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REPRESENTATION THEOREM FOR HARMONIC BERGMAN AND BLOCH FUNCTIONS

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Abstract

In this paper, we give the representation theorem for harmonic Bergman functions and harmonic Bloch functions on smooth bounded domains. As an application, we discuss Toeplitz operators.

1. Introduction

Let Ω be a smooth bounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$, i.e., for every boundary point $\eta \in \partial\Omega$, there exist a neighborhood V of η in $\mathbb{R}^n$ and a C^∞-diffeomorphism $f\colon V \to f(V) \subset \mathbb{R}^n$ such that $f(\eta) = 0$ and $f(\Omega \cap V) = \{(y_1, \dots, y_n) \in \mathbb{R}^n;\ y_n > 0\} \cap f(V)$. For $1 \leq p < \infty$, we denote by $b^p = b^p(\Omega)$ the harmonic Bergman space on Ω, i.e., the set of all real-valued harmonic functions f on Ω such that $\|f\|_p := \left(\int_\Omega |f|^p\,dx\right)^{1/p} < \infty$, where dx denotes the usual n-dimensional Lebesgue measure on Ω. As is well-known, b^p is a closed subspace of $L^p = L^p(\Omega)$ and hence, b^p is a Banach space (for example see [1]). Especially, when $p = 2$, b^2 is a Hilbert space, which has the reproducing kernel, i.e., there exists a unique symmetric function $R(\,\cdot\,,\,\cdot\,)$ on $\Omega \times \Omega$ such that for any $f \in b^2$ and any $x \in \Omega$,

$$f(x) = \int_\Omega R(x, y) f(y)\,dy. \tag{1}$$

The function $R(\,\cdot\,,\,\cdot\,)$ is called the harmonic Bergman kernel of Ω. When Ω is the open unit ball B, an explicit form is known:

$$R(x, y) = R_B(x, y) = \frac{(n-4)|x|^4|y|^4 + (8x \cdot y - 2n - 4)|x|^2|y|^2 + n}{n|B|(1 - 2x \cdot y + |x|^2|y|^2)^{1+n/2}},$$

where $x \cdot y$ denotes the Euclidean inner product in $\mathbb{R}^n$ and $|B|$ is the Lebesgue measure

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of B. We denote by P the corresponding integral operator

$$P\psi(x) := \int_\Omega R(x,y)\psi(y)\,dy \tag{2}$$

for $x \in \Omega$. It is known that $P\colon L^p \to b^p$ is bounded for $1 < p < \infty$; see Theorem 4.2 in [6].

The following result is shown in [8].

Theorem A. *Let $1 < p < \infty$ and let Ω be a smooth bounded domain. Then we can choose a sequence $\{\lambda_i\}$ in Ω satisfying the following property: For any $f \in b^p(\Omega)$, there exists a sequence $\{a_i\} \in l^p$ such that*

$$f(x) = \sum_{i=1}^{\infty} a_i R(x,\lambda_i) r(\lambda_i)^{(1-1/p)n}, \tag{3}$$

where $r(x)$ denotes the distance between x and $\partial\Omega$.

The equation (3) is called an atomic decomposition of f. The above theorem shows the existence of a sequence $\{\lambda_i\} \subset \Omega$ permitting an atomic decomposition for every $f \in b^p$.

Theorem A does not refer to the case $p = 1$. This deeply comes from the fact that $P : L^1 \to b^1$ is not bounded. In the present paper, we give an atomic decomposition for $p = 1$ by using a modified reproducing kernel $R_1(\cdot\,,\,\cdot)$, introduced in [3].

Theorem 1. *Let $1 \le p < \infty$ and let Ω be a smooth bounded domain. Then we can choose a sequence $\{\lambda_i\}$ in Ω satisfying the following property: For any $f \in b^p(\Omega)$, there exists a sequence $\{a_i\} \in l^p$ such that*

$$f(x) = \sum_{i=1}^{\infty} a_i R_1(x,\lambda_i) r(\lambda_i)^{(1-1/p)n}.$$

Also, we consider the harmonic Bloch space. We define the harmonic Bloch space $\mathcal{B}$ by

$$\mathcal{B} := \{ f : \Omega \to \mathbb{R} : f \text{ is harmonic and } \|f\|_{\mathcal{B}} < \infty\},$$

where

$$\|f\|_{\mathcal{B}} := \sup\{r(x)|\nabla f(x)| : x \in \Omega\}$$

and ∇ denotes the gradient operator $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. Note that $\|\cdot\|_{\mathcal{B}}$ is a semi-norm on $\mathcal{B}$. We fix a reference point $x_0 \in \Omega$. $\mathcal{B}$ can be made into a Banach space by introducing the norm

$$\|f\| := |f(x_0)| + \|f\|_{\mathcal{B}}.$$

Also, $\tilde{\mathcal{B}}$ denotes the space of all Bloch functions f such that $f(x_0) = 0$. Then, $(\tilde{\mathcal{B}}, \|\cdot\|_{\mathcal{B}})$ is a Banach space. Using a kernel

$$\tilde{R}_1(x, y) = R_1(x, y) - R_1(x_0, y),$$

we have the following theorem.

