} \, dh + C \int_{0 < |h| < 1} |x + h|^\delta \cdot |h|^{2-d-\alpha} \, dh
$$
$$
\le C(|x|^\delta + 2) \cdot c_d \int_0^1 u^{1-\alpha} \, du + 2C \int_{0 < |h| < 1} (|x|^\delta + |h|^\delta) \cdot |h|^{2-d-\alpha} \, dh
$$
$$
= \frac{Cc_d}{2 - \alpha}(|x|^\delta + 2) + 2Cc_d \int_0^1 (|x|^\delta + u^\delta) \cdot u^{1-\alpha} \, du
$$

$$\leq \frac{Cc_d}{2-\alpha}(|x|^\delta + 2) + 2Cc_d\left(\frac{|x|^\delta}{2-\alpha} + \frac{1}{2+\delta-\alpha}\right),$$

$$\text{(II)} = C\int_{|x-y|\geq 1}(|x|^\delta + |y|^\delta + 2)|x-y|^{-d-\alpha}\,dy$$

$$= C(|x|^\delta + 2)\int_{|h|\geq 1}|h|^{-d-\alpha}\,dh + C\int_{|h|\geq 1}|x+h|^\delta\cdot|h|^{-d-\alpha}\,dh$$

$$\leq Cc_d(|x|^\delta + 2)\int_1^\infty u^{-1-\alpha}\,du + 2^{\delta-1}Cc_d\int_1^\infty(|x|^\delta + u^\delta)u^{-1-\alpha}\,du$$

$$= \frac{Cc_d}{\alpha}(|x|^\delta + 2) + 2^{\delta-1}Cc_d\left(\frac{|x|^\delta}{\alpha} + \frac{1}{\alpha-\delta}\right).$$

Here we used the inequality: $|x+h|^\delta \leq 2^{\delta-1}(|x|^\delta + |h|^\delta)$ for any $x,h \in \mathbb{R}^d$ in estimating the term (II). Thus $M_s \in L^1_{\text{loc}}(\mathbb{R}^d)$ holds.

(J.2): We first show that $\sup_{x\in\mathbb{R}^d}\int_{|x-y|\geq 1}|k_a(x,y)|\,dy < \infty$:

$$\sup_{x\in\mathbb{R}^d} C\int_{|x-y|\geq 1}\left||x|^\delta - |y|^\delta\right|\cdot|x-y|^{-d-\alpha}\,dy$$

$$\leq \sup_{x\in\mathbb{R}^d} C\int_{|x-y|\geq 1}|x-y|^\delta\cdot|x-y|^{-d-\alpha}\,dy$$

$$= Cc_d\int_0^\infty u^{-1+\delta-\alpha}\,du = \frac{Cc_d}{\alpha-\delta} < \infty.$$

Next we see

$$\sup_{x\in\mathbb{R}^d}\int_{0<|x-y|<1}|k_a(x,y)|^\gamma\,dy$$

$$\leq \sup_{x\in\mathbb{R}^d} C^\gamma\int_{0<|x-y|<1}\left(\left||x|^\delta - |y|^\delta\right|\cdot|x-y|^{-d-\alpha}\right)^\gamma\,dty$$

$$\leq \sup_{x\in\mathbb{R}^d} C^\gamma\int_{0<|x-y|<1}|x-y|^{(\delta-d-\alpha)\gamma}\,dy \leq c'\int_0^1 u^{(\delta-d-\alpha)\gamma+d-1}\,du < \infty$$

when $\gamma < d/(d+\alpha-\delta)$. In obtaining the first inequality, we used the Hölder continuity of the function $x \mapsto |x|^\delta$ for $0 < \delta \leq 1$: $\left||x|^\delta - |y|^\delta\right| \leq |x-y|^\delta$, $x,y \in \mathbb{R}^d$. Moreover, for such γ,

$$\sup_{0<|x-y|<1}\frac{|k_a(x,y)|^{2-\gamma}}{k_s(x,y)} = C^{1-\gamma}\cdot\sup_{0<|x-y|<1}\frac{\left(\left||x|^\delta - |y|^\delta\right|\cdot|x-y|^{-d-\alpha}\right)^{2-\gamma}}{(|x|^\delta + |y|^\delta + 2)|x-y|^{-d-\alpha}}$$

