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SCALE-INVARIANT BOUNDARY HARNACK PRINCIPLE IN INNER UNIFORM DOMAINS

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Abstract

We prove a scale-invariant boundary Harnack principle in inner uniform domains in the context of non-symmetric local, regular Dirichlet spaces. For inner uniform Euclidean domains, our results apply to divergence form operators that are not necessarily symmetric, and complement earlier results by H. Aikawa and A. Ancona.

Introduction

The boundary Harnack principle is a property of a domain that provides control over the ratio of two harmonic functions in that domain near some part of the boundary where the two functions vanish. Whether a given domain satisfies the boundary Harnack principle depends on the geometry of its boundary and, in fact, there is more than one kind of boundary Harnack principle. For a Euclidean domain Ω, two versions found in the literature are as follows.
(i) We say that the *boundary Harnack principle* holds on Ω if, for any domain V and any compact $K \subset V$ intersecting the boundary $\partial\Omega$, there exists a positive constant $A = A(\Omega, V, K)$ such that for any two positive functions u and v that are harmonic in Ω and vanish continuously (except perhaps on a polar set) along $V \cap \partial\Omega$, we have

$$\frac{u(x)}{u(x')} \leq A \frac{v(x)}{v(x')}, \quad \forall x, x' \in K \cap \Omega.$$

(ii) We say that the *scale-invariant boundary Harnack principle* holds on Ω, if there exist positive constants A_0, A_1 and R, depending only on Ω, with the following property. Let $\xi \in \partial\Omega$ and $r \in (0, R)$. Then for any two positive functions u and v that are harmonic in $B(\xi, A_0 r) \cap \Omega$ and vanish continuously (except perhaps on a polar set) along $B(\xi, A_0 r) \cap \partial\Omega$, we have

$$\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')}, \quad \forall x, x' \in B(\xi, r) \cap \Omega.$$

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A third version, important for our purpose and perhaps more natural, would replace the Euclidean balls in (ii) by the inner balls of the domain Ω.

A property similar to (i) was first introduced by Kemper ([20]). The scale-invariant boundary Harnack principle (ii) on Lipschitz domains was proved independently in [4, 5] and [36], a not scale invariant version was proved in [11].

Bass and Burdzy ([9]) used probabilistic arguments to prove property (i) on so-called twisted Hölder domains of order $\alpha \in (1/2, 1]$. Aikawa ([1]) proved the scale-invariant boundary Harnack principle on uniform domains in Euclidean space. This result was extended to inner uniform domains in [3]. Ancona gave a different proof for inner uniform domains in [6]. Moreover, Aikawa ([2]) proved that (inner) uniform domains are in fact *characterized* by the scale-invariant boundary Harnack principle. Other works on the boundary Harnack principle include [7, 8].

In [15], Gyrya and Saloff-Coste generalized Aikawa's reasoning to uniform domains in symmetric strongly local Dirichlet spaces of Harnack-type that admit a carré du champ. Moreover, they deduced that the boundary Harnack principle also holds on inner uniform domains, by considering the inner uniform domain as a uniform domain in a different metric space, namely the completion of the inner uniform domain with respect to its inner metric.

In this paper, we extend the result of [15] in two directions. First, we consider Dirichlet forms that allow lower order terms and non-symmetry. We do not assume the existence of a carré du champ. Second, we prove the boundary Harnack principle directly on inner uniform domains.

We follow Aikawa's reasoning, but with the Euclidean distance replaced by the inner distance of the domain. A crucial Lemma in our proof concerns the relation between balls in the inner metric and connected components of balls in the metric of the ambient space, see Lemma 3.7. This relation was already used in [6] to prove a boundary Harnack principle on inner uniform domains in Euclidean space. Ancona ([6]) also treated second order uniformly elliptic operators with some lower order terms, under the additional condition that the domain is uniformly regular. Following Aikawa's line of reasoning, we do not need the domain to be uniformly regular.

Our main result is Theorem 4.2. We now explain how it applies to Euclidean space. Formally, let

$$Lf = \sum_{i,j=1}^{n} \partial_j (a_{i,j} \partial_i f) - \sum_{i=1}^{n} b_i \partial_i f + \sum_{i=1}^{n} \partial_i (d_i f) - cf. \tag{1}$$

Assume that the coefficients $a = (a_{i,j})$, $b = (b_i)$, $d = (d_i)$, c are smooth and satisfy $c - \operatorname{div} b \geq 0$, $c - \operatorname{div} d \geq 0$, and, $\forall \xi \in \mathbb{R}^n$, $\sum_{i,j} a_{i,j} \xi_i \xi_j \geq \epsilon |\xi|^2$, $\epsilon > 0$.

Theorem 0.1. *Let L be the operator defined above and let $\Omega \subset \mathbb{R}^n$ be an inner uniform domain. There exists $C = C(\Omega) > 0$ and for each $R \in (0, C \cdot \operatorname{diam}(\Omega))$ there exist $A_0, A_1 \in (0, \infty)$, depending on Ω, R and on the coefficients a, b, c and d, such* *that the scale-invariant boundary Harnack principle holds in the following form. For any* $\xi \in \partial_{\tilde{\Omega}}\Omega$, $r \in (0, R)$ *and any two positive functions* u *and* v *that are local weak solutions of* $Lu = 0$ *in* $B_{\tilde{\Omega}}(\xi, A_0 r) \cap \Omega$ *and vanish weakly along* $B_{\tilde{\Omega}}(\xi, A_0 r) \cap \partial_{\tilde{\Omega}}\Omega$, *we have*

$$\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')}, \quad \forall x, x' \in B_{\tilde{\Omega}}(\xi, r).$$

Moreover, if $b = d = c = 0$ *then the constants* A_0 *and* A_1 *are independent of* R.

Here, by a *local weak solution* u on a domain $U \subset \mathbb{R}^n$ we mean a function that is locally in the Sobolev space $W^1(U)$ of all functions in $L^2(U)$ whose distributional first derivatives can be represented by functions in $L^2(U)$, and satisfies $\int Lu\,\psi = 0$ for all test functions ψ in $W^1_0(U)$, the closure of $C^\infty_0(U)$ (the space of all smooth, compactly supported functions on U) in the W^1-norm $\|\cdot\|_2^2 + \|\nabla\cdot\|_2^2$. A weak solution u *vanishes weakly* along $U \cap \partial_{\tilde{\Omega}}\Omega$ if u is locally in $W^1_0(\Omega)$ near $U \cap \partial\Omega$. See Section 1.1. The definition of a ball $B_{\tilde{\Omega}}$ in the inner metric is given in Section 3.3, $\partial_{\tilde{\Omega}} B_{\tilde{\Omega}}$ denotes the boundary of the ball with respect to its completion in the inner metric.

In Sections 1 and 2, we review some general properties of Dirichlet spaces and describe the conditions that we impose on the space. Moreover, we state a localized version of the parabolic Harnack inequality for local weak solutions of the heat equation for second-order differential operators with lower order terms. In Section 3 we prove estimates for the heat kernel on balls and for the capacity of balls. After recalling the definition and some properties of inner uniform domains, we give estimates for Green functions on inner balls intersected with an inner uniform domain. In Section 4, we give a proof of the boundary Harnack principle.

1. Preliminaries

1.1. Local weak solutions. Let X be a connected locally compact separable metrizable space, and let μ be a positive Radon measure with full support. Let $(\mathcal{E}, \mathcal{F})$ be a strongly local regular symmetric Dirichlet form on $L^2(X, \mu)$. We denote by $(L, D(L))$ and $(P_t)_{t\geq 0}$ the infinitesimal generator and the semigroup, respectively, associated with $(\mathcal{E}, \mathcal{F})$. See [13].

There exists a measure-valued quadratic form $d\Gamma$ defined by

$$\int f\, d\Gamma(u, u) = \mathcal{E}(uf, u) - \frac{1}{2}\mathcal{E}(f, u^2), \quad \forall f, u \in \mathcal{F} \cap L^\infty(X),$$

and extended to unbounded functions by setting $\Gamma(u,u) = \lim_{n\to\infty} \Gamma(u_n,u_n)$, where $u_n = \max\{\min\{u, n\}, -n\}$. Using polarization, we obtain a bilinear form $d\Gamma$. In particular,

$$\mathcal{E}(u, v) = \int d\Gamma(u, v), \quad \forall u, v \in \mathcal{F}.$$

Let $U \subset X$ be open. Set

$$\mathcal{F}_{\mathrm{loc}}(U) = \{ f \in L^2_{\mathrm{loc}}(U) \colon \forall \text{open rel. compact } A \subset U,\ \exists f^\sharp \in \mathcal{F} \text{ such that } f|_A = f^\sharp|_A\ \mu\text{-a.e.}\},$$

where $L^2_{\mathrm{loc}}(U)$ is the space of functions that are locally in $L^2(U,\mu)$. For $f,g \in \mathcal{F}_{\mathrm{loc}}(U)$ we define $\Gamma(f,g)$ locally by $\Gamma(f,g)|_A = \Gamma(f^\sharp, g^\sharp)|_A$, where $A \subset U$ is open and relatively compact and $f^\sharp, g^\sharp$ are functions in $\mathcal{F}$ such that $f = f^\sharp$, $g = g^\sharp$ μ-a.e. on A.

The *intrinsic distance* $d := d_{\mathcal{E}}$ induced by $(\mathcal{E}, \mathcal{F})$ is defined as

$$d_{\mathcal{E}}(x,y) := \sup\{ f(x) - f(y) \colon f \in \mathcal{F}_{\mathrm{loc}}(X) \cap C(X),\ d\Gamma(f,f) \le d\mu\},$$

for all $x, y \in X$, where $C(X)$ is the space of continuous functions on X. Consider the following properties of the intrinsic distance that may or may not be satisfied. They are discussed in [33, 31].

(A1) The intrinsic distance d is finite everywhere, continuous, and defines the original topology of X.

(A2) (X, d) is a complete metric space.

Note that if (A1) holds true, then (A2) is by [33, Theorem 2] equivalent to

(A2′) $\forall x \in X,\ r > 0$, the open ball $B(x,r)$ is relatively compact in(X,d).

Moreover, (A1) and (A2) imply that (X,d) is a geodesic space, i.e. any two points in X can be connected by a minimal geodesic in X. See [33, Theorem 1]. If (A1) and (A2) hold true, then by [31, Proposition 1],

$$d(x,y) := \sup\{ f(x) - f(y) \colon f \in \mathcal{F} \cap C_{\mathrm{c}}(X),\ d\Gamma(f,f) \le d\mu\}, \quad x, y \in X.$$

It is sometimes sufficient to consider property (A2′) only on an open subset $Y \subset X$, that is,

(A2-Y) For any ball $B(x,2r) \subset Y$, $B(x,r)$ is relatively compact.

For a domain U in X, define

$$\mathcal{F}(U) = \left\{ u \in \mathcal{F}_{\mathrm{loc}}(U) \colon \int_U |u|^2\, d\mu + \int_U d\Gamma(u,u) < \infty \right\},$$

$$\mathcal{F}_{\mathrm{c}}(U) = \{ u \in \mathcal{F}(U) \colon \text{The essential support of } u \text{ is compact in } U\},$$

$$\mathcal{F}^0(U) = \text{the closure of } \mathcal{F}_{\mathrm{c}}(U) \text{ for the norm } \left(\mathcal{E}(u,u) + \int_U u^2\, d\mu \right)^{1/2}.$$

Note that $\mathcal{F}_c(U)$ is a linear subspace of $\mathcal{F}$.

The *inner distance* d_U on U is defined as

$$d_U(x, y) = \inf\{\mathrm{length}(\gamma) \mid \gamma : [0, 1] \to U \text{ continuous}, \ \gamma(0) = x, \ \gamma(1) = y\},$$

where

$$\mathrm{length}(\gamma) = \sup\left\{\sum_{i=1}^{n} d(\gamma(t_i), \gamma(t_{i-1})) : n \in \mathbb{N}, \ 0 \le t_0 < \cdots < t_n \le 1\right\}.$$

REMARK 1.1. Suppose (A1), (A2-Y) are satisfied. Let $U \subset Y$ be open. Then $d_U = d_{\mathcal{E}_U^D}$, where $\mathcal{E}_U^D$ is the Dirichlet-type form on U defined in Definition 3.1 below. See, e.g., [15].

Let $\tilde{U}$ be the completion of U in the inner metric.

DEFINITION 1.2. Let V be an open subset of U. Set

$$\begin{aligned}&\mathcal{F}^0_{\mathrm{loc}}(U, V)\\ &= \{ f \in L^2_{\mathrm{loc}}(V, \mu) : \forall \text{open } A \subset V \text{ rel. compact in } \overline{U} \text{ with}\\ &\qquad\qquad d_U(A, U \setminus V) > 0, \ \exists f^\sharp \in \mathcal{F}^0(U) : \ f^\sharp = f \ \mu\text{-a.e. on } A\},\end{aligned}$$

where

$$d_U(A, U \setminus V) = \inf\{d_U(x, y) : x \in A, y \in U \setminus V\}.$$

DEFINITION 1.3. Let $V \subset U$ be open. A function $u : V \to \mathbb{R}$ is called *harmonic* or a *local weak solution* of $Lu = 0$ in V, if

(i) $u \in \mathcal{F}_{\mathrm{loc}}(V)$,

(ii) For any function $\phi \in \mathcal{F}_c(V)$, $\mathcal{E}(u, \phi) = \int f\phi \, d\mu$.

If in addition

$$u \in \mathcal{F}^0_{\mathrm{loc}}(U, V),$$

then u is a local weak solution with *Dirichlet boundary condition* along $\tilde{V}^{d_U} \setminus U$, where $\tilde{V}^{d_U}$ is the completion of V under d_U.

