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DEFORMATIONS OF SPECIAL LEGENDRIAN SUBMANIFOLDS WITH BOUNDARY

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Abstract

In this paper, for a compact special Legendrian submanifold with smooth boundary of contact Calabi–Yau manifolds we study the deformation of it with boundary confined in an appropriately chosen contact submanifold of codimension two which we also call a scafford (Definition 2.3) by analogy with Butsher [1]. Our first result shows that it cannot be deformed, and the second claims that deformations of such a special Legendrian submanifold forms a one-dimensional smooth manifold under suitably weaker boundary confinement conditions. They may be viewed as supplements of the closed case considered by Tomassini and Vezzoni [17].

1. Introduction and main results

The calibrated geometry was invented by Harvey and Lawson in their seminal paper [5]. A class of important calibrated submanifolds is special Lagrangian submanifolds in Calabi–Yau manifolds. Let (M, J, ω, Ω) be a real $2n$-dimensional Calabi–Yau manifold. A special Lagrangian submanifold of it is a submanifold L with $\omega|_L = 0$ and $\mathrm{Im}(\Omega)|_L = 0$. In 1996 McLean [10] developed the deformation theory of special Lagrangian submanifolds (and other special calibrated submanifolds) and showed:

McLean theorem ([10]). *A normal vector field V to a compact special Lagrangian submanifold L without boundary in (M, J, ω, Ω) is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the corresponding 1-form $(JV)^\flat$ on L is harmonic. There are no obstructions to extending a first order deformation to an actual deformation and the tangent space to such deformations can be identified through the cohomology class of the harmonic form with $H^1(L; \mathbb{R})$.*

Since then the theory is generalized to various situations. See [6, 7, 13] and references therein. For example, S. Salur [14] generalized McLean theorem to symplectic manifolds. We here only list those closely related to ours. The first one is the case of compact special Lagrangian submanifolds with nonempty boundary considered by

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Butsher [1]. He called a submanifold L in the Calabi–Yau manifold (M, J, ω, Ω) *minimal Lagrangian* if $\omega|_L = 0$ and $\mathrm{Im}(e^{i\theta}\Omega)|_L = 0$ for some $\theta \in \mathbb{R}$. If L is a Lagrangian submanifold of (M, ω) with nonempty boundary ∂L and $N \in \Gamma(T_{\partial L}L)$ is the inward unit normal vector field of ∂L in L, he defined a *scaffold* for L to be a submanifold W of M such that $\partial L \subset W$, the bundle $(TW)^\omega$ is trivial, and that N is a smooth section of the bundle $(T_{\partial L}W)^\omega$.

Butsher theorem ([1]). *Let L be a special Lagrangian submanifold of a compact Calabi–Yau manifold M with non-empty boundary ∂L and let W be a symplectic, codimension two scaffold for L. Then the space of minimal Lagrangian submanifolds sufficiently near L (in a suitable $C^{1,\beta}$ sense) but with boundary on W is finite dimensional and is parametrized over the harmonic 1-forms of L satisfying Neumann boundary conditions.*

The work inspired Kovalev and Lotay [8] to study the analogous deformation problem of a compact coassociative 4-fold with boundary inside a particular fixed 6-dimensional submanifold with a compatible Hermitian symplectic structure in a 7-manifold with closed G_2-structures. Recently Gayet and Witt [3] also investigated the deformation of a compact associative submanifold with boundary in a coassociative submanifold in a topological G_2-manifold.

As a natural generalization of the Calabi–Yau manifolds in the context of contact geometry Tomassini and Vezzoni [17, Definition 3.1] introduced the notion of a contact Calabi–Yau manifold, cf. Definition 2.1. Let (M, η, J, ϵ) be a $(2n+1)$-dimensional contact Calabi–Yau manifold, and $j : L \hookrightarrow M$ be a compact special Legendrian submanifold without boundary (cf. Definition 2.2). Two special Legendrian submanifolds $j_0 \colon L \hookrightarrow M$ and $j_1 \colon L \hookrightarrow M$ are called *deformation equivalent* if there exists a smooth map $F \colon L \times [0, 1] \to M$ such that

- $F(\cdot\,, t) \colon L \times \{t\} \to M$ is a special Legendrian embedding for any $t \in [0, 1]$;
- $F(\cdot\,, 0) = j_0$, $F(\cdot\,, 1) = j_1$.

(cf. [17, Definition 4.4]). If there exists a diffeomorphism $\phi \in \mathrm{Diff}(L)$ such that $j_1 = j_0 \circ \phi$ we say j_0 and j_1 to be *equivalent*. This yields an equivalent relation $\sim$ among all embeddings from L to M. Let $\tilde{\mathfrak{M}}(L)$ be the set of special Legendrian submanifolds of (M, α, J, ϵ) which are deformation equivalent to $j \colon L \hookrightarrow M$. Call $\mathfrak{M}(L) := \tilde{\mathfrak{M}}(L)/\sim$ the *moduli space of special Legendrian submanifolds* which are deformation equivalent to $j \colon L \hookrightarrow M$. Tomassini and Vezzoni [17, Theorem 4.5] proved:

Tomassini–Vezzoni theorem ([17, Theorem 4.5]). Let (M, η, J, ϵ) be a contact Calabi–Yau manifold of dimension $2n+1$, and $L \subset M$ be a compact special Legendrian submanifold without boundary. Then the moduli space $\mathfrak{M}(L)$ is a smooth one-dimensional manifold.

Motivated by the above works, we study in this paper the local deformations of compact special Legendrian submanifolds with (nonempty) boundary. (The boundary is always assumed to be smooth throughout this paper.) Different from the case $\partial L = \emptyset$ considered by Tomassini and Vezzoni [17], it is showed in Remark 5.1 that the moduli space $\mathfrak{M}(L)$ is infinite dimensional.

In order to get interesting results it is necessary to add some boundary conditions. Inspired by [1, Definition 1] we introduce a notion of *scaffold* for L in Definition 2.3, which is a suitable contact submanifold W. Two special Legendrian submanifolds $j_0 \colon L \hookrightarrow M$ and $j_1 \colon L \hookrightarrow M$ with $j_0(\partial L) \subset W$ and $j_1(\partial L) \subset W$ are called *deformation equivalent* if there exists a smooth map $F \colon L \times [0, 1] \to M$ such that

- $F(\,\cdot\,, t) \colon L \times \{t\} \to M$ is a special Legendrian embedding with $F(\partial L, t) \subset W$ for any $t \in [0, 1]$;
- $F(\,\cdot\,, 0) = j_0$, $F(\,\cdot\,, 1) = j_1$.

The *moduli space of special Legendrian submanifolds* which are deformation equivalent to $j \colon L \hookrightarrow M$ with $j(\partial L) \subset W$ is defined as

$$\begin{aligned}\mathfrak{M}(L, W) := \{&\text{special Legendrian submanifolds of } (M, \alpha, J, \epsilon)\\ &\text{which are deformation equivalent to } j \colon L \hookrightarrow M\\ &\text{with } j(\partial L) \subset W \text{ and are near } j\}/\!\sim.\end{aligned}$$

Our first result is

Theorem 1.1. *Let (M, J, α, ϵ) be a contact Calabi–Yau manifold, and L be a compact special Legendrian submanifold with nonempty boundary ∂L inside a scaffold W of codimension two. Then L cannot be deformed as a special Legendrian submanifold with boundary confined in W. In other words $\mathfrak{M}(L, W)$ only consists of the class of j.*

This is in contrast with the case of compact special Legendrian submanifolds without boundary considered in Tomassini–Vezzoni theorem. Such a local rigidity is similar to the case of a compact simply connected special Lagrangian submanifold without boundary in McLean theorem, and Simons' rigidity result of stable minimal submanifolds with fixed boundary in [16].

Now we turn to consider weaker boundary conditions. Let (M, α, J, ϵ) be a $(2n+1)$-dimensional contact Calabi–Yau manifold, and $L \subset M$ be a compact special Legendrian submanifold with (non-empty) boundary. A normal vector field V to L is called *boundary α-constant* if $\alpha(V)|_{\partial L}$ is constant. The following result, which is stated in a similar way to McLean theorem above, is similar to that of Tomassini and Vezzoni [17].

