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# NECESSARY AND SUFFICIENT CONDITIONS FOR THE SOLVABILITY AND MAXIMAL REGULARITY OF ABSTRACT DIFFERENTIAL EQUATIONS OF MIXED TYPE IN HÖLDER SPACES

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**Abstract**

In this paper we study and obtain some necessary and sufficient conditions on the data for the existence, uniqueness of the strict solution and maximal regularity for some second-order differential equations with mixed boundary conditions whose forcing term belongs to Hölder continuous spaces. A few illustrative examples related to the interpolation theory are discussed.

## 1. Introduction

Let us consider, in a complex Banach space $X$, the second order abstract differential equation

$$u''(x) + Au(x) = f(x), \quad x \in (0, 1) \tag{1}$$

with the Dirichlet–Neumann boundary conditions

$$u(0) = d_0, \quad u'(1) = n_1. \tag{2}$$

Here $d_0$ and $n_1$ are given elements in $X$ and $A$ is a closed linear operator of domain $D(A)$ not necessarily dense in $X$.

We assume throughout the paper the following ellipticity hypothesis

$$\forall \lambda \geq 0,\ \exists (A - \lambda I)^{-1} \in L(X)\colon \|(A - \lambda I)^{-1}\|_{L(X)} \leqslant \frac{C}{1 + \lambda}. \tag{3}$$

Our study will show the existence, uniqueness and regularity of the solution under the assumption above in the case

$$f \in C^{\theta}([0, 1]; X), \quad 0 < \theta < 1.$$

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In fact, we prove that there exists a strict unique solution of problem (1)–(2), that is

$$u \in C^2([0, 1]; X) \cap C([0, 1]; D(A)),$$

if and only if

$$\begin{cases} d_0 \in D(A), \quad n_1 \in D(\sqrt{-A}), \\ Ad_0 - f(0) \in \overline{D(A)} \quad \text{and} \quad \sqrt{-A}n_1 \in \overline{D(A)}, \end{cases}$$

and that

$$u'', \, Au \in C^\theta([0, 1]; X) \tag{4}$$

if and only if

$$\begin{cases} d_0 \in D(A), \quad n_1 \in D(\sqrt{-A}), \\ Ad_0 - f(0), \, \sqrt{-A}n_1 \in (D(A), X)_{1-\theta/2,\infty}. \end{cases}$$

where for all $\theta \in ]0, 1[$ and $p \in [1, \infty]$, $(D(A), H)_{1-\theta,\infty}$ is the well known interpolation space, see Lions–Peetre [13]. The property (4) is called the maximal regularity.

This work is based fundamentally on an explicit representation of the solution using the square root of $-A$ and the Krein's method. We then analyze carefully all the components of the solution, by using the Sinestrari method [17], the Lions's reiteration theorem [13], the semigroup theory and some techniques applied in [6].

The square root of the operator $-A$ will appear naturally in this paper. When we have to study equation (1) just with Dirichlet's boundary conditions, the use of this square root is not necessary, see Labbas [12]. In our case, we must carefully use this square root since the density of $D(A)$ is not assumed. For this end, we use the paper on fractional powers of non-densely defined operators by Martinez–Sanz [15].

In the last decades, many researchers have been interested in the resolution of problem (1). Many of them studied (1) as an abstract problem of elliptic type, i.e. under assumption (3), with different boundary conditions in both cases $f$ Hölder continuous or $f$ in $L^p(0, 1; X)$ by using fractional powers of operators or Dunford functional calculus. We cite at first, the pioneer Da Prato and Grisvard theory on the sum of operators [4]. Such a method yields interesting results by Labbas–Terreni [10], [11], on more complicated situations, for instance, the case of variable coefficients operators $A(x)$. In [12] we find a complete study of (1) under Dirichlet's boundary conditions and in variable coefficients operators case, where the author has used the Green's kernels techniques.