For a time interval I and a separable Hilbert space H, let $L^2(I \to H)$ be the Hilbert space of those functions $v : I \to H$ such that

$$\|v\|_{L^2(I \to H)} = \left(\int_I \|v(t)\|_H^2 \, dt\right)^{1/2} < \infty.$$

Let $W^1(I \to H) \subset L^2(I \to H)$ be the Hilbert space of those functions $v: I \to H$ in $L^2(I \to H)$ whose distributional time derivative v' can be represented by functions in $L^2(I \to H)$, equipped with the norm

$$\|v\|_{W^1(I\to H)} = \left(\int_I \|v(t)\|_H^2 + \|v'(t)\|_H^2 \, dt\right)^{1/2} < \infty.$$

Identifying $L^2(X,\mu)$ with its dual space and using the dense embeddings $\mathcal{F} \subset L^2(X,\mu) \subset \mathcal{F}'$, we set

$$\begin{aligned} \mathcal{F}(I \times X) &= L^2(I \to \mathcal{F}) \cap W^1(I \to \mathcal{F}'), \\ \mathcal{F}^0(I \times U) &= L^2(I \to \mathcal{F}^0(U)) \cap W^1(I \to \mathcal{F}^0(U)'), \end{aligned}$$

where $\mathcal{F}'$ and $\mathcal{F}^0(U)'$ denote the dual spaces of $\mathcal{F}$ and $\mathcal{F}^0(U)$, respectively. It is well-known that $L^2(I \to L^2(X, d\mu))$ can be identified with $L^2(I \times X, dt \times d\mu)$. Let

$$\mathcal{F}_{\mathrm{loc}}(I \times U)$$

be the set of all functions $u: I \times U \to \mathbb{R}$ such that for any open interval J that is relatively compact in I, and any open subset A relatively compact in U, there exists a function $u^\sharp \in \mathcal{F}(I \times X)$ such that $u^\sharp = u$ a.e. in $J \times A$.

Let

$$\begin{aligned} \mathcal{F}_{\mathrm{c}}(I \times U) = \{u \in \mathcal{F}(I \times X) \colon &\text{ There is a compact set } K \subset U \text{ that contains} \\ &\text{ the supports of } u(t, \cdot) \text{ for a.e. } t \in I\}. \end{aligned}$$

For an open subset $V \subset U$, let $Q = I \times V$ and let

$$\mathcal{F}^0_{\mathrm{loc}}(U, Q)$$

be the set of all functions $u: Q \to \mathbb{R}$ such that for any open interval J that is relatively compact in I, and any open set $A \subset V$ relatively compact in $\overline{U}$ with $d_U(A, U \setminus V) > 0$, there exists a function $u^\sharp \in \mathcal{F}^0(I \times U)$ such that $u^\sharp = u$ a.e. in $J \times A$.

DEFINITION 1.4. Let I be an open interval and $V \subset U$ open. Set $Q = I \times V$. A function $u: Q \to \mathbb{R}$ is a *local weak solution* of the heat equation $\partial_t u = Lu$ in Q, if

(i) $u \in \mathcal{F}_{\mathrm{loc}}(Q)$,

(ii) For any open interval J relatively compact in I,

$$\forall \phi \in \mathcal{F}_{\mathrm{c}}(Q), \quad \int_J \left\langle \frac{\partial}{\partial t} u, \phi \right\rangle_{\mathcal{F}',\mathcal{F}} dt + \int_J \mathcal{E}(u(t, \cdot), \phi(t, \cdot))\, dt = 0.$$

If in addition

$$u \in \mathcal{F}^0_{\mathrm{loc}}(U, Q),$$

then u is a local weak solution with *Dirichlet boundary condition* along $\tilde{V}^{d_U} \setminus U$.

REMARK 1.5. We will abuse notation in writing $\int \partial_t u\phi \, d\mu$ for the pairing $\langle (\partial/\partial)u, \phi\rangle_{\mathcal{F}',\mathcal{F}}$.

1.2. Volume doubling, Poincaré inequality, and Harnack inequality. Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be as in the previous section. Let $Y \subset X$ be open.

We say that (X, μ) satisfies the *volume doubling property* on Y, if there exists a constant $D_Y \in (0, \infty)$ such that for every ball $B(x, 2r) \subset Y$,

$$V(x, 2r) \le D_Y V(x, r), \tag{VD}$$

where $V(x, r) = \mu(B(x, r))$ denotes the volume of $B(x, r)$.

The symmetric Dirichlet space $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfies the *Poincaré inequality* on Y if there exists a constant $P_Y \in (0, \infty)$ such that for any ball $B(x, 2r) \subset Y$,

$$\forall f \in \mathcal{F}, \quad \int_{B(x,r)} |f - f_B|^2 \, d\mu \le P_Y r^2 \int_{B(x,2r)} d\Gamma(f, f), \tag{PI}$$

where $f_B = \int_{B(x,r)} f \, d\mu / V(x, r)$ is the mean of f over $B(x, r)$.

For any $s \in \mathbb{R}$, $\tau > 0$, $\delta \in (0, 1)$ and $B(x, 2r) \subset Y$, define

$$\begin{aligned}
I &= (s - \tau r^2, s),\\
B &= B(x, r,),\\
Q &= I \times B,\\
Q_- &= \left(s - \frac{(3+\delta)\tau r^2}{4}, s - \frac{(3-\delta)\tau r^2}{4}\right) \times \delta B,\\
Q_+ &= \left(s - \frac{(1+\delta)\tau r^2}{4}, s\right) \times \delta B.
\end{aligned}$$

DEFINITION 1.6. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the *parabolic Harnack inequality* on Y if, for any $\tau > 0$, $\delta \in (0, 1)$, there exists a constant $H_Y(\tau, \delta) \in (0, \infty)$ such that for any ball $B(x, 2r) \subset Y$, any $s \in \mathbb{R}$, and any positive function $u \in \mathcal{F}_{\mathrm{loc}}(Q)$ with $\partial_t u = Lu$ weakly in Q, the following inequality holds,

$$\sup_{z \in Q_-} u(z) \le H_Y \inf_{z \in Q_+} u(z). \tag{PHI}$$

Here both the supremum and the infimum are essential, i.e. computed up to sets of measure zero.

The parabolic Harnack inequality implies the *elliptic Harnack inequality*,

$$\sup_{z \in B(x,r)} u(z) \le H'_Y \inf_{z \in B(x,r)} u(z), \tag{EHI}$$

where u is any positive function in $\mathcal{F}_{\mathrm{loc}}(Q)$ that is a local weak solution of $Lu = 0$ in $B(x, 2r)$. Also, (PHI) implies the Hölder continuity of local weak solutions.

Theorem 1.7. *Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a strongly local regular symmetric Dirichlet space. Assume that the intrinsic distance $d_{\mathcal{E}}$ satisfies* (A1) *and* (A2). *Then the following properties are equivalent*:
(i) *$(\mathcal{E}, \mathcal{F})$ satisfies the parabolic Harnack inequality on X.*
(ii) *The volume doubling condition and the Poincaré inequality are satisfied on X.*
(iii) *The semigroup $(P_t)_{t>0}$ admits an integral kernel $p(t, x, y)$, $t > 0$, $x, y \in X$, and there exist constants $c_1, c_2, c_3, c_4 > 0$ such that*

$$\frac{c_1}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\mathcal{E}}(x, y)^2}{c_2 t}\right) \leq p(t, x, y) \leq \frac{c_3}{V(x, \sqrt{t})} \exp\left(-\frac{d_{\mathcal{E}}(x, y)^2}{c_4 t}\right)$$

for all $x, y \in X$ and all $t > 0$.

Proof. For a detailed discussion see [31], [32], [34], and [30]. □

The following theorem is a special case of Theorem 2.8 below.

Theorem 1.8. *Let $(X, \mu, \mathcal{E}, \mathcal{F})$ be a strongly local regular symmetric Dirichlet space and $Y \subset X$. Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies* (A1), (A2-Y), *the volume doubling property* (VD) *on Y and the Poincaré inequality* (PI) *on Y. Then $(\mathcal{E}, \mathcal{F})$ satisfies the parabolic Harnack inequality on Y. The Harnack constant depends only on D_Y, P_Y, τ, δ.*

DEFINITION 1.9. If each point $x \in X$ has a neighborhood Y_x for which the hypotheses of the above theorem are satisfied, then we say that the space is *locally of Harnack-type*.

EXAMPLES 1.10. (i) Let (M, g) be a Riemannian manifold of dimension n. Since M is locally Euclidean, it is locally of Harnack-type. Suppose the Ricci curvature of M is bounded below, that is, there is a constant $K \geq 0$ so that the Ricci tensor is bounded below by $\mathcal{R} \geq -Kg$. Then the volume doubling condition and the Poincaré inequality hold uniformly on all balls $Y_x = B(x, r)$, $x \in M$, $r \in (0, R)$, with constants D_M and P_M depending on $\sqrt{K}R$, hence the parabolic Harnack inequality holds. See [30, Section 5.6.3]. In particular, if $K = 0$ then volume doubling and Poincaré inequality hold true globally with scale-invariant constants.
(ii) Let M be a complete locally compact length-metric space of finite Hausdorff dimension $n \geq 2$. M is called an *Alexandrov space*, if its curvature is bounded below by some $K \in \mathbb{R}$ in the following sense. For any two points $x, y \in M$, let γ_{xy} be a minimal

geodesic joining x to y with parameter proportional to the arc-length. Then for any triangle $\triangle xyz$ consisting of the three geodesics γ_{xy}, γ_{yz}, γ_{zx}, there exists a comparison triangle $\triangle\tilde{x}\tilde{y}\tilde{z}$ in a simply connected space of constant curvature K such that

$$d(x, y) = d(\tilde{x}, \tilde{y}), \quad d(y, z) = d(\tilde{y}, \tilde{z}), \quad d(z, x) = d(\tilde{z}, \tilde{x})$$

and

$$d(\gamma_{xy}(s), \gamma_{xz}(t)) \geq d(\gamma_{\tilde{x}\tilde{y}}(s), \gamma_{\tilde{x}\tilde{z}}(t)) \quad \text{for any} \quad s, t \in [0, 1].$$

Alexandrov spaces arise naturally as limits (in the Gromov–Hausdorff topology) of sequences of closed Riemannian manifolds $M(n, K, D)$ of dimension n, diameter at most D, and with sectional curvature bounded below by $K \in \mathbb{R}$.

On any Alexandrov space there is a canonical strongly local regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mathcal{H}^n)$, where $\mathcal{H}^n$ is the Hausdorff measure in dimension n, given by

$$\mathcal{E}(f, g) = \int_M \langle \nabla f, \nabla g\rangle \, d\mathcal{H}^n,$$
$$\mathcal{F} = W_0^1(M).$$

The inner product $\langle \cdot, \cdot \rangle$, the gradient ∇ and the Sobolev space $W_0^1(M)$ are Riemannian like objects that are provided by the Alexandrov space structure. Concrete descriptions of these objects as well as of the associated infinitesimal generator (Laplacian) are given in [21].

Let $Y \subset M$ be open and relatively compact. Like in the case of a manifold with Ricci curvature bounded below, it is proved in [21] that the Dirichlet form $(\mathcal{E}, \mathcal{F})$ satisfies the volume doubling condition and the Poincaré inequality on Y, as well as conditions (A1) and (A2-Y).

(iii) Let Ω be an open, connected subset of $\mathbb{R}^n$. Let X_i, $0 \leq i \leq k$, be smooth vector fields on R^n which satisfy Hörmander's condition, that is, there is an integer N such that at any point x in Ω, the vectors $X_i(x)$ and all their brackets of order less than $N + 1$ span the tangent space at x. Let ω be a smooth positive function on $\mathbb{R}^n$ such that ω and ω^{-1} are bounded. Then the symmetric Dirichlet form

$$\mathcal{E}(f, g) = \int_\Omega \sum_{i=1}^k X_i f X_i g \omega \, d\mu, \quad f, g \in \mathcal{F},$$

where the domain $\mathcal{F}$ is the closure of $C_0^\infty(\Omega)$ in the $(\mathcal{E}(\cdot, \cdot) + \|\cdot\|_2)$-norm, is sub-elliptic. That is, for any relatively compact set U there exist constants c, ϵ such that

$$\mathcal{E}(f, f) \geq c\|f\|_{2,\epsilon}^2, \quad f \in C_0^\infty(\Omega),$$

where $\|f\|_{2,\epsilon}^2 = \int |\hat{f}(\xi)|^2(1 + |\xi|^2)^\epsilon \, d\xi$. See [17].

The distance $d_{\mathcal{E}}$ induced by $(\mathcal{E}, \mathcal{F})$ satisfies conditions (A1) and (A2), see [19]. Moreover, the Poincaré inequality, [18], and the volume doubling condition, [28], hold true locally.

2. The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$

2.1. Non-symmetric forms.

DEFINITION 2.1. Let $(\mathcal{E}, \mathcal{F})$ be a bilinear form on $L^2(X, \mu)$. Let $\mathcal{E}^{\mathrm{sym}}(u, v) = (1/2)(\mathcal{E}(u, v) + \mathcal{E}(v, u))$ be its symmetric part and $\mathcal{E}^{\mathrm{skew}}(u, v) = (1/2)(\mathcal{E}(u, v) - \mathcal{E}(v, u))$ its skew-symmetric part. Then $(\mathcal{E}, \mathcal{F})$ is a *coercive closed form*, if
(i) $\mathcal{F}$ is a dense linear subspace of $L^2(X, \mu)$,
(ii) $(\mathcal{E}^{\mathrm{sym}}, \mathcal{F})$ is a positive definite, closed form on $L^2(X, \mu)$,
(iii) $(\mathcal{E}, \mathcal{F})$ satisfies the *sector condition*, i.e. there exists a constant $C_0 > 0$ such that

$$|\mathcal{E}^{\mathrm{skew}}(u, v)| \leq C_0(\mathcal{E}_1(u, u))^{1/2}(\mathcal{E}_1(v, v))^{1/2},$$

for all $u, v \in \mathcal{F}$, where $\mathcal{E}_1(f, g) = \mathcal{E}(f, g) + \int_X fg\, d\mu$.

Coercive closed forms are discussed in [25]. Every coercive closed form $(\mathcal{E}, \mathcal{F})$ is associated uniquely with an infinitesimal generator $(L, D(L))$ and a strongly continuous contraction semigroup $(P_t)_{t>0}$. Furthermore, the form

$$\mathcal{E}^*(f, g) := \mathcal{E}(g, f),$$
$$D(\mathcal{E}^*) := \mathcal{F}.$$

is also a coercive closed form. Its infinitesimal generator $(L^*, D(L^*))$ is the adjoint operator of $(L, D(L))$, and for its semigroup $(P_t^*)_{t>0}$, P_t^* is the adjoint of P_t for each $t > 0$. If these semigroups admit continuous kernels p^* and p, respectively, then the kernels are related by

$$p^*(t, x, y) = p(t, y, x), \quad \forall t > 0,\ \forall x, y \in X.$$

Throughout the paper we will use the notation $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a,b\}$ for $a,b \in \mathbb{R}$. For any $f \in L^2(X,\mu)$, let $f^+ = \max\{f,0\}$ and $f \wedge 1 = \min\{f,1\}$.