Theorem 2. *Let Ω be a smooth bounded domain. Then we can choose a sequence $\{\lambda_i\}$ in Ω satisfying the following property: For any $f \in \tilde{\mathcal{B}}$, there exists a sequence $\{a_i\} \in l^\infty$ such that*

$$f(x) = \sum_{j=1}^{\infty} a_j \tilde{R}_1(x, \lambda_j) r(\lambda_j)^n.$$

In case that a domain Ω is the unit ball or the upper half space, preceding results are obtained in [5] and [4].

We often abbreviate inessential constants involved in inequalities by writing $X \lesssim Y$, if there exists an absolute constant $C > 0$ such that $X \leq CY$.

2. Preliminaries

In this section, we will introduce some results in [6] and [3]. Those results play important roles in this paper.

First, we introduce some estimates for the harmonic Bergman kernel. These estimates are obtained by H. Kang and H. Koo [6]. We use the following notations. We put $d(x, y) := r(x) + r(y) + |x - y|$ for $x, y \in \Omega$, where $r(x)$ denotes the distance between x and $\partial\Omega$. For an n-tuple $\alpha := (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers, called a multi-index, we denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $D_x^\alpha := (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$. We also use $D_i := \partial/\partial x_i$ and $D_{ij} := \partial^2/\partial x_i \partial x_j$.

Theorem B (H. Kang and H. Koo [6]). *Let α, β be multi-indices.*

(1) *There exists a constant $C > 0$ such that*

$$|D_x^\alpha D_y^\beta R(x, y)| \leq \frac{C}{d(x, y)^{n+|\alpha|+|\beta|}}$$

for every $x, y \in \Omega$.

(2) *There exists a constant $C > 0$ such that*

$$R(x, x) \geq \frac{C}{r(x)^n}$$

for every $x \in \Omega$.

Second, we explain the modified reproducing kernel $R_1(x, y)$ introduced by B.R. Choe, H. Koo and H. Yi [3]. We call $\eta \in C^\infty(\bar{\Omega})$ a defining function if η satisfies the conditions that

$$\Omega = \{x \in \mathbb{R}^n \mid \eta(x) > 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^n \mid \eta(x) = 0\}$$

and $\nabla\eta$ does not vanish on $\partial\Omega$. Here, we choose a defining function η with condition that

$$|\nabla\eta|^2 = 1 + \eta\omega \tag{4}$$

for some $\omega \in C^\infty(\bar{\Omega})$. We can easily construct the above defining function, because $\partial\Omega$ is smooth. Remark that $r(x)$ is comparable to $\eta(x)$.

We define a differential operator K_1 by

$$K_1 f := f - \frac{1}{2}\Delta(\eta^2 f) \tag{5}$$

for $f \in C^\infty$. We also define a kernel $R_1(x, y)$ by

$$R_1(x, y) := K_1(R_x)(y)$$

for $x, y \in \Omega$, where $R_x(y) := R(x, y)$, and denote by P_1 the corresponding integral operator

$$P_1 f(x) := \int_\Omega R_1(x, y) f(y)\, dy.$$

We call $R_1(x, y)$ the modified reproducing kernel. This kernel satisfies the reproducing property and has the following estimates.

Theorem C (B.R. Choe, H. Koo and H. Yi [3]). *Let Ω be a smooth bounded domain. Then*

(1) *R_1 has the reproducing property, i.e., $P_1 f = f$ for $f \in b^1$.*

(2) *Let α be multi-index. Then there exists $C > 0$ such that for $x, y \in \Omega$*

$$|D_x^\alpha R_1(x, y)| \leq C \frac{r(y)}{d(x, y)^{n+1+|\alpha|}} \tag{6}$$

and

$$|\nabla_y R_1(x, y)| \leq \frac{C}{d(x, y)^{n+1}}. \tag{7}$$

(3) *$P_1 \colon L^p \to b^p$ is bounded for $1 \leq p < \infty$.*

Finally, we prepare some lemmas.

Lemma 2.1 (Lemma 4.1 in [6]). *Let s be a nonnegative real number and $t < 1$. If $s + t > 0$, then there exists a constant $C > 0$ such that*

$$\int_\Omega \frac{dy}{d(x, y)^{n+s} r(y)^t} \le \frac{C}{r(x)^{s+t}}$$

for every $x \in \Omega$.