Other researchers focused their attention to the resolution of

$$u''(x) + Bu'(x) + Au(x) = f(x), \quad x \in (0, 1), \tag{5}$$

when $X$ is any complex Banach space and

$$f \in C^\theta([0, 1]; X) \quad \text{or} \quad f \in L^p(0, 1; X), \quad 0 < \theta < 1, \; 1 < p < \infty.$$

Very interesting approaches to (5), where $A$ is even substituted by $A + \lambda I$, with $\lambda$ a complex parameter, are described in the paper by S. Yakubov and Y. Yakubov [20]. They have worked in a Hilbert space $H$, $-A$ is supposed to be a positive operator in $H$, $D(A)$ being compactly embedded into $H$, and $B$ is a closed operator in $H$ whose domain is related to $(D(A), H)_{1/2,1}$.

A recent work by Arendt [1] proved that the problem

$$u''(x) + B(x)u'(x) + A(x)u(x) = f(x), \quad x \in (0, \delta)$$

with boundary conditions $u(0) = x$, $u'(0) = y$, has a unique solution $u$ such that

$$u \in W^{2,p}(0, \delta; X) \cap L^p(0, \delta; D(A)) \quad \text{and} \quad u' \in L^p(0, \delta; D(B)),$$

in the case where $D(A)$ and $D(B)$ are Banach spaces which embed continuously and densely into $X$ and $f$ belongs to $L^p(0, \delta; X)$.

At last, a new approach, based on the semigroup techniques by Krein [9] and fractional powers of operators, has been developed by Favini, Labbas, Maingot, Tanabe and Yagi [5], [6] concerning the complete equation (5) under Dirichlet boundary conditions. In our work we have been inspired by this last reference.

In this paper, we are interested in the resolution of problem (1) with Dirichlet–Neumann boundary conditions. The latter conditions make our study difficult, especially when the operator $A$ is not densely defined. We then give, necessary and sufficient conditions on the data to have existence, uniqueness and maximal regularity of the strict solution. We also obtain some a priori estimates. Moreover, the cross-regularity is proved i.e.:

$$Au(\cdot) - f(\cdot) \in B([0, 1]; (D(A), X)_{1-\theta/2,\infty}).$$

Some interpolation results come as applications to our results.

Here is an outline of the paper. In Sections 2 and 3 of this work, we will recall some basic properties of analytic semigroups. We also give some technical lemmas which are useful to give a precise analysis of the representation of the solution $u$. Section 4 is devoted to the existence, uniqueness and maximal regularity of the strict solution. In Section 5 we give some a priori estimates.

Finally, Section 6 contains some new examples related to interpolation theory.

## 2. Technical results

REMARK 1. Hypothesis (3) implies that the operator $(-\sqrt{-A})$ generates an analytic semigroup denoted by $(e^{-\sqrt{-A}x})_{x\geq 0}$ on $X$, see for instance Balakrishnan [2].

We put throughout the paper

$$B = \sqrt{-A}$$

and

$$Z = e^{-2B}.$$

**Proposition 2.** *Assume* (3). *The operator $I - Z$ has a bounded inverse given by*

$$(I - Z)^{-1} = \frac{1}{2\pi i} \int_{\gamma_\#} \frac{e^{2z}}{1 - e^{2z}} (zI + B)^{-1}\, dz + I,$$

*where $\gamma_\#$ is a suitable curve in the complex plane.*

Proof. Since the imaginary axis is contained in the resolvent set $\rho(-B)$, we then can adapt the Lunardi's proof [14], p.59 by choosing an appropriate curve $\gamma_\#$ on account of the fact that $-B$ generates an analytic semigroup. □

