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A FAKE HOMFLY POLYNOMIAL OF A KNOT

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Abstract

We introduce a fake HOMFLY polynomial of a knot and show existence of such polynomials of a given knot.

1. Introduction

In this paper, we deal with the HOMFLY polynomial [2, 10, 13] of a knot.

The HOMFLY polynomial $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ of an oriented link L is an invariant of the isotopy type of L, which is defined by the following formulas:
(1) $P(U; v, z) = 1$;
(2) $v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z)$,
where U is the trivial knot and L_+, L_- and L_0 are three links that are identical except near one point where they are as in Fig. 1.

The reduced polynomial $P(L; 1, z)$ of L is called the Conway polynomial [1] of L and denoted by $\nabla(L; z)$. The Jones polynomial $V(L; t)$ [7] of L is defined as the reduced polynomial $P(L; t, t^{1/2} - t^{-1/2})$.

By [10], the HOMFLY polynomial of an oriented knot K is of the form

$$P(K; v, z) = \sum_{j \geq 0} P_{2j}(L; v) z^{2j},$$

where each Laurent polynomial $P_{2j}(K; v) \in \mathbb{Z}[v^{\pm 2}]$ is called the $2j$*-th coefficient polynomial* of $P(K; v, z)$ in z or the $2j$*-th HOMFLY coefficient polynomial*. The $2j$-th coefficient polynomial of a knot is said to be *trivial* if it coincides with that of the trivial knot.

In [11], the author shows that there are infinitely many 2-bridge knots with trivial 0-th HOMFLY coefficient polynomial. The purpose of the paper is to explore a little further into HOMFLY coefficient polynomials of a knot.

Let K and K' be oriented knots. The HOMFLY polynomial $P(K'; v, z)$ of K' is said to be a *fake HOMFLY polynomial of K with identical order* $2m$ if $P(K'; v, z) \equiv$

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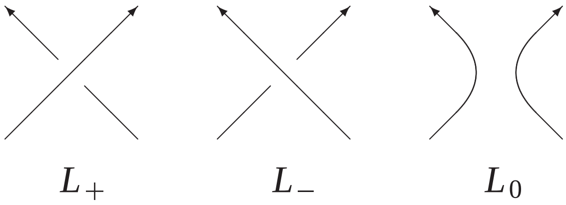

Fig. 1. A skein triple.

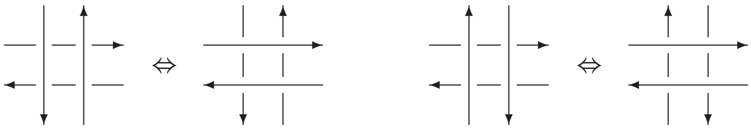

Fig. 2. Pass-moves.

$P(K; v, z) \pmod{z^{2m+2}}$, that is $P_{2j}(K'; v, z) = P_{2j}(K; v, z)$, $0 \leq j \leq m$, in terms of HOMFLY coefficient polynomials of K and K'.

REMARK 1.1. If the HOMFLY polynomial of K' is a fake HOMFLY polynomial of K with identical order $2m$, then it is also a fake HOMFLY polynomial of K with identical order $2j$, $0 \leq j < m$.

In order to state our main theorems we give some definitions.

The Gordian distance from K to K' is defined to be the minimum number of crossing changes needed to transform K into K'. We denote it by $d_G(K, K')$.

A pass-move [8] is a local move on a diagram of an oriented knot as in Fig. 2. Two knots are pass-equivalent if one can be obtained from the other by a combination of Reidemeister moves and pass-moves. If K and K' are pass-equivalent, then the pass distance from K to K' is defined to be the minimum number of pass-moves needed to change K into K'. We denote it by $d_{\text{pass}}(K, K')$.

Let f be an isotopy type invariant of an oriented link, which takes values in an abelian group. Then, f can be uniquely extended to a singular link invariant by the Vassiliev skein relation:

$$f(L_\times) = f(L_+) - f(L_-),$$

where $L_\times$ is a singular link with a double point x, and L_+ and L_- are ones obtained from $L_\times$ by replacing x with a positive and a negative crossing, respectively. We call f a finite type invariant if there exists an integer q such that $f(L) = 0$ for any singular link L with more than q double points. The smallest of such integers is the order of f. We denote a finite type invariant of L with order q by $f_q(L)$.

The signature of K, $\sigma(K)$, is a cobordism invariant of a knot; see [12], Section 8F in [14].

Theorem 1.2 ([9]). *Let m, l and q be integers with $m \geq 0$ and $l, q > 0$. For an oriented knot K, there exist infinitely many knots $\{K_n; n \in \mathbb{N}\}$ with the following properties*:
(1) *The HOMFLY polynomial of K_n is a fake HOMFLY polynomial of K with identical order $2m$*;
(2) $P(K_n; v, z) \equiv P(K; v, z) \bmod (v^{2l} - 1)$;
(3) $\nabla(K_n; z) = \nabla(K; z)$;
(4) *K_n is a band sum of K and the trivial knot, and so K_n is cobordant to K and* $\sigma(K_n) = \sigma(K)$;
(5) $d_G(K_n, K) = 1$;
(6) $d_{\text{pass}}(K_n, K) = 1$;
(7) $f_j(K_n) = f_j(K)$, $0 \leq j \leq q$.

REMARK 1.3. Kawauchi informed the author that Theorem 1.2 could be accomplished by using imitation theory, that is, appropriate almost identical link imitations of a given knot had the properties in the theorem. However, in this paper, the proof of the theorem is given by an elementary method of construction which shows explicitly diagrams of knots.

Corollary 1.4. *For a non-negative integer m, there exist infinitely many knots $\{K_n; n \in \mathbb{N}\}$ whose $2j$-th HOMFLY coefficient polynomials, $0 \leq j \leq m$, are trivial.*

Since an almost identical link imitation of a knot has the same Alexander polynomial as the knot, imitation theory does not work on the following theorem.

Theorem 1.5. *Let m be a non-negative integer and q a positive integer. For an oriented knot K, there exist infinitely many knots $\{K_n; n \in \mathbb{N}\}$ with the following properties*:
(1) *The HOMFLY polynomial of K_n is a fake HOMFLY polynomial of K with identical order $2m$*;
(2) $\nabla(K_n; z) \neq \nabla(K; z)$, *and so K_n is not an imitation of K*;
(3) $d_G(K_n, K) = 1$;
(4) $d_{\text{pass}}(K_n, K) = 1$;
(5) $\sigma(K_n) = \sigma(K)$;
(6) $f_j(K_n) = f_j(K)$, $0 \leq j \leq q$.

To prove theorems, we make use of polynomials derived from a tangle which comes from decomposition of a knot. We introduce them in Section 2. The proofs of theorems are given in Sections 4 and 6 after preliminaries in Sections 3 and 5.

2. The normal coordinates of a tangle

A tangle T is a pair (B^3, t) of a 3-ball B^3 and a proper 1-submanifold t with $\partial t \neq \emptyset$. T is said to be a 2-string tangle if T consists of two arcs and some circle components. Each of ∂t is called an endpoint of T. T is called *properly oriented* if each arc of T is oriented as in Fig. 3.

The numerator (resp. denominator) of T denoted by $N(T)$ (resp. $D(T)$) is a link obtained from T by connecting four endpoints of T by two arcs outside T as in the left (resp. right) figure of Fig. 4.

A tangle T is said to be of *type* $N_{\mu(N(T))}$ (resp. *type* $D_{\mu(D(T))}$) or an $N_{\mu(N(T))}$*-tangle* (resp. a $D_{\mu(D(T))}$*-tangle*) if T is a properly oriented 2-string tangle and $\mu(N(T)) < \mu(D(T))$ (resp. $\mu(N(T) > \mu(D(T)))$, where $\mu(L)$ denotes the number of components of a link L.

We denote by E_{2n}, $n \in \mathbb{Z}$, and E_∞ tangles of type D_1 and of type N_1 as in Fig. 5, respectively. E_{2n} has $2|n|$ positive (resp. negative) crossings if $n > 0$ (resp. $n < 0$) and E_0 means horizontal parallel strings without crossings.

Let $L(T)$, $L(E_0)$ and $L(E_\infty)$ be three links identical outside a ball and inside are a properly oriented 2-string tangle T, the D_1-tangle E_0 and the N_1-tangle E_∞, respectively.