We define the associated integral operator I_s by

$$I_s f(x) := \int_\Omega \frac{r(y)^s}{d(x, y)^{n+s}} f(y)\, dy.$$

Lemma 2.2. *If $s = 0$, then $I_s\colon L^p \to L^p$ is bounded for $1 < p < \infty$ and if $s > 0$, then $I_s\colon L^p \to L^p$ is bounded for $1 \le p < \infty$.*

Proof. When $s \ge 0$ and $1 < p < \infty$, the L^p-boundedness of I_s is shown by Schur's test; see Lemma 2.6 in [8]. We have only to show that $I_s\colon L^1 \to L^1$ is bounded for $s > 0$. By Lemma 2.1, we have

$$\begin{aligned} \|I_s f\|_{L^1} &\le \int_\Omega \int_\Omega \frac{r(y)^s}{d(x, y)^{n+s}} |f(y)|\, dy\, dx \\ &\le \int_\Omega |f(y)| r(y)^s \int_\Omega \frac{1}{d(x, y)^{n+s}}\, dx\, dy \\ &\le C\|f\|_{L^1}. \end{aligned}$$

This completes the proof. □

3. Representation theorem for harmonic Bergman functions

In this section, we give a proof of Theorem 1. We need to take sequences $\{\lambda_i\}_i \subset \Omega$ with the following property in the same similar way in [8].

Lemma 3.1. *There exists a number $c > 0$ such that for each $0 < \delta < 1/4$, we can choose a sequence $\{\lambda_i\}_i \subset \Omega$ and a disjoint covering $\{E_i\}_i$ of Ω satisfying the following conditions*:

(a) *E_i is measurable for each $i \in \mathbb{N}$ and $\{E_i\}_i$ are mutually disjoint*;

(b) *$B(\lambda_i, c\delta r(\lambda_i)) \subset E_i \subset B(\lambda_i, \delta r(\lambda_i))$ for each $i \in \mathbb{N}$.*

In what follow, $\{\lambda_i\}_i$, $\{E_i\}_i$ are taken in Lemma 3.1. We define operators A, U and S as follows:

$$A\{a_i\}(x) := \sum_{i=1}^{\infty} a_i R_1(x, \lambda_i) r(\lambda_i)^{(1-1/p)n}, \tag{8}$$

$$Uf := \{|E_i| f(\lambda_i) r(\lambda_i)^{-(1-1/p)n}\}_i, \tag{9}$$

and

$$Sf(x) := \sum_{i=1}^{\infty} R_1(x, \lambda_i) f(\lambda_i)|E_i|. \tag{10}$$

Theorem 1 means that A: $l^p \to b^p$ is onto for $1 \leq \infty$. First, we show the boundedness of the operators A, U and S.

Lemma 3.2. *Let $1 \leq p < \infty$. Then $U: b^p \to l^p$, $A: l^p \to b^p$ and $S: b^p \to b^p$ are bounded.*

Proof. First, we show that U is bounded. For any $f \in b^p$, by using the condition (b) in Lemma 3.1, we have

$$\begin{aligned}
\|Uf\|_{l^p}^p &= \sum_{i=1}^{\infty} \left| |E_i| f(\lambda_i) r(\lambda_i)^{-(1-1/p)n} \right|^p \\
&\lesssim \sum_{i=1}^{\infty} | f(\lambda_i)|^p r(\lambda_i)^n \\
&\lesssim \sum_{i=1}^{\infty} \int_{E_i} | f(\lambda_i)|^p \, dy \\
&\lesssim \sum_{i=1}^{\infty} \int_{E_i} | f(y)|^p \, dy = \| f\|_p^p.
\end{aligned}$$

Next, we show that A is bounded. For any $\{a_i\} \in l^p$ and any $x \in \Omega$, by Theorem C, we have

$$\begin{aligned}
|A\{a_i\}(x)| &\lesssim \sum_i |a_i| r(\lambda_i)^{(1-1/p)n} \frac{r(\lambda_i)}{d(x, \lambda_i)^{n+1}} \\
&= \sum_i |a_i| r(\lambda_i)^{(1-1/p)n} |E_i|^{-1} \int_{E_i} \frac{r(\lambda_i)}{d(x, \lambda_i)^{n+1}} \, dy \\
&\lesssim \sum_i |a_i| r(\lambda_i)^{(1-1/p)n} |E_i|^{-1} \int_{E_i} \frac{r(y)}{d(x, y)^{n+1}} \, dy \\
&= I_1 g(x),
\end{aligned}$$

where $g(x) := \sum_i |a_i| r(\lambda_i)^{(1-1/p)n} |E_i|^{-1} \chi_{E_i}(x)$ and χ_{E_i} denotes the characteristic function of E_i. Since $B(\lambda_i, c\delta r(\lambda_i)) \subset E_i$, we have

$$r(\lambda_i)^{(1-1/p)n} |E_i|^{-1} \leq \frac{1}{(c\delta)^n} |E_i|^{-1/p}. \tag{11}$$