**Corollary 3.** *Under hypothesis* (3), *the operator* $(I + Z)$ *has a bounded inverse.*

Proof. We have

$$(I - e^{-2B})(I + e^{-2B}) = I - e^{-4B}$$

then

$$(I + e^{-2B}) = (I - e^{-2B})^{-1}(I - e^{-4B}).$$

Therefore,

$$(I + e^{-2B})^{-1} = (I - e^{-4B})^{-1}(I - e^{-2B}).$$

For $d_0 \in X$, consider the following abstract function

$$\begin{aligned} ]0, 1] &\to X, \\ x &\mapsto D_0(x, \sqrt{-A})d_0, \end{aligned}$$

where

$$D_0(x, \sqrt{-A})d_0 = (I + Z)^{-1}(I + e^{-2\sqrt{-A}(1-x)})e^{-\sqrt{-A}x} d_0.$$

We have the following result □

**Lemma 4.** *We have*:

1. $D_0(\,\cdot\,, \sqrt{-A})d_0 \in C^\infty(]0, 1]; D(A^k)),\ k \in \mathbb{N}$,
2. $\forall x \in ]0, 1],\ D_0''(x, \sqrt{-A})d_0 + AD_0(x, \sqrt{-A})d_0 = 0$,

3. $\exists C > 0,\ \forall x \in ]0, 1],\ \|D_0(x, \sqrt{-A})d_0\|_X \leq C\|d_0\|_X$.

Proof. 1. Let $x > 0$ and $d_0 \in X$. It is not difficult to see that

$$(I \pm Z)^{-1}e^{-Bx} = e^{-Bx}(I \pm Z)^{-1},$$

therefore

$$D_0(x, B)d_0 = e^{-Bx}(I + e^{-2B(1-x)})(I + Z)^{-1}d_0.$$

Hence we deduce the first statement using [17, Proposition 1.1].

2. For $x \in ]0, 1]$, it holds that

$$\begin{aligned} D_0'(x, B)d_0 &= (I + Z)^{-1}[(2Be^{-2B(1-x)})e^{-Bx}d_0 - (I + e^{-2B(1-x)})Be^{-Bx}d_0], \\ D_0''(x, B)d_0 &= (I + Z)^{-1}[(4(-A)e^{-2B(1-x)})e^{-Bx}d_0 - (2Be^{-2B(1-x)})Be^{-Bx}d_0] \\ &\quad - (I + Z)^{-1}[(2Be^{-2B(1-x)})Be^{-Bx}d_0 + (I + e^{-2B(1-x)})Ae^{-Bx}d_0] \\ &= -(I + Z)^{-1}(I + e^{-2B(1-x)})Ae^{-Bx}d_0. \end{aligned}$$

Therefore

$$\begin{aligned} &D_0''(x, B)d_0 + AD_0(x, B)d_0 \\ &= -(I + Z)^{-1}(I + e^{-2B(1-x)})Ae^{-Bx}d_0 + A(I + Z)^{-1}(I + e^{-2B(1-x)})e^{-Bx}d_0 \\ &= -(I + Z)^{-1}[(I + e^{-2B(1-x)})Ae^{-Bx}d_0 - (I + e^{-2B(1-x)})Ae^{-Bx}d_0] = 0. \end{aligned}$$

3. It is well known (see Tanabe [18, (3.27)]) that there exists a constant $M > 0$ such that for any $x > 0$, $d_0 \in X$,

$$\left\|e^{-Bx}d_0\right\|_X \leq M\|d_0\|_X.$$

Thus, $\exists C > 0$:

$$\begin{aligned} \|D_0(x, B)d_0\|_X &= \|(I + Z)^{-1}(I + e^{-2B(1-x)})e^{-Bx}d_0\|_X \\ &\leq C\|d_0\|_X. \end{aligned}$$

□

Let us specify the behavior of $D_0(\,\cdot\,, B)$ near 0.