Theorem 1.2. *Let (M, α, J, ϵ) be a $(2n+1)$-dimensional contact Calabi–Yau manifold, and and $L \subset M$ be a compact special Legendrian submanifold with (non-empty) boundary. A boundary α-constant normal vector field V to L is the deformation vector field to a normal deformation through special Legendrian submanifolds if and only if*

$\alpha(V)$ is constant. Moreover the tangent space to such deformations is given by $\mathbb{R}R_\alpha$, where R_α is the Reeb vector field of α.

Similar to the case L being compact and without boundary considered in Theorem 4.5 of [17] the deformation in Theorem 1.2 is also given by the isometries generated by the Reeb vector field, which is completely different from the deformation without boundary constraints as proved in Remark 5.1.

The key points in the proofs of Theorems 1.1 and 1.2 are to find a suitable definition of scaffold for a special Legendrian submanifold with boundary and to prove a corresponding result with Lemma 5 of [1], Lemma 3.1. For the former we propose and study it in Section 2. The proof of the latter will be given in Section 3 and is more troublesome because we need to use not only contact neighborhood theorem but also symplectic neighborhood theorem. In Sections 4 and 5, we complete the proofs of Theorems 1.1 and 1.2 respectively.

2. Preliminaries

2.1. Contact Calabi–Yau manifolds and special Legendrian submanifolds. Let (M,α) be a contact manifold with contact distribution $\xi = \ker\alpha$ and Reeb vector field R_α. Then $\kappa := d\alpha/2$ restricts to a symplectic vector bundle structure on $\xi \to M$, $\kappa|_\xi$, and every compatible complex structure $J \in \mathcal{J}(\xi, \kappa|_\xi)$ gives a Riemannian metric g_J on the bundle $\xi \to M$, $g_J(u,v) = \kappa(u, Jv)$ for $u, v \in \xi$. By setting $J(R_\alpha) = 0$ we can extend J to an endomorphism of TM, *also denoted by* J without special statements. Clearly

$$J^2 = -\mathbf{I} + \alpha \otimes R_\alpha, \quad \text{and} \quad g := g_J + \alpha \otimes \alpha \tag{2.1}$$

is a Riemannian metric g on M, where $\mathbf{I}$ is the identity endomorphism on TM. Define a Nijenhuis tensor of J by

$$N_J(X,Y) = [JX, JY] - J[X, JY] - J[JX, Y] + J^2[X,Y]$$

for all $X, Y \in TM$. If $N_J = -d\alpha \otimes R_\alpha$ then the pair (α, J) is a *Sasakian structure* on M, and the triple (M, α, J) is called a *Sasakian manifold*. On such a manifold it holds that $d\Lambda^r_B(M) \subset \Lambda^r_B(M)$ and $J(\Lambda^r_B(M)) = \Lambda^r_B(M)$, where $\Lambda^r_B(M)$ is the set of all differential r-form γ on M with $\iota_{R_\alpha}\gamma = 0$ and $\mathcal{L}_{R_\alpha}\gamma = 0$. So we have a split

$$\Lambda^r_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \Lambda^{p,q}_J(\xi)$$

and $\kappa = (1/2)\, d\alpha \in \Lambda^{1,1}_J(\xi)$.

DEFINITION 2.1 ([17, Definition 2.1]). A *contact Calabi–Yau manifold* is a quadruple (M,α,J,ϵ) consisting of a $(2n+1)$-dimensional Sasakian manifold (M,α,J) and a nowhere vanishing basic form $\epsilon \in \Lambda_J^{n,0}(\xi)$ such that

$$\epsilon \wedge \bar{\epsilon} = c_n \frac{\kappa^n}{n!}$$

and

$$d\epsilon = 0,$$

where $c_n = (-1)^{n(n+1)/2}(2i)^n$ and $\kappa = (1/2)\, d\alpha$.

DEFINITION 2.2 ([17, Definition 4.2]). Let $(M^{2n+1},\alpha,J,\epsilon)$ be a contact Calabi–Yau manifold. An embedding $p\colon L \to M$ is called a *special Legendrian submanifold* if $\dim L = n$, $p^*\alpha = 0$ and $p^* \operatorname{Im}\epsilon = 0$.

Clearly, $p^*\epsilon = p^*(\operatorname{Re}\epsilon)$ is a volume form on L. Thus every special Legendrian submanifold has a natural orientation. By [10, p. 722] or [2, Proposition 2.6] we have

$$p^*(\iota_Y \operatorname{Im}\epsilon) = -\star(p^*(\iota_Y\kappa)) = -\frac{1}{2}\star(p^*(\iota_Y d\alpha)) \tag{2.2}$$

for any section $Y\colon L \to p^*\xi$, where the star operator $\star$ is computed with respect to $p^*(g_J) = p^*(\kappa \circ (\mathrm{id} \times J))$ and the volume form $\mathrm{Vol}(L) := p^*\epsilon = p^*(\operatorname{Re}\epsilon)$.

For any n-dimensional manifold N, the cotangent bundle T^*N has a canonical 1-form λ_{can}. The 1-jet bundle $J^1N = \mathbb{R} \times T^*N$ is a contact manifold with contact form $\alpha = \pi_1^*(dt) - \pi_2^*(\lambda_{\mathrm{can}})$ and Reeb vector field $\partial/\partial t$, where $t \in \mathbb{R}$ is the real parameter and π_i is the projection from $\mathbb{R} \times T^*N$ onto the i-th factor, $i = 1, 2$. (See [9, Example 3.44]).

2.2. Boundary conditions. Corresponding to [1, Definition 1] we introduce:

DEFINITION 2.3. Let L be a submanifold of the contact manifold $(M,\xi = \ker\alpha)$ with boundary ∂L and let $N \in \Gamma(T_{\partial L}L)$ be the inward unit normal vector field of ∂L in L. A contact submanifold (W,ξ') of (M,ξ) is called a *scaffold* for L if
(i) $\partial L \subset W$,
(ii) $N \in \Gamma(\xi'^{\perp}|_{\partial L})$, and
(iii) the bundle $\xi'^{\perp}$ is trivial, where $\xi'^{\perp}$ is the symplectically orthogonal complement of ξ' in $(\xi|_W, \kappa|_{\xi|_W})$.

Given a contact manifold (M,α) let J and g be as in (2.1). If (W,ξ') is a contact submanifold of $(M,\xi = \ker\alpha)$, that is, $T_xW \cap \xi_x = \xi'_x$ for all $x \in W$, the following claim shows that the condition (iii) of Definition 2.3 is equivalent to one that $(TW)^{\perp_g}$ is trivial, where $(TW)^{\perp_g}$ denotes the orthogonal complementary bundle of TW in T_WM with respect to the metric g.

Claim 2.4. $(TW)^{\perp_g} = (J\xi')^{\perp} = J(\xi'^{\perp})$.

Proof. For $x \in W$, since $\xi_x'^{\perp} \subset \xi_x$ and J_x restricts to a complex structure on ξ_x we have

$$\begin{aligned}\xi_x'^{\perp} &= \{v \in \xi_x \mid \kappa(v, u) = 0 \ \forall u \in \xi_x'\}\\ &= \{v \in \xi_x \mid \kappa(Jv, Ju) = 0 \ \forall u \in \xi_x'\}\\ &= \{v \in \xi_x \mid g_J(Jv, u) = 0 \ \forall u \in \xi_x'\}\\ &= \{v \in \xi_x \mid g(Jv, bR_\alpha + u) = 0 \ \forall bR_\alpha + u \in R_\alpha\mathbb{R} + \xi_x'\}\\ &= \{v \in \xi_x \mid g(Jv, Y) = 0 \ \forall Y \in T_xW\}.\end{aligned}$$

This implies $J\xi'^{\perp} = (TW)^{\perp_g}$ or $\xi'^{\perp} = J(TW)^{\perp_g}$. Moreover, both $J\xi'^{\perp}$ and $\xi'^{\perp}$ are contained in $\xi|_W$, and ξ is J-invariant. It is easy to check that $J\xi'^{\perp} = (J\xi')^{\perp}$. □

Proposition 2.5. *Let L be a Legendrian submanifold of the contact manifold $(M, \xi = \ker\alpha)$ with (nonempty) boundary ∂L and let W be a scaffold for L. Then ∂L is a Legendrian submanifold of (W, ξ').*

Proof. Since L is the Legendrian submanifold of (M,ξ), $TL \subset \xi|_L$. Moreover the definition of the scaffold implies that $T\partial L \subset T_{\partial L}W$ and thus $T\partial L \subset T_{\partial L}W \cap \xi|_{\partial L} = \xi'|_{\partial L}$. This shows that the boundary ∂L is a Legendrian submanifold of (W, ξ'). □