Hence, we have

$$\|g\|_{L^p}^p \lesssim \int_\Omega \sum_i |a_i|^p \, |E_i|^{-1} \chi_{E_i}(x)\, dx \leq \|\{a_i\}\|_{l^p}^p.$$

Therefore, by Lemma 2.1, we have

$$\|A\{a_i\}\|_{b^p} \lesssim \|I_1 g\|_{L^p} \lesssim \|\{a_i\}\|_{l^p}.$$

S is bounded, because $S = A \circ U$. This completes the proof. □

The next lemma is essential for the proof of main theorem.

Lemma 3.3. *Let $1 \leq p < \infty$. Then there exist $\{\lambda_i\}_i \subset \Omega$ and $\{E_i\}_i$ such that $S\colon b^p \to b^p$ is bijective.*

Proof. For $0 < \delta < 1/4$, we take $\{\lambda_i\}_i$ and $\{E_i\}$ in Lemma 3.1. We have only to show that $\|I - S\| < 1$ for a sufficiently small $\delta > 0$. By the condition of $\{E_i\}$, for $f \in b^p$ we have

$$\begin{aligned}
(I - S) f(x) &= \int_\Omega f(y) R_1(x, y)\, dy - \sum_{i=1}^\infty R_1(x, \lambda_i) f(\lambda_i) |E_i| \\
&= \sum_{i=1}^\infty \int_{E_i} f(y)(R_1(x, y) - R_1(x, \lambda_i))\, dy \\
&\quad + \sum_{i=1}^\infty \int_{E_i} (f(y) - f(\lambda_i)) R_1(x, \lambda_i)\, dy \\
&=: F_1(x) + F_2(x) \quad \text{say.}
\end{aligned}$$

First, we estimate $F_1(x)$. By (7), we have

$$\begin{aligned}
|F_1(x)| &\lesssim \sum_{i=1}^\infty \int_{E_i} |f(y)|\, |y - \lambda_i|\, |\nabla_y R_1(x, \bar{y})|\, dy \\
&\lesssim \delta \sum_{i=1}^\infty \int_{E_i} |f(y)| r(\lambda_i) \frac{1}{d(x, \bar{y})^{n+1}}\, dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \delta \sum_{i=1}^{\infty} \int_{E_i} \frac{r(y)}{d(x,y)^{n+1}} |f(y)|\, dy \\
&= \delta I_1 |f|(x).
\end{aligned}
$$

Next, we estimate $F_2(x)$. For any $y \in E_i$, by the mean-value property, we have

$$
|R_1(y,z) - R_1(\lambda_i, z)| \leq \delta r(\lambda_i) |\nabla_x R_1(\bar{y}, z)| \tag{12}
$$

for some $\bar{y}$ on the line segment between y and λ_i. Therefore, by (12) and (6), we have

$$
\begin{aligned}
|f(y) - f(\lambda_i)| &\leq \int_\Omega |R_1(y,z) - R_1(\lambda_i, z)|\, |f(z)|\, dz \\
&\lesssim \delta \int_\Omega \frac{r(\lambda_i) r(z)}{d(\bar{y}, z)^{n+2}} |f(z)|\, dz \\
&\lesssim \delta I_1 |f|(y).
\end{aligned}
$$

Hence, by Theorem C, we have

$$
\begin{aligned}
|F_2(x)| &\leq \sum_{i=1}^{\infty} \int_{E_i} |f(y) - f(\lambda_i)|\, |R_1(x, \lambda_i)|\, dy \\
&\lesssim \delta \int_{E_i} \frac{r(y)}{d(x,y)^{n+1}} I_1 |f|(y)\, dy \\
&= \delta \int_\Omega \frac{r(y)}{d(x,y)^{n+1}} I_1 |f|(y)\, dy \\
&= \delta I_1 \circ I_1 |f|(x).
\end{aligned}
$$

By Lemma 2.2, we have $\|(I - S) f\|_{b^p} \leq \delta C \|f\|_{b^p}$. Remark that this constant C is independent of δ. Hence, if we choose $\delta < C^{-1}$, then we obtain $\|(I - S)\| < 1$. This completes the proof. $\square$

Proof of Theorem 1. By Lemma 3.3, we choose a sequence $\{\lambda_j\}$ such that $S\colon b^p \to b^p$ is bijective. Hence, $A\colon l^p \to b^p$ is onto, which implies Theorem 1. $\square$

4. Representation theorem for harmonic Bloch functions

In this section, we give a proof of Theorem 2. We need to recall a pointwise estimate for $\mathcal{B}$ (see [3]):

$$
|f(x)| \lesssim \|f\|_{\mathcal{B}} (1 + \log^+ r(x)^{-1}) \tag{13}
$$

for any $x \in \Omega$ and any $f \in \mathcal{B}$. We need some operators discussed in [3]. Let $\mathcal{F}_1$ denote the class of all differential operators F of form

$$F = \omega_0 + \sum_{i=1}^{n} \omega_i \eta D_i \tag{14}$$

for some real functions $\omega_i \in C^\infty(\bar{\Omega})$. We put

$$F(x, y) := F(R_x)(y)$$

for $F \in \mathcal{F}_1$. The following theorem is shown [3].