DEFINITION 2.2. A *Dirichlet form* $(\mathcal{E},\mathcal{F})$ is a coercive closed form such that for all $u \in \mathcal{F}$ we have $u^+ \wedge 1 \in \mathcal{F}$ and the following two inequalities hold,

$$\begin{aligned} &\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0, \\ &\mathcal{E}(u - u^+ \wedge 1, u + u^+ \wedge 1) \geq 0. \end{aligned} \tag{2}$$

This definition is equivalent to the property that the semigroup $(P_t)_{t>0}$ associated with the coercive closed form $(\mathcal{E}, \mathcal{F})$ and its adjoint $(P_t^*)_{t>0}$ are both sub-Markovian.

The symmetric part $\mathcal{E}^{\mathrm{sym}}$ of a local, regular Dirichlet form can be written uniquely as

$$\mathcal{E}^{\mathrm{sym}}(f,g) = \mathcal{E}^{\mathrm{s}}(f,g) + \int fg\,d\kappa, \quad \text{for all} \quad f,g \in \mathcal{F},$$

where $\mathcal{E}^{\mathrm{s}}$ is strongly local and κ is a positive Radon measure. Let Γ be the energy measure of the strongly local part $\mathcal{E}^{\mathrm{s}}$.

EXAMPLE 2.3. On Euclidean space, consider the form

$$\mathcal{E}(f,g) = \int \sum_{i,j=1}^{n} a_{i,j}\partial_i f \partial_j g\,dx + \int \sum_{i=1}^{n} b_i \partial_i f g\,dx + \int \sum_{i=1}^{n} f\, d_i \partial_i g\,dx + \int cfg\,dx,$$

where the coefficients $a = (a_{i,j})$, $b = (b_i)$, $d = (d_i)$, c are bounded and measurable with $c - \operatorname{div} b \geq 0$ and $c - \operatorname{div} d \geq 0$ in the distribution sense, and, $\forall \xi \in \mathbb{R}^n$, $\sum_{i,j} a_{i,j}\xi_i\xi_j \geq \epsilon|\xi|^2$, $\epsilon > 0$. Then $(\mathcal{E},\mathcal{F})$ with domain $\mathcal{F} = W_0^1(\mathbb{R}^n)$ is a Dirichlet form.

Set $\tilde{a}_{i,j} := (a_{i,j} + a_{j,i})/2$ and $\check{a}_{i,j} = (a_{i,j} - a_{j,i})/2$. Then the symmetric part of $\mathcal{E}$ is

$$\begin{aligned}\mathcal{E}^{\mathrm{sym}}(f,g) = &\int \sum_{i,j=1}^{n} \tilde{a}_{i,j}\partial_i f \partial_j g\,dx + \int \sum_{i=1}^{n} \frac{b_i + d_i}{2} \partial_i f g\,dx \\ &+ \int \sum_{i=1}^{n} f \frac{b_i + d_i}{2} \partial_i g\,dx + \int cfg\,dx,\end{aligned}$$

while the skew-symmetric part of $\mathcal{E}$ is

$$\begin{aligned}\mathcal{E}^{\mathrm{skew}}(f,g) = &\int \sum_{i,j=1}^{n} \check{a}_{i,j}\partial_i f \partial_j g\,dx + \int \sum_{i=1}^{n} \frac{b_i - d_i}{2} \partial_i f g\,dx \\ &+ \int \sum_{i=1}^{n} f \frac{-b_i + d_i}{2} \partial_i g\,dx.\end{aligned}$$

The symmetric part $\mathcal{E}^{\mathrm{sym}}$ can be decomposed into its strongly local part

$$\mathcal{E}^{\mathrm{s}}(f,g) = \sum_{i,j=1}^{n} \int \tilde{a}_{i,j}\partial_i f \partial_j g\,dx$$

and its killing part, where κ is given by

$$\int \psi\,d\kappa = \frac{1}{2}\int (c - \operatorname{div} b + c - \operatorname{div} d)\psi\,dx, \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

2.2. Assumptions on the forms. We fix a symmetric strongly local regular Dirichlet form $(\hat{\mathcal{E}}, \mathcal{F})$ on $L^2(X, \mu)$ with energy measure $\hat{\Gamma}$. Let $Y \subset X$ and assume that the intrinsic metric $d = d_{\hat{\mathcal{E}}}$ satisfies (A1)–(A2-Y).

Let $(\mathcal{E}, D(\mathcal{E}))$ be a (possibly non-symmetric) local bilinear form on $L^2(X, \mu)$.

ASSUMPTION 1. (i) $(\mathcal{E}, D(\mathcal{E}))$ is a local, regular Dirichlet form. Its domain $D(\mathcal{E})$ is the same as the domain of the form $(\hat{\mathcal{E}}, \mathcal{F})$, that is, $D(\mathcal{E}) = \mathcal{F}$. Let C_0 be the constant in the sector condition for $(\mathcal{E}, \mathcal{F})$.
(ii) There is a constant $C_1 \in (0, \infty)$ so that for all $f, g \in \mathcal{F}_{\mathrm{loc}}(Y)$ with $fg \in \mathcal{F}_{\mathrm{c}}(Y)$,

$$C_1^{-1} \int f^2 \, d\hat{\Gamma}(g, g) \leq \int f^2 \, d\Gamma(g, g) \leq C_1 \int f^2 \, d\hat{\Gamma}(g, g),$$

where Γ is the energy measure of $\mathcal{E}^{\mathrm{s}}$.
(iii) There are constants $C_2, C_3 \in [0, \infty)$ so that for all $f \in \mathcal{F}_{\mathrm{loc}}(Y)$ with $f^2 \in \mathcal{F}_{\mathrm{c}}(Y)$,

$$\int f^2 \, d\kappa \leq 2\left(\int f^2 \, d\mu\right)^{1/2} \left(C_2 \int d\hat{\Gamma}(f, f) + C_3 \int f^2 \, d\mu\right)^{1/2}$$

(iv) There are constants $C_4, C_5 \in [0, \infty)$ such that for all $f \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y)$, $g \in \mathcal{F}_{\mathrm{c}}(Y) \cap L^\infty(Y)$,

$$\left|\mathcal{E}^{\mathrm{skew}}(f, fg^2)\right| \leq 2\left(\int f^2 \, d\hat{\Gamma}(g, g)\right)^{1/2} \left(C_4 \int g^2 \, d\hat{\Gamma}(f, f) + C_5 \int f^2 g^2 \, d\mu\right)^{1/2}.$$

ASSUMPTION 2. There are constants $C_6, C_7 \in [0, \infty)$ such that

$$\begin{aligned} |\mathcal{E}^{\mathrm{skew}}(f, f^{-1}g^2)| &\leq 2\left(\int d\hat{\Gamma}(g, g)\right)^{1/2} \left(C_6 \int g^2 \, d\hat{\Gamma}(\log f, \log f)\right)^{1/2} \\ &\quad + 2\left(\int d\hat{\Gamma}(g, g) + \int g^2 \, d\hat{\Gamma}(\log f, \log f)\right)^{1/2} \left(C_7 \int g^2 \, d\mu\right)^{1/2}, \end{aligned}$$

for all $0 \leq f \in \mathcal{F}_{\mathrm{loc}}(Y)$ with $f + f^{-1} \in L^\infty_{\mathrm{loc}}(Y)$, and all $g \in \mathcal{F}_{\mathrm{c}}(Y) \cap L^\infty(Y)$.

REMARK 2.4. (i) Assumptions 1 and 2 are more restrictive than Assumptions 1 and 2 in [23]. Here, we assume in addition that $(\mathcal{E}, \mathcal{F})$ is a time-independent Dirichlet form. In particular, $(\mathcal{E}, \mathcal{F})$ is positive definite and Markovian.
(ii) Assumption 1 (ii) holds if and only if for all $f \in \mathcal{F}_{\mathrm{c}}(Y)$,

$$C_1^{-1} \hat{\mathcal{E}}(f, f) \leq \mathcal{E}^{\mathrm{s}}(f, f) \leq C_1 \hat{\mathcal{E}}(f, f).$$

See, e.g., [27].
(iii) $\mathcal{E}$ satisfies the above assumptions if and only if the adjoint $\mathcal{E}^*(f, g) := \mathcal{E}(g, f)$ satisfies them.

(iv) If Assumption 1 (iv) is satisfied with $C_4 = 0$, then Assumption 2 is satisfied with $C_6 = 0$. To see this, apply Assumption 1(iv) to $\mathcal{E}_t^{\text{skew}}(f, f^{-1}g^2) = \mathcal{E}_t^{\text{skew}}(f, f(f^{-1}g)^2)$.
(v) Assumptions 1 and 2 are satisfied by the classical forms on Euclidean space associated with the example given in the introduction. The constants C_4, C_6 can be taken to be equal to 0 only if $a_{i,j}$ is symmetric for all i, j, and C_2, C_5, C_7 can be taken to be equal to 0 only if $b_i = d_i = 0$ for all i (i.e., if there is no drift term).

Let

$$C_8 := C_2 + C_3^{1/2} + C_5 + C_7.$$

2.3. Parabolic Harnack inequality. Let $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ be a strongly local regular symmetric Dirichlet space and $Y \subset X$. Assume (A1)-(A2-Y). Let $(\mathcal{E}, \mathcal{F})$ satisfy Assumptions 1 and 2. Let $(L, D(L))$ be the infinitesimal generator associated with $(\mathcal{E}, \mathcal{F})$.

DEFINITION 2.5. Let $V \subset U \subset X$ be open subsets. A function $u\colon V \to \mathbb{R}$ is a *local weak solution* of $Lu = 0$ in V, if
(i) $u \in \mathcal{F}_{\text{loc}}(V)$,
(ii) for any function $\phi \in \mathcal{F}_{\text{c}}(V)$, $\mathcal{E}(u, \phi) = 0$.
If in addition

$$u \in \mathcal{F}^0_{\text{loc}}(U, V),$$

then u is a local weak solution with *Dirichlet boundary condition* along $\tilde{V}^{d_U} \setminus U$.

DEFINITION 2.6. Let I be an open interval and $V \subset U$ open. Set $Q = I \times V$. A function $u\colon Q \to \mathbb{R}$ is a *local weak solution* of the heat equation $\partial_t u = Lu$ in Q, if
(i) $u \in \mathcal{F}_{\text{loc}}(Q)$,
(ii) For any open interval J relatively compact in I,

$$\forall \phi \in \mathcal{F}_{\text{c}}(Q), \quad \int_J \int_V \frac{\partial}{\partial t} u\phi \, d\mu \, dt + \int_J \mathcal{E}(u(t,\ \cdot), \phi(t,\ \cdot)) \, dt = 0.$$

If in addition

$$u \in \mathcal{F}^0_{\text{loc}}(U, Q),$$

then u is a local weak solution with *Dirichlet boundary condition* along $\tilde{V}^{d_U} \setminus U$.

Analogously to Definition 1.6, we can describe the elliptic and parabolic Harnack inequalities for local weak solutions of $Lu = 0$ and $\partial_t u = Lu$, respectively.

Lemma 2.7. *Suppose* $(\mathcal{E}, \mathcal{F})$ *satisfies* (A1), (A2). *A function* $u\colon I \to \mathcal{F}$ *is a local weak solution of* $\partial_t u = Lu$ *on* $Q = I \times U$ *if and only if*

(i) $u \in L^2(I \to \mathcal{F})$,

(ii)

$$-\int_I \left\langle \frac{\partial}{\partial t}\phi, u \right\rangle_{\mathcal{F}',\mathcal{F}} dt + \int_I \mathcal{E}(u(t, \cdot), \phi(t, \cdot))\, dt = 0, \tag{3}$$

for all $\phi \in \mathcal{F}(Q)$ *with compact support in* $I \times U$.

Proof. See [12, Lemma 5.1]. □

Theorem 2.8. *Let* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ *and* $(\mathcal{E}, \mathcal{F})$ *be as above and* $Y \subset X$. *Suppose that* $(\mathcal{E}, \mathcal{F})$ *satisfies* Assumptions 1, 2, *and* $(\hat{\mathcal{E}}, \mathcal{F})$ *satisfies* (A1), (A2-Y), *the volume doubling property* (VD) *on* Y *and the Poincaré inequality* (PI) *on* Y. *Then* $(\mathcal{E}, \mathcal{F})$ *satisfies the parabolic Harnack inequality* (PHI) *on* Y. *The Harnack constant depends only on* D_Y, P_Y, τ, δ, C_1–C_7 *and an upper bound on* $C_8 r^2$.

Proof. See [23]. □

Corollary 2.9. *Let* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$, $(\mathcal{E}, \mathcal{F})$ *and* $Y \subset X$ *be as in* Theorem 2.8. *Fix* $\tau > 0$ *and* $\delta \in (0, 1)$. *Then there exist* $\beta \in (0, 1)$ *and* $H \in (0, \infty)$ *such that for any* $B(x, 2r) \subset Y$, $s > 0$, *any local weak solution of* $\partial_t u = Lu$ *in* $Q = (s - \tau r^2, s) \times B(x, r)$ *has a continuous representative which satisfies*

$$\sup_{(t,y),(t',y')\in Q_-} \left\{ \frac{|u(t, y) - u(t', y')|}{[|t - t'|^{1/2} + d_{\mathcal{E}}(y, y')^{\beta}]} \right\} \leq \frac{H}{r^{\beta}} \sup_Q |u|$$

where $Q_- = (s - (3 + \delta)\tau r^2/4, s - (3 - \delta)\tau r^2/4) \times B(x, \delta r)$. *The constant* H *depends only on* D_Y, P_Y, τ, δ, C_1–C_7 *and an upper bound on* $C_8 r^2$.

Proof. See, e.g., [30]. □

3. Green functions estimates and inner uniformity

3.1. Dirichlet-type form. For the rest of the paper, we fix a symmetric strongly local regular Dirichlet space $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ and an open subset $Y \subset X$. Suppose (A1)–(A2-Y), the volume doubling condition (VD) on Y and the Poincaré inequality (PI) on Y hold. Let $(\mathcal{E}, \mathcal{F})$ be a bilinear form which satisfies Assumptions 1 and 2. Recall that by Theorem 2.8, L and L^* satisfy (PHI) on Y.

DEFINITION 3.1. Let U be an open subset of X. The Dirichlet-type form on U is defined as

$$\mathcal{E}_U^D(f, g) := \mathcal{E}(f, g), \quad f, g \in \mathcal{F}^0(U).$$

The form $(\mathcal{E}_U^D, \mathcal{F}^0(U))$ is associated with a semigroup $P_U^D(t)$, $t > 0$. Using the reasoning in [32, Section 2.4], one can show that, if $U \subset Y$, then the semigroup $P_U^D(t)$ admits a continuous integral kernel $p_U^D(t, x, y)$. Moreover, the map $y \mapsto p_U^D(t, x, y)$ is in $\mathcal{F}^0(U)$.