$$\leq C^{1-\gamma}\cdot\sup_{0<|x-y|<1}|x-y|^{(\delta-d-\alpha)(2-\gamma)+d+\alpha} < \infty$$

provided that $(\delta - d - \alpha)(2 - \gamma) + d + \alpha \geq 0$, that is, $\gamma \geq (d + \alpha - 2\delta)/(d + \alpha - \delta)$. Hence if we take

$$\frac{d + \alpha - 2\delta}{d + \alpha - \delta} \leq \gamma < \frac{d}{d + \alpha - \delta},$$

then (J.2) is satisfied and this can happen in the case when $\alpha/2 < \delta \leq 1$.

Now we define

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^{d} x_i x_j \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left(\frac{d \cdot x_i}{2} + 1 \right) \cdot \frac{\partial u(x)}{\partial x_i}$$

$$+ \frac{C}{2} \int_{y \neq x} (u(y) - u(x) - \nabla u(x) \cdot (y - x) \mathbf{1}_{F(x)}(y - x)) \frac{(|x|^\delta + 1)}{|x - y|^{d+\alpha}} \, dy$$

for $f \in D(\mathcal{L}) := C_0^2(\mathbb{R}^d)$, where $F(x) := \{h \in \mathbb{R}^d \colon 0 < |h| \leq \sqrt{1 + |x|^2}\}$, $x \in \mathbb{R}^d$. Then we see that the restriction of the generator of the form η to $D(\mathcal{L})$ on $L^2(\mathbb{R}^d)$ coincides with $(\mathcal{L}, D(\mathcal{L}))$. In fact, the form of the local part is easily seen from the corresponding part of the Dirichlet form. As for the nonlocal part, since the nonlocal part of generator of the Dirichlet form is defined through the limit of the following integrals:

$$\mathcal{L}^{(j)}u(x) := \lim_{n \to \infty} \mathcal{L}^{(j,n)}u(x) := \lim_{n \to \infty} \frac{1}{2} \int_{|x-y|>1/n} (u(y) - u(x)) \frac{C(|x|^\delta + 1)}{|x - y|^{d+\alpha}} \, dy$$

by (1.1) and the integral $\int_{|x-y|>1/n} \nabla u(x) \cdot (y-x) \mathbf{1}_{F(x)}(y-x) C(|x|^\delta + 1)/(|x - y|^{d+\alpha}) \, dy$ disappears for any $n \in \mathbb{N}$, it follows that

$$\mathcal{L}^{(j)}u(x) = \lim_{n \to \infty} \frac{1}{2} \int_{|x-y|>1/n} (u(y) - u(x)) \frac{C(|x|^{\delta+1} + 1)}{|x - y|^{d+\alpha}} \, dy$$

$$= \lim_{n \to \infty} \frac{1}{2} \int_{|x-y|>1/n} (u(y) - u(x) - \nabla u(x) \cdot (y - x) \mathbf{1}_{F(x)}) \frac{C(|x|^\delta + 1)}{|x - y|^{d+\alpha}} \, dy$$

$$= \frac{1}{2} \int_{x \neq y} (u(y) - u(x) - \nabla u(x) \cdot (y - x) \mathbf{1}_{F(x)}) \frac{C(|x|^\delta + 1)}{|x - y|^{d+\alpha}} \, dy.$$

We also see that $\mathcal{L}^{(j)}u \in L^2(\mathbb{R}^d)$ for $u \in C_0^2(\mathbb{R}^d)$. It is easily seen that the conditions (L.1)–(L.3) are satisfied for $(\mathcal{L}, C_0^2(\mathbb{R}^d))$.

Take a smooth function w defined on $[0, +\infty)$ so that

$$w(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq 1, \\ 0 & \text{if } t \geq 2 \end{cases}$$

and set $f_n(x) = w(|x|/n)$, $x \in \mathbb{R}^d$, $n = 1, 2, \ldots$. Then we show that (L.4) holds for the sequence $\{f_n\}$. To this end, we follow an argument developed in [20]. Since the

function w is constant outside the annulus $\{1 \le |x| \le 2\}$, the supports of $\partial_i f_n$ and $\partial_i \partial_j f_n$ are included in the set $K_n = \{n \le |x| \le 2n\}$ for $i, j = 1, 2, \ldots, d$. Moreover, noting that w, w' and w'' are continuous functions having support compact, it follows

$$(6.1) \qquad c := \sup_{x \in \mathbb{R}^d} (1 + |x|^2)\{|w(|x|)| + |w'(|x|)| + |w''(|x|)|\} < \infty.$$