Theorem D. *For $c_1 > 0$ and $F_1 \in \mathcal{F}_1$, we put $H_1 := c_1(K_1 - G_1)$, where K_1 is the differential operator defined in* (5) *and $G_1\psi(x) := (1/4)\int_\Omega \psi(y)F_1(x, y)\eta(y)\,dy$. We can choose a constant $c_1 > 0$ and $F_1 \in \mathcal{F}_1$ with the following properties*:
(a) $H_1 : b^p \to L^p$ *is bounded for each* $1 \le p < \infty$;
(b) $H_1 : \mathcal{B} \to L^\infty$ *is bounded and* $H_1(\mathcal{B}_0) \subset C_0 + \mathcal{B}_0 \cap b^\infty$;
(c) $P_1 H_1 f = f$ *for* $f \in b^1$.

REMARK. Recall $R_1(x, y) = R_1(x, y) - R_1(x_0, y)$ where x_0 is a fixed reference point. Denote by $\tilde{P}_1$ the corresponding operator $\tilde{P}_1 f(x) := \int_\Omega \tilde{R}_1(x, y) f(y)\,dy$. From Theorem D, we easily have

$$\tilde{P}_1 H_1 f = f \tag{15}$$

for any $f \in \tilde{B}$.

We give the estimates for H_1.

Lemma 4.1. *Let $0 < \delta < 1$ and $x \in \Omega$. Then*

$$|H_1 f(y) - H_1 f(x)| \lesssim \delta \| f \|_{\mathcal{B}} \tag{16}$$

for any $f \in \mathcal{B}$ and $y \in B(x, \delta r(x))$.

Proof. To obtain the estimate for H_1, we show the properties on $\mathcal{F}_1$ and G_1. First, we give the estimate for $\mathcal{F}_1$.

STEP 1. *Let $F \in \mathcal{F}_1$, $f \in \mathcal{B}$ and $x \in \Omega$. Then*

$$|Ff(x) - Ff(y)| \lesssim \delta \| f \|_{\mathcal{B}} \tag{17}$$

for $0 < \delta < 1$ and $y \in B(x, \delta r(x))$.

Proof of Step 1. Let $f \in \mathcal{B}$. By the mean-value property, for $y \in B(x, \delta r(x))$ we have

$$
\begin{aligned}
|Ff(y) - Ff(x)| &\leq |\omega_0|\,|f(y) - f(x)| + \sum_{i=1}^{n} |\omega_i|\,\eta|D_i(f(y) - f(x))| \\
&\lesssim |\omega_0|\,\delta r(\bar{y})|\nabla f(\bar{y})| + \sum_{i=1}^{n} |\omega_i|\,\delta r(\bar{y})^2|\nabla D_i f(\bar{y})| \\
&\lesssim \delta \| f\|_{\mathcal{B}}.
\end{aligned}
$$

The proof of Step 1 finished.

We put $\tilde{K}_1 f = -2(\Delta\eta + \omega)\eta f - 4\eta\nabla\eta \cdot \nabla f$. Then, $\tilde{K}_1 \in \mathcal{F}_1$ and $K_1 f = \tilde{K}_1 f$ for any harmonic function f. In particular, we have

$$
|K_1 f(y) - K_1 f(x)| \lesssim \delta \| f\|_{\mathcal{B}} \tag{18}
$$

for any $f \in \mathcal{B}$, $x \in \Omega$ and $y \in B(x, \delta r(x))$.

STEP 2. *Let F_1 and G_1 satisfy the conditions of* Theorem D. *Then*

$$
|G_1 f(y) - G_1 f(x)| \lesssim \delta \| f\|_{\mathcal{B}} \tag{19}
$$

for any $f \in \mathcal{B}$, $x \in \Omega$ and $y \in B(x, \delta r(x))$.