Lemma 2.1. *Let T be a tangle of type $D_{\mu(D(T))}$. Then, there exists a unique pair $(e_0(T; v, z), e_\infty(T; v, z))$ of polynomials for T so that*

$$\begin{aligned} &P(L(T); v, z) \\ &= (vz)^{1-\mu(D(T))}\{e_0(T; v, z)P(L(E_0); v, z) + vze_\infty(T; v, z)P(L(E_\infty); v, z)\}, \end{aligned}$$

where $e_0(T; v, z),\ e_\infty(T; v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$.

Proof. Linear skein theory gives a unique pair $(e_0(T; v, z), e_\infty(T; v, z))$ of polynomials for T so that

$$\begin{aligned} &P(L(T); v, z) \\ &= (vz)^{1-\mu(D(T))}\{e_0(T; v, z)P(L(E_0); v, z) + vze_\infty(T; v, z)P(L(E_\infty); v, z)\}, \end{aligned}$$

where $e_0(T; v, z), e_\infty(T; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. We only have to show that $e_0(T; v, z)$ and $e_\infty(T; v, z)$ are elements of $\mathbb{Z}[v^{\pm 2}, z^2]$. Considering the HOMFLY polynomials of the numerator and the denominator of T, we have

$$P(D(T); v, z) = (vz)^{1-\mu(D(T))}h(D(T); v, z)$$

and

$$P(N(T); v, z) = (vz)^{1-\mu(D(T))}(vz)^{-1}h(N(T); v, z),$$

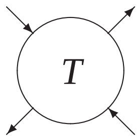

Fig. 3. A properly oriented 2-string tangle.

Fig. 4. The numerator and the denominator.

E_{2n} E_∞

Fig. 5. Trivial tangles.

where $h(D(T); v, z), h(N(T); v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$, because $\mu(N(T)) = \mu(D(T)) + 1$. Thus,

$$h(D(T); v, z) = e_0(T; v, z)P(D(E_0); v, z) + vze_\infty(T; v, z)P(D(E_\infty); v, z)$$

and

$$h(N(T); v, z) = vze_0(T; v, z)P(N(E_0); v, z) + v^2z^2e_\infty(T; v, z)P(N(E_\infty); v, z).$$

Since $D(E_0)$ and $N(E_\infty)$ are trivial knots and $N(E_0)$ and $D(E_\infty)$ are 2-component trivial links, we obtain

$$h(D(T); v, z) = e_0(T; v, z) + (1 - v^2)e_\infty(T; v, z)$$

and

$$h(N(T); v, z) = (1 - v^2)e_0(T; v, z) + v^2z^2e_\infty(T; v, z).$$

From these equalities, we have

$$\{(1 - v^2)^2 - v^2z^2\}e_\infty(T; v, z) = (1 - v^2)h(D(T); v, z) - h(N(T); v, z).$$

It follows that $e_\infty(T; v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$, which leads to $e_0(T; v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$. □

Lemma 2.2. *Let T be a tangle of type $N_{\mu(N(T))}$. Then, there exists a unique pair $(e_0(T; v, z), e_\infty(T; v, z))$ of polynomials for T so that*

$$\begin{aligned} &P(L(T); v, z) \\ &= (vz)^{1-\mu(N(T))}\{vze_0(T; v, z)P(L(E_0); v, z) + e_\infty(T; v, z)P(L(E_\infty); v, z)\}, \end{aligned}$$

where $e_0(T; v, z), e_\infty(T; v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$.

Proof. The proof of the lemma is similar to that of Lemma 2.1. □

The polynomials $e_0(T; v, z)$ and $e_\infty(T; v, z)$ which appear in Lemmas 2.1 or 2.2 are essentially determined by the tangle T only. So, a pair $(e_0(T; v, z), e_\infty(T; v, z))$ of the polynomials is called the *normal coordinates* of T.

Let M_D and M_N be 2×2 matrices whose entries are in $\mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ defined by $\begin{pmatrix} (v^{-1} - v)z^{-1} & vz \\ 1 & 1 - v^2 \end{pmatrix}$ and $\begin{pmatrix} 1 - v^2 & 1 \\ vz & (v^{-1} - v)z^{-1} \end{pmatrix}$, respectively.

Let 3_1 [14] be the trefoil knot. We put $\lambda(v,z) = -(1-v^2)^2 + v^2z^2 = P(3_1!; v,z) - 1$, where $K!$ denotes the mirror image of a knot K.

REMARK 2.3. $\det M_D = \det M_N = -v^{-1}z^{-1}\lambda(v, z) \neq 0$.

The following two lemmas are corollaries of Lemmas 2.1 and 2.2.

Lemma 2.4. *Let T be a tangle of type $D_{\mu(D(T))}$. Then, the normal coordinates $(e_0(T; v, z), e_\infty(T; v, z))$ of T are expressed as follows*:

$$\begin{aligned} \begin{pmatrix} e_0(T; v, z) \\ e_\infty(T; v, z) \end{pmatrix} &= (vz)^{\mu(D(T))-1} M_D^{-1} \begin{pmatrix} P(N(T); v, z) \\ P(D(T); v, z) \end{pmatrix} \\ &= \frac{(vz)^{\mu(D(T))-1}}{\lambda(v, z)} \begin{pmatrix} -v(1 - v^2)zP(N(T); v, z) + v^2z^2P(D(T); v, z) \\ vzP(N(T); v, z) - (1 - v^2)P(D(T); v, z) \end{pmatrix}. \end{aligned}$$

Proof. Let $\nu = (vz)^{\mu(D(T))-1}$. By Lemma 2.1 and equalities $P(N(E_0); v, z) = P(D(E_\infty); v, z) = (v^{-1} - v)z^{-1}$ and $P(N(E_\infty); v, z) = P(D(E_0); v, z) = 1$, we have

$$\begin{aligned} \nu \begin{pmatrix} P(N(T); v, z) \\ P(D(T); v, z) \end{pmatrix} &= \begin{pmatrix} P(N(E_0); v, z) & vzP(N(E_\infty); v, z) \\ P(D(E_0); v, z) & vzP(D(E_\infty); v, z) \end{pmatrix} \begin{pmatrix} e_0(T; v, z) \\ e_\infty(T; v, z) \end{pmatrix} \\ &= M_D \begin{pmatrix} e_0(T; v, z) \\ e_\infty(T; v, z) \end{pmatrix}. \end{aligned}$$

Thus, we obtain the first expression of the normal coordinates of T. Since $M_D^{-1} = (-vz/\lambda(v, z))\begin{pmatrix} 1 - v^2 & -vz \\ -1 & (v^{-1} - v)z^{-1} \end{pmatrix}$, we have the second equality. □

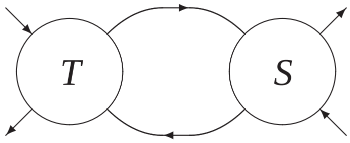

Fig. 6. Addition of tangles.

Lemma 2.5. *Let T be a tangle of type $N_{\mu(N(T))}$. Then, the normal coordinates $(e_0(T; v, z), e_\infty(T; v, z))$ of T are expressed as follows*:

$$\begin{pmatrix} e_0(T; v, z) \\ e_\infty(T; v, z) \end{pmatrix} = (vz)^{\mu(N(T))-1} M_N^{-1} \begin{pmatrix} P(N(T); v, z) \\ P(D(T); v, z) \end{pmatrix}$$
$$= \frac{(vz)^{\mu(N(T))-1}}{\lambda(v, z)} \begin{pmatrix} -(1-v^2)zP(N(T); v, z) + vzP(D(T); v, z) \\ v^2z^2P(N(T); v, z) - v(1-v^2)P(D(T); v, z) \end{pmatrix}.$$

Proof. The proof of the lemma is similar to that of Lemma 2.4. □

Let T and S be properly oriented 2-string tangles. We define addition of tangles T and S by connecting endpoints of T and S as in Fig. 6 and denote it by $T + S$.

If T is a D_1-tangle and S is an N_1-tangle, then $T + S$ is an N_1-tangle.