Under the assumptions of Proposition 2.5, let $f_t\colon L \to M$ be a deformation of L satisfying $f_t(\partial L) \subset W$ for all t, and let $V = (d/dt)f_t|_{t=0}$ be the corresponding deformation vector field. Clearly, $V(x) \in T_xW$ for any $x \in \partial L$. Since L is a Legendrian submanifold, we have $TL \subset \xi|_L$. Note that $N(x) \in T_xL$ for any $x \in \partial L$. Then the condition (ii) of Definition 2.3 implies that $N(x) \in \xi_x'^{\perp}$, and so $N(x) \in T_xL \cap \xi_x'^{\perp}$ and

$$J_xN(x) \in J_x(T_xL \cap \xi_x'^{\perp}) \subset J_x\xi_x'^{\perp} \subset J_x\xi_x = \xi_x.$$

Since W is a contact submanifold, we may write $V(x) = Y + aR_\alpha(x)$, where $Y \in \xi_x'$. By Claim 2.4, $J_xN(x) \in J_x\xi_x'^{\perp} = (T_xW)^{\perp_g}$ and thus

$$0 = g(J_xN(x), V(x)) = g_J(J_xN(x), Y) = \kappa(J_xN(x), J_xY) = \kappa(N(x), Y).$$

Note that $Y = V(x) - \alpha(V(x))R_\alpha(x)$ and that $\iota_{R_\alpha}d\alpha = 0$. We get

Claim 2.6. *If $f_t\colon L \to M$ be a deformation of L satisfying $f_t(\partial L) \subset W$ for all t, then the corresponding deformation vector field V satisfies* Neumann boundary condition: $d\alpha(N(x), V(x)) = 0 \ \forall x \in \partial L$.

The Neumann boundary condition implies $\alpha(V|_{\partial L}) = 0$, see Remark 3.5.

EXAMPLE 2.7. It is not hard to construct an example satisfying the boundary conditions of Theorems 1.1 and 1.2. Let $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ denote the standard Euclidean coordinate in $\mathbb{R}^{2n+1}$. The standard contact Calabi–Yau structure (α, J, ϵ) on $\mathbb{R}^{2n+1}$ is given by

$$\alpha = 2\,dz - 2\sum_{j=1}^{n} y_j\,dx_j, \quad \epsilon = (dx_1 + i\,dy_1) \wedge \cdots \wedge (dx_n + i\,dy_n)$$

and

$$J : \xi = \operatorname{Ker}(\alpha) = \operatorname{Span}(\{y_1\partial_z + \partial_{x_1}, \dots, y_n\partial_z + \partial_{x_n}, \partial_{y_1}, \dots, \partial_{y_n}\}) \to \xi$$

where J is given by $JX_r = Y_r = \partial_{y_r}$ and $JY_r = -X_r = -y_r\partial_z - \partial_{x_r}$, $r = 1, \dots, n$. (See [17, Example 3.2]). Observe that this structure is invariant under the action of the subgroup $\mathbb{Z}^n \times \{0\}^{n+1}$ of $\mathbb{Z}^{2n+1}$. It descends to such a structure on $M = \mathbb{R}^{2n+1}/(\mathbb{Z}^n \times \{0\}^{n+1}) = \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^{n+1}$, also denoted by (α, J, ϵ) without occurs of confusions. As usual we write the point of M as $([x_1], \dots, [x_n], y_1, \dots, y_n, z)$. Let $n \geq 2$. Consider the contact submanifold of (M, α), $W = W_0 \cup W_1$,

$$W_k = \left\{([x_1], \dots, [x_n], y_1, \dots, y_{n-1}, 0, z) \in M \;\middle|\; x_n = \frac{k+1}{3}\right\}, \quad k = 0, 1.$$

Since the contact form on it is $\alpha' = \alpha|_W = 2\,dz - 2\sum_{j=1}^{n-1} y_j\,dx_j$, it is easy to see that the symplectically orthogonal complementary bundle $\xi'^{\perp}$ of $\xi' = \operatorname{Ker}(\alpha')$ in $(\xi|_W, \kappa|_{\xi|_W})$ is trivial. In fact, we have

$$\begin{aligned}
\xi' &= \operatorname{Span}(\{y_1\partial_z + \partial_{x_1}, \dots, y_{n-1}\partial_z + \partial_{x_{n-1}}, \partial_{y_1}, \dots, \partial_{y_{n-1}}\}),\\
\xi'^{\perp} &= \operatorname{Span}(\{y_n\partial_z + \partial_{x_n}, \partial_{y_n}\}).
\end{aligned}$$

Consider $L = \{([x_1],\dots,[x_n],0,\dots,0) \in M \mid 1/3 \leq x_n \leq 2/3\}$. It is a compact Legendrian submanifold with boundary $\partial L = \partial_0 L \cup \partial_1 L$, where

$$\partial_k L = \{([x_1], \dots, [x_n], 0, \dots, 0) \in M \mid x_n = (k+1)/3\}, \quad k = 0, 1.$$

Clearly, $\partial_k L \subset W_k$, $k = 0, 1$, and thus $\partial L \subset W$. By (2.1) the metric $g = g_J + \alpha \otimes \alpha$ satisfies: $g(R_\alpha, R_\alpha) = 1$, $g(X_r, X_s) = g(Y_r, Y_s) = \delta_{rs}$ and $g(X_r, Y_s) = g(X_r, R_\alpha) = g(Y_r, R_\alpha) = 0$ for $r, s = 1, \dots, n$. For $p = ([x_1], \dots, [x_n], 0, \dots, 0) \in \partial_0 L$ we have

$$T_p L = \operatorname{Span}(\{\partial_{x_1}|_p, \dots, \partial_{x_n}|_p\}), \quad T_p\partial_0 L = \operatorname{Span}(\{\partial_{x_1}|_p, \dots, \partial_{x_{n-1}}|_p\}).$$

Since $X_j|_p = \partial_{x_j}|_p$, $j = 1, \dots, n$, it follows that $X_n|_p$ is the inward unit normal vector at p of ∂L in L. Similarly, for $p = ([x_1], \dots, [x_n], 0, \dots, 0) \in \partial_1 L$ the inward unit normal vector at p of ∂L in L is $-X_n|_p$. Namely the inward unit normal vector field N of ∂L in L belongs to $\Gamma(\xi'^{\perp}|_{\partial L})$. Hence W is a scaffold for L.

3. Constructing a new metric

In the study of the deformation of the special Legendrian submanifold L without boundary by Tomassini and Vezzoni [17], the deformations of L are parameterized by sections of the normal bundle $N(L)$ using the exponent map $\exp(V)\colon L \to M$. However, in our case, since W is generally not totally geodesic, it cannot be assured that the image of ∂L under $\exp(V)$ sits in W. In order to fix out the problem we shall follow the ideas in [1] to construct a new metric $\hat{g}$ such that the image of ∂L under the corresponding exponent map is contained in W, that is, such that W is totally geodesic near ∂L. The following is an analogue of [1, Lemma 5].

Lemma 3.1. *Let L be a compact Legendrian submanifold of the contact manifold (M, J, α) with (nonempty) boundary ∂L and let W be a scaffold for it of codimension two. Then there is a neighborhood $\mathscr{U} = \mathscr{U}(\partial L, M)$ of ∂L in M and a contact embedding $\phi\colon \mathscr{U} \to \mathbb{R} \times T^*(\partial L) \times \mathbb{R}^2$ such that the following conditions hold*:

(i) $\phi(W \cap \mathscr{U}) \subset \mathbb{R} \times T^*(\partial L) \times \{(0, 0)\}$,

(ii) $\phi(\partial L) = \{0\} \times \partial L \times \{0, 0\}$,

(iii) $(t, x, v, s_1, s_2) \in \phi(\mathscr{U}) \to (t, x, v, 0, 0) \in \phi(\mathscr{U})$,

(iv) *for any nowhere zero smooth section $V\colon W \to \xi'^{\perp}|_W$, ϕ can be required to satisfy $\phi_*(V(p)) = (\partial/\partial s_1)|_{\phi(p)}$ for any $p \in \partial L$, where (s_1, s_2) the coordinate functions of $\mathbb{R}^2$.*

Note that the condition (iv) is slightly weaker than the corresponding one of [1, Lemma 5 (4)]. It is sufficient for us to construct a suitable metric in Proposition 3.2. Even so our proof uses not only contact neighborhood theorem but also symplectic neighborhood theorem in contrast with the proof of [1, Lemma 5 (4)]. It is a key of our proof.