The extended Dirichlet space $\mathcal{F}^0(U)_e$ is defined as the family of all measurable, almost everywhere finite functions u such that there exists an approximating sequence $(u_n) \in \mathcal{F}^0(U)$ that is a Cauchy sequence with respect to the norm $\| f\|_{\mathcal{F}^0(U)_e} := \mathcal{E}_U^D(f, f)^{1/2}$, and $u = \lim u_n$ μ-almost everywhere. If $(\mathcal{E}_U^D, \mathcal{F}^0(U))$ is transient then $\mathcal{F}^0(U)_e$ is complete by [13, Lemma 1.5.5].

The Green function on U is defined as

$$G_U^D(x, y) := \int_0^\infty p(t, x, y)\, dt, \quad x, y \in U.$$

3.2. Capacity. The potential theory for symmetric regular Dirichlet forms is developed in [13, Chapter 2]. The potential theory of non-symmetric Dirichlet forms is treated in [25]. In this section, we recall some definitions and facts that we are going to use.

Let $U \subset Y$ be open. Assume that $(\mathcal{E}_U^D,\mathcal{F}^0(U))$ is transient. For any open set $A \subset U$ define

$$\mathcal{L}_{A,U} = \{w \in D(\mathcal{E}_U^D)\colon w \geq 1 \text{ a.e. on } A\}.$$

If $\mathcal{L}_{A,U} \neq \emptyset$, then there exist unique functions $e_{A,1}, \hat{e}_{A,1} \in \mathcal{L}_{A,U}$ such that for all $w \in \mathcal{L}_{A,U}$ it holds

$$\mathcal{E}_1(e_{A,1}, w) \geq \mathcal{E}_1(e_{A,1}, e_{A,1}) \quad \text{and} \quad \mathcal{E}_1(w, \hat{e}_{A,1}) \geq \mathcal{E}_1(\hat{e}_{A,1}, \hat{e}_{A,1}). \tag{4}$$

Notice that this implies that $\mathcal{E}_1(e_{A,1},\hat{e}_{A,1}) = \mathcal{E}_1(e_{A,1},e_{A,1}) = \mathcal{E}_1(\hat{e}_{A,1},\hat{e}_{A,1})$. Moreover, for any open $A \subset U$ such that $\mathcal{L}_{A,U} \neq \emptyset$, $e_{A,1}$ is the smallest function u on U such that $u \wedge 1$ is a 1-excessive function in $D(\mathcal{E}_U^D)$ and $u \geq 1$ on A. See [25, Proposition III.1.5].

The 1-*capacity* (with respect to $(\mathcal{E}, \mathcal{F})$) of A in U is defined by

$$\mathrm{Cap}_{U,1}(A) = \begin{cases} \mathcal{E}_1(e_{A,1}, e_{A,1}), & \mathcal{L}_{A,U} \neq \emptyset, \\ +\infty, & \mathcal{L}_{A,U} = \emptyset. \end{cases}$$

The 1-capacity is extended to non-open sets $A \subset U$ by

$$\mathrm{Cap}_{U,1}(A) = \inf\{\mathrm{Cap}_{U,1}(B)\colon A \subset B \subset U,\ B \text{ open}\}.$$

The 0-*capacity* is defined similarly, with $\mathcal{E}_1$ replaced by $\mathcal{E}$ and $\mathcal{F}^0(U)$ replaced by the extended Dirichlet space $\mathcal{F}^0(U)_e$.

Now assume $A \subset X$ is closed. By [10, Proposition VI.4.3], $e_{A,0} = \mathcal{G}_U \nu_A$ is a potential. Hence, for the equilibrium measure ν_A it holds

$$\mathrm{Cap}_{U,0}(A) = \mathcal{E}(e_{A,0}, e_{A,0}) = \mathcal{E}(\mathcal{G}_U \nu_A, e_{A,0}) = \int \widetilde{e_{A,0}}\, d\nu_A = \nu_A(U).$$

Let $\widetilde{\mathrm{Cap}}_{U,1}(A) = \mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}})$ be the 1-capacity with respect to the strongly local part $\mathcal{E}^{\mathrm{s}}$ of the symmetric part $\mathcal{E}^{\mathrm{sym}}$.

Lemma 3.2. *For any subset* $A \subset U \subset Y$,

$$\widetilde{\mathrm{Cap}}_{U,1}(A) \le \mathrm{Cap}_{U,1}(A) \le C\, \widetilde{\mathrm{Cap}}_{U,1}(A),$$

where $C = (1 + C_0)^2(2 + C_1C_2 + 2C_3^{1/2})$.

Proof. It suffices to consider an open set $A \subset U$. By (4), the Cauchy–Schwarz inequality, the sector condition and Assumption 1,

$$\begin{aligned}\mathcal{E}_1(e_{A,1}, e_{A,1}) &\le \mathcal{E}_1(e_{A,1}, e_{A,1}^{\mathrm{s}})\\ &\le (1 + C_0)(\mathcal{E}_1(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}))^{1/2}(\mathcal{E}_1(e_{A,1}, e_{A,1}))^{1/2}\\ &\le (1 + C_0)((2 + C_1C_2 + 2C_3^{1/2})\mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}))^{1/2}(\mathcal{E}_1(e_{A,1}, e_{A,1}))^{1/2}.\end{aligned}$$

Hence,

$$\mathrm{Cap}_{U,1}(A) = \mathcal{E}_1(e_{A,1}, e_{A,1}) \le C\mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}) = C\, \widetilde{\mathrm{Cap}}_{U,1}(A),$$

where $C = (1 + C_0)^2(2 + C_1C_2 + 2C_3^{1/2})$, On the other hand, by (4) and the Cauchy–Schwarz inequality,

$$\begin{aligned}\mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}) &\le \mathcal{E}_1^{\mathrm{s}}(e_{A,1}, e_{A,1}^{\mathrm{s}}) \le (\mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}))^{1/2}(\mathcal{E}_1^{\mathrm{s}}(e_{A,1}, e_{A,1}))^{1/2}\\ &\le (\mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}))^{1/2}(\mathcal{E}_1(e_{A,1}, e_{A,1}))^{1/2}.\end{aligned}$$

Therefore,

$$\widetilde{\mathrm{Cap}}_{U,1}(A) = \mathcal{E}_1^{\mathrm{s}}(e_{A,1}^{\mathrm{s}}, e_{A,1}^{\mathrm{s}}) \le \mathcal{E}_1(e_{A,1}, e_{A,1}) = \mathrm{Cap}_{U,1}(A). \qquad \square$$

For a ball $B(x, 2R) \subset Y$, let

$$\lambda_R := \inf_{0 \ne f \in \mathcal{F}^0(B(x,R))} \frac{\mathcal{E}^D_{B(x,R)}(f, f)}{\int f^2\, d\mu} > 0$$

be the lowest Dirichlet eigenvalue of $-L^{\mathrm{sym}}$ on $B(x, R)$. Note that $\lambda_R \ge C/R^2$ for some constant $C > 0$ depending on D_Y and P_Y (see, e.g., [16, Theorem 2.6]). For any $f \in \mathcal{F}^0(B(x, R))$, we have

$$\mathcal{E}^D_{B(x,R)}(f, f) \le \mathcal{E}^D_{B(x,R),1}(f, f) \le (1 + \lambda_R^{-1})\mathcal{E}^D_{B(x,R)}(f, f). \tag{5}$$

Let $f \in \mathcal{F}^0(B(x,R))_e$. Then there is an approximating sequence (f_n) in $\mathcal{F}^0(B(x,R))$ such that $\mathcal{E}^D_{B(x,R)}(f_n - f_m, f_n - f_m) \to 0$ as $n, m \to \infty$, and $f_n \to f$ almost everywhere. Thus,

$$\mathcal{E}^D_{B(x,R),1}(f_n - f_m, f_n - f_m) \le (1 + \lambda_R^{-1})\mathcal{E}^D_{B(x,R)}(f_n - f_m, f_n - f_m) \to 0,$$

hence $\mathcal{F}^0(B(x,R))_e = \mathcal{F}^0(B(x,R))$. In particular, $e_{B(x,R),0} \in \mathcal{F}^0(B(x,R))$.

Theorem 3.3. *Suppose* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ *satisfies* (A1)–(A2-Y), (VD) *on* Y *and* (PI) *on* Y, *and* $(\mathcal{E}, \mathcal{F})$ *satisfies* Assumptions 1 *and* 2. *Then there are constants* $a, A \in (0, \infty)$ *such that for any* $r \in (0, R)$ *and any ball* $B(x, 2R) \subset Y$ *we have*

$$A^{-1}\int_r^R \frac{s}{V(x,s)}\,ds \le (\mathrm{Cap}_{B(x,R),0}(B(x,r)))^{-1} \le A\int_r^R \frac{s}{V(x,s)}\,ds. \tag{6}$$

The constant A *depends only on* D_Y, P_Y, C_0, $C_1C_2 + 2C_3^{1/2}$ *and an upper bound on* λ_R^{-1}, *where* λ_R *is the smallest Dirichlet eigenvalue of* $-L^{\mathrm{sym}}$ *on* $B(x,R)$.

Proof. Let $r \in (0, R)$ and $B = B(x, r)$. First, consider the estimate

$$A^{-1}\int_r^R \frac{s}{V(x,s)}\,ds \le (\widetilde{\mathrm{Cap}}_{B(x,R),0}(B(x,r)))^{-1} \le A\int_r^R \frac{s}{V(x,s)}\,ds. \tag{7}$$

The lower bound is proved in [33, Theorem 1] using the strong locality of $\mathcal{E}^{\mathrm{s}}$. The upper bound can be proved as in [14, Lemma 4.3] using the heat kernel estimates of Theorem 3.9 below.

If $(\mathcal{E}, \mathcal{F})$ is symmetric and strongly local, then $\mathrm{Cap}_{B(x,R),0}(B)$ is the same as $\widetilde{\mathrm{Cap}}_{B(x,R),0}(B)$, hence the assertion follows. Otherwise, we show that the two 0-capacities are comparable. In view of Lemma 3.2, it suffices to show that

$$\frac{1}{C}\,\mathrm{Cap}_{B(x,R),0}(B) \le \mathrm{Cap}_{B(x,R),1}(B) \le C\,\mathrm{Cap}_{B(x,R),0}(B)$$

and

$$\frac{1}{C'}\,\widetilde{\mathrm{Cap}}_{B(x,R),0}(B) \le \widetilde{\mathrm{Cap}}_{B(x,R),1}(B) C'\,\widetilde{\mathrm{Cap}}_{B(x,R),0}(B)$$

for some constants $C, C' \in (0, \infty)$.

$$\begin{aligned}\mathcal{E}(e_{B,0}, e_{B,0}) &\le \mathcal{E}(e_{B,0}, e_{B,1}) \le (1 + C_0)\mathcal{E}_1(e_{B,0}, e_{B,0})^{1/2}\mathcal{E}_1(e_{B,1}, e_{B,1})^{1/2}\\ &\le (1 + C_0)(1 + \lambda_R^{-1})^{1/2}\mathcal{E}(e_{B,0}, e_{B,0})^{1/2}\mathcal{E}_1(e_{B,1}, e_{B,1})^{1/2}.\end{aligned}$$

Hence,

$$\mathrm{Cap}_{B(x,R),0}(B) \le (1 + C_0)^2(1 + \lambda_R^{-1})\,\mathrm{Cap}_{B(x,R),1}(B).$$

Similarly, we get

$$\mathcal{E}_1(e_{B,1}, e_{B,1}) \le (1 + C_0)^2 \mathcal{E}_1(e_{B,0}, e_{B,0}) \le (1 + C_0)^2 (1 + \lambda_R^{-1}) \mathcal{E}(e_{B,0}, e_{B,0}),$$

and hence,

$$\mathrm{Cap}_{B(x,R),1}(B) \le (1 + C_0)^2 (1 + \lambda_R^{-1}) \,\mathrm{Cap}_{B(x,R),0}(B).$$

By similar arguments, it follows that $\widetilde{\mathrm{Cap}}_{B(x,R),0}(B)$ and $\widetilde{\mathrm{Cap}}_{B(x,R),1}(B)$ are comparable. $\square$

From now on, we only consider the 0-capacity, and thus drop the index 0.

3.3. (Inner) uniformity. Let $\Omega \subset X$ be open and connected. Recall that the *inner metric* on Ω is defined as

$$d_\Omega(x, y) = \inf\{\mathrm{length}(\gamma) \mid \gamma : [0, 1] \to \Omega \text{ continuous}, \ \gamma(0) = x, \ \gamma(1) = y\},$$

and $\tilde{\Omega}$ is the completion of Ω with respect to d_Ω. Whenever we consider an inner ball $B_{\tilde{\Omega}}(x, R) := \{y \in \tilde{\Omega} : d_\Omega(x, y) < R\}$ or $B_\Omega(x, R) := B_{\tilde{\Omega}}(x, R) \cap \Omega$, we assume that its radius is minimal in the sense that $B_{\tilde{\Omega}}(x, R) \neq B_{\tilde{\Omega}}(x, r)$ for all $r < R$.

For an open set $B \subset \Omega$ let $\partial_{\tilde{\Omega}} B = \overline{B}^{d_\Omega} \setminus B$ be the boundary of B with respect to its completion for the metric d_Ω. This should not be confused with the boundary $\partial_X B = \overline{B} \setminus B$ in (X, d). Let $\partial_\Omega B = \Omega \cap \partial_{\tilde{\Omega}} B$ be the part of the boundary that lies in Ω. If x is a point in Ω, denote by $\delta_\Omega(x) = d(x, X \setminus \Omega)$ the distance from x to the boundary of Ω. For a subset $A \subset \tilde{\Omega}$, let $\mathrm{diam}_\Omega(A)$ be its diameter in $(\tilde{\Omega}, d_\Omega)$.

DEFINITION 3.4. (i) Let $\gamma : [\alpha, \beta] \to \Omega$ be a rectifiable curve in Ω and let $c \in (0, 1)$, $C \in (1, \infty)$. We call γ a (c, C)*-uniform* curve in Ω if

$$\delta_\Omega(\gamma(t)) \ge c \cdot \min\{d(\gamma(\alpha), \gamma(t)), d(\gamma(t), \gamma(\beta))\}, \quad \text{for all} \quad t \in [\alpha, \beta], \tag{8}$$

and if

$$\mathrm{length}(\gamma) \le C \cdot d(\gamma(\alpha), \gamma(\beta)).$$

The domain Ω is called (c, C)*-uniform* if any two points in Ω can be joined by a (c, C)-uniform curve in Ω.
(ii) *Inner uniformity* is defined analogously by replacing the metric d on X with the inner metric d_Ω on Ω.
(iii) The notion of (*inner*) (c, C)*-length-uniformity* is defined analogously by replacing $d(\gamma(a), \gamma(b))$ by $\mathrm{length}(\gamma|_{[a,b]})$.