Lemma 2.6. *Let T be a D_1-tangle and S an N_1-tangle. Let $(e_0(T;v,z), e_\infty(T;v,z))$ and $(e_0(S; v, z), e_\infty(S; v, z))$ be the normal coordinates of T and S, respectively. Then, the normal coordinates $(e_0(T+S;v,z), e_\infty(T+S;v,z))$ of the tangle $T+S$ are expressed as follows*:

$$e_0(T + S; v, z) = e_0(T; v, z)e_0(S; v, z)$$

and

$$e_\infty(T + S; v, z) = e_0(T; v, z)e_\infty(S; v, z) + v^2z^2e_\infty(T; v, z)e_0(S; v, z)$$
$$+ (1 - v^2)e_\infty(T; v, z)e_\infty(S; v, z).$$

Proof. Note that $L(E_0 + S) = L(S)$ and $L(E_\infty + S)$ is the connected sum of $L(E_\infty)$ and $D(S)$. By using Lemma 2.1, we have

$$\begin{aligned} P(L(T + S); v, z) &= e_0(T; v, z)P(L(E_0 + S); v, z) \\ &\quad + vze_\infty(T; v, z)P(L(E_\infty + S); v, z) \\ &= e_0(T; v, z)P(L(S); v, z) \\ &\quad + vze_\infty(T; v, z)P(D(S); v, z)P(L(E_\infty); v, z). \end{aligned}$$

Since

$$P(D(S); v, z) = vze_0(S; v, z)P(D(E_0); v, z) + e_\infty(S; v, z)P(D(E_\infty); v, z)$$
$$= vze_0(S; v, z) + (v^{-1} - v)z^{-1}e_\infty(S; v, z),$$

we obtain

$$\begin{aligned}
&P(L(T+S); v, z)\\
&= e_0(T; v, z)\{vze_0(S; v, z)P(L(E_0); v, z) + e_\infty(S; v, z)P(L(E_\infty); v, z)\}\\
&\quad + vze_\infty(T; v, z)\{vze_0(S; v, z) + (v^{-1} - v)z^{-1}e_\infty(S; v, z)\}P(L(E_\infty); v, z)\\
&= vze_0(T; v, z)e_0(S; v, z)P(L(E_0); v, z)\\
&\quad + \{e_0(T; v, z)e_\infty(S; v, z) + v^2z^2e_\infty(T; v, z)e_0(S; v, z)\\
&\qquad + (1 - v^2)e_\infty(T; v, z)e_\infty(S; v, z)\}P(L(E_\infty); v, z).
\end{aligned}$$

This completes the proof. □

Lemma 2.7. *Under the same assumption as* Lemma 2.6,

$$\begin{aligned}
P(N(T+S); v, z) = {}&(1 - v^2)e_0(T; v, z)e_0(S; v, z) + e_0(T; v, z)e_\infty(S; v, z)\\
&+ v^2z^2e_\infty(T; v, z)e_0(S; v, z) + (1 - v^2)e_\infty(T; v, z)e_\infty(S; v, z).
\end{aligned}$$

Proof. Since $P(N(E_0); v, z) = (v^{-1} - v)z^{-1}$ and $P(N(E_\infty); v, z) = 1$, Lemma 2.6 shows the claim. □

Proposition 2.8. *Under the same assumption as* Lemma 2.6,

$$P(N(T+S); v, z) = P(D(T); v, z)P(N(S); v, z) + \lambda(v, z)e_\infty(T; v, z)e_0(S; v, z).$$

Proof. Since $P(D(T); v, z) = e_0(T; v, z) + (1 - v^2)e_\infty(T; v, z)$ and $P(N(S); v, z) = (1 - v^2)e_0(S; v, z) + e_\infty(S; v, z)$, by Lemma 2.7,

$$\begin{aligned}
&P(D(T); v, z)P(N(S); v, z)\\
&= (1 - v^2)e_0(T; v, z)e_0(S; v, z) + e_0(T; v, z)e_\infty(S; v, z)\\
&\quad + (1 - v^2)^2e_\infty(T; v, z)e_0(S; v, z) + (1 - v^2)e_\infty(T; v, z)e_\infty(S; v, z)\\
&= P(N(T+S); v, z) + \{(1 - v^2)^2 - v^2z^2\}e_\infty(T; v, z)e_0(S; v, z),
\end{aligned}$$

completing the proof. □

Combining Proposition 2.8 with Lemmas 2.4 and 2.5, we easily obtain the following.

Corollary 2.9. *Let T be a D_1-tangle and S an N_1-tangle. Then,*

$$\begin{aligned}&\lambda(v, z)P(N(T+S); v, z)\\&= -v(1-v^2)z\{P(N(T); v, z)P(N(S); v, z) + P(D(T); v, z)P(D(S); v, z)\}\\&\quad + v^2z^2\{P(N(T); v, z)P(D(S); v, z) + P(D(T); v, z)P(N(S); v, z)\}.\end{aligned}$$

Lemma 2.10. *Let T and S be tangles of type D_1. Let $(e_0(T; v, z), e_\infty(T; v, z))$ and $(e_0(S; v, z), e_\infty(S; v, z))$ be normal coordinates of T and S, respectively. Then, the normal coordinates $(e_0(T+S; v, z), e_\infty(T+S; v, z))$ of the D_1-tangle $T+S$ are expressed as follows*:

$$e_0(T+S; v, z) = e_0(T; v, z)e_0(S; v, z)$$

and

$$\begin{aligned}e_\infty(T+S; v, z) &= e_0(T; v, z)e_\infty(S; v, z)\\&\quad + e_\infty(T; v, z)\{e_0(S; v, z) + (1-v^2)e_\infty(S; v, z)\}.\end{aligned}$$

Proof. Since $L(E_0+S) = L(S)$, and $L(E_\infty+S)$ is the connected sum of $L(E_\infty)$ and $D(S)$, by Lemma 2.1, we have

$$\begin{aligned}P(L(T+S); v, z) &= e_0(T; v, z)P(L(E_0+S); v, z)\\&\quad + vze_\infty(T; v, z)P(L(E_\infty+S); v, z)\\&= e_0(T; v, z)P(L(S); v, z)\\&\quad + vze_\infty(T; v, z)P(D(S); v, z)P(L(E_\infty); v, z).\end{aligned}$$

Since $P(L(S); v, z) = e_0(S; v, z)P(L(E_0); v, z) + vze_\infty(S; v, z)P(L(E_\infty); v, z)$, the last expression is $e_0(T; v, z)e_0(S; v, z)P(L(E_0); v, z) + vz\{e_0(T; v, z)e_\infty(S; v, z) + e_\infty(T; v, z)P(D(S); v, z)\}P(L(E_\infty); v, z)$. Thus, we obtain

$$e_0(T+S; v, z) = e_0(T; v, z)e_0(S; v, z)$$

and

$$e_\infty(T+S; v, z) = e_0(T; v, z)e_\infty(S; v, z) + e_\infty(T; v, z)P(D(S); v, z).$$

Since

$$\begin{aligned}P(D(S); v, z) &= e_0(S; v, z)P(D(E_0); v, z) + vze_\infty(S; v, z)P((E_\infty); v, z)\\&= e_0(S; v, z) + (1-v^2)e_\infty(S; v, z),\end{aligned}$$

we have the result. □

3. Preliminaries for the proof of Theorem 1.2

Let X and Y be properly oriented 2-string tangles and $R(X, Y)$ the properly oriented 2-string tangle illustrated in Fig. 7.

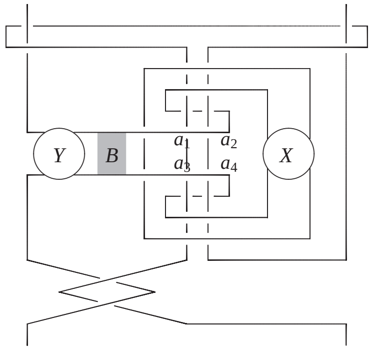

Fig. 7. Tangle $R(X, Y)$.

Note that $N(R(X, Y)) = D(Y + X)$. It is easy to see the following.