Proof of Lemma 3.1. Since ∂L is a compact Legendrian submanifold of $\mathbb{R} \times T^*(\partial L)$ without boundary, from the Neighborhood Theorem for Legendrian (cf. Corollary 2.5.9 in [4]) it follows that there exists a contactomorphism ϕ_0 from a neighborhood $\mathcal{U}_0(\partial L, W)$ of ∂L in W to one $\mathscr{V}_0(0_{\partial L})$ of the zero section of $T^*(\partial L)$ in $\mathbb{R} \times T^*(\partial L)$ such that

$$\phi_0(x) = (0, x) \quad \forall x \in \partial L. \tag{3.1}$$

Fix a Riemannian metric on the bundle $T^*(\partial L)$, and then take a sufficiently small $\epsilon > 0$ such that

$$M_1' := \{(t, x, v)\colon |t| \le \epsilon,\ v \in T_x^*(\partial L) \text{ with } |v| \le \epsilon\} \subset \mathscr{V}_0(0_{\partial L}). \tag{3.2}$$

We get another neighborhood of ∂L in W,

$$M_0' := \phi_0^{-1}(M_1') \subset \mathscr{U}_0(\partial L, W) \subset W. \tag{3.3}$$

Then $\phi_0\colon M_0' \to M_1'$ is a contactomorphism. Obverse that M_0' and M_1' are compact contact submanifolds of W and $T^*(\partial L)\times\mathbb{R}$ with boundary and of codimension zero, respectively.

Let λ_{can} denote the canonical 1-form on $T^*\partial L$. Recall that the contact form and Reeb vector field on $J^1\partial L = \mathbb{R}\times T^*(\partial L)$ are

$$\tilde{\beta} = dt - \lambda_{\text{can}} \quad \text{and} \quad R_{\tilde{\beta}} = \frac{\partial}{\partial t}. \tag{3.4}$$

Assume that s_1, s_2 are the coordinate functions of $\mathbb{R}^2$. We have a contact form on $J^1\partial L\times\mathbb{R}^2 = \mathbb{R}\times T^*(\partial L)\times\mathbb{R}^2$,

$$\beta = \tilde{\beta} - s_1\,ds_2 = dt - \lambda_{\text{can}} - s_1\,ds_2, \tag{3.5}$$

whose Reeb vector field is given by $R_\beta = \partial/\partial t$. Denote by $(\ker(\tilde{\beta}))^\perp$ the symplectically orthogonal complement of $\ker(\tilde{\beta})$ in $\ker(\beta)$ (with respect to $d\beta$). It is easily checked that it is equal to the trivial bundle

$$\operatorname{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right) \to J^1\partial L\times\mathbb{R}^2.$$

Define $M_0 := M$, $M_1 := \mathbb{R}\times T^*(\partial L)\times\mathbb{R}^2$, and M_0' and M_1' as above. (Identify $M_1' \equiv M_1'\times\{(0,0)\}\subset J^1L\times\mathbb{R}^2$). Since $\xi'^\perp$ is trivial we can pick two vector fields V_1, V_2 such that V_1, V_2 form a basis of $\xi'^\perp$ and satisfy $d\alpha(V_1,V_2)=0$. There exists an obvious symplectic vector bundle isomorphism

$$\xi'^\perp|_{M_0'} \to \operatorname{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\Bigg|_{M_1'}$$

given by

$$\Phi(V_1(x)) = \left.\frac{\partial}{\partial s_1}\right|_{(\phi_0(x),0,0)}$$

and

$$\Phi(V_2(x)) = \left.\frac{\partial}{\partial s_2}\right|_{(\phi_0(x),0,0)}$$

for any $x\in M_0'$. By Theorem 2.5.15 of [4], we may extend ϕ_0 into a contactomorphism ϕ_1 from a neighborhood $\mathscr{U}(M_0')$ of M_0' in $M_0 = M$ to that $\mathscr{U}(M_1')$ of $M_1'\equiv M_1'\times\{(0,0)\}$ in M_1 such that $T\phi_1|_{\xi'^\perp|_{M_0'}}$ and Φ are bundle homotopic (as symplectic bundle isomorphisms) up to a conformality. (*Note*: From the proof of [4, Theorem 2.5.15] it is not hard to see that the theorem still holds if compact contact submanifold M_i' have boundary and $M_i'\subset \operatorname{Int}(M_i)$.)

Actually, we may assume that $\mathscr{U}(M_1')$ has the following form:

$$\mathscr{U}(M_1') = \{(t, x, v)\colon |t| < \varepsilon,\ v \in T_x^*(\partial L) \text{ with } |v| < \varepsilon'\} \times \{(s_1, s_2) \in \mathbb{R}^2\colon |s_1|, |s_2| < \delta\}, \tag{3.6}$$

where $0 < \varepsilon' < \varepsilon$ and $\delta > 0$, and

$$\mathscr{U}(M_0') := \phi_1^{-1}(\mathscr{U}(M_1')).$$

By suitably shrinking $\mathscr{U}(M_0')$ and $\mathscr{U}(M_1')$ if necessary, we can require

$$W_0 := W \cap \mathscr{U}(M_0') \subset \mathscr{U}_0(\partial L, W), \tag{3.7}$$

$$\phi_1(W_0) \subset \mathbb{R} \times T^*\partial L \times \{(0, 0)\}, \tag{3.8}$$

$$(t, x, v, s_1, s_2) \in \mathscr{U}(M_1') \Rightarrow \phi_1^{-1}(t, x, v, 0, 0) \in W_0.$$

Clearly, $\mathscr{U}(M_0')$ and ϕ_1 satisfy the conditions (i)–(iii) in Lemma 3.1.

For (iv) we need to modify ϕ_1 and $\mathscr{U}(M_0')$. Since ϕ_1 is a contactomorphism,

$$\phi_{1*}(\xi'^{\perp}|_{W_0}) \subset \operatorname{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\bigg|_{\phi_1(W_0)}.$$

It follows that there exist smooth real functions $f_1, f_2\colon W_0 \to \mathbb{R}$ such that

$$\phi_{1*}(V(x)) = f_1(x)\frac{\partial}{\partial s_1}\bigg|_{\phi(x)} + f_2(x)\frac{\partial}{\partial s_2}\bigg|_{\phi(x)}$$

and

$$|f_1(x)| + |f_2(x)| \neq 0$$

for any $x \in W_0$, where $V\colon W \to \xi'^{\perp}|_W$ is the given nowhere zero smooth section in Lemma 3.1 (iv).

Take $\epsilon > 0$ sufficiently small so that

$$R_\epsilon := \{(t, x, v, 0, 0) \in \mathbb{R} \times T^*\partial L \times \mathbb{R}^2\colon |t| \le \epsilon,\ |v| \le \epsilon\} \subset \phi_1(W_0).$$

Consider the compact symplectic submanifold of $(T^*\partial L \times \mathbb{R}^2, -d\lambda_{\mathrm{can}} - ds_1 \wedge ds_2)$,

$$S_\epsilon := \{(x, v, 0, 0) \in T^*\partial L \times \mathbb{R}^2\colon |v| \le \epsilon\}. \tag{3.9}$$

Its symplectic normal bundle is

$$\operatorname{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\bigg|_{S_\epsilon},$$

and $\phi_{1*}(V)$ restricts to a nowhere zero smooth section

$$p \mapsto f_1 \circ \phi^{-1}(p) \frac{\partial}{\partial s_1}\bigg|_p + f_2 \circ \phi^{-1}(p) \frac{\partial}{\partial s_2}\bigg|_p. \tag{3.10}$$

Obverse that there exists an obvious symplectic vector bundle isomorphism

$$\Psi\colon \mathrm{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\bigg|_{S_\epsilon} \to \mathrm{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\bigg|_{S_\epsilon}$$

which sends the section in (3.10) to one

$$p \mapsto \frac{\partial}{\partial s_1}\bigg|_p.$$

Hence the symplectic neighborhood theorem[1] (cf. [9, Theorem 3.30]) yields a symplectomorphism between neighborhoods of S_ϵ in $(T^*\partial L \times \mathbb{R}^2, -d\lambda_{\mathrm{can}} - ds_1 \wedge ds_2)$,

$$\varphi\colon \mathcal{N}_0(S_\epsilon) \to \mathcal{N}_1(S_\epsilon)$$

such that

$$\varphi(p) = p \quad \text{and} \quad d\varphi(p) = \Psi_p \tag{3.11}$$

for any $p \in S_\epsilon$. In particular, we have

$$d\varphi(p)(\phi_{1*}(V)|_p) = \frac{\partial}{\partial s_1}\bigg|_p \quad \forall p \in S_\epsilon. \tag{3.12}$$

Since (3.5) implies

$$\ker(\beta)|_{(t,x,v,s_1,s_2)} = T_{(x,v)}T^*\partial L \times \mathrm{Span}\left(\left\{\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}\right\}\right)\bigg|_{(s_1,s_2)},$$

the map

$$\phi_2\colon \mathbb{R} \times \mathcal{N}_0(S_\epsilon) \to \mathbb{R} \times \mathcal{N}_1(S_\epsilon), \quad (t, p) \mapsto (t, \varphi(p)) \tag{3.13}$$

must be a contactomorphism with respect to the induced contact structure from $(\mathbb{R} \times T^*(\partial L) \times \mathbb{R}^2, \beta)$.