For any $i, j = 1, 2, \ldots, d$ with $i \ne j$,

$$\partial_i \partial_i f_n(x) = \frac{1}{n} \frac{|x|^2 - x_i^2}{|x|^3} w'\left(\frac{|x|}{n}\right) + \frac{1}{n^2} \frac{x_i^2}{|x|^2} w''\left(\frac{|x|}{n}\right), \quad x \in K_n$$

and

$$\partial_i \partial_j f_n(x) = \frac{1}{n^2} \frac{x_i x_j}{|x|^2} w''\left(\frac{|x|}{n}\right), \quad x \in K_n.$$

So

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} |x_i x_j \partial_i \partial_j f_n(x)| \le \frac{|x|}{n} \left|w'\left(\frac{|x|}{n}\right)\right| + \frac{|x|^2}{n^2} \left|w''\left(\frac{|x|}{n}\right)\right| \le 2c,$$

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}^d} \left|\left(\frac{d x_i}{2} + 1\right) \partial_i f_n(x)\right| \le \left(\frac{d|x_i|}{2} + 1\right) \cdot \frac{|x_i|}{n} \cdot \left|w'\left(\frac{|x|}{n}\right)\right| \le \frac{dc}{2}$$

and

$$(6.2) \qquad \lim_{n \to \infty} x_i x_j \partial_i \partial_j f_n(x) = \lim_{n \to \infty} \left(\frac{d x_i}{2} + 1\right) \partial_i f_n(x) = 0.$$

On the other hand, we also see

$$|\partial_i \partial_j f_n(x)| \le \frac{1}{n^2} \left\{\left|w'\left(\frac{|x|}{n}\right)\right| + \left|w''\left(\frac{|x|}{n}\right)\right|\right\}, \quad \text{for} \quad x \in \mathbb{R}^d, \ i, j = 1, 2, \ldots, d.$$

Hence by the Taylor theorem applied to f_n, we find

$$|f_n(x + h) - f_n(x) - \nabla f_n(x) \cdot h|$$

$$= \frac{1}{2} \left|\sum_{i,j=1}^{d} \partial_i \partial_j f_n(x + \theta h) h_i h_j\right|$$

$$\le \frac{d^2}{2n^2} |h|^2 \left\{\left|w'\left(\frac{|x + \theta h|}{n}\right)\right| + \left|w''\left(\frac{|x + \theta h|}{n}\right)\right|\right\}, \quad for \quad x, h \in \mathbb{R}^d$$

with some constant $\theta = \theta(x, h) \in (0, 1)$. A simple calculation tells us that

$$(1 + |x|^2) \le 3n^2 \left(1 + \left|\frac{x + \theta h}{n}\right|^2\right),$$

for $x, h \in \mathbb{R}^d$ with $0 < |h| \le \sqrt{1 + |x|^2}$ and $0 < \theta \le 1$.

So we have

$$\frac{3d^2}{2(1+|x|^2)}|h|^2\left(1+\left|\frac{x+\theta h}{n}\right|^2\right)\cdot\left\{\left|w'\left(\frac{|x+\theta h|}{n}\right)\right|+\left|w''\left(\frac{|x+\theta h|}{n}\right)\right|\right\}$$
$$\leq\frac{3cd^2}{2(1+|x|^2)}|h|^2$$

and this implies that

$$\sup_{n\geq1}\sup_{x\in\mathbb{R}^d}\left|\int_{0<|h|<\sqrt{1+|x|^2}}(f_n(x+h)-f_n(x)-\nabla f_n(x)\cdot h)\frac{C(1+|x|^\delta)}{|h|^{d+\alpha}}\,dh\right|<\infty$$

and

$$(6.3)\qquad\lim_{n\to\infty}\int_{0<|h|<\sqrt{1+|x|^2}}(f_n(x+h)-f_n(x)-\nabla f_n(x)\cdot h)\frac{C(1+|x|^\delta)}{|h|^{d+\alpha}}\,dh=0.$$