Lemma 3.1. *Let U and U_2 be the trivial knot and the 2-component trivial link respectively. Then, $R(E_0, E_0) = E_\infty$, $R(E_0, E_\infty) = E_\infty \sqcup U$, $R(E_\infty, E_\infty) = E_\infty \sqcup U_2$.*

Lemma 3.2. *Let X and Y be tangles of type D_1. Let $(e_0(X; v, z), e_\infty(X; v, z))$, $(e_0(Y; v, z), e_\infty(Y; v, z))$ and $(e_0(R(E_\infty, E_0); v, z), e_\infty(R(E_\infty, E_0); v, z))$ be the normal coordinates of X, Y and $R(E_\infty, E_0)$, respectively. Then, the normal coordinates $(e_0(R(X, Y); v, z), e_\infty(R(X, Y); v, z))$ of the tangle $R(X, Y)$ are expressed as follows*:

$$e_0(R(X, Y); v, z) = e_\infty(X; v, z)e_0(Y; v, z)e_0(R(E_\infty, E_0); v, z)$$

and

$$\begin{aligned} e_\infty(R(X, Y); v, z) = {} & e_0(X; v, z)e_0(Y; v, z) + (1 - v^2)e_0(X; v, z)e_\infty(Y; v, z) \\ & + (1 - v^2)^2 e_\infty(X; v, z)e_\infty(Y; v, z) \\ & + e_\infty(X; v, z)e_0(Y; v, z)e_\infty(R(E_\infty, E_0); v, z). \end{aligned}$$

Proof. By using normal coordinates of X and Y, we have

$$\begin{aligned} P(L(R(X, Y)); v, z) &= e_0(X; v, z)P(L(R(E_0, Y); v, z) \\ &\quad + vze_\infty(X; v, z)P(L(R(E_\infty, Y)); v, z) \\ &= e_0(X; v, z)\{e_0(Y; v, z)P(L(R(E_0, E_0)); v, z) \\ &\qquad + vze_\infty(Y; v, z)P(L(R(E_0, E\infty)); v, z)\} \\ &\quad + vze_\infty(X; v, z)\{e_0(Y; v, z)P(L(E_\infty, E_0)); v, z) \\ &\qquad + vze_\infty(Y; v, z)P(L(R(E_\infty, E_\infty)); v, z)\} \end{aligned}$$

$$\begin{aligned}&= e_0(X; v, z)e_0(Y; v, z)P(L(R(E_0, E_0)); v, z)\\&\quad + vze_0(X; v, z)e_\infty(Y; v, z)P(L(R(E_0, E_\infty)); v, z)\\&\quad + vze_\infty(X; v, z)e_0(Y; v, z)P(L(R(E_\infty, E_0)); v, z)\\&\quad + v^2z^2e_\infty(X; v, z)e_\infty(Y; v, z)P(L(R(E_\infty, E_\infty)); v, z).\end{aligned}$$

By Lemma 3.1, we obtain

$$\begin{aligned}P(L(R(E_0, E_0)); v, z) &= P(L(E_\infty); v, z),\\P(L(R(E_0, E_\infty)); v, z) &= (v^{-1} - v)z^{-1}P(L(R(E_\infty)); v, z)\end{aligned}$$

and

$$P(L(R(E_\infty, E_\infty)); v, z) = \{(v^{-1} - v)z^{-1}\}^2P(L(R(E_\infty)); v, z).$$

Since $R(E_\infty, E_0)$ is an N_2-tangle, we see that

$$\begin{aligned}P(L(R(E_\infty, E_0)); v, z) &= e_0(R(E_\infty, E_0); v, z)P(L(E_0); v, z)\\&\quad + (vz)^{-1}e_\infty(R(E_\infty, E_0); v, z)P(L(E_\infty); v, z).\end{aligned}$$

From these equalities, we find that $P(L(R(X, Y)); v, z)$ is equal to

$$\begin{aligned}&vze_\infty(X; v, z)e_0(Y; v, z)e_0(R(E_\infty, E_0); v, z)P(L(E_0); v, z)\\&+ \{e_0(X; v, z)e_0(Y; v, z)\\&\quad + (1 - v^2)e_0(X; v, z)e_\infty(Y; v, z)\\&\quad + (1 - v^2)^2e_\infty(X; v, z)e_\infty(Y; v, z)\\&\quad + e_\infty(X; v, z)e_0(Y; v, z)e_\infty(R(E_\infty, E_0); v, z)\}P(L(E_\infty); v, z).\end{aligned}$$

Since $R(X, Y)$ is an N_1-tangle, we obtain the claim. □

The proof of the following lemma is straightforward.

Lemma 3.3.

$$P(N(R(E_\infty, E_0)); v, z) = (v^{-1} - v)z^{-1}$$

and

$$\begin{aligned}P(D(R(E_\infty; E_0)); v, z) = v^{-2}(v^{-1} - v)^2\{&v^2z^{-2} + 2(v^{-1} - v)^2\\&+ (5v^{-2} + 12 + 5v^2)z^2\\&+ (4v^{-2} - 13 + 4v^2)z^4\\&+ (v^{-2} - 6 + v^2)z^6 - z^8\}.\end{aligned}$$

Lemma 3.4.

$$e_0(R(E_\infty, E_0); v, z) = -v^{-2}(v^{-1} - v)^2 z^2 (1 + z^2)^2 (2 + z^2).$$

Proof. Note that $R(E_\infty, E_0)$ is an N_2-tangle. Put $T = R(E_\infty, E_0)$. By Lemma 2.5, we have

$$e_0(T; v, z) = \frac{vz}{\lambda(v, z)}\{-(1 - v^2)P(N(T); v, z) + vzP(D(T); v, z)\}.$$

Lemma 3.3 ensures the claim. □

For a properly oriented 2-string tangle T, we denote by $T^\perp$ the properly oriented 2-string tangle obtained from T by reversing orientations of strings after rotating through angle $\pi/2$ in the anticlockwise direction about an axis perpendicular to the projective plane. We call it the *rotation* of T.

Lemma 3.5. *Let* $(e_0(T; v, z), e_\infty(T; v, z))$ *and* $(e_0(T^\perp; v, z), e_\infty(T^\perp; v, z))$ *be the normal coordinates of an* N_1*-tangle* T *and its rotation* $T^\perp$*. Then,*

$$(e_0(T^\perp; v, z), e_\infty(T^\perp; v, z)) = (e_\infty(T; v, z), e_0(T; v, z)).$$

Proof. Let Ψ be a resolution tree for $L(T)$ obtained by switching or smoothing crossings of T. We may assume that each end node $L(S_j)$, $1 \le j \le n$, of Ψ is disjoint union of $L(E_0)$ or $L(E_\infty)$ with some circle components. Then, we obtain a resolution tree for $L(T^\perp)$ from Ψ by replacing each end node $L(S_j)$ of Ψ with $L(S_j^\perp)$. From the two resolution trees, we see that the HOMFLY polynomials of $L(T)$ and $L(T^\perp)$ can be written as

$$P(L(T); v, z) = \sum_{j=1}^{n} h(L(S_j); v, z)P(L(S_j); v, z)$$

and

$$P(L(T^\perp); v, z) = \sum_{j=1}^{n} h(L(S_j); v, z)P(L(S_j^\perp); v, z),$$

where $h(L(S_j); v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. Thus, if

$$P(L(T); v, z) = g_0(L(T); v, z)P(L(E_0); v, z) + g_\infty(L(T); v, z)P(L(E_\infty); v, z),$$

where $g_0(L(T); v, z), g_\infty(L(T); v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, then we obtain

$$P(L(T^\perp); v, z) = g_\infty(L(T); v, z)P(L(E_0); v, z) + g_0(L(T); v, z)P(L(E_\infty); v, z).$$

Since the tangle $T^\perp$ is of type D_1, we have the claim. □

Let X_0 be an N_1-tangle and $\{Y_n;\, n \in \mathbb{N}\}$ a set of D_1-tangles. We define a sequence $\{W_n;\, n \in \mathbb{N}\}$ of N_1-tangles by the following recursive formulas:
(1) $W_1 = R(X_0^{\perp}, Y_1)$;
(2) $W_n = R(W_{n-1}^{\perp}, Y_n)$, $n \geq 2$.