Take a neighborhood $\mathscr{U}$ of ∂L in M such that

$$\mathscr{U} \subset \mathscr{U}(M_0') \quad \text{and} \quad \phi_1(\mathscr{U}) \subset \mathbb{R} \times \mathcal{N}_0(S_\epsilon),$$
$$(t, x, v, s_1, s_2) \in \phi_2(\phi_1(\mathscr{U})) \Rightarrow (t, x, v, 0, 0) \in \phi_2(\phi_1(\mathscr{U})).$$

[1] From the proof of [9, Theorem 3.30] it is not hard to see that the theorem still holds if compact symplectic submanifold Q_j have boundary and $Q_j \subset \mathrm{Int}(M_j)$.

Then the composition $\phi := \phi_2 \circ (\phi_1|_{\mathscr{U}})$ is a contact embedding from $\mathscr{U}$ into $(\mathbb{R} \times T^*(\partial L) \times \mathbb{R}^2, \beta)$ such that the condition (iii) is satisfied. By (3.8) and (3.11) it is easy to see that (i) is satisfied for ϕ and $\mathscr{U}$, i.e.

$$\phi(W \cap \mathscr{U}) \subset \mathbb{R} \times T^*\partial L \times \{(0, 0)\}.$$

From (3.1) and (3.11) it follows that $\phi(\partial L) = \{0\} \times \partial L \times \{0, 0\}$. That is, (i) holds. Finally, (3.12) implies that ϕ satisfies the condition (iv), i.e.

$$d\phi(p)(V(p)) = \left.\frac{\partial}{\partial s_1}\right|_{\phi(p)} \quad \forall p \in \partial L. \qquad \square$$

As in [1], with Lemma 3.1 we may construct the desired metric $\hat{g}$ as follows.

STEP 1. Recall that N the inward unit normal vector field of ∂L in L and $N \in \Gamma(\xi'^{\perp}|_{\partial L})$. Let $\mathscr{U}$ and ϕ be as in the Lemma 3.1 with $\phi_*(N(p)) = (\partial/\partial s_1)|_{\phi(p)}$ for any $p \in \partial L$. By shrinking W we assume that N has been extended into a nowhere zero section in $\Gamma(\xi'^{\perp}|_W)$. Hence using Lemma 3.1 (iii) we may define a metric g' on $\phi(\mathscr{U})$ as follows:

$$g'(t, x, v, s_1, s_2) := (\phi^{-1})^*(g|_W(\phi^{-1}(t, x, v, 0, 0))) + ds_1 \otimes ds_1 + ds_2 \otimes ds_2$$

for every $(t, x, v, s_1, s_2) \in \phi(\mathscr{U})$.

STEP 2. Consider the metric $g_1 := \phi^* g'$ on $\mathscr{U}$. Take a neighborhood $\mathscr{V}$ of ∂L in M such that the closure of $\mathscr{V}$ is contained in $\mathscr{U}$. Let $\rho \colon M \to \mathbb{R}$ be a smooth function such that $\rho = 1$ on a neighborhood $\mathscr{V}$, and $\rho = 0$ outside $\mathscr{U}$. We then define the metric $\hat{g}$ by

$$\hat{g} := \rho g' + (1 - \rho) g.$$

The following two propositions correspond to Propositions 6 and 7 in [1], respectively.

Proposition 3.2. *For the neighborhood $\mathscr{V}$ of ∂L in* Step 2, *$W \cap \mathscr{V}$ is totally geodesic with respect to the metric $\hat{g}$.*

Proof. For any $p \in W \cap \mathscr{V}$, Lemma 3.1 gives a local contact coordinate system around it,

$$\mathcal{O}(p) \to \mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}^2, \quad q \mapsto (t(q), z_1(q), \dots, z_{2n-2}(q), s_1(q), s_2(q))$$

such that

- for some smooth function $h \colon \mathcal{O}(p) \to \mathbb{R}$ it holds that

$$\alpha|_{\mathcal{O}(p)} = e^h\left(dt - \sum_{k=1}^{n-1} z_{n-1+k}\, dz_k - s_2\, ds_1\right), \tag{3.14}$$

and the Reeb field $R_\alpha = \partial/\partial t$;
- $W \cap \mathcal{O}(p) \ni q \mapsto (t(q), z_1(q), \dots, z_{2n-2}(q))$ is a a local contact coordinate system around p in the relatively open neighborhood $W \cap \mathcal{O}(p)$ and

$$\alpha|_{W\cap\mathcal{O}(p)} = e^{h_0}\left(dt - \sum_{k=1}^{n-1} z_{n-1+k}\, dz_k\right), \tag{3.15}$$

where $h_0 = h|_{W\cap\mathcal{O}(p)}$. Moreover the Reeb field of $\alpha|_{W\cap\mathcal{O}(p)}$ is given by the restriction of $\partial/\partial t$ to $W \cap \mathcal{O}(p)$.

For convenience we write t as z_0. In the corresponding local coordinate vector fields

$$\frac{\partial}{\partial z_0} = \frac{\partial}{\partial t}, \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{2n-2}}, \frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}$$

we have

$$\hat{g} = \sum_{k,l=0}^{n-1} (g|_W)_{kl}\, dz_k \otimes dz_l + ds_1 \otimes ds_2. \tag{3.16}$$

It is easily computed that

$$\hat{g}\left(\nabla_{\partial/\partial z_k} \frac{\partial}{\partial z_l}, \frac{\partial}{\partial s_i}\right) = \frac{1}{2}(\hat{g}_{z_k s_i, z_l} + \hat{g}_{z_l s_i, z_k} - \hat{g}_{z_k z_l, s_i}) = 0.$$

So the second fundamental form of $W \cap \mathscr{V}$ with respect to $\hat{g}$ vanishes, that is, $W \cap \mathscr{V}$ is totally geodesic. □

Proposition 3.3. *Let L be a compact Legendrian submanifold with boundary of the contact manifold (M,α), and let W be a codimension two scaffold for L. Denote by $\hat{N}(L)$ the normal bundle of L with respect to $\hat{g}$. For $p \in \partial L$, suppose that $\hat{V} \in \hat{N}_p(L)$ satisfies the boundary condition*

$$(d\alpha)_p(N(p), \hat{V}) = 0.$$

Then $\hat{V} \in \xi'_p$. (In fact we have proved

$$\{\hat{V} \in \hat{N}_p(L) \mid (d\alpha)_p(N(p),\ \hat{V}) = 0\} \cap T_p\partial L = \{0\}.)$$

Proof. For any point $p \in \partial L$, take the local coordinate system around it as in the proof of Proposition 3.2. By composing with a suitable linear contactomorphism of form

$$\mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}^2,$$
$$(z_0, z_1, \dots, z_{2n-2}, s_1, s_2) \mapsto (z_0, \mathcal{A}(z_1, \dots, z_{2n-2}), s_1, s_2),$$

we may assume that

$$\frac{\partial}{\partial z_1}\bigg|_p, \dots, \frac{\partial}{\partial z_{n-1}}\bigg|_p$$

forms a basis of $T_p\partial L$, and that they are also orthogonal to vectors

$$\frac{\partial}{\partial z_n}\bigg|_p, \dots, \frac{\partial}{\partial z_{2n-2}}\bigg|_p$$

with respect to $\hat{g}$. (*Note*: Such a transformation does not change the Reeb field, i.e. we have still $R_\alpha = \partial/\partial t$.) Since the normal vector field N of ∂L in L in the local coordinate system is equal to $\partial/\partial s_1$, we get an orthogonal basis of T_pL,

$$\frac{\partial}{\partial z_1}\bigg|_p, \dots, \frac{\partial}{\partial z_{n-1}}\bigg|_p, \frac{\partial}{\partial s_1}\bigg|_p.$$