Proof of Step 2. For $f \in \mathcal{B}$, by the mean-value property, for $y \in B(x, \delta r(x))$ we have

$$
|G_1 f(y) - G_1 f(x)| \lesssim \int_\Omega |f(z)|r(z)|y - x|\,|\nabla F_1 R_{\bar{y}}(z)|\,dz
$$

for some $\bar{y}$ on the line segment between x and y. Because $r(x)$ comparable to $r(\bar{y})$, by (6) and (13), we have

$$
\begin{aligned}
|G_1 f(y) - G_1 f(x)| &\lesssim \int_\Omega |f(z)|r(z)|y - x|\,|\nabla F_1 R_{\bar{y}}(z)|\,dz \\
&\lesssim \delta \| f\|_{\mathcal{B}} \int_\Omega r(\bar{y}) \frac{1}{d(\bar{y}, z)^{n+1}}\,dz \\
&\lesssim \delta \| f\|_{\mathcal{B}}.
\end{aligned}
$$

The proof of Step 2 finished. By (18) and Step 2, we obtain Lemma 4.1. □

Again, for $0 < \delta < 1/4$ we choose a sequence $\{\lambda_j\}$ in Ω and a disjoint covering $\{E_j\}$ of Ω obtained by Lemma 3.1. We define the operators $\tilde{A}\colon l^\infty \to \tilde{\mathcal{B}}$, $\tilde{S}\colon \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$

and $\tilde{U}\colon \tilde{\mathcal{B}} \to l^\infty$ by

$$\tilde{A}\{a_i\}(x) := \sum_{j=1}^{\infty} a_j \tilde{R}_1(x, \lambda_j)|E_j|, \tag{20}$$

$$\tilde{S} f(x) := \sum_{j=1}^{\infty} H_1 f(\lambda_j) \tilde{R}_1(x, \lambda_j)|E_j|, \tag{21}$$

and

$$\tilde{U} f := \{H_1 f(\lambda_j)\}_j. \tag{22}$$

In the similar manner as in the proof of Theorem 1. We begin with showing that $\tilde{A}$, $\tilde{U}$ and $\tilde{S}$ are bounded.

Lemma 4.2. *$\tilde{A}\colon l^\infty \to \tilde{\mathcal{B}}$, $\tilde{U}\colon \tilde{\mathcal{B}} \to l^\infty$ and $\tilde{S}\colon \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ are bounded.*

Proof. It is obvious that $\tilde{U}\colon \tilde{\mathcal{B}} \to l^\infty$ is bounded by Theorem D. Since $\tilde{S} = \tilde{A}\tilde{U}$, we have only to show that $\tilde{A}\colon l^\infty \to \tilde{\mathcal{B}}$ is bounded. Taking $\{a_i\} \in l^\infty$, by (6) and Lemma 2.1, we have

$$\begin{aligned}
|r(x)\nabla(\tilde{A}\{a_i\})(x)| &= r(x)\left|\sum_{j=1}^{\infty} a_j \nabla_x \tilde{R}_1(x, \lambda_j)|E_j|\right| \\
&\lesssim \|\{a_i\}\|_{l^\infty} \sum_{j=1}^{\infty} r(x)\frac{r(\lambda_j)}{d(x, \lambda_j)^{n+1}}|E_j| \\
&= \|\{a_i\}\|_{l^\infty} \int_\Omega \frac{r(x)r(y)}{d(x, y)^{n+2}}\, dy \\
&\lesssim \|\{a_i\}\|_{l^\infty},
\end{aligned}$$

which implies $\tilde{A}\colon l^\infty \to \tilde{\mathcal{B}}$ is bounded. $\square$

Finally, we state an important lemma for the representation theorem.

Lemma 4.3. *There exists $\{\lambda_i\}_i \subset \Omega$ and $\{E_i\}_i$ such that $\tilde{S}\colon \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is bijective.*

Proof. For $0 < \delta < 1/4$, we take $\{\lambda_i\}_i$ and $\{E_i\}$ in Lemma 3.1. We show that $\|I - \tilde{S}\| < 1$ for a sufficiently small $\delta > 0$. By Theorem D, we have

$$\begin{aligned}
(I - \tilde{S}) f(x) &= \sum_{j=1}^{\infty} \int_{E_j} H_1 f(y)\tilde{R}_1(x, y)\, dy - \sum_{j=1}^{\infty} \int_{E_j} H_1 f(\lambda_j)\tilde{R}_1(x, \lambda_j)\, dy \\
&= \sum_{j=1}^{\infty} \int_{E_j} \tilde{R}_1(x, y)(H_1 f(y) - H_1 f(\lambda_j))\, dy \\
&\quad + \sum_{j=1}^{\infty} \int_{E_j} H_1 f(\lambda_j)(\tilde{R}_1(x, y) - \tilde{R}_1(x, \lambda_j))\, dy.
\end{aligned}$$

We calculate $r(x)|\nabla(I-\tilde{S})f(x)|$ to estimate the Bloch norm.