Proposition 3.6. *For any positive integer n,*

$$e_0(W_n; v, z) = e_0(X_0; v, z)\left(\prod_{j=1}^{n} e_0(Y_j; v, z)\right) e_0(R(E_\infty, E_0); v, z)^n.$$

Proof. The proof is by an induction on the number n. If $n = 1$, then Lemmas 3.2 and 3.5 show that

$$\begin{aligned} e_0(W_1; v, z) &= e_0(R(X_0^{\perp}, Y_1); v, z) \\ &= e_\infty(X_0^{\perp}; v, z) e_0(Y_1; v, z) e_0(R(E_\infty, E_0); v, z) \\ &= e_0(X_0; v, z) e_0(Y_1; v, z) e_0(R(E_\infty, E_0); v, z). \end{aligned}$$

Thus, the claim is true. Suppose that $n > 1$. By Lemmas 3.2 and 3.5 and the inductive hypothesis, we have

$$\begin{aligned} e_0(W_n; v, z) &= e_0(R(W_{n-1}^{\perp}, Y_n); v, z) \\ &= e_\infty(W_{n-1}^{\perp}; v, z) e_0(Y_n; v, z) e_0(R(E_\infty, E_0); v, z) \\ &= e_0(W_{n-1}; v, z) e_0(Y_n; v, z) e_0(R(E_\infty, E_0); v, z) \\ &= e_0(X_0; v, z)\left(\prod_{j=1}^{n} e_0(Y_j; v, z)\right) e_0(R(E_\infty, E_0); v, z)^n. \end{aligned}$$

This completes the proof. □

The normal coordinates of the tangle E_{2n}, $n \in \mathbb{Z}$, are the following.

Lemma 3.7. $(e_0(E_{2n}; v, z), e_\infty(E_{2n}; v, z)) = (v^{2n}, (v^{2n} - 1)/(v^2 - 1))$.

Proof. Let $\varepsilon_n = n/|n|$. Since $P(L(E_{2(n-\varepsilon)}); v, z) = e_0(E_{2(n-\varepsilon)}; v, z) P(L(E_0); v, z) + vze_\infty(E_{2(n-\varepsilon)}; v, z) P(L(E_\infty); v, z)$, we have

$$\begin{aligned} P(L(E_{2n}); v, z) &= v^{2\varepsilon_n} P(L(E_{2(n-\varepsilon)}); v, z) + \varepsilon_n v^{\varepsilon_n} z P(L(E_\infty); v, z) \\ &= v^{2\varepsilon_n} e_0(E_{2(n-\varepsilon_n)}; v, z) P(L(E_0); v, z) \\ &\quad + vz\{v^{2\varepsilon_n} e_\infty(E_{2(n-\varepsilon_n)}; v, z) + \varepsilon_n v^{1-\varepsilon_n}\} P(L(E_\infty); v, z). \end{aligned}$$

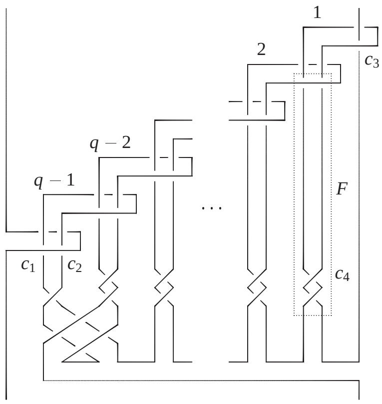

Fig. 8. Tangle H_q.

Thus, we obtain

$$\begin{pmatrix} e_0(E_{2n}; v, z) \\ e_\infty(E_{2n}; v, z) \end{pmatrix} = v^{2n} \begin{pmatrix} e_0(E_0; v, z) \\ e_\infty(E_0; v, z) \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{v^{2n}-1}{v^2-1} \end{pmatrix}.$$

Since $e_0(E_0; v, z) = 1$ and $e_\infty(E_0; v, z) = 0$, we have the desired polynomials. □

Lemmas 3.5 and 3.7 give the normal coordinates of $E_{2n}^\perp$, $n \in \mathbb{Z}$.

Corollary 3.8. $(e_0(E_{2n}^\perp; v, z), e_\infty(E_{2n}^\perp; v, z)) = ((v^{2n}-1)/(v^2-1), v^{2n})$.

Let n and l be positive integers. We denote an n-tuple $(r_1, r_2, \dots, r_n)$ of integers $r_1, r_2, \dots, r_n$ by $\langle r \rangle_n$. We denote by $W_{(n,l,\langle r \rangle_n)}$ the N_1-tangle obtained from W_n by putting $X_0 = E_{2l}^\perp$ and $Y_j = E_{2r_j}$, $1 \le j \le n$.

By Proposition 3.6, Lemmas 3.4 and 3.7, and Corollary 3.8, we have the following.

Lemma 3.9.

$$\begin{aligned} &e_0(W_{(n,l,\langle r \rangle_n)}; v, z) \\ &= (-1)^{n+1} v^{2(\sum_{j=1}^n r_j - 2n)} (v^{2l}-1)(1-v^2)^{2n-1} z^{2n} (1+z^2)^{2n} (2+z^2)^n. \end{aligned}$$

Let T be a properly oriented 2-string tangle and c a crossing on T. We denote by S_cT the tangle obtained from T by switching the crossing c. We also denote by Z_cT the tangle obtained from T by smoothing the crossing c.

Let H_q, $q \ge 3$, be the N_1-tangle depicted in Fig. 8.

Note that $N(H_q)$ is the trivial knot. We also see that $N(S_{c_1}S_{c_2}H_q)$ and $N(S_{c_3}H_q)$ are trivial knots because $S_{c_1}S_{c_2}H_q = E_\infty$ and $S_{c_3}H_q = E_\infty$.

Let K be an oriented knot and T_K a D_1-tangle with $D(T_K) = K$. For a positive integer q, $q \geq 2$, we denote the N_1-tangle $R(H^{\perp}_{q+1}, T_K)$ by $Q_{(K,q)}$. Note that $N(Q_{(K,q)}) = D(T) = K$. We also denote a knot $N(Q^{\perp}_{(K,q)} + W_{(n,l,\langle r\rangle_n)})$ by $K[n,l,\langle r\rangle_n,q]$, where $W_{(n,l,\langle r\rangle_n)}$ denotes the N_1-tangle introduced after Corollary 3.8.

Proposition 3.10.

$$\begin{aligned} &P(K[n, l, \langle r\rangle_n, q]; v, z) - P(K; v, z) \\ &= \lambda(v, z)e_0(R(E_\infty, E_0); v, z)e_0(H_{q+1}; v, z)e_0(T_K; v, z)e_0(W_{(n,l,\langle r\rangle_n)}; v, z). \end{aligned}$$

Proof. By Proposition 2.8, we have

$$\begin{aligned} P(K[n, l, \langle r\rangle_n, q]; v, z) &= P(D(Q^{\perp}_{(K,q)}); v, z)P(N(W_{(n,l,\langle r\rangle_n)}; v, z) \\ &\quad + \lambda(v, z)e_\infty(Q^{\perp}_{(K,q)}; v, z)e_0(W_{(n,l,\langle r\rangle_n)}; v, z) \\ &= P(N(Q_{(K,q)}); v, z)P(N(W_{(n,l,\langle r\rangle_n)}; v, z) \\ &\quad + \lambda(v, z)e_0(Q_{(K,q)}; v, z)e_0(W_{(n,l,\langle r\rangle_n)}; v, z). \end{aligned}$$

Since $P(N(Q_{(K,q)}); v, z) = P(K; v, z)$, $P(N(W_{(n,l,\langle r\rangle_n)}); v, z) = P(N(E^{\perp}_{2n}); v, z) = P(U; v, z) = 1$, and $e_0(Q_{(K,q)}; v, z) = e_0(H_{q+1}; v, z)e_0(T_K; v, z)e_0(R(E_\infty, E_0); v, z)$ from Proposition 3.6, we obtain the result. □

Lemma 3.11. $e_0(T_K; v, z) \neq 0$.

Proof. By Lemma 2.4 and $D(T_K) = K$, we obtain

$$e_0(T_K; v, z) = \frac{1}{\lambda(v, z)}\{-v(1 - v^2)zP(N(T_K); v, z) + v^2z^2P(K; v, z)\}.$$

It follows that $e_0(T_K; 1, z) = \nabla(K; z)$. Since K is a knot, we have $\nabla(K; z) \neq 0$, and thus, $e_0(T_K; v, z) \neq 0$. □

Lemma 3.12. *For* $q \geq 3$, $\nabla(D(H_q); z) = (-1)^q z^{2q-1}$.