It is easy to see that for some $\lambda \in \mathbb{R}$ the vector fields

$$\frac{\partial}{\partial z_n}\bigg|_p, \dots, \frac{\partial}{\partial z_{2n-2}}\bigg|_p, \frac{\partial}{\partial s_2}\bigg|_p + \lambda\frac{\partial}{\partial t}\bigg|_p$$

spans $\hat{N}_p(L)$. Let

$$\hat{V} := a_n\frac{\partial}{\partial z_n}\bigg|_p + \cdots + a_{2n-2}\frac{\partial}{\partial z_{2n-2}}\bigg|_p + b\left(\frac{\partial}{\partial s_2}\bigg|_p + \lambda\frac{\partial}{\partial t}\bigg|_p\right) \in \hat{N}_p(L)$$

satisfy $(d\alpha)_p(N(p), \hat{V}) = 0$. Since $N(p) = (\partial/\partial s_1)|_p$ and $R_\alpha(p) = (\partial/\partial t)|_p$ we have

$$(d\alpha)_p\left(\frac{\partial}{\partial s_1}\ \bigg|_p, a_n\frac{\partial}{\partial z_n}\ \bigg|_p + \cdots + a_{2n-2}\frac{\partial}{\partial z_{2n-2}}\ \bigg|_p + b\frac{\partial}{\partial s_2}\ \bigg|_p\right) = 0. \tag{3.17}$$

By (3.14) it is easy computed that

$$\begin{aligned} d\alpha|_{\mathcal{O}(p)} = e^h&\left(-\sum_{k=1}^{n-1} dz_{n-1+k}\wedge dz_k - ds_2\wedge ds_1\right)\\ &+ e^h\, dh\wedge\left(dt - \sum_{k=1}^{n-1} z_{n-1+k}\, dz_k - s_2\, ds_1\right). \end{aligned} \tag{3.18}$$

Note that $s_1 = s_2 = 0$ at p. It follows from (3.17)–(3.18) that $be^h|_p = 0$ and thus $b = 0$. This shows

$$\hat{V} = a_n\frac{\partial}{\partial z_n}\bigg|_p + \cdots + a_{2n-2}\frac{\partial}{\partial z_{2n-2}}\bigg|_p \in \xi'_p.$$

Clearly, when $\hat{V} \neq 0$ we have also $\hat{V} \notin T_p\partial L$. $\square$

REMARK 3.4. Let $\widehat{\exp}$ be the exponent map of the metric $\hat{g}$. For any $p \in \partial L$ and $v \in \hat{N}_p(L)$ with $d\alpha(N(p), v) = 0$, Proposition 3.2 and Proposition 3.3 show that $\widehat{\exp}(p, v) \in W \cap \mathscr{U}$ if $|v|$ is small enough.

REMARK 3.5. Here we give a proof for the statement just after Claim 2.6. For $x \in \partial L$ we can decompose $V(x)$ into a sum $\hat{V}(x) + \check{V}(x)$, where $\hat{V}(x) \in \hat{N}_x(L)$ and $\check{V}(x) \in T_xL$. Then $(d\alpha)_x(N(x), V(x)) = (d\alpha)_x(N(x), \hat{V}(x)) + (d\alpha)_x(N(x), \check{V}(x))$. Since $N(x)$ sits in T_xL, $(d\alpha)_x(N(x), \check{V}(x)) = 0$ and hence $(d\alpha)_x(N(x), V(x)) = (d\alpha)_x(N(x), \hat{V}(x))$. Suppose $(d\alpha)_x(N(x), V(x)) = 0$. Then $(d\alpha)_x(N(x), \hat{V}(x)) = 0$ and thus $\hat{V}(x) \in \xi'_x$ by Proposition 3.3. It follows that $\alpha_x(V(x)) = \alpha_x(\hat{V}(x)) + \alpha_x(\check{V}(x)) = 0$.

4. The proof of Theorem 1.1

Let us start with a brief review of notations in Hodge theory (cf. [12, 15] for details) and then proceed to the proof of Theorem 1.1.

For $k \in \mathbb{N} \cup \{0\}$, $1 \leq p < \infty$ and $0 < a < 1$, let $W^{k,p}\Omega^r(L)$ (resp. $C^{k,a}\Omega^r(L)$) denote the space of r-forms of class $W^{k,p}$ (resp. $C^{k,a}$) as usual (cf. [12, 15]). Each form ω of them has a "tangential component" $\mathbf{t}\omega$ and a "normal component" $\mathbf{n}\omega$ (cf. [11, Definition 4.2] or [15, (2.25)]), which satisfy

$$\mathbf{t}(\star\omega) = \star(\mathbf{n}\omega) \quad \text{and} \quad \mathbf{n}(\star\omega) = \star(\mathbf{t}\omega) \tag{4.1}$$

by Lemma 4.2 of [11], where $\star$ is the Hodge star operator of the metric $\hat{g}$. Set

$$C^{k,a}\Omega^r_{\mathbf{D}}(L) := \{\omega \in C^{k,a}\Omega^r(L)\colon \mathbf{t}\omega = 0\},$$
$$C^{k,a}\Omega^r_{\mathbf{N}}(L) := \{\omega \in C^{k,a}\Omega^r(L)\colon \mathbf{n}\omega = 0\}$$

and

$$\mathcal{H}C^{k,a}\Omega^r(L) := \{\omega \in C^{k,a}\Omega^r(L)\colon d\omega = \delta\omega = 0\}.$$

Replacing $C^{k,a}$ by $W^{k,p}$ gives corresponding spaces $W^{k,p}\Omega^r_{\mathbf{D}}(L)$, $W^{k,p}\Omega^r_{\mathbf{D}}(L)$ and $\mathcal{H}W^{k,p}\Omega^r(L)$. Clearly, for $S^r_{\mathbf{N}} = C^{k,\alpha}\Omega^r_{\mathbf{N}}(L)$ and $S^r_{\mathbf{D}} = C^{k,\alpha}\Omega^r_{\mathbf{D}}(L)$ (or $S^r_{\mathbf{N}} = W^{k,p}\Omega^r_{\mathbf{N}}(L)$ and $S^r_{\mathbf{D}} = W^{k,p}\Omega^r_{\mathbf{D}}(L)$), (4.1) implies

$$\star(S^r_{\mathbf{N}}) \subset S^{n-r}_{\mathbf{D}} \quad \text{and} \quad \star(S^r_{\mathbf{D}}) \subset S^{n-r}_{\mathbf{N}}. \tag{4.2}$$

By the definition of the co-differential δ, for any r-form ω it holds that

$$\star(\star\omega) = (-1)^{r(n-r)}\omega, \quad \star\delta\omega = (-1)^r d\star\omega, \quad \star d\omega = (-1)^{r+1}\delta\star\omega. \tag{4.3}$$

For $k \in \mathbb{N} \cup \{0\}$ the closure $C^{k,a}(d\Omega^r(L))$ of $d\Omega^r(L)$ in $C^{k,a}\Omega^{r+1}(L)$ is contained $\{d\eta\colon \eta \in C^{k+1,a}\Omega^r(L)\}$ by the Poincaré lemma (cf. §3.1 of [1]).

Since the exponent map $\widehat{\exp}$ of the metric $\hat{g}$ is a local diffeomorphism, (by tubular neighborhood theorem) the sufficiently small neighborhood of the zero section of $\hat{N}(L)$ satisfying the boundary condition corresponds to the deformations of submanifold L with boundary ∂L confined in W in one-to-one way.