**Lemma 5.** 1. *Let $d_0 \in X$. Then*

$$D_0(\,\cdot\,, \sqrt{-A})d_0 \in C([0, 1]; X) \quad \textit{if and only if} \quad d_0 \in \overline{D(A)}.$$

2. *Let $d_0 \in D(A)$. Then*

$$D_0(\,\cdot\,, \sqrt{-A})d_0 \in C([0, 1]; D(A)) \quad \textit{if and only if} \quad Ad_0 \in \overline{D(A)}.$$

Proof. The result is obtained by the commutativity of $(I + Z)^{-1}$ and $A$ on $D(A)$ and [17, Proposition 1.2]. We also use the fact that

$$\overline{D(\sqrt{-A})} = \overline{D(A)},$$

see Haase [8, Corollary 3.1.11]. □

Now, for $n_1 \in X$, consider the following abstract function

$$\begin{aligned} &[0, 1[ \to X,\\ &x \mapsto N_1(x, \sqrt{-A})n_1, \end{aligned}$$

where

$$N_1(x, \sqrt{-A})n_1 = (I + Z)^{-1}(I - e^{-2\sqrt{-A}x})e^{-\sqrt{-A}(1-x)}(-A)^{-1/2}n_1.$$

We have the following result

**Lemma 6.** *We have*:
1. $N_1(\cdot, \sqrt{-A})n_1 \in C^\infty([0, 1[; D(A^k)),\ k \in \mathbb{N}$,
2. $\forall x \in [0, 1[,\ N_1''(x, \sqrt{-A})n_1 + AN_1(x, \sqrt{-A})n_1 = 0$,
3. $\exists C > 0,\ \forall x \in [0, 1[,\ \|N_1(x, \sqrt{-A})n_1\|_X \le C\|n_1\|_X$.

Proof. The proof is not difficult. It suffices to replace $x$ by $1 - x$. □

**Lemma 7.** (1) *Let $n_1 \in X$. Then*

$$N_1(\cdot, \sqrt{-A})n_1 \in C([0, 1]; X) \quad \textit{if and only if} \quad n_1 \in \overline{D(A)}.$$

(2) *Let $n_1 \in D(\sqrt{-A})$. Then*

$$N_1(\cdot, \sqrt{-A})n_1 \in C([0, 1]; D(A)) \quad \textit{if and only if} \quad \sqrt{-A}n_1 \in \overline{D(A)}.$$

Proof. The proof of this lemma is the same as Lemma 5. □

## 3. Representation of the solution

We assume here that (3) holds.

Let us suppose that problem (1)–(2) has a strict solution $u$ and set

$$u(1) = u_1.$$

Then $u$ is the strict solution of the following problem

$$\begin{cases} u''(x) - B^2u(x) = f(x),\\ u(0) = d_0,\\ u(1) = u_1. \end{cases} \tag{6}$$

Therefore, $u$ is represented (see [5]) by

$$u(x) = e^{-xB}\xi_0 + e^{-(1-x)B}\xi_1 - \frac{1}{2}B^{-1}\int_0^x e^{-(x-s)B} f(s)\,ds - \frac{1}{2}B^{-1}\int_x^1 e^{-(s-x)B} f(s)\,ds,$$

where

$$\begin{aligned}\xi_0 &= (I-Z)^{-1}(d_0 - e^{-B}u_1)\\ &\quad + \frac{1}{2}(I-Z)^{-1}B^{-1}\left(\int_0^1 e^{-sB} f(s)ds - \int_0^1 e^{-(2-s)B} f(s)\,ds\right),\\ \xi_1 &= (I-Z)^{-1}(-e^{-B}d_0 + u_1)\\ &\quad + \frac{1}{2}(I-Z)^{-1}B^{-1}\left(\int_0^1 e^{-(1-s)B} f(s)\,ds - \int_0^1 e^{-(1+s)B} f(s)\,ds\right).\end{aligned}$$