Proof. The proof is by an induction on the number q. If $q = 3$, then we see that $\nabla(D(H_3); z) = -z^5$ by a direct calculation. Suppose that $q > 3$. Let c_3 and c_4 be crossings in H_q as in Fig. 8. From recursive formulas for skein triples $(D(S_{c_3}H_q) = U_2, D(H_q), D(Z_{c_3}H_q))$ and $(D(S_{c_4}Z_{c_3}H_q), D(Z_{c_3}H_q), D(Z_{c_4}Z_{c_3}H_q) = U_2)$, we have $\nabla(D(H_q); z) = -z\nabla(D(S_{c_4}Z_{c_3}H_q); z)$ because $\nabla(U_2; z) = 0$. Since $D(S_{c_4}Z_{c_3}H_q)$ is the connected sum of $D(H_q)$ and a diagram of the positive Hopf link, we obtain $\nabla(D(H_q); z) = -z^2\nabla(D(H_{q-1}))$. By the inductive hypothesis, we have the claim. □

Lemma 3.13. *For* $q \geq 3$, $e_0(H_q; v, z) \neq 0$.

Proof. By Lemma 2.5, we have

$$e_0(H_q; v, z) = \frac{1}{\lambda(v, z)}\{-(1 - v^2)P(N(H_q); v, z) + vzP(D(H_q); v, z)\},$$

and thus, $e_0(H_q; 1, z) = z^{-1}P(D(H_q); 1, z) = z^{-1}\nabla(D(H_q); z)$. Lemma 3.12 completes the proof. □

4. Proof of Theorem 1.2

For a 2-variable polynomial $h \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$, we denote the minimal degree of h in z by $\min \deg_z h$.

Let K be an oriented knot. Let m, l and q be integers with $m \geq 0$, $l > 0$ and $q > 1$. Let $\langle r \rangle_m \in \mathbb{Z}^m$.

Proposition 4.1. $K[m, l, \langle r \rangle_m, q]$ *is distinct from* K *and the HOMFLY polynomial of* $K[m, l, \langle r \rangle_m, q]$ *is a fake HOMFLY polynomial of* K *with identical order* $2m$.

Proof. Recall that $\lambda(v, z) \neq 0$. Proposition 3.10 and Lemmas 3.4, 3.9, 3.11 and 3.13 show that $P(K[m, l, \langle r \rangle_m, q]; v, z) \neq P(K; v, z)$. It follows that $K[m, l, \langle r \rangle_m, q]$ is distinct from K. From the definitions of $\lambda(v, z)$ and the normal coordinates of a tangle, we have

$$\min \deg_z \lambda(v, z) = 0, \quad \min \deg_z e_0(H_q; v, z) \geq 0 \quad \text{and} \quad \min \deg_z e_0(T_K; v, z) \geq 0.$$

By Lemmas 3.4 and 3.9, we obtain

$$\min \deg_z e_0(R(E_\infty, E_0); v, z) = 2 \quad \text{and} \quad \min \deg_z e_0(W_{(m, l, \langle r \rangle_m)}; v, z) \geq 2m.$$

From Proposition 3.10, it follows that

$$\min \deg_z (P(K[m, l, \langle r \rangle_m, q]; v, z) - P(K; v, z)) \geq 2m + 2,$$

completing the proof. □

Lemma 4.2. $P(K[m, l, \langle r \rangle_m, q]; v, z) \equiv P(K; v, z) \bmod (v^{2l} - 1)$.

Proof. Since $e_0(W_{(m, l, \langle r \rangle_m)}; v, z) \equiv 0 \bmod (v^{2l} - 1)$ from Lemma 3.9, the claim is given by Proposition 3.10. □

Lemma 4.3. $\nabla(K[m, l, \langle r \rangle_m, q]; z) = \nabla(K; z)$.

Proof. By Lemma 4.2, we have

$$\nabla(K[m, l, \langle r\rangle_m, q]; z) = P(K[m, l, \langle r\rangle_m, q]; 1, z) = P(K; 1, z) = \nabla(K; z). \qquad \square$$

Lemma 4.4. $d_G(K[m, l, \langle r\rangle_m, q], K) = 1$.

Proof. Since the tangle H_{q+1} can be changed into E_∞ by switching the crossing c_3 in H_{q+1} indicated in Fig. 8, the tangle $R(H_{q+1}^{\perp}, T_K)$ becomes the tangle $R(E_0, T_K)$ by a single crossing change. Then, the knot $K[m, l, \langle r\rangle_m, q]$ which is equal to $N(Q_{(K,q)}^{\perp} + W_{(m,l,\langle r\rangle_m)}) = N(R(H_{q+1}^{\perp}, T_K)^{\perp} + W_{(m,l,\langle r\rangle_m)})$ becomes a knot $N(E_0, T_K)^{\perp} + W_{(m,l,\langle r\rangle_m)}) = D(T_K) \# N(W_{(m,l,\langle r\rangle_m)}) = K \# N(W_{(m,l,\langle r\rangle_m)})$. Since $N(W_{(m,l,\langle r\rangle_m)}) = N(E_{2l}^{\perp}) = D(E_{2l}) = U$, we have the result. $\square$

Lemma 4.5. *$K[m, l, \langle r\rangle_m, q]$ is a band sum of K and the trivial knot.*

Proof. Since $N(W_{(m,l,\langle r\rangle_m)}) = U$ and $N(H_{q+1}) = U$, $K[m,l,\langle r\rangle_m,q]$ can be changed into $K \sqcup U$ by a hyperbolic transformation along the band B in the tangle $Q_{(K,q)}$ as in Fig. 7. This completes the proof. $\square$

Lemma 4.6. *$K[m, l, \langle r\rangle_m, q]$ can be changed into K by a single pass-move.*

Proof. Let a_1, a_2, a_3 and a_4 be crossings in the tangle $Q_{(K,q)}$ depicted in Fig. 7. If the four crossings are switched simultaneously, then $K[m, l, \langle r\rangle_m, q]$ is changed into the connected sum $N(W_{(m,l,\langle r\rangle_m)}) \# D(T_K) \# N(H_{q+1})$, which is equivalent to K because $N(W_{(m,l,\langle r\rangle_m)})$ and $N(H_{q+1})$ are trivial knots. Since such an operation is a pass-move, the claim is true. $\square$

Lemma 4.7. *Let f be a finite type invariant with order less than $q + 1$. Then, $f(K[m, l, \langle r\rangle_m, q]) = f(K)$.*

Proof. Let c_1 and c_2 be crossings in H_{q+1} as shown in Fig. 8. Since $S_{c_1} S_{c_2} H_{q+1} = E_\infty$, we see that $S_{c_1} S_{c_2} K[m,l,\langle r\rangle_m,q] = K$. Since switching the crossings c_1 and c_2 can be realized by applying a C_{q+1}-move [5] in H_{q+1}, K and $K[m, l, \langle r\rangle_m, q]$ are C_{q+1}-equivalent. Thus, by [4, 6] we have the result. $\square$

We are now ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. For distinct integers s and t, let $\langle r\rangle_m = (s, 0, \ldots, 0)$ and $\langle r'\rangle_m = (t, 0, \ldots, 0)$, where $\langle r\rangle_m, \langle r'\rangle_m \in \mathbb{Z}^m$. Then, by Lemma 3.9, we obtain

$$\begin{aligned} &e_0(W_{(m,l,\langle r\rangle_m)}; v, z) - e_0(W_{(m,l,\langle r'\rangle_m)}; v, z) \\ &= (-1)^{m+1} v^{-4m} (v^{2s} - v^{2t})(v^{2l} - 1)(1 - v^2)^{2m-1} z^{2m} (1 + z^2)^{2m} (2 + z^2)^m \neq 0. \end{aligned}$$

Thus, by Proposition 3.10 and Lemmas 3.4, 3.11 and 3.13, we have

$$\begin{aligned}&\frac{P(K[m,l,\langle r\rangle_m,q];v,z)-P(K[m,l,\langle r'\rangle_m,q];v,z)}{\lambda(v,z)e_0(R(E_\infty,E_0);v,z)e_0(H_{q+1};v,z)e_0(T_K;v,z)}\\&=e_0(W_{(m,l,\langle r\rangle_m)};v,z)-e_0(W_{(m,l,\langle r'\rangle_m)};v,z)\neq 0.\end{aligned}$$

Let $K_n=K[m,l,\langle r\rangle_m,q]$, where $n\in\mathbb{N}$ and $\langle r\rangle_m=(n,0,\dots,0)\in\mathbb{Z}^m$. Then, K_i and K_j, $i\neq j$, are distinct because of the above argument. Proposition 4.1 and Lemmas 4.2–4.7 show that the knots $\{K_n;\ n\in\mathbb{N}\}$ are desired ones. □

For $\langle r\rangle_m\in\mathbb{Z}^m$, we denote $\sum_{j=1}^m r_j$ by $\|r\|_m$. Then, we have the following.