Let $C^{2,a}(\Gamma(\hat{N}(L)))$ denote the Banach space of $C^{2,a}$-sections of the bundle $\hat{N}(L)$. Define the Banach space

$$\mathcal{X} := \{V \in C^{2,a}(\Gamma(\hat{N}(L))) \colon d\alpha(N, V|_{\partial L}) = 0\},$$

and denote by $\mathscr{U}$ a neighborhood of 0 in $\mathcal{X}$. For $V \in \mathscr{U}$ define $\widehat{\exp}_V \colon L \to M$, $x \mapsto \widehat{\exp}_V(x) := \widehat{\exp}_x(V(x))$. Set

$$\begin{aligned} F \colon \mathscr{U} &\to C^{1,a}\Omega^1(L) \oplus C^{0,a}\Omega^n(L), \\ V &\mapsto ((\widehat{\exp}_V)^*\alpha, 2(\widehat{\exp}_V)^* \operatorname{Im}\epsilon). \end{aligned} \tag{4.4}$$

It is C^1 as done in [1, 17]. Clearly, $\widehat{\exp}_V$ is homotopic to the inclusion $j \colon L \hookrightarrow M$ via $\widehat{\exp}_{tV}$, and hence they induce the same homomorphisms between the de Rham cohomology groups. It follows that the de Rham cohomology classes

$$[\widehat{\exp}_V^*(\operatorname{Im}\epsilon)] = \widehat{\exp}_V^*[\operatorname{Im}\epsilon] = j^*[\operatorname{Im}\epsilon] = [j^*(\operatorname{Im}\epsilon)] \in H^n(L, \mathbb{R}) \quad \text{vanish.}$$

This shows that

$$\operatorname{Im} F \subseteq C^{1,a}\Omega^1(L) \oplus dC^{1,a}\Omega^{n-1}(L).$$

Consider F as a map to $C^{1,a}\Omega^1(L) \oplus dC^{1,a}\Omega^{n-1}(L)$.

To compute the differential of F at 0, for $V \in \mathcal{X}$ we set $f = \alpha(V)$ and $Y := V - fR_\alpha$. Then $f \in C^{2,a}(L)$ and $Y \in C^{1,a}(\Gamma(\xi|_L))$. By Proposition 3.3, $V(p) \in \xi'_p$ for any $p \in \partial L$, and so $f(p) = 0 \forall p \in \partial L$. Now $V = fR_\alpha + Y$. By the Cartan formula one can compute the linearization of F at 0,

$$\begin{aligned} F'(0)(V) &= \frac{d}{dt}(\widehat{\exp}_{tV}^*\,\alpha, 2\,\widehat{\exp}_{tV}^* \operatorname{Im}\epsilon)|_{t=0} \\ &= (\mathcal{L}_V\alpha, 2\mathcal{L}_V \operatorname{Im}\epsilon)|_L \\ &= (d\iota_{fR_\alpha+Y}\alpha + \iota_{fR_\alpha+Y}d\alpha, 2d\iota_{fR_\alpha+Y} \operatorname{Im}\epsilon)|_L \\ &= (df + \iota_Y d\alpha, 2d\iota_Y \operatorname{Im}\epsilon)|_L \\ &= (df + \iota_Y d\alpha, -d \star \iota_Y d\alpha)|_L \\ &= (d(f \circ j) + j^*(\iota_Y d\alpha), -d \star j^*(\iota_Y d\alpha)). \end{aligned} \tag{4.5}$$

Here the fifth equality comes from (2.2). In order to show that $F'(0)$ is surjective, we need to write each

$$(\eta, d\zeta) \in C^{1,\alpha}\Omega^1(L) \oplus dC^{1,\alpha}\Omega^{n-1}(L)$$

as a convenient form.

Note that $\mathbf{t}(d\omega) = d(\mathbf{t}\omega)$ and $\mathbf{n}(\delta\omega) = \delta(\mathbf{n}\omega)$ for any C^1-form ω on L (cf. [15, Proposition 1.2.6 (b)]). Since $C^{0,a}\Omega^{n-1}(L) \subset L^2\Omega^{n-1}(L)$, by [11, Theorem 5.7, 5.8] or [12, Theorem 7.7.7, 7.7.8] we may write $\zeta = \delta_n\gamma' + d\gamma'' + h(\zeta)$, where

$$\gamma' \in C^{1,a}\Omega^n_{\mathbf{N}}(L)), \quad \gamma'' \in C^{1,a}\Omega^{n-2}_{\mathbf{D}}(L), \quad h(\zeta) \in \mathcal{H}C^{0,a}\Omega^{n-1}(L).$$

Moreover (4.2) and $\mathbf{t}(d\omega) = d(\mathbf{t}\omega)$ imply

$$\delta_n = (-1)^{n(n+1)+1} \star d_0\star\colon C^{2,\alpha}\Omega^n_{\mathbf{N}}(L) \to C^{1,\alpha}\Omega^{n-1}_{\mathbf{N}}(L).$$

We may assume

$$d\zeta = -d \star du \quad \text{with} \quad u \in C^{2,\alpha}\Omega^0_{\mathbf{D}}(L).$$

Similarly, we have

$$\eta = \delta v + d\beta + h(\eta),$$

where

$$v \in C^{2,a}\Omega^2_{\mathbf{N}}(L)), \quad \beta \in C^{2,a}\Omega^0_{\mathbf{D}}(L), \quad h(\eta) \in \mathcal{H}C^{1,a}\Omega^1(L)).$$

By (4.3), $d \star \delta v = (-1)^2 d(d \star v) = 0$ and $d(\star h(\eta)) = (-1) \star \delta h(\eta) = 0$. We get

$$\begin{aligned}(\eta, d\zeta) &= (d\beta - du + du + \delta v + h(\eta), -d \star (du + \delta v + h(\eta)) \\ &= (d\chi + \omega, -d \star \omega),\end{aligned}$$

where

$$\begin{aligned}\chi &:= \beta - u \in C^{2,a}\Omega^0_{\mathbf{D}}(L), \\ \omega &:= du + \delta v + h(\eta) \in C^{1,a}\Omega^1(L).\end{aligned} \tag{4.6}$$

Take $f = \chi$. We need to find a $Y \in C^{1,a}(\Gamma(\xi|_L))$ such that

$$fR_\alpha + Y \in C^{1,a}(\Gamma(\hat{N}(L)))$$

and

$$j^*(\iota_Y d\alpha) = \omega.$$

Since $j^*(\iota_{Y+fR_\alpha} d\alpha) = j^*(\iota_Y d\alpha) = \omega$, it suffices to find a $Z \in C^{1,a}(\Gamma(\hat{N}(L)))$ such that

$$j^*(\iota_Z d\alpha) = \omega. \tag{4.7}$$

To this goal, consider the symplectic vector bundle $(\xi|_L, d\alpha|_{\xi|_L})$ with a Lagrangian subbundle TL. Let $TL_\xi^{\perp_{\hat{g}}}$ be the orthogonal complementary bundle of TL in $\xi|_L$ with respect to $\hat{g}$. Then $TL_{\xi}^{\perp_{\hat{g}}} = \xi \cap \hat{N}(L)$. So $\xi|_L = TL \oplus_{\hat{g}} (\xi \cap \hat{N}(L))$. Note that ω may be viewed as a section of the bundle $\mathrm{Hom}(TL, \mathbb{R})$. We may extend it into a section of $\mathrm{Hom}(\xi|_L, \mathbb{R})$, $\hat{\omega}$, by defining

$$\hat{\omega}_p(u+v) = \omega_p(u)$$

for any $p \in L$ and $u + v \in T_pL \oplus_{\hat{g}} (\xi \cap \hat{N}(L))_p$, where $u \in T_pL$ and $v \in (\xi \cap \hat{N}(L))_p$. Note that $\hat{\omega} \in C^{1,a}(\Gamma(\mathrm{Hom}(\xi|_L, \mathbb{R})))$. The non-degeneracy of $d\alpha$ on ξ implies that there exists a unique section $Z\colon L \to \xi|_L$ such that

$$(d\alpha)_p(Z(p), A) = \hat{\omega}_p(A) \quad \forall\, p \in L \text{ and } A \in \xi_p.$$

Clearly, $Z \in C^{1,a}(\xi|_L)$. Since $\xi|_L = TL \oplus_{\hat{g}} (\xi \cap \hat{N}(L))$ we get a unique decomposition $Z = Z_1 + Z_2$, where $Z_1 \in C^{1,a}(\Gamma(TL))$ and $Z_2 \in C^{1,a}(\Gamma(\xi \cap \hat{N}(L)))$. Obverse that

$$j^*(\iota_{Z_1}\, d\alpha) = 0.$$

In fact, for any $p \in L$ and $u \in T_pL$ it holds that

$$(j^*(\iota_{Z_1} d\alpha))_p(u) = (\iota_{Z_1} d\alpha)_{j(p)}(j_* u) = (d\alpha)_p(Z_1(p), u) = 0$$

since T_pL is a Lagrangian subspace of $(\xi_p, (d\alpha)_p)$. Hence we get

$$(d\alpha)_p(Z_2(p), A) = \hat{\omega}_p(A) \quad \forall\, p \in L \text{ and } A \in \xi_p.$$