The next proposition is taken from [15, Proposition 3.3]. See also [26, Lemma 2.7].

Proposition 3.5. *Assume that (X, d) is a complete, locally compact length metric space with the property that there exists a constant D such that for any $r > 0$, the maximal number of disjoint balls of radius $r/4$ contained in any ball of radius r is bounded above by D. Then any connected open subset $U \subset X$ is uniform if and only if it is length-uniform.*

Let Ω be a (c_u, C_u)-inner uniform domain in (X, d).

Lemma 3.6. *For every ball $B = B_{\tilde{\Omega}}(x, r)$ in $(\tilde{\Omega}, d_\Omega)$ with minimal radius, there exists a point $x_r \in B$ with $d_\Omega(x, x_r) = r/4$ and $d(x_r, X \setminus \Omega) \geq c_u r/8$.*

Proof. This is immediate, see [15, Lemma 3.20]. □

The following lemma is crucial for the proof of the boundary Harnack principle on inner uniform domains, rather than uniform domains. Similar results were already used in [3] and [6] to prove a boundary Harnack principle on inner uniform domains in Euclidean space.

Let $p\colon \tilde{\Omega} \to \overline{\Omega}$ be the natural projection, namely the unique continuous map such that $p|_\Omega$ is the identity map on Ω. For any $x \in \tilde{\Omega}$ and any ball $D = B(p(x), r)$, let D' be the connected component of $p^{-1}(D \cap \overline{\Omega})$ that contains x. It follows that $D' \cap \Omega$ is the connected component of $D \cap \Omega$ whose closure in $\tilde{\Omega}$ contains x.

Lemma 3.7. *Suppose μ has the volume doubling property on $Y \subset X$. Then there exists a positive constant C_Ω such that for any ball $D = B(p(x), r)$ with $x \in \tilde{\Omega}$ and $B(p(x), 4r) \subset Y$,*

$$B_{\tilde{\Omega}}(x, r) \subset D' \subset B_{\tilde{\Omega}}(x, C_\Omega r).$$

The constant C_Ω depends only on D_Y and the inner uniformity constants c_u, C_u of Ω.

REMARK 3.8. (i) For any $x \in \Omega$, $r > 0$,

$$D' \cap \Omega = \{y \in \Omega\colon d_{\mathrm{diam}}(x, y) \leq r\},$$

where the *inner diameter metric* d_{diam} is defined as

$$d_{\mathrm{diam}}(x, y) := \inf\{\mathrm{diam}(\gamma)\colon \gamma \text{ path from } x \text{ to } y \text{ in } \Omega\},$$

and the diameter is taken in the metric d of the underlying space (X, d).

In the context of Euclidean space, [35, Theorem 3.4] states that the inner diameter metric and the inner (length) metric are equivalent, a statement that is slightly stronger than the conclusion of Lemma 3.7. The proof given in [35] extends to the present setting. We include a proof of Lemma 3.7 for the convenience of the reader.

(ii) The hypothesis that Ω is inner uniform can be relaxed to the hypothesis that any two points in $B_\Omega(x, C_\Omega r)$ can be connected by a path that is inner uniform in Ω.

Proof of Lemma 3.7. Clearly, $B_\Omega(x,r) \subset D'$. To show the second inclusion, we follow the line of reasoning given in [35, Proof of Theorem 3.4]. Replacing r by a slightly larger radius, we may assume that $x \in \Omega$. Let $y \in D' \cap \Omega$ and let α be a path in $D' \cap \Omega$ connecting x to y. Note that this path does not need to be an inner uniform path. Nevertheless, there exist finitely many points $x = x_1, x_2, \dots, x_N = y$ on the path α so that $d_\Omega(x_{j-1}, x_j) = d(x_{j-1}, x_j)$ for all $2 \le j \le N$. Let $M \le 2r$ be the diameter of α in (X, d). By Lemma 3.6 each x_j can be joined to a point $y_j \in \Omega$ with $d_\Omega(y_j, \partial_{\tilde{\Omega}}\Omega) \ge c_u M/4$ by an inner uniform path α_j of length at most $M/2$. Set $U^* = \{y_j : 1 \le j \le N\}$ and

$$U = \bigcup_j B_\Omega\left(y_j, \frac{c_u M}{4}\right).$$

Let w be the number of connected components of U. There exists a constant $C = C(D_Y, c_u)$ such that for each j, we have

$$\mu\left(B_\Omega\left(y_j, \frac{c_u M}{4}\right)\right) = \mu\left(B\left(y_j, \frac{c_u M}{4}\right)\right) \ge C\mu\left(B\left(y_j, \frac{3M}{2}\right)\right) \ge C\mu(B(x, M)).$$

Hence $w \cdot C\mu(B(x, M)) \le \mu(U) \le \mu(B(x, 2M))$ and

$$w \le C'.$$

We claim that if $z, z_* \in U^*$ are in the same connected component W of U, then there exists a path β connecting z to z_* in W such that $\text{length}(\beta) \le c_1 M$ for some constant $c_1 > 0$ depending only on c_u and D_Y. Since W is a connected component of U, there is a finite sequence $z = z_0, \dots, z_k = z_*$ of points in U^* such that $B_\Omega(z_{i-1}) \cap B_\Omega(z_i) \neq \emptyset$ for all $1 \le i \le k$, where $B(z_i) := B(z_i, c_u M/4) = B_\Omega(z_i, c_u M/4)$. We may assume that the balls $B(z_i)$ with even i are disjoint (otherwise consider a subsequence of (z_i)). Since there are $\lfloor k/2 \rfloor$ of these balls and, for each i, $\mu(B(z_i)) \asymp \mu(B(z))$, we get

$$\left\lfloor \frac{k}{2} \right\rfloor \mu(B(z)) \le C'' \sum_{1 \le j \le k/2} \mu(B(z_{2j})) \le \mu(W) \le \mu(U) \le \mu(B(x, 2M)),$$

so $k \le C'''$. For each i, we can connect z_{i-1} to z_i by a path β_i in Ω of length at most $c_u M/2$. Now the conjunction of the paths β_i is a path β of length at most

$$\text{length}(\beta) \le \frac{k c_u M}{2} \le c_1 M. \tag{9}$$

We define integers $0 = j_0 < j_1 < \cdots < j_s = N$ and distinct connected components $W_1, \ldots, W_s$ of U as follows. Let W_1 be the connected component that contains y_1. Assuming that j_{n-1} and W_{n-1} are defined, we iteratively define j_n to be the largest number j such that $y_j \in W_{n-1}$, and let W_n be the component that contains y_{j_n+1}.

For each $1 \le i \le s$ we have shown above that there exists a path β_{j_i} connecting $y_{j_{i-1}+1}$ to y_{j_i}. Let γ be the conjunction of these paths, of the geodesic segments $[x_{j_i}, x_{j_i+1}]$, $1 \le i \le s-1$, and of the paths α_m for $m = 1, j_1, j_1+1, j_2, j_2+1, \ldots, j_s = N$. Then γ is path in $\tilde{\Omega}$ that connects x to y and has length

$$\text{length}(\gamma) \le sc_1M + sM + \frac{sM}{2} \le C'\left(c_1 + \frac{3}{2}\right)M.$$

This means that $D' \subset B_{\tilde{\Omega}}(x, C_\Omega r)$ with $C_\Omega = C'(2c_1 + 3)$. □

3.4. Green function estimates. Recall that for an open set $U \subset X$, G_U is the Green function and p_U^D is the heat kernel associated with $(\mathcal{E}_U^D, \mathcal{F}^0(U))$.

Theorem 3.9. *Suppose* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ *satisfies* (A1)–(A2-Y), (VD) *on* Y *and* (PI) *on* Y, *and* $(\mathcal{E}, \mathcal{F})$ *satisfies* Assumptions 1 *and* 2. *Let* $B = B(a, R)$ *with* $B(a, 2R) \subset Y$.
(i) *For any fixed* $\epsilon \in (0, 1)$ *there are constants* $c, C \in (0, \infty)$ *such that for any* $x, y \in B(a, (1-\epsilon)R)$ *and* $0 < \epsilon t \le R^2$, *the Dirichlet heat kernel* p_B^D *is bounded below by*

$$p_B^D(t, x, y) \ge \frac{c}{V(x, \sqrt{t} \wedge R_x)} \exp\left(-C\frac{d(x, y)^2}{t}\right),$$

where $R_x = d(x, \partial_X B)/2$.
(ii) *For any fixed* $\epsilon \in (0,1)$ *there are constants* $c, C \in (0,\infty)$ *such that for any* $x, y \in B$, $t \ge (\epsilon R)^2$, *the Dirichlet heat kernel* p_B^D *is bounded above by*

$$p_B^D(t, x, y) \le \frac{C}{V(a, R)} \exp\left(-\frac{ct}{R^2}\right).$$

(iii) *There exist constants* $c, C \in (0, \infty)$ *such that for any* $x, y \in B$, $t > 0$, *the Dirichlet heat kernel* p_B^D *is bounded above by*

$$p_B^D(t, x, y) \le C\frac{\exp(-c(d(x, y)^2/t) + C_8 t)}{V(x, \sqrt{t} \wedge R)^{1/2} V(y, \sqrt{t} \wedge R)^{1/2}}. \tag{10}$$

All the constants c, C *above depend only on* D_Y, P_Y, C_1–C_7 *and an upper bound on* $C_8 R^2$.

Proof. See [23]. □

Lemma 3.10. *Let $B(a, 2R) \subset Y$. Then for any relatively compact, open set $V \subset B(a, R)$, the Green function $y \mapsto G_V(x, y)$ is in $\mathcal{F}^0_{\mathrm{loc}}(V, V \setminus \{x\})$ for any fixed $x \in V$.*

Proof. We follow [15, Lemma 4.7]. Recall that the map $y \mapsto p^D_V(t, x, \cdot)$ is in $\mathcal{F}^0(V)$. The heat kernel upper bounds of Theorem 3.9 imply that $\psi G_V(x, \cdot) \in L^2(X, \mu)$ for any continuous function ψ with compact support K in $X \setminus \{x\}$. Indeed, by the set monotonicity of the kernel and Theorem 3.9, there are constants $c, C \in (0, \infty)$, depending on R, such that for all $t \geq R^2$ and $z, y \in V$,

$$p^D_V(t, z, y) \leq C e^{-ct/R^2}, \tag{11}$$

and there are constants $c', C' \in (0, \infty)$ depending on R such that for all $t > 0$ and $z, y \in V$,

$$p^D_V(t, z, y) \leq C' e^{-c'/t}. \tag{12}$$

This shows that the integral $\psi G_V(x, \cdot) = \int_0^\infty \psi\, p^D_V(t, x, \cdot)\, dt$ converges at 0 and ∞ in $L^2(X, \mu)$. Hence $\psi G_V(x, \cdot)$ is in $L^2(X, \mu)$.

Next, we show that the integral also converges in $\mathcal{F}^0(V)$. Let ψ be as above with the additional property that $d\Gamma(\psi, \psi) \leq d\mu$ on X. For fixed $0 < a < b < \infty$, set $g = \int_a^b p^D_V(t, x, \cdot)\, dt$ and observe that $\psi g, \psi^2 g \in \mathcal{F}^0(V)$. By the Cauchy–Schwarz inequality and Assumption 1,

$$\begin{aligned}
\mathcal{E}(\psi g, \psi g) &\leq \int_V g^2\, d\Gamma(\psi, \psi) + \int_V d\Gamma(g, \psi^2 g) + \int_V \psi^2 g^2\, d\kappa \\
&\leq C \int_V (-Lg) g\, d\mu + C \int_{K \cap V} g^2\, d\mu \\
&= C \int_{K \cap V} \psi^2 g (p^D_V(a, x, \cdot) - p^D_V(b, x, \cdot))\, d\mu + C \int_{K \cap V} g^2\, d\mu \\
&\leq C \int_{K \cap V} g p^D_V(a, x, \cdot)\, d\mu + C \int_{K \cap V} g^2\, d\mu.
\end{aligned}$$

for some constant $C > 0$ depending on $\sup \psi^2$. Now, observe that (11) and (12) imply that

$$\int_{K \cap V} g^2\, d\mu = \int_{K \cap V} \left(\int_a^b p^D_V(t, x, \cdot)\, dt \right)^2 d\mu$$

tends to 0 when a, b tend to infinity or when a, b tend to 0 (this is indeed the argument we used above to show that $G_V(x, \cdot)$ is in $L^2(X, d\mu)$). The same estimates (11) and (12) imply that $\int_{K \cap V} g p^D_V(a, x, \cdot)\, d\mu$ tends to 0 when a, b tend to infinity or when a, b tend to 0. This implies that the integral $\psi G_V(x, y) = \psi \int_0^\infty p^D_V(t, x, \cdot)\, dt$ converges in $\mathcal{F}^0(V)$ as desired. □

Lemma 3.11. (i) *There is a constant C depending only on D_Y, P_Y, C_1–C_7 and an upper bound on C_8R^2, such that for any ball $B(z,2R) \subset Y$,*

$$\forall x, y \in B(z, R), \quad G_{B(z,R)}(x, y) \leq C \int_{d(x,y)^2/2}^{2R^2} \frac{ds}{V(x, \sqrt{s})}. \tag{13}$$

(ii) *Fix $\theta \in (0, 1)$. There is a constant C depending only on θ, D_Y, P_Y, C_1–C_7 and an upper bound on C_8R^2, such that for any ball $B(z,2R) \subset Y$,*

$$\forall x, y \in B(z, \theta R), \quad G_{B(z,R)}(x, y) \geq C \int_{d(x,y)^2/2}^{2R^2} \frac{ds}{V(x, \sqrt{s})}. \tag{14}$$

Proof. See [15, Lemma 4.8] and use the estimates of Theorem 3.9. □

Recall that for an open set $U \subset X$, $B_U(x, r) = \{y \in U : d_U(x, y) < r\}$, where d_U is the inner metric of the domain U. Let $G_{B_U(x,r)}$ be the Green function on $B_U(x, r)$.