Lemma 4.8. $P(K[m,l,\langle r\rangle_m,q];v,z)=P(K[m,l,\langle r'\rangle_m,q];v,z)$ *if* $\|r\|_m=\|r'\|_m$.

Proof. By Lemma 3.9 and the assumption of the lemma, we find that

$$e_0(W_{(m,l,\langle r\rangle_m)};v,z)-e_0(W_{(m,l,\langle r'\rangle_m)};v,z)=0.$$

From Proposition 3.10, we obtain the result. □

Conjecture 4.9. $K[m,l,\langle r\rangle_m,q]$ *and* $K[m,l,\langle r'\rangle_m,q]$ *are distinct even though* $\|r\|_m=\|r'\|_m$.

REMARK 4.10. Almost identical link imitations which give a solution of Theorem 1.2 have the same HOMFLY polynomial.

Proposition 4.11. *For an oriented knot* K, *there exist infinitely many knots with the same reduced HOMFLY polynomial at* $z=\sqrt{-2}$ *as* K. *In particular, there exist infinitely many knots* $\{K_n\colon n\in\mathbb{N}\}$ *with* $P(K_n;v,\sqrt{-2})=1$.

Proof. From Lemma 3.9 and Proposition 3.10, it follows that knots in Theorem 1.2 have the same reduced HOMFLY polynomial at $z=\sqrt{-2}$ as K. □

Proposition 4.12. *Let* K *be an oriented knot. Let* p *be a positive integer and* ξ *a primitive* $2p$*-th root of unity. Then, there exist infinitely many knots* $\{K_n;\ n\in\mathbb{N}\}$ *with* $V(K_n;\xi)=V(K;\xi)$.

Proof. We show that knots in Theorem 1.2 have the same value of the Jones polynomial at $t=\xi$ as K. From the second property in Theorem 1.2, there exists a polynomial $g(v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ so that $P(K_n; v, z) - P(K; v, z) = (v^{2p} - 1)g(v, z)$. Thus, we have

$$\begin{aligned} V(K_n; \xi) - V(K; \xi) &= P(K_n; \xi, \xi^{1/2} - \xi^{-1/2}) - P(K; \xi, \xi^{1/2} - \xi^{-1/2}) \\ &= (\xi^{2p} - 1)g(\xi, \xi^{1/2} - \xi^{-1/2}) \\ &= 0, \end{aligned}$$

completing the proof. □

5. Preliminaries for the proof of Theorem 1.5

Let X be a properly oriented 2-string tangle and $R(X)$ the properly oriented 2-string tangle illustrated in Fig. 9.

Note that $N(R(X)) = D(X)$, and that $R(X)$ is an N_1-tangle if X is a D_1-tangle.

The following lemma is easily obtained.

Lemma 5.1. $R(E_0) = E_\infty$.

It is easy to see that $N(R(E_\infty))$ is the trivial 2-component link. Since it is shown that $D(R(E_\infty))$ is the link 9^3_{18} [14] with appropriate orientations, we have the following.

Lemma 5.2. $P(N(R(E_\infty)); v, z) = (v^{-1} - v)z^{-1}$ *and* $P(D(R(E_\infty)); v, z) = (v^{-2} - 2 + v^2)z^{-2} + (-v^{-4} + 3v^{-2} - 3 + v^2) + (-v^{-4} + 3v^{-2} - 2)z^2 + v^{-2}z^4$.

Let $\alpha(v, z) = v^{-2}z^2(1 - v^2 + z^2)$. Since $R(E_\infty)$ is a tangle of type N_2, Lemma 2.5 gives the following.

Lemma 5.3. $(e_0(R(E_\infty); v, z), e_\infty(R(E_\infty); v, z)) = (\alpha(v, z), (1 - v^2)(1 - \alpha(v, z))$.

Lemma 5.4. *Let X be a D_1-tangle and $(e_0(X; v, z), e_\infty(X; v, z))$ the normal coordinates of X. Then, the normal coordinates $(e_0(R(X); v, z), e_\infty(R(X); v, z)$ of the tangle $R(X)$ are expressed as follows*:

$$e_0(R(X); v, z) = e_\infty(X; v, z)e_0(R(E_\infty); v, z)$$

and

$$e_\infty(R(X); v, z) = e_0(X; v, z) + e_\infty(X; v, z)e_\infty(R(E_\infty); v, z).$$

Proof. Since X is a tangle of type D_1, by using the normal coordinates of X, we have

$$P(L(R(X); v, z) = e_0(X; v, z)P(L(R(E_0)); v, z) + vze_\infty(X; v, z)P(L(R(E_\infty)); v, z).$$

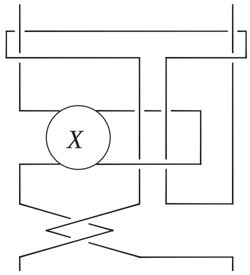

Fig. 9. Tangle $R(X)$.

Lemma 5.1 shows that $P(L(R(E_0)); v, z) = P(L(E_\infty); v, z)$. Since $R(E_\infty)$ is an N_2-tangle, Lemma 2.2 gives

$$P(L(R(E_\infty)); v, z) = e_0(R(E_\infty); v, z)P(L(E_0); v, z) \\ + v^{-1}z^{-1}e_\infty(R(E_\infty); v, z)P(L(E_\infty); v, z).$$

Thus, we obtain

$$P(L(R(X)); v, z) = vze_\infty(X; v, z)e_0(R(E_\infty); v, z)P(L(E_0); v, z) \\ + \{e_0(X; v, z) + e_\infty(X; v, z)e_\infty(R(E_\infty); v, z)\}P(L(E_\infty); v, z).$$

Since $R(X)$ is an N_1-tangle, we have the desired formulas. □

Let q, $q \geq 2$, be an integer. We define a sequence $\{W_{(n,q)}; n \in \mathbb{N}\}$ of N_1-tangles by the following recursive formulas:
(1) $W_{(1,q)} = R(H^{\perp}_{q+1})$;
(2) $W_{(n,q)} = R(W^{\perp}_{(n-1,q)})$, $n \geq 2$.

REMARK 5.5. $N(W_{(n,q)}) = D(H^{\perp}_{q+1}) = N(H_{q+1}) = U$.

Proposition 5.6. *For any positive integer n,*

$$e_0(W_{(n,q)}; v, z) = e_0(H_{q+1}; v, z)e_0(R(E_\infty); v, z)^n.$$

Proof. The proof is by an induction on the number n. If $n = 1$, then Lemmas 3.5 and 5.4 show that

$$e_0(W_{(1,q)}; v, z) = e_\infty(H^{\perp}_{q+1}; v, z)e_0(R(E_\infty); v, z) = e_0(H_{q+1}; v, z)e_0(R(E_\infty); v, z).$$

Thus, the claim is true. Suppose that $n > 1$. By the inductive hypothesis and Lemmas 3.5 and 5.4, we have

$$\begin{aligned} e_0(W_{(n,q)}; v, z) &= e_\infty(W^\perp_{(n-1,q)}; v, z) e_0(R(E_\infty); v, z) \\ &= e_0(W_{(n-1,q)}; v, z) e_0(R(E_\infty); v, z) \\ &= e_0(H_{q+1}; v, z) e_0(R(E_\infty); v, z)^n. \end{aligned}$$

This completes the proof. □

Let K be an oriented knot and T_K a D_1-tangle with $D(T_K) = K$.