$$r(x)|\nabla(I-\tilde{S})f(x)| \le r(x)\left|\nabla_x \sum_{j=1}^{\infty}\int_{E_j}\tilde{R}_1(x,y)(H_1 f(y)-H_1 f(\lambda_j))\,dy\right| + r(x)\left|\nabla_x \sum_{j=1}^{\infty}\int_{E_j} H_1 f(\lambda_j)(\tilde{R}_1(x,y)-\tilde{R}_1(x,\lambda_j))\,dy\right|.$$

Because $\nabla_x \tilde{R}_1(x,y)=\nabla_x R_1(x,y)$, Lemma 4.1 shows that the first term is bounded by

$$\delta\|f\|_{\mathcal{B}} r(x)\sum_{j=1}^{\infty}\int_{E_j}|\nabla_x \tilde{R}_1(x,y)|\,dy \lesssim \delta\|f\|_{\mathcal{B}} r(x)\int_{\Omega}\frac{r(y)}{d(x,y)^{n+2}}\,dy \lesssim \delta\|f\|_{\mathcal{B}}.$$

The second term can be estimated by

$$r(x)\|H_1 f\|_{L^\infty}\sum_{j=1}^{\infty}\int_{E_j}|\nabla_x R_1(x,y)-\nabla_x R_1(x,\lambda_j)|\,dy \lesssim \delta\|f\|_{\mathcal{B}} r(x)\int_{\Omega}\frac{r(y)}{d(x,y)^{n+2}}\,dy \lesssim \delta\|f\|_{\mathcal{B}}.$$

Thus, there exist a constant $C>0$ such that $\|(I-\tilde{S})f\|_{\mathcal{B}} \le C\delta\|f\|_{\mathcal{B}}$. Hence, if we take $\delta < C^{-1}$, then $\tilde{S}\colon \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is bijective. This completes the proof. □

Finally, we give a proof of Theorem 2.

Proof of Theorem 2. We put $f \in \tilde{\mathcal{B}}$. By Lemma 4.3, we can choose a constant $\delta > 0$ such that $\tilde{S}\colon \tilde{\mathcal{B}} \to \tilde{\mathcal{B}}$ is bijective. This implies $\tilde{A}\colon l^\infty \to \tilde{\mathcal{B}}$ is surjective. Hence, for any $f \in \tilde{\mathcal{B}}$, we can find a sequence $\{a_i'\} \in l^\infty$ such that $\tilde{A}\{a_i'\} = f$. Therefore, if we put $a_j := a_j' E_j / r(\lambda_j)^n$, then $\{a_i\}$ is in l^∞ and satisfies $f(x) = \sum_{j=1}^{\infty} a_j R(x,\lambda_j) r(\lambda_j)^n$. This completes the proof. □

5. Application

In this section, we analyze positive Toeplitz operators on b^2 by using Theorem 1. We define several operators and functions. $M(\Omega)$ denotes the space of all complex Berel measures on Ω. For $\mu \in M(\Omega)$, the corresponding Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu f(x) := \int_{\Omega} R(x,y) f(y)\,d\mu(y) \quad (x \in \Omega). \tag{23}$$

Let μ be a finite positive Borel measure. For $\delta \in (0, 1)$, the averaging function $\hat{\mu}_\delta$ is defined by

$$\hat{\mu}_\delta(x) := \frac{\mu(B(x, \delta r(x)))}{V(B(x, \delta r(x)))} \quad (x \in \Omega). \tag{24}$$

We recall Schatten σ-class operators. A compact operator T on a separable Hilbert space is called Schatten σ-class operator, if the following norm is finite;

$$\|T\|_{S_\sigma(X)} := \left(\sum_{m=1}^{\infty} |s_m(T)|^\sigma\right)^{1/\sigma} \tag{25}$$

where $\{s_m(T)\}_m$ is the sequence of all singular value of T. Let S_σ be the space of all Schatten σ-class operators on b^2. In [2], B.R. Choe, Y.J. Lee and K. Na studied conditions that positive Toeplitz operators are bounded, compact and in Schatten σ-class on b^2 for $1 \leq \sigma < \infty$. We would like to discuss a condition that positive Toeplitz operators are in Schatten σ-class on b^2.

Theorem 3. *Let $2(n-1)/(n+2) < \sigma$ and μ be a finite positive Borel measure. Choose a sequence $\{\lambda_j\}$ in* Theorem 1. *Then, if $\sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma < \infty$, then $T_\mu \in S_\sigma$.*

We recall a general property; see for example [7].