Lemma 3.12. *Fix $\theta \in (0, 1)$. Let $U \subset X$ be an open set.*
(i) *There is a constant C depending only on θ, D_Y, P_Y, C_1–C_7 and an upper bound on C_8R^2 such that for any $B(z, 2R) \subset Y$,*

$$G_{B_U(z,R)}(x, y) \leq G_{U \cap B(z,R)}(x, y) \leq C \frac{R^2}{V(x, R)}, \tag{15}$$

for all $x, y \in U \cap B(z, R)$ with $d(x, y) \geq \theta R$.
(ii) *Let U be an open subset so that $\overline{U} \subset Y$. Consider a ball $B_U(z, 2R) \subset Y$ and suppose that any two points in $B_U(z, \delta R)$ can be connected by a (c_u, C_u)-inner uniform curve in U, for some $\delta < 1/3$. Then there is a constant C depending only on θ, D_Y, P_Y, c_u, C_u, C_1–C_7 and an upper bound on C_8R^2, such that*

$$G_{B_U(z,R)}(x, y) \geq C \frac{R^2}{V(x, R)}, \tag{16}$$

for all $x, y \in B_U(z, \delta R)$ with $d(x, X \setminus U)$, $d(y, X \setminus U) \in (\theta R, \infty)$ and $d_U(x, y) \leq \delta R/C_u$.

Proof. We follow the line of reasoning of [15, Lemma 4.9]. Set $B = B(z, R)$, $W = U \cap B(z, R)$. The upper bound (15) follows easily from Lemma 3.11 and the monotonicity inequality $G_W \leq G_B$. By assumption, there is an $\epsilon_1 > 0$ such that for any x, y as in (ii), there is a path in U from x to y of length less than $C_u d_U(x, y) \leq \delta R$ that stays at distance at least $\epsilon_1 R$ from $X \setminus U$. Since $x, y \in B_U(z, \delta R)$ and $\delta < 1/3$, this path is contained in

$$B_U(z, R) \cap \{\zeta \in U : d(\zeta, X \setminus U) > \epsilon_1 R\}.$$

Using this path, the Harnack inequality easily reduces the lower bound (16) to the case when y satisfies $d(x, y) = \eta R$ for some arbitrary fixed $\eta \in (0, \epsilon_1)$ small enough. Pick $\eta > 0$ so that, under the conditions of the lemma, the ball $B(x, 2\eta R)$ is contained in $B_U(z, R)$. Let $W = B_U(z, R)$. Then the monotonicity property of Green functions implies that $G_W(x, y) \geq G_{B(x,\eta R)}(x, y)$. Lemma 3.11 and the volume doubling property then yield

$$G_W(x, y) \geq C \frac{R^2}{V(x, R)}.$$

This is the desired lower bound. □

4. Boundary Harnack principle

4.1. Reduction to Green functions estimates. Let $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ be a symmetric strongly local regular Dirichlet space and $Y \subset X$. Suppose (A1)–(A2-Y), the volume doubling condition (VD) on Y and the Poincaré inequality (PI) on Y hold. Suppose that $(\mathcal{E}, \mathcal{F})$ satisfies Assumptions 1 and 2. We obtain that under these assumptions, local weak solutions of $Lu = 0$ (resp. $L^*u = 0$) in Y are harmonic functions for the associated Markov process and, hence, satisfy the maximum principle. This can be proved following the line of reasoning given in [13, Theorem 4.3.2, Lemma 4.3.2] and using [25, Proposition V.1.6, Proof of Lemma III.1.4]. See also [22].

Let Ω be a domain so that $\overline{\Omega} \subset Y$. For $\xi \in \partial_{\tilde{\Omega}}\Omega$, set $B_\Omega(\xi, r) := B_{\tilde{\Omega}}(\xi, r) \cap \Omega$. Let $c_u \in (0, 1)$ and $C_u \in (1, \infty)$. Let $A_3 = 12((2 + 2C_u) \vee C_\Omega)$, $A_0 = A_3 + 7$, $A_7 = 2/c_u + 1$, and $A_8 = 2(A_0 \vee 7A_7)$. Recall that $p : \tilde{\Omega} \to \overline{\Omega}$ is the natural projection ($p(x) = x$ for $x \in \Omega$) and C_Ω is the constant defined in Section 3.3. For $\xi \in \partial_{\tilde{\Omega}}\Omega$, let R_ξ be the largest radius so that

(i) $B(p(\xi), A_8 R_\xi) \subsetneq Y$,
(ii) $(A_0 \vee 26/c_u) R_\xi \leq \operatorname{diam}_\Omega(\Omega)/2$ if Ω is a bounded domain,
(iii) any two points in $B_{\tilde{\Omega}}(\xi, (A_0 + 8/c_u) R_\xi)$ can be connected by a curve that is (c_u, C_u)-inner uniform in Ω.

Theorem 4.1. *There exists a constant $A_1' \in (1, \infty)$ such that for any $\xi \in \partial_{\tilde{\Omega}}\Omega$ with $R_\xi > 0$ and any*

$$0 < r < R \leq \inf\{R_{\xi'} : \xi' \in B_{\tilde{\Omega}}(\xi, 7R_\xi) \setminus \Omega\},$$

we have

$$\frac{G_{Y'}(x, y)}{G_{Y'}(x', y)} \leq A_1' \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')},$$

for all $x, x' \in B_\Omega(\xi, r)$ and $y, y' \in \partial_\Omega B_\Omega(\xi, 6r)$. Here $Y' = B_\Omega(\xi, A_0 r)$. The constant A_1' depends only on D_Y, P_Y, c_u, C_u, C_0–C_7, and an upper bound on $C_8 R^2$.

The proof of this theorem is the content of Section 4.2 below. It is based on the estimates for the Green functions in Section 3.4.

Theorem 4.2. *Let* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ *be a strongly local regular symmetric Dirichlet space that satisfies* (A1), (A2-Y), (VD) *and* (PI) *on* $Y \subset X$. *Suppose* $(\mathcal{E}, \mathcal{F})$ *satisfies* Assumptions 1 *and* 2. *Let* $\Omega \subset Y$ *be a bounded inner uniform domain in* (X, d). *There exists a constant* $A_1 \in (1, \infty)$ *such that for any* $\xi \in \partial_{\tilde{\Omega}}\Omega$ *with* $R_\xi > 0$ *and any*

$$0 < r < R \leq \inf\{R_{\xi'} : \xi' \in B_{\tilde{\Omega}}(\xi, 7R_\xi) \setminus \Omega\},$$

and any two non-negative weak solutions u, v *of* $Lu = 0$ *in* $Y' = B_\Omega(\xi, 12C_\Omega r)$ *with weak Dirichlet boundary condition along* $B_{\tilde{\Omega}}(\xi, 12C_\Omega r) \setminus \Omega$, *we have*

$$\frac{u(x)}{u(x')} \leq A_1 \frac{v(x)}{v(x')},$$

for all $x, x' \in B_\Omega(\xi, r)$. *The constant* A_1 *depends only on the volume doubling constant* D_Y, *the Poincaré constant* P_Y, *the constants* C_0–C_7 *which give control over the skew-symmetric part and the killing part of the Dirichlet form, the inner uniformity constants* c_u, C_u, *and an upper bound on* C_8R^2.

REMARK 4.3. (i) The hypothesis that $R_\xi > 0$ can be understood as "local inner uniformity". Clearly, $R_\xi > 0$ holds true at every boundary point ξ of an inner uniform domain. Since the statement of Theorem 4.2 is local, it is natural to only require that points near ξ can be connected by inner uniform curves.
(ii) A consequence of Theorem 4.2 is that the ratio u/v of the two local weak solutions u and v is Hölder continuous.
(iii) As an application of the geometric boundary Harnack principle of Theorem 4.1, two-sided estimates of the Dirichlet heat kernel on inner uniform domains have been obtained in the companion paper [24].

Theorem 4.4. *Let* $(X, \mu, \hat{\mathcal{E}}, \mathcal{F})$ *be a strongly local regular symmetric Dirichlet space that satisfies* (A1), (A2-Y), (VD) *and* (PI) *on* $Y \subset X$. *Suppose* $(\mathcal{E}, \mathcal{F})$ *satisfies* Assumptions 1 *and* 2. *Let* $\Omega \subset Y$ *be a bounded inner uniform domain in* (X, d). *Then the Martin compactification relative to* $(\mathcal{E}, \mathcal{F})$ *of* Ω *is homeomorphic to* $\tilde{\Omega}$ *and each boundary point* $\xi \in \tilde{\Omega} \setminus \Omega$ *is minimal.*

Proof. The assertion can be proved along the line of [3, Theorem 1.1] using the boundary Harnack principle of Theorem 4.2. □

Proposition 4.5. *Let* $\xi \in \tilde{\Omega} \setminus \Omega$ *with* $R_\xi > 0$. *Let* $0 < r \leq R_\xi$. *Let* f *be non-negative harmonic on* $B_\Omega(\xi, 2C_\Omega r)$ *with Dirichlet boundary condition along* $(\partial_{\tilde{\Omega}}\Omega) \cap$

$B_{\tilde{\Omega}}(\xi, 2C_\Omega r)$. *Then there exists a positive Radon measure* ν_f *such that*

$$\tilde{f}(x) = \int_{\partial_\Omega B_\Omega(\xi,r)} G_{B_\Omega(\xi,R)}(x,y)\, d\nu_f(y), \quad \forall x \in B_\Omega(\xi, r),\ R \geq 2C_\Omega r, \tag{17}$$

where $\tilde{f}$ *is a modification of* f *that is continuous on* $B_\Omega(\xi, 2r)$.

Proof. Let $\psi \in \mathcal{F}^0(B(p(\xi), 2r))$, $0 \leq \psi \leq 1$, be a cutoff function that is 1 on $B(p(\xi), r)$, where $p\colon \tilde{\Omega} \to \overline{\Omega}$ is the natural projection. Let B' be the connected component of $p^{-1}(\overline{\Omega} \cap B(p(\xi), 2r))$ which contains ξ. By Lemma 3.7, we have $B' \subset B_{\tilde{\Omega}}(\xi, 2C_\Omega r)$. Let $B'_\Omega = B' \cap \Omega$. Set

$$u := f\psi 1_{B'_\Omega}$$

and observe that $\hat{u} \in \mathcal{F}^0(B_\Omega(\xi, 2C_\Omega r))$. Let $R \geq 2C_\Omega r$, $V = B_\Omega(\xi, R)$, $A = \{x \in \Omega\colon d_\Omega(\xi, x) \leq r\}$ and $F = \partial_\Omega B_\Omega(\xi, r)$. Let $u \in \mathcal{F}^0(B_\Omega(\xi, 2C_\Omega r))$ be a function that equals u on A and is superharmonic on V. By the 0-order version of [29, Theorem 1.4.1, Theorem 2.3.1], u is a potential.

Let u_A and u_F be the reduced functions of u on A and F, respectively. Since u is harmonic on A, it follows from the 0-order version of [29, Theorem 2.4.2 and p. 62] that $u = u_A = u_F$ a.e. on A. Let u_A and u_F be the reduced functions of u on A and F, respectively. Since u is harmonic on A, it follows from the 0-order version of [29, Theorem 2.4.2] that $u = u_A = u_F$ a.e. on A. Let μ_F be the 0-sweeping out of μ on F, that is, μ_F is a positive Radon measure with support contained in F and $u_F = U\mu_F$. By the 0-order version of [29, Theorem 2.3.5],

$$\mathcal{E}^D_Y(u_F, v) = \int_F \tilde{v}(x)\mu_F(dx), \quad \forall v \in \mathcal{F}^0(V)_e.$$

Applying this to $v = G^*_V\phi$ for suitable test functions ϕ, we obtain

$$\begin{aligned}\int_V u_F(x)\phi(x)\, d\mu(x) = \mathcal{E}^D_V(u_F, v) &= \int_F \int_V G^*_V(y,x)\phi(x)\, d\mu(x)\, d\mu_F(y) \\ &= \int_V \left(\int_F G_Y(x,y)\, d\mu_F(y)\right)\phi(x)\, d\mu(x).\end{aligned}$$

Hence,

$$u_F(x) = \int_F G_Y(y,x)\, d\mu_F(y) \quad \text{for} \quad \mu\text{-a.e. } x \in V.$$

Since $f(x) = u(x) = u_F(x)$ for μ-a.e. $x \in B_\Omega(\xi, r)$, the assertion follows for μ-a.e. $x \in B_\Omega(\xi, r)$. Since f is harmonic, it satisfies EHI, hence admits a continuous modification $\tilde{f}$. Also, the Green function is continuous. Hence, the assertion follows. $\square$

Proof of Theorem 4.2. Fix $\xi \in \partial_{\tilde{\Omega}}\Omega$ and $0 < r < R$ as in the theorem. Let $Y' = B_\Omega(\xi, A_0 r)$. Let u, v be local weak solutions u of $Lu = 0$ in $B_\Omega(\xi, 12C_\Omega r)$ with weak Dirichlet boundary condition along $B_{\tilde{\Omega}}(\xi, 12C_\Omega r) \setminus \Omega$. By Proposition 4.5, there exists a Borel measure ν_u such that

$$u(x) = \int_{\partial_\Omega B_\Omega(\xi, 6r)} G_{Y'}(x, y)\, d\nu_u(y), \quad \forall x \in \cap B_\Omega(\xi, 6r). \tag{18}$$

By Theorem 4.1, there exists a constant $A_1' \in (1, \infty)$ such that for all $x, x' \in B_\Omega(\xi, r)$ and all $y, y' \in \partial_\Omega B_\Omega(\xi, 6r)$, we have

$$\frac{G_{Y'}(x, y)}{G_{Y'}(x', y)} \le A_1' \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')}.$$

For any (fixed) $y' \in \partial_\Omega B_\Omega(\xi, 6r)$, we find that

$$\begin{aligned}\frac{1}{A_1'} u(x) &\le \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} \int_{\partial_\Omega B_\Omega(\xi, 6r)} G_{Y'}(x', y)\, d\nu_u(y)\\ &= \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} u(x') \le A_1' u(x).\end{aligned}$$

We get a similar inequality for v. Thus, for all $x, x' \in B_\Omega(\xi, r)$,

$$\frac{1}{A_1'} \frac{u(x)}{u(x')} \le \frac{G_{Y'}(x, y')}{G_{Y'}(x', y')} \le A_1' \frac{v(x)}{v(x')}. \tag{19}$$

□

4.2. Proof of Theorem 4.1. We follow closely [1] and [15]. Notice that the estimates for the Green function G in Section 3.4 and the results in this section also hold for the adjoint G^*. Let Ω, Y be as above and fix $\xi \in \partial_{\tilde{\Omega}}\Omega$ with $R_\xi > 0$.