This implies $j^*(\iota_{Z_2}\, d\alpha) = \omega$. In summary we have proved:

Claim 4.1. *There exists a unique section $Z\colon L \to \xi|_L \cap \hat{N}(L)$ such that* (4.7) *is satisfied. Moreover, Z is also of class $C^{1,a}$. As a consequence the map $F'(0)$ is surjective.*

Next let us compute $\ker(F'(0))$. Let $V \in \mathcal{X}$ sit in $\ker(F'(0))$. As above we may write $V = fR_\alpha + Y$, where $f = \alpha(V)$ and $Y \in C^{k+1,a}(\Gamma(\xi|_L))$. (4.5) yields

$$df + j^*(\iota_Y d\alpha) = 0, \tag{4.8}$$

$$-d \star j^*(\iota_Y d\alpha) = 0. \tag{4.9}$$

From (4.8) we get

$$\begin{aligned} 0 = \delta(df + j^*(\iota_Y\, d\alpha)) &= \delta\, df + \delta(j^*(\iota_Y\, d\alpha)) \\ &= \delta\, df + (-1)^{2n+1} \star d \star (j^*(\iota_Y\, d\alpha)) \\ &= \delta\, df \end{aligned}$$

because of (4.9). Hence $\triangle f = 0$. By Proposition 3.3 the boundary condition $d\alpha(N, V|_{\partial L}) = 0$ implies $V(p) \in \xi'_p \subset \xi_p$ for any $p \in \partial L$, and thus $f|_{\partial L} = 0$. Since ∂L is a non-empty closed manifold, the maximum principle leads to $f \equiv 0$. Hence $V = Y$. By (4.8) we have

$$(d\alpha)_p(Y(p), u) = 0 \quad \forall p \in L \text{ and } u \in T_pL.$$

This means that $Y(p)$ belongs to $T_pL \subset \xi_p$ since T_pL is a Lagrangian subspace in $(\xi_p, (d\alpha)_p)$. Moreover, $Y(p) = V(p) \in \hat{N}_p(L)$, and $T_pL \cap (\xi_p \cap \hat{N}_p(L)) = \{0\}$ $\forall p \in L$. We get $V(p) = 0$ for any $p \in L$. It shows $\ker F'(0) = 0$. Combing this with Claim 4.1 we prove that the differential

$$F'(0)\colon T_0\mathcal{X} \to C^{1,a}\Omega^1(L) \oplus dC^{1,a}\Omega^{n-1}(L)$$

is a Banach space isomorphism. The inverse function theorem implies that there exists a neighborhood of 0 in $\mathcal{X}$, $\mathscr{U}_0 \subset \mathscr{U}$, such that $F^{-1}(0) \cap \mathscr{U}_0 = \{0\}$. This completes the proof of Theorem 1.1. □

5. The proof of Theorem 1.2

Let $\langle R_\alpha\rangle$ denote the real line bundle generated by $R_\alpha|_L$. Then the normal bundle of L with respect to the metric g, $N(L)$, is equal to $\langle R_\alpha\rangle \oplus_g JTL$. For a small section $V\colon L \to N(L)$, the exponent map of g yields a map

$$\exp_V\colon L \to M, \quad x \mapsto \exp_x(V(x)).$$

Thus there exists a neighborhood $\mathscr{V}$ of 0 in

$$\mathcal{Y} := \{V \in C^{2,a}(\Gamma(\langle R_\alpha\rangle)) \oplus C^{1,a}(\Gamma(JTL))\colon \alpha(V)|_{\partial L} = \text{const}\}$$

so that the following map is well-defined:

$$G\colon \mathscr{V} \to C^{1,a}\Omega^1(L) \oplus C^{0,a}\Omega^n(L), \qquad V \mapsto (\exp_V^* \alpha, 2\exp_V^* \operatorname{Im}\epsilon). \tag{5.1}$$

It is C^1 ([17]), and $\operatorname{Im}(G) \subseteq C^{1,a}\Omega^1(L) \oplus dC^{1,a}\Omega^{n-1}(L)$ as above since $\exp_V$ is homotopic to the inclusion $j\colon L \hookrightarrow M$ via $\exp_{tV}$.

Considering G as a map to $C^{1,a}\Omega^1(L) \oplus dC^{1,a}\Omega^{n-1}(L)$, and writing $V = JX + fR_\alpha$, we may get

$$\begin{aligned} G'(0)(V) &= \frac{d}{dt}(\exp_{tV}^* \alpha, 2\exp_{tV}^* \operatorname{Im}\epsilon)|_{t=0} \\ &= (\mathcal{L}_V\alpha, 2\mathcal{L}_V \operatorname{Im}\epsilon)|_L \\ &= (df + \iota_{JX}\, d\alpha, -d * \iota_{JX}\, d\alpha)|_L \end{aligned} \tag{5.2}$$

as above. Moreover, each $(\eta, d\zeta) \in C^{1,\alpha}\Omega^1(L) \oplus dC^{1,\alpha}\Omega^{n-1}(L)$ may be written as $(\eta, d\zeta) = (d\chi + \omega, -d \star \omega)$, where χ and ω are as in (4.6). Take $f = \chi$, and one easily find $X \in C^{1,\alpha}(\Gamma(TL))$ such that $j^*(\iota_{JX} d\alpha) = \omega$. Clearly, such a $V = fR_\alpha + JX$ satisfies $\alpha(V)|_{\partial L} = 0$. Hence $G'(0)$ is surjective.

Assume that $V = fR_\alpha + JX$ sits in $\ker(G'(0))$. Then f and JX satisfy

$$df + j^*(\iota_{JX}\, d\alpha) = 0, \quad -d \star j^*(\iota_{JX}\, d\alpha) = 0.$$

It follows that $\triangle f = \delta\, df = 0$. Recall that $f = \alpha(V)$ is equal to a constant c on ∂L. By the maximum principle we get $f \equiv c$, and hence

$$j^*(\iota_{JX}\, d\alpha) = 0.$$

From this we derive $JX = 0$ as above. This prove $\ker(G'(0)) = \{cR_\alpha \mid c \in \mathbb{R}\}$. Hence $(0, 0)$ is a regular value of the restriction of G to a small neighborhood $\mathscr{V}_0$ of $0 \in \mathscr{V}$, and thus the moduli space $\mathfrak{M}(L)$ is a 1-dimensional smooth manifold by the implicit function theorem. □

Since $\iota_{R_\alpha}\epsilon = 0$ and $\mathcal{L}_{R_\alpha}\epsilon = 0$ we have $\psi_t(\mathrm{Im}\,\epsilon) = \mathrm{Im}\,\epsilon \forall t$, where ψ_t is the flow of R_α. For special Legendrian embedding (submanifold) $p\colon L \to M$ we obtain $p_t^*\alpha = 0$ and $p_t^*\,\mathrm{Im}\,\epsilon = 0$ with $p_t = \psi_t \circ p$ for any t. So the deformation in Theorem 1.2 is actually given by the isometries generated by the Reeb vector field.

REMARK 5.1. If we replace $\mathscr{V}$ by a neighborhood $\mathscr{W}$ of 0 in

$$C^{2,a}(\Gamma(\langle R_\alpha \rangle)) \oplus C^{1,a}(\Gamma(JTL)),$$

then the map

$$\hat{G}\colon \mathscr{W} \to C^{1,a}(\Lambda^1(L)) \oplus C^{0,a}(\Lambda^n(L)), \quad V \mapsto (\exp_V^*\alpha,\, 2\exp_V^*\,\mathrm{Im}\,\epsilon),$$

is still C^1 and has the image $\mathrm{Im}(\hat{G}) \subseteq C^{1,a}(\Lambda^1(L)) \oplus dC^{1,a}(\Lambda^{n-1}(L))$. From the above proof it is easy to see that $\hat{G}'(0)$ is surjective. If $V = fR_\alpha + JX$ belongs to $\ker(\hat{G}'(0))$, we have $\triangle f = 0$ as above. But ∂L is a nonempty closed manifold, by Theorem 3.4.6 of [15] each $b \in C^\infty(\partial L)$ corresponds to a unique $f \in C^\infty(L)$ satisfying $\triangle f = 0$ and $f|_{\partial L} = b$. It follows that $\ker(\hat{G}'(0))$ must be of infinite dimension.

The corresponding problems with [1, Corollary 9] and [17, Theorem 4.8] can also be considered similarly.

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