Lemma 5.1 ([7]). *If T is a compact operator on a Hilbert space H and $0 < \sigma \leq 2$, then for any orthonormal basis $\{e_n\}$, we have*

$$\|T\|_{S_\sigma(H)}^\sigma \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\langle Te_n, e_k\rangle|^\sigma. \tag{26}$$

Proof of Theorem 3. When $1 \leq \sigma < \infty$, the statement of Theorem 3 is shown in [2]. Hence, we assume $\sigma < 1$. We put a sequence $\{\lambda_j\}$ satisfying the assumption of Theorem 3. We show the following inequality

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle A^* T_\mu A e_i, e_j\rangle|^\sigma \lesssim \sum_{j=1}^{\infty} \hat{\mu}_\delta(\lambda_j)^\sigma, \tag{27}$$

where $\{e_n\}$ is an orthonormal basis for l^2 and A is the operator of the atomic decomposition obtained by Theorem 1. First, we calculate $\langle A^* T_\mu A e_i, e_j\rangle$.

$$Ae_i(x) = R_1(x, \lambda_i) r(\lambda_i)^{n/2}$$

and

$$T_\mu A e_i(x) = r(\lambda_i)^{n/2} \int_\Omega R(x, y) R_1(y, \lambda_i)\, d\mu(y).$$

Therefore, we have

$$\langle A^* T_\mu A e_i, e_j \rangle = r(\lambda_i)^{n/2} r(\lambda_j)^{n/2} \int_\Omega R_1(y, \lambda_i) R_1(y, \lambda_j)\, d\mu(y).$$

Then, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle A^* T_\mu A e_i, e_j \rangle|^\sigma \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| r(\lambda_i)^{n/2} r(\lambda_j)^{n/2} \int_\Omega R_1(x, \lambda_i) R_1(x, \lambda_j)\, d\mu(x) \right|^\sigma \\
&\lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} r(\lambda_i)^{n\sigma/2} r(\lambda_j)^{n\sigma/2} \left(\sum_{k=1}^{\infty} \int_{B(\lambda_k, \delta r(\lambda_k))} \frac{r(\lambda_i)}{d(x, \lambda_i)^{n+1}} \frac{r(\lambda_j)}{d(x, \lambda_j)^{n+1}}\, d|\mu|(x) \right)^\sigma \\
&\lesssim \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} r(\lambda_i)^{n\sigma/2} r(\lambda_j)^{n\sigma/2} \left(\sum_{k=1}^{\infty} |\mu|(B(\lambda_k, \delta r(\lambda_k))) \frac{r(\lambda_i)}{d(\lambda_k, \lambda_i)^{n+1}} \frac{r(\lambda_j)}{d(\lambda_k, \lambda_j)^{n+1}} \right)^\sigma .
\end{aligned}$$

By $\sigma < 1$, we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} r(\lambda_i)^{n\sigma/2} r(\lambda_j)^{n\sigma/2} \left(\sum_{k=1}^{\infty} |\mu|(B(\lambda_k, \delta r(\lambda_k))) \frac{r(\lambda_i)}{d(\lambda_k, \lambda_i)^{n+1}} \frac{r(\lambda_j)}{d(\lambda_k, \lambda_j)^{n+1}} \right)^\sigma \\
&\lesssim \sum_{k=1}^{\infty} \hat{\mu}_\delta(\lambda_k)^\sigma r(\lambda_k)^{n\sigma} \left(\sum_{i=1}^{\infty} \frac{r(\lambda_i)^{n\sigma/2+\sigma}}{d(\lambda_k, \lambda_i)^{(n+1)\sigma}} \right)^2 \\
&= \sum_{k=1}^{\infty} \hat{\mu}_\delta(\lambda_k)^\sigma \left(\sum_{i=1}^{\infty} \frac{r(\lambda_k)^{n\sigma/2} r(\lambda_i)^{n\sigma/2+\sigma}}{d(\lambda_k, \lambda_i)^{(n+1)\sigma}} \right)^2 .
\end{aligned}$$

By Lemma 2.1 and the condition of σ, we have

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{r(\lambda_k)^{n\sigma/2} r(\lambda_i)^{n\sigma/2+\sigma}}{d(\lambda_k, \lambda_i)^{(n+1)\sigma}} &\lesssim \sum_{i=1}^{\infty} \int_{B(\lambda_i, \delta\lambda_i)} \frac{r(\lambda_k)^{n\sigma/2} r(\lambda_i)^{n\sigma/2+\sigma-n}}{d(\lambda_k, \lambda_i)^{n+(n+1)\sigma-n}}\, dy \\
&\lesssim \int_\Omega \frac{r(\lambda_k)^{n\sigma/2} r(y)^{n\sigma/2+\sigma-n}}{d(\lambda_k, y)^{n+(n+1)\sigma-n}}\, dy \lesssim 1.
\end{aligned}$$

By an estimate (26) and Lemma 5.1, we obtained $A^* T_\mu A \in S_\sigma$. By Lemma 3.3, there exists $S^{-1}\colon b^2 \to b^2$ and S^{-1} is bounded. Because $T_\mu = (US^{-1})^* A^* T_\mu A (US^{-1})$, we obtain $T_\mu \in S_\sigma$. This completes the proof. □

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