DEFINITION 4.6. For $\eta \in (0, 1)$ and any open set $U \subset X$, define the *capacitary width* $w_\eta(U)$ by

$$w_\eta(U) = \inf\left\{ r > 0 \colon \forall x \in U,\ \frac{\operatorname{Cap}_{B(x,2r)}(\overline{B(x, r)} \setminus U)}{\operatorname{Cap}_{B(x,2r)}(\overline{B(x, r)})} \ge \eta \right\},$$

where $\inf \emptyset := +\infty$ (e.g., when $\operatorname{Cap}_{B(x,2r)}(\overline{B(x, r)}$ is not well-defined.)

Note that $w_\eta(U)$ is an increasing function of $\eta \in (0, 1)$ and an increasing function of the set U.

Lemma 4.7. *There are constants* $A_7 \in (0, \infty)$ *and* $\eta \in (0, 1)$ *depending only on* D_Y, P_Y, c_u, C_u, C_0–C_7, *and an upper bound on* $C_8 R^2$, *such that for all* $0 < r < R \le 2R_\xi$,

$$w_\eta(\{y \in B_{\tilde{\Omega}}(\xi, R) \colon d_\Omega(y, \partial_{\tilde{\Omega}}\Omega) < r\}) \le A_7 r.$$

Proof. We follow [15, Lemma 4.12]. Let $Y_r = \{y \in B_{\tilde{\Omega}}(\xi, R) \colon d_\Omega(y, \partial_{\tilde{\Omega}}\Omega) < r\}$ and $y \in Y_r$. Since $r < c_u \operatorname{diam}_\Omega(\Omega)/12$, there exists a point $x \in \Omega$ such that $d_\Omega(x, y) = 4r/c_u$. By assumption, there is an inner uniform curve connecting y to x in Ω. Let $z \in \partial_\Omega B_\Omega(y, 2r/c_u)$ be a point on this curve and note that $d_\Omega(y, z) = 2r/c_u \le d_\Omega(x, y) - d_\Omega(y, z) \le d_\Omega(x, z)$. Hence,

$$d_\Omega(z, \partial_{\tilde{\Omega}}\Omega) \ge c_u \min\{d_\Omega(y, z), d_\Omega(z, x)\} = 2r.$$

So for any $y \in Y_r$ there exists a point $z \in \partial_\Omega B_\Omega(y, 2r/c_u)$ with $d_\Omega(z, \partial_{\tilde{\Omega}}\Omega) \ge 2r$. Thus, $B(z, r) \subset B(y, A_7 r) \setminus Y_r$ if $A_7 = 2/c_u + 1$. The capacity of $B(y, A_7 r) \setminus Y_r$ in $B(y, 2A_7 r)$ is larger than the capacity of $B(z, r)$ in $B(y, 2A_7 r)$, which is larger than the capacity of $B(z, r)$ in $B(z, 3A_7 r)$. Thus, by Theorem 3.3, we have

$$\frac{\operatorname{Cap}_{B(y,2A_7 r)}(\overline{B(y, A_7 r)} \setminus Y_r)}{\operatorname{Cap}_{B(y,2A_7 r)}(\overline{B(y, A_7 r)})} \ge \frac{\operatorname{Cap}_{B(z,3A_7 r)}(\overline{B(z, r)})}{\operatorname{Cap}_{B(y,2A_7 r)}(\overline{B(y, A_7 r)})} \ge \eta,$$

for some $\eta \in (0, 1)$. Hence, for this η, we have $w_\eta(Y_r) \le A_7 r$. □

Write $w(U) := w_\eta(U)$ for the capacitary width of an open set $U \subset \Omega$, where η is the same constant as in Lemma 4.7.

The following lemma relates the capacitary width to the L-harmonic measure ω. A similar inequality holds for the L^*-harmonic measure ω^*. We write $f \asymp g$ to indicate that $cg \le f \le Cg$, for some constants $c, C \in (0, \infty)$ that depend only on D_Y, P_Y, c_u, C_u, C_0–C_7, and an upper bound on $C_8 R^2$.

Lemma 4.8. *There is a constant $a_1(D_Y, P_Y, C_0 - C_7, C_8 R^2)$ such that for any non-empty open set $U \subset X$ and any ball $B(x, 3r) \subset Y$ with $x \in U$, $0 < r < R$, we have*

$$\omega_{U \cap B(x,r)}(x, U \cap \partial_X B(x, r)) \le \exp\left(2 - \frac{a_1 r}{w(U)}\right).$$

Proof. We follow [1, Lemma 1] and [15, Lemma 4.13]. We may assume that $r/w(U) > 2$. For any $\kappa \in (0, 1)$, we can pick $w(U) \le s < w(U) + \kappa$ so that

$$\frac{\operatorname{Cap}_{B(y,2s)}(\overline{B(y, s)} \setminus U)}{\operatorname{Cap}_{B(y,2s)}(\overline{B(y, s)})} \ge \eta \quad \forall y \in U.$$

Consider a point $y \in U$ such that $B(y, 3s) \subset Y$ and let $E = \overline{B(y, s)} \setminus U$. Let ν_E be the equilibrium measure of E in $B = B(y, 2s)$. We claim that there exists $A_2 > 0$ such that

$$G_B \nu_E \ge A_2 \eta \quad \text{on} \quad B(y, 3s/2). \tag{20}$$

Let $F = \overline{B(y,s)}$ and ν_F be the equilibrium measure of F in B. Then, by the Harnack inequality, for any z with $d(y,z) = 3s/2$, we have

$$G_B(z,\zeta) \asymp G_B(z,y) \quad \forall \zeta \in B(y,s).$$

Hence,

$$G_B\nu_F(z) = \int_F G_B(z,\zeta)\nu_F(d\zeta) \asymp G_B(z,y)\nu_F(F)$$

and

$$G_B\nu_E(z) = \int_E G_B(z,\zeta)\nu_E(d\zeta) \asymp G_B(z,y)\nu_E(E).$$

Moreover, since $\nu_F(F) = \mathrm{Cap}_B(F)$, the two-sided inequality (6) and Lemma 3.11 yield that $G_B\nu_F(z) \simeq 1$. Hence, by choice of s, for any $z \in \partial_X B(y, 3s/2)$,

$$G_B\nu_E(z) \asymp \frac{G_B\nu_E(z)}{G_B\nu_F(z)} \asymp \frac{\nu_E(E)}{\nu_F(F)} \asymp \frac{\mathrm{Cap}_B(E)}{\mathrm{Cap}_B(F)} \geq \eta.$$

This proves (20).

Now, fix $x \in U$ such that $B(x, 3r) \subset Y$. For simplicity, write

$$\omega(\,\cdot\,) = \omega_{U\cap B(x,r)}(\,\cdot\,, U \cap \partial_X B(x,r)).$$

Let k be the integer such that $2kw(U) < r < 2(k+1)w(U)$, and pick $s > w(U)$ so close to $w(U)$ that $2ks < r$. We claim that

$$\sup_{U\cap\overline{B(x,r-2js)}} \{\omega\} \leq (1 - A_2\eta)^j \tag{21}$$

for $j = 0, 1, \ldots, k$ with A_2, η as in (20). Note that for $j = k$, (21) yields the inequality stated in this lemma:

$$\omega(x) \leq (1 - A_2\eta)^k \leq \exp(\log((1 - A_2\eta)^{r/(2w(U))})) \leq e^2 \exp\left(\frac{-a_1 r}{w(U)}\right),$$

with $a_1 = -(\log(1 - A_2\eta))/2$.

Inequality (21) is proved by induction, starting with the trivial case $j = 0$. Assume that (21) holds for $j - 1$. By the maximum principle, it suffices to prove

$$\sup_{U\cap\partial_X B(x,r-2js)} \{\omega\} \leq (1 - A_2\eta)^j. \tag{22}$$

Let $y \in U \cap \partial_X B(x, r - 2js))$. Then $\overline{B(y,2s)} \subset \overline{B(x, r - 2(j-1)s)}$ so that the induction hypothesis implies that

$$\omega \leq (1 - A_2\eta)^{j-1} \quad \text{on} \quad U \cap \overline{B(y,2s)}.$$

Since ω vanishes (quasi-everywhere) on $(\partial_X U) \cap B(x,r) \supset (\partial_X U) \cap \overline{B(y,2s)}$, the mean value property implies that

$$\begin{aligned}\omega(b) &= \int_{\partial_X(U\cap B(y,2s))} \omega(a)\omega_{U\cap B(y,2s)}(b,da)\\ &\leq (1-A_2\eta)^{j-1}\omega_{U\cap B(y,2s)}(b, U\cap \partial_X B(y,2s))\end{aligned}$$

for any $b \in V \cap B(y,2s)$. To estimate

$$u = \omega_{U\cap B(y,2s)}(\,\cdot\,, U\cap \partial_X B(y,2s)),$$

on $U \cap B(y,2s)$, we compare it to

$$v = 1 - G_{B(y,2s)}\nu_E,$$

where, as above, ν_E denotes the equilibrium measure of $E = \overline{B(y,s)} \setminus U$ in $B(y,2s)$. Both functions are L-harmonic in $U \cap B(y,2s)$, and it holds $u \leq v$ on $\partial_X(U \cap B(y,2s))$ quasi-everywhere (in the limit sense). By (20), this implies

$$u \leq v \leq 1 - A_2\eta$$

on $U \cap B(y,s)$. Hence,

$$\omega \leq (1-A_2\eta)^j \quad \text{on} \quad U\cap B(y,s).$$

Since this holds for any $y \in U \cap \partial_X B(x, r-2js)$, (22) is proved. □

Lemma 4.9. *There exists a constant $A_2 \in (0,\infty)$ depending only on D_Y, P_Y, C_0–C_7, c_u, C_u, and an upper bound on C_8R^2, such that for any $0 < r < R \leq R_\xi$ and any $x \in B_\Omega(\xi,r)$, we have*

$$\omega(x, \partial_\Omega B_\Omega(\xi,2r), B_\Omega(\xi,2r)) \leq A_2\frac{V(\xi,r)}{r^2}G_{B_\Omega(\xi,C_\Omega A_3 r)}(x,\xi_{16r}).$$

Here ξ_{16r} is any point in Ω with $d_\Omega(\xi,\xi_{16r}) = 4r$ and

$$d(\xi_{16r}, X\setminus\Omega) = d(\xi_{16r}, X\setminus Y') \geq 2c_u r.$$

A similar estimate holds for the L^*-harmonic measure ω^*.

Proof. We follow [1, Lemma 2] and [15, Lemma 4.14]. Recall that $A_3 \geq 2(12 + 12C_u)$ so that all (c_u, C_u)-inner uniform paths connecting two points in $B_\Omega(\xi, 12r)$ stay in $B_\Omega(\xi, A_3r/2)$. Recall that $Y' = B_\Omega(\xi, A_0r)$, where $A_0 = A_3 + 7$. For any $z \in B_\Omega(\xi, A_3r)$, set

$$G'(z) = G_{B_\Omega(\xi,A_3r)}(z,\xi_{16r}).$$

Let $s = \min\{c_u r, 5r/C_u\}$. Since

$$B_\Omega(\xi_{16r}, s) \subset B_\Omega(\xi, A_3 r) \setminus B_\Omega(\xi, 2r),$$

the maximum principle yields

$$\forall y \in B_\Omega(\xi, 2r), \quad G'(y) \leq \sup_{z \in \partial_\Omega B_\Omega(\xi_{16r}, s)} G'(z).$$

Lemma 3.12 and the volume doubling condition yield

$$\sup_{z \in \partial_\Omega B_\Omega(\xi_{16r}, s)} G'(z) \leq C \frac{r^2}{V(\xi, r)},$$

for some constant $C > 0$. Hence, there exists $\epsilon_1 > 0$ such that

$$\forall y \in B_\Omega(\xi, 2r), \quad \epsilon_1 \frac{V(\xi, r)}{r^2} G'(y) \leq e^{-1}.$$

Write

$$B_\Omega(\xi, 2r) = \bigcup_{j \geq 0} U_j \cap B_\Omega(\xi, 2r),$$

where

$$U_j = \left\{ x \in Y' \colon \exp(-2^{j+1}) \leq \epsilon_1 \frac{V(\xi, r)}{r^2} G'(x) < \exp(-2^j) \right\}.$$

Let $V_j = \left(\bigcup_{k \geq j} U_k \right)$. We claim that

$$w_\eta(V_j \cap B_\Omega(\xi, 2r)) \leq A_4 r \exp\left(\frac{-2^j}{\sigma} \right) \tag{23}$$

for some constants $A_4, \sigma \in (0, \infty)$.

Suppose $x \in V_j$. Observe that for $z \in \partial_\Omega B_\Omega(\xi_{16r}, s)$, by the inner uniformity of the domain, the length of the Harnack chain of balls in $B_\Omega(\xi, A_3 r) \setminus \{\xi_{16r}\}$ connecting x to z is at most $A_5 \log(1 + A_6 r / d(x, X \setminus Y'))$ for some constants $A_5, A_6 \in (0, \infty)$. Hence, there are constants $\epsilon_2, \epsilon_3, \sigma$ such that

$$\begin{aligned} \exp(-2^j) > \epsilon_1 \frac{V(\xi, r)}{r^2} G'(x) &\geq \epsilon_2 \frac{V(\xi, r)}{r^2} G'(z) \left(\frac{d(x, X \setminus Y')}{r} \right)^\sigma \\ &\geq \epsilon_3 \left(\frac{d(x, X \setminus Y')}{r} \right)^\sigma. \end{aligned}$$

The last inequality is obtained by applying Lemma 3.12 with $R = A_3 r$ and $\delta = 5/A_3$. Now we have that for any $x \in V_j \cap B_\Omega(\xi, 2r)$,

$$d(x, X \setminus V_j) \leq d(x, X \setminus Y') \leq \left(\epsilon_3^{-1/\sigma} \exp\left(\frac{-2^j}{\sigma} \right) r \right) \wedge 2r.$$

This together with Lemma 4.7 yields (23).