For all $x\in\mathbb{R}^d$,

$$\int_{|h|\geq\sqrt{1+|x|^2}}(f_n(x+h)-f_n(x))\frac{C(1+|x|^\delta)}{|h|^{d+\alpha}}\,dh\leq 2C(1+|x|^\delta)\int_{|h|\geq\sqrt{1+|x|^2}}\frac{dh}{|h|^{d+\alpha}}$$
$$=2Cc_d(1+|x|^\delta)\int_{\sqrt{1+|x|^2}}^\infty u^{-1-\alpha}\,du=\frac{2Cc_d}{\alpha}(1+|x|^\delta)\cdot(1+|x|^2)^{-\alpha/2}.$$

Since $\delta\leq1\leq\alpha$, we see

$$\sup_{n\in\mathbb{N}}\sup_{x\in\mathbb{R}^d}\left|\int_{|h|\geq\sqrt{1+|x|^2}}(f_n(x+h)-f_n(x))C(|x|^\delta+1)|h|^{-d-\alpha}\,dh\right|<\infty$$

and

$$(6.4)\qquad\lim_{n\to\infty}\int_{|h|\geq\sqrt{1+|x|^2}}(f_n(x+h)-f_n(x))C(|x|^\delta+1)|h|^{-d-\alpha}\,dh=0.$$

Hence, combining the calculations above with (6.2)–(6.4), we find that $\{\mathcal{L}f_n\}$ is uniformly bounded and the sequence $\mathcal{L}f_n$ converges to 0. This implies that (L.4) is satisfied and then we can conclude the process is conservative. Thus we obtain the following proposition:

Proposition 6.1. *Take $1\leq\alpha<2$, $\alpha/2<\delta\leq1$ and $C>0$ is a sufficiently large real number.*

Define the following quadratic form η on $L^2(\mathbb{R}^d)$:

$$\eta(u, v) = \frac{1}{2} \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} x_i x_j \frac{\partial u(x)}{\partial x_i} \frac{\partial u(x)}{\partial x_j} \, dx + \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u(x)}{\partial x_i} v(x) \, dx$$

$$+ \frac{C}{2} \iint_{x \neq y} (u(x) - u(y))(v(x) - v(y)) \frac{|x|^\delta + 1}{|x - y|^{d+\alpha}} \, dx \, dy$$

$$+ \frac{C}{2} \iint_{x \neq y} (u(x) - u(y))v(x)(|x|^\delta - |y|^\delta)|x - y|^{-d-\alpha} \, dx \, dy$$

for $u, v \in C_0^2(\mathbb{R}^d)$. Then $(\eta, C_0^2(\mathbb{R}^d))$ is closable on $L^2(\mathbb{R}^d)$ and its closure $(\eta, \mathcal{F})$ is a regular lower bounded semi-Dirichlet form on $L^2(\mathbb{R}^d)$. Moreover the associated Hunt process is conservative.

REMARK 6.1. (i) In [24], Takeda and Trutnau recently showed the conservativeness of non-symmetric diffusion processes (without the jump part) by using forward and backward martingale decomposition which is a generalization of the so-called Lyons–Zheng decomposition of the Dirichlet form. Their conditions on the diffusion data are a sharp and they also treated the case where the diffusion coefficients are not necessarily smooth, but different from the diffusion processes case, our processes involve the jump part and the tool of the martingale additive functional may not be applicable to obtain a sharp result.

(ii) Similar to [20], writing down a precise form of the generator of a lower bounded semi-Dirichlet form on some nice functions space, we can also show the conservativeness of the associated Hunt process under the following conditions in addition to the assumptions imposed in Theorem 4.1: there exists a constant $C > 0$ so that for any $x \in \mathbb{R}^d$ and $i, j = 1, 2, \ldots, d$,

- $|a_{ij}(x)| \vee |\partial a_{ij}(x)/\partial x_i| \leq C(1 + |x|^2) \log(|x| + 2)$
- $\int_{0 < |h| < \sqrt{|x|^2 + 4}/2} |h|^2 k_s(x, x + h) \, dh \leq C(1 + |x|^2) \log(|x| + 2)$
- $\int_{0 < |h| < \sqrt{|x|^2 + 4}/2} |h| \cdot |k_a(x, x + h) - k_a(x, x - h)| \, dh \leq C(1 + |x|) \log(|x| + 2)$.

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