Lemma 5.7. *The normal coordinates of the tangle $T_K + E_{2l}$, $l \in \mathbb{Z}$, are*

$$\begin{aligned} &(e_0(T_K + E_{2l}; v, z), e_\infty(T_K + E_{2l}; v, z)) \\ &= \left(v^{2l} e_0(T_K; v, z), \frac{v^{2l} - 1}{v^2 - 1} e_0(T_K; v, z) + e_\infty(T_K; v, z) \right). \end{aligned}$$

Proof. Since Lemma 3.7 gives $e_0(E_{2l}; v, z) + (1 - v^2) e_\infty(E_{2l}; v, z) = 1$, the lemma immediately comes from Lemma 2.10. □

We denote a knot $N((T_K + E_{2l}) + W_{(n,q)})$ by $K[n, q, l]$.

Proposition 5.8.

$$\begin{aligned} &P(K[n, q, l]; v, z) - P(K; v, z) \\ &\quad = \lambda(v, z) e_0(H_{q+1}; v, z) e_0(R(E_\infty); v, z)^n \left(\frac{v^{2l} - 1}{v^2 - 1} e_0(T_K; v, z) + e_\infty(T_K; v, z) \right). \end{aligned}$$

Proof. By Proposition 2.8, we have

$$\begin{aligned} P(K[n, q, l]; v, z) = {} & P(D(T_K + E_{2l}); v, z) P(N(W_{(n,q)}); v, z) \\ & + \lambda(v, z) e_\infty(T_K + E_{2l}; v, z) e_0(W_{(n,q)}; v, z). \end{aligned}$$

Since $D(T_K + E_{2l}) = D(T_K) = K$ and $N(W_{(n,q)}) = U$, the first term on the right-hand side of the above equality is equal to $P(K; v, z)$. Proposition 5.6 and Lemma 5.7 lead to the claim. □

Corollary 5.9.

$$\nabla(K[n, q, l]; z) = \nabla(K; z) + (-1)^{q+1} z^{4n+2q+2} (l \nabla(K; z) + e_\infty(T_K; 1, z)).$$

Proof. Note that

$$e_0(T_K; 1, z) = P(K; 1, z) = \nabla(K; z)$$

and

$$e_0(H_{q+1}; 1, z) = z^{-1}P(D(H_{q+1}); 1, z) = z^{-1}\nabla(D(H_{q+1}); z).$$

By Lemma 5.3, we have the result. □

From the definition of the normal coordinates of a tangle, $e_\infty(T_K; 1, 2\sqrt{-1})$ is an integer.

Lemma 5.10.

$$\frac{\nabla(K[n, q, l]; 2\sqrt{-1})}{\nabla(K; 2\sqrt{-1})} > 1$$

if $l > |e_\infty(T_K; 1, 2\sqrt{-1})/\nabla(K; 2\sqrt{-1})|$.

Proof. Note that $n \geq 1$ and $q \geq 2$. By Corollary 5.9, we obtain

$$\frac{\nabla(K[n, q, l]; 2\sqrt{-1})}{\nabla(K; 2\sqrt{-1})} = 1 + 4^{2n+q+1}\left(l + \frac{e_\infty(T_K; 1, 2\sqrt{-1})}{\nabla(K; 2\sqrt{-1})}\right).$$

Since $|e_\infty(T_K; 1, 2\sqrt{-1})/\nabla(K; 2\sqrt{-1})| + e_\infty(T_K; 1, 2\sqrt{-1})/\nabla(K; 2\sqrt{-1}) \geq 0$, we have the desired inequality. □

6. Proof of Theorem 1.5

Let K be an oriented knot and T_K a D_1-tangle with $D(T_K) = K$. Let m, l and q be integers with $m \geq 0$, $l > l_0 = |e_\infty(T_K; 1, 2\sqrt{-1})/\nabla(K; 2\sqrt{-1})|$ and $q > 1$.

The following is an immediate consequence of Lemma 5.10.

Lemma 6.1. $\nabla(K[m+1, q, l]; z) \neq \nabla(K; z)$, *that is,* $K[m+1, q, l]$ *is distinct from* K.

Proposition 6.2. *The HOMFLY polynomial of* $K[m+1, q, l]$ *is a fake HOMFLY polynomial of* K *with identical order* $2m$.

Proof. Note that Lemma 6.1 shows $P(K[m+1, q, l]; v, z) \neq P(K; v, z)$. From the definitions of $\lambda(v, z)$ and the normal coordinates of a tangle, we obtain four inequalities: $\min\deg_z \lambda(v, z) = 0$, $\min\deg_z e_0(H_{q+1}; v, z) \geq 0$, $\min\deg_z e_0(T_K; v, z) \geq 0$ and $\min\deg_z e_\infty(T_K; v, z) \geq 0$. By Lemma 5.3, we have $\min\deg_z e_0(R(E_\infty); v, z) = 2$. From Proposition 5.8, it follows that

$$\min\deg_z(P(K[m+1, q, l]; v, z) - P(K; v, z)) \geq 2m + 2,$$

completing the proof. □

Lemma 6.3. $d_G(K[m+1, q, l], K) = 1$.

Proof. Since the tangle H_{q+1} can be changed into E_∞ by switching the crossing c_3 in H_{q+1} indicated in Fig. 8, the tangle $R(H_{q+1}^{\perp})$ becomes $R(E_0) = E_\infty$ by a single crossing change. Hence, the tangle $W_{(m+1,q)}$ is changed into E_∞ by a single crossing change, and thus, $K[m+1, q, l]$ becomes $N(T_K + E_{2l} + E_\infty) = D(T_K + E_{2l}) = D(K) = K$. □

The value $\nabla(K; 2\sqrt{-1})$, whose absolute value is equal to the determinant of K, is an integer. By [3], it is known that the signature $\sigma(K)$ of K has the following properties:

(1) $\nabla(K; 2\sqrt{-1})/|\nabla(K; 2\sqrt{-1})| = \sqrt{-1}^{\sigma(K)}$;

(2) $|\sigma(K) - \sigma(K')| \leq 2$,

where K' is a knot obtained from K by switching a crossing of K.

Lemma 6.4. $\sigma(K[m+1, q, l]) = \sigma(K)$.

Proof. Since two integers $\nabla(K[m+1, q, l]; 2\sqrt{-1})$ and $\nabla(K; 2\sqrt{-1})$ have the same signature by Lemma 5.10, we have $\sigma(K[m+1, q, l]) \equiv \sigma(K)(\text{mod } 4)$. Since $d_G(K[m+1, q, l], K) = 1$ by Lemma 6.3, we obtain $|\sigma(K[m+1, q, l]) - \sigma(K)| \leq 2$. These two relations give the claim. □

Lemma 6.5. *$K[m+1, q, l]$ can be changed into K by a single pass-move.*

Proof. We consider the 3-string tangle in the disk F bounded by dotted segments depicted in the tangle H_{q+1} as in Fig. 8. If we apply a Γ-move [8] to the 3-string tangle, then we find that the tangle H_{q+1} can be changed into the tangle E_∞. Thus, $K[m+1, q, l]$ can be changed into K by a single Γ-move as the proof of Lemma 6.3. Since a Γ-move can be accomplished by a combination of Reidemeister moves and a pass-move, we complete the proof. □

Lemma 6.6. *Let f be a finite type invariant with order less than $q+1$. Then, $f(K[m+1, q, l]) = f(K)$.*

Proof. The proof is similar to that of Lemma 4.7. □

Proof of Theorem 1.5. For two integers l and l' with $l' > l > l_0$, by Corollary 5.9, we obtain

$$\nabla(K[m+1, q, l']; z) - \nabla(K[m+1, q, l]; z) = (-1)^{q+1} z^{4(m+1)+2q+2} \nabla(K; z)(l' - l) \neq 0.$$

Hence, $K[m+1, q, l']$ and $K[m+1, q, l]$ are distinct. Proposition 6.2 and Lemmas 6.1 and 6.3–6.6 show that knots $\{K[m+1, q, l];\ l_0 < l \in \mathbb{N}\}$ are desired ones. □

Proposition 6.7. *For an oriented knot K, there exist infinitely many knots with the same reduced HOMFLY polynomial at $z = \sqrt{v^2-1}$ as K. In particular, there exist infinitely many knots $\{K_n;\ n \in \mathbb{N}\}$ with $P(K_n; v, \sqrt{v^2-1}) = 1$.*

Proof. From Lemma 5.3 and Proposition 5.8, it follows that the knots K_n in Theorem 1.5 have the same reduced HOMFLY polynomial at $z = \sqrt{v^2-1}$ as K. □

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