Let $R_0 = 2r$ and for $j \geq 1$,

$$R_j = \left(2 - \frac{6}{\pi^2}\sum_{k=1}^{j}\frac{1}{k^2}\right)r.$$

Then $R_j \downarrow r$ and

$$\begin{aligned}\sum_{j=1}^{\infty}\exp\left(2^{j+1} - \frac{a_1(R_{j-1}-R_j)}{A_4 r\exp(-2^j/\sigma)}\right) &= \sum_{j=1}^{\infty}\exp\left(2^{j+1} - \frac{6a_1}{A_4\pi^2}j^{-2}\exp\left(\frac{2^j}{\sigma}\right)\right)\\ &\leq \sum_{j=1}^{\infty}\exp\left(2^{j+1} - \frac{3a_1}{C_\Omega A_4\pi^2}j^{-2}\exp\left(\frac{2^j}{\sigma}\right)\right)\\ &< C < \infty.\end{aligned}\tag{24}$$

Let $\omega_0 = \omega(\,\cdot\,, \partial_\Omega B_\Omega(\xi, 2r), B_\Omega(\xi, 2r))$ and

$$d_j = \begin{cases}\sup\left\{\dfrac{r^2\omega_0(x)}{V(\xi, r)G'(x)} : x \in U_j \cap B_\Omega(\xi, R_j)\right\}, & \text{if } U_j \cap B_\Omega(\xi, R_j) \neq \emptyset,\\ 0, & \text{if } U_j \cap B_\Omega(\xi, R_j) = \emptyset.\end{cases}$$

Since the sets $U_j \cap B_\Omega(\xi, 2r)$ cover $B_\Omega(\xi, 2r)$ and $B_\Omega(\xi, r) \subset B_\Omega(\xi, R_k)$ for each k, to prove Lemma 4.9, it suffices to show that

$$\sup_{j\geq 0} d_j \leq A_2 < \infty$$

where A_2 is as in Lemma 4.9.

We proceed by iteration. Since $\omega_0 \leq 1$, we have by definition of U_0,

$$d_0 = \sup_{U_0\cap B_\Omega(\xi,2r)}\frac{r^2\omega_0(x)}{V(\xi, r)G'(x)} \leq \epsilon_1 e^2.$$

Let $j > 0$. For $x \in U_{j-1} \cap B_\Omega(\xi, R_{j-1})$, we have by definition of d_{j-1} that

$$\omega_0(x) \leq d_{j-1}\frac{V(\xi, r)}{r^2}G'(x).$$

Also, $\omega_0 \leq 1$. Thus, the maximum principle yields that, for $x \in V_j \cap B_\Omega(\xi, R_j)$,

$$\omega_0(x) \leq \omega(x, V_j \cap \partial_X B_\Omega(\xi, R_{j-1}), V_j \cap B_\Omega(\xi, R_{j-1})) + d_{j-1}\frac{V(\xi, r)}{r^2}G'(x).\tag{25}$$

For $x \in V_j \cap B_\Omega(\xi, R_j)$, let $D = B(p(x), C_\Omega^{-1}(R_{j-1} - R_j))$ and let D' be the connected component of $p^{-1}(D \cap \overline{\Omega})$ that contains x. Then by Lemma 3.7,

$$D' \cap \Omega \subset B_\Omega(x, R_{j-1} - R_j) \subset B_\Omega(\xi, R_{j-1}),$$

hence $D' \cap \Omega \cap V_j \cap \partial_X B_\Omega(\xi, R_{j-1}) = \emptyset$. Thus, the first term on the right hand side of (25) is not greater than

$$\begin{aligned}
&\omega\left(x, V_j \cap D' \cap \partial_X B\left(p(x), \frac{R_{j-1} - R_j}{C_\Omega}\right), V_j \cap D' \cap B\left(p(x), \frac{R_{j-1} - R_j}{C_\Omega}\right)\right) \\
&\le \exp\left(2 - \frac{a_1}{C_\Omega} \frac{R_{j-1} - R_j}{w_\eta(V_j \cap D')}\right) \\
&\le \exp\left(2 - \frac{a_1}{C_\Omega} \frac{R_{j-1} - R_j}{w_\eta(V_j)}\right) \\
&\le \exp\left(2 - \frac{a_1}{C_\Omega A_4} \exp\left(\frac{2^j}{\sigma}\right) \frac{R_{j-1} - R_j}{r}\right) \\
&\le \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\sigma}\right)\right)
\end{aligned}$$

by Lemma 4.8, monotonicity of $U \mapsto w_\eta(U)$ and (23). Here $\epsilon_6 = 6a_1/(\pi^2 A_4 C_\Omega)$. Moreover, by definition of U_j,

$$\epsilon_1 \frac{V(\xi, r)}{r^2} G'(x) \ge \exp(-2^{j+1})$$

for $x \in U_j$. Hence, for $x \in U_j \cap B_\Omega(\xi, R_j)$, (25) becomes

$$\begin{aligned}
\omega_0(x) &\le \exp\left(2 - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\sigma}\right)\right) + d_{j-1} \frac{V(\xi, r)}{r^2} G'(x) \\
&\le \left(\epsilon_1 \exp\left(2 + 2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\sigma}\right)\right) + d_{j-1}\right) \frac{V(\xi, r)}{r^2} G'(x).
\end{aligned}$$

Dividing both sides by $(V(\xi, r)/r^2) G'(x)$ and taking the supremum over all points $x \in U_j \cap B_\Omega(\xi, R_j)$,

$$d_j \le \epsilon_1 \exp\left(2 + 2^{j+1} - \epsilon_6 j^{-2} \exp\left(\frac{2^j}{\sigma}\right)\right) + d_{j-1},$$

and hence for every integer $i > 0$,

$$d_i \le \epsilon_1 e^2 \left(1 + \sum_{j=1}^{\infty} \exp\left(2^{j+1} - \frac{6a_1}{\pi^2 A_4 C_\Omega} j^{-2} \exp\left(\frac{2^j}{\sigma}\right)\right)\right) = \epsilon_1 e^2 (1 + C) < \infty$$

by (24). □

Proof of Theorem 4.1. We follow [15, Theorem 4.5] and [1, Lemma 3]. Recall that $A_0 = A_3 + 7 = 2(12 + 12C_u) + 7$. Fix $\xi \in \partial_{\tilde{\Omega}} \Omega$ with $R_\xi > 0$, let $0 < r < R \le$

$\inf\{R_{\xi'} : \xi' \in B_\Omega(\xi, 7R_\xi) \setminus \Omega\}$ and set $Y' = B_\Omega(\xi, A_0 r)$. Note that any two points in $B_\Omega(\xi, 12r)$ can be connected by a (c_u, C_U)-inner uniform path that stays in $B_\Omega(\xi, A_3 r/2)$.

Fix $x^* \in B_\Omega(\xi, r)$, $y^* \in \partial_\Omega B_\Omega(\xi, 6r)$ such that $c_1 r \leq d(x^*, \partial_{\tilde{\Omega}}\Omega) \leq r$ and $6c_0 r \leq d(y^*, \partial_{\tilde{\Omega}}\Omega) \leq 6r$, for some constants $c_0, c_1 \in (0, 1)$ depending on c_u and C_u. Existence of x^* and y^* follows from the inner uniformity of Ω. It suffices to show that for all $x \in B_\Omega(\xi, r)$ and $y \in \partial_\Omega B_\Omega(\xi, 6r)$ we have

$$G_{Y'}(x, y) \asymp \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*). \tag{26}$$

Fix $y \in \partial_\Omega B_\Omega(\xi, 6r)$, and call u (v, respectively) the left(right)-hand side of (26), viewed as a function of x. Then u is positive and L^*-harmonic in $Y' \setminus \{y\}$, whereas v is positive and L^*-harmonic in $Y' \setminus \{y^*\}$. Both functions vanish quasi-everywhere on the boundary of Y'.

Since $y^* \in \partial_\Omega B_\Omega(\xi, 6r)$ and $6c_0 r \leq d(y^*, \partial_{\tilde{\Omega}}\Omega) \leq 6r$, it follows that the ball $B_\Omega(y^*, 3c_0 r)$ is contained in $B_\Omega(\xi, 9r) \setminus B_\Omega(\xi, 3r)$. Let $z \in \partial_\Omega B_\Omega(y^*, c_0 r)$. By a repeated use of Harnack inequality (a finite number of times, depending only on c_u and C_u), one can compare the value of v at z and at x^*, so that by Lemma 3.12 (notice that $d(x^*, y) \geq c_1 r$) and the volume doubling property,

$$v(z) \leq C v(x^*) = C G_{Y'}(x^*, y) \leq C' \frac{r^2}{V(\xi, r)}.$$

Now, if $y \in B_\Omega(y^*, 2c_0 r)$, then by Lemma 3.12 (notice that $d_\Omega(z, y) \leq 3r \leq A_0 r/(6C_u)$ and $z, y \in B_\Omega(\xi, A_0 r/6)$) and the volume doubling property,

$$u(z) = G_{Y'}(z, y) \geq c \frac{r^2}{V(\xi, r)},$$

so that we have $u(z) \geq c' v(z)$ in this case for some $c' > 0$. If instead $y \in \Omega \setminus B_\Omega(y^*, 2c_0 r)$, then we can connect z and x^* by a path of length comparable to r that stays away (at scale r) from both $\partial_{\tilde{\Omega}}\Omega$ and the point y. Hence, the Harnack inequality implies that $u(z) \asymp u(x^*) = v(x^*) \asymp v(z)$ in this case. This shows that we always have

$$u(z) \geq \epsilon_3 v(z) \quad \forall z \in \partial_\Omega B_\Omega(y^*, c_0 r).$$

By the maximum principle, we obtain

$$u \geq \epsilon_3 v \quad \text{on} \quad Y' \setminus B_\Omega(y^*, c_0 r).$$

Since $B_\Omega(\xi, r) \subset Y' \setminus B_\Omega(y^*, c_0 r)$, we have proved that $u \geq \epsilon_3 v$ on $B_\Omega(\xi, r)$, that is,

$$G_{Y'}(x, y) \geq \epsilon_3 \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*) \tag{27}$$

for all $x \in B_\Omega(\xi, r)$ and $y \in \partial_\Omega B_\Omega(\xi, 6r)$. This is one half of (26).

We now focus on the other half of (26), that is,

$$\epsilon_4 G_{Y'}(x, y) \le \frac{G_{Y'}(x^*, y)}{G_{Y'}(x^*, y^*)} G_{Y'}(x, y^*), \tag{28}$$

for all $x \in B_\Omega(\xi, r)$ and $y \in \partial_\Omega B_\Omega(\xi, 6r)$.

For $x \in B_\Omega(\xi, 2r)$ and $z \in B_\Omega(\xi, 9r) \setminus B_\Omega(\xi, 3r)$, Lemma 3.12 and the volume doubling condition yield

$$G_{Y'}(x, z) \le C \frac{r^2}{V(\xi, r)}.$$

Regarding $G_{Y'}(x, z)$ as L-harmonic function of x, the maximum principle gives

$$G_{Y'}(\,\cdot\,, z) \le C \frac{r^2}{V(\xi, r)} \omega(\,\cdot\,, \partial_\Omega B_\Omega(\xi, 2r), B_\Omega(\xi, 2r)) \quad \text{on} \quad B_\Omega(\xi, 2r).$$

Using Lemma 4.9 (note that $A_0 > A_3$) and the Harnack inequality (to move from ξ_{16r} to y^*), we get for $x \in B_\Omega(\xi, r)$ and $z \in B_\Omega(\xi, 9r) \setminus B_\Omega(\xi, 3r)$, that

$$G_{Y'}(x, z) \le C A_2 \frac{r^2}{V(\xi, r)} \frac{V(\xi, r)}{r^2} G_{Y'}(x, \xi_{16r}) \le C' G_{Y'}(x, y^*), \tag{29}$$

for some constant $C' \in (0, \infty)$. Fix $x \in B_\Omega(\xi, r)$ and $y \in \partial_\Omega B_\Omega(\xi, 6r)$. If $d_\Omega(y, \partial_{\tilde{\Omega}} \Omega) \ge c_0 r/2$, then $G_{Y'}(x, y) \asymp G_{Y'}(x, y^*)$ and $G_{Y'}(x^*, y) \asymp G_{Y'}(x^*, y^*)$ by the Harnack inequality, so that (28) follows. Hence we now assume that $y \in \partial_\Omega B_\Omega(\xi, 6r)$ satisfies $d_\Omega(y, \partial_{\tilde{\Omega}} \Omega) < c_0 r/2$. Let $\xi' \in \partial_{\tilde{\Omega}} \Omega$ be a point such that $d_\Omega(y, \xi') < c_0 r/2$. It follows that $y \in B_\Omega(\xi', r)$. Also,

$$B_\Omega(\xi', 2r) \subset B_\Omega(y, 3r) \subset B_\Omega(\xi, 9r) \setminus B_\Omega(\xi, 3r).$$

We apply inequality (29) to get $G_{Y'}(x, z) \le C_4 G_{Y'}(x, y^*)$ for any $z \in B_\Omega(\xi', 2r)$. Regarding $G_{Y'}(x, y) = G^*_{Y'}(y, x)$ as L^*-harmonic function of y, we obtain

$$G_{Y'}(x, y) \le C_4 G_{Y'}(x, y^*) \omega^*(y, \partial_\Omega B_\Omega(\xi', 2r), B_\Omega(\xi', 2r)). \tag{30}$$

Let us apply Lemma 4.9 with ξ replaced by ξ'. This yields

$$\begin{aligned} \omega^*(y, \partial_\Omega B_\Omega(\xi', 2r), B_\Omega(\xi', 2r)) &\le A_2 \frac{V(\xi', r)}{r^2} G^*_{B_\Omega(\xi', C_\Omega A_3 r)}(y, \xi'_{16r}) \\ &\le A'_2 \frac{V(\xi, r)}{r^2} G_{Y'}(\xi'_{16r}, y), \end{aligned} \tag{31}$$

where $\xi'_{16r} \in \Omega$ is any point such that $d_\Omega(\xi'_{16r}, \xi') = 4r$ and $d(\xi'_{16r}, X \setminus \Omega) \ge 2c_u r$. Observe that we have used the volume doubling property as well as the set monotonicity of the Green function, and that $B_\Omega(\xi', A_3 r) \subset B_\Omega(\xi, A_0 r)$ because $A_0 = A_3 + 7$ and $d_\Omega(\xi, \xi') \le 7r$. Now, (30) and (31) give

$$G_{Y'}(x, y) \le C_5 \frac{V(\xi, r)}{r^2} G_{Y'}(\xi'_{16r}, y) G_{Y'}(x, y^*). \tag{32}$$

By construction, $d_\Omega(\xi'_{16r}, y) \ge d(\xi'_{16r}, \xi') - d_\Omega(\xi', y) \ge 2r$ and $d_\Omega(x^*, y) \ge d_\Omega(\xi, y) - d_\Omega(\xi, x^*) \ge 5r$. Using the inner uniformity of Ω, we find a chain of balls, each of radius $\asymp r$ and contained in $Y' \setminus \{y\}$, going from x^* to ξ'_{16r}, so that the length of the chain is uniformly bounded in terms of c_u, C_u. Applying the Harnack inequality repeatedly thus yields $G_{Y'}(\xi'_{16r}, y) \asymp G_{Y'}(x^*, y)$. As Lemma 3.12 gives $G_{Y'}(x^*, y^*) \asymp r^2/V(\xi, r)$, inequality (32) implies (28). This completes the proof. $\square$

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