We deduce that

$$\begin{aligned}n_1 &= u'(1)\\ &= -2(I-Z)^{-1}Be^{-B}d_0 + (I-Z)^{-1}(I+Z)Bu_1\\ &\quad - \frac{1}{2}e^{-B}(I-Z)^{-1}\left(\int_0^1 e^{-sB} f(s)\,ds - \int_0^1 e^{-(2-s)B} f(s)\,ds\right)\\ &\quad + \frac{1}{2}(I-Z)^{-1}\left(-\int_0^1 e^{-(1+s)B} f(s)\,ds + Z\int_0^1 e^{-(1-s)B} f(s)\,ds\right).\end{aligned}$$

Then

$$\begin{aligned}u_1 &= (I+Z)^{-1}(2e^{-B}d_0 + (I-Z)B^{-1}n_1)\\ &\quad + (I+Z)^{-1}B^{-1}\left(-\int_0^1 e^{-(1-s)B} f(s)\,ds + \int_0^1 e^{-(1+s)B} f(s)\,ds\right).\end{aligned} \tag{7}$$

Therefore $u$ is formally given by

$$\begin{aligned}u(x) &= (I+Z)^{-1}[(e^{-xB} + e^{-(2-x)B})d_0 + (e^{-(1-x)B} - e^{-(1+x)B})B^{-1}n_1]\\ &\quad + \frac{1}{2}(I+Z)^{-1}B^{-1}\left[\int_0^1 e^{-(x+s)B} f(s)\,ds + \int_0^1 e^{-(2-x+s)B} f(s)\,ds\right]\\ &\quad + \frac{1}{2}(I+Z)^{-1}B^{-1}\left[\int_0^1 e^{-(2+x-s)B} f(s)\,ds - \int_0^1 e^{-(2-x-s)B} f(s)\,ds\right]\\ &\quad - \frac{1}{2}B^{-1}\int_0^x e^{-(x-s)B} f(s)\,ds - \frac{1}{2}B^{-1}\int_x^1 e^{-(s-x))B} f(s)\,ds.\end{aligned} \tag{8}$$

## 4. Existence, uniqueness and maximal regularity

Now, consider problem (1)–(2). Its solution is given by (8).

**Theorem 8.** *Under* (3) *let* $f \in C^{\theta}([0,1];X)$, $0 < \theta < 1$. *Then the following assertions are equivalent.*

1. *Problem* (1)–(2) *has a unique strict solution* $u$, *that is*

$$u \in C^2([0,1];X) \cap C([0,1];D(A))$$

*and* $u$ *satisfies* (1)–(2).

2. *For* $d_0 \in D(A)$, $Ad_0 - f(0) \in \overline{D(A)}$, *let* $u$ *be given by the formula* (8). *Then*

$$n_1 \in D(\sqrt{-A}) \quad \text{and} \quad \sqrt{-A}n_1 \in \overline{D(A)}.$$

Proof. Suppose that statement 1 holds. Then

$$d_0 = u(0) \in D(A),$$

and

$$Ad_0 - f(0) = -u''(0) \in \overline{D(A)},$$
$$Au(1) - f(1) = -u''(1) \in \overline{D(A)}.$$

Now let us prove that the solution is necessarily represented by (8) for $x \in (0,1)$. Put

$$\begin{aligned} L(f)(x) &= \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(x+s)B}f(s)\,ds + \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x+s)B}f(s)\,ds \\ &\quad + \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2+x-s)B}f(s)\,ds - \frac{1}{2}(I+Z)^{-1}\int_0^1 B^{-1}e^{-(2-x-s)B}f(s)\,ds \\ &\quad - \frac{1}{2}\int_0^x B^{-1}e^{-(x-s)B}f(s)\,ds - \frac{1}{2}\int_x^1 B^{-1}e^{-(s-x))B}f(s)\,ds. \end{aligned}$$

Writing $f(x) = Au(x) + u''(x)$ we obtain

$$\begin{aligned} &L(f)(x) \\ &= L(Au)(x) + L(u'')(x) \\ &= \sum_{i=1}^{6} H_i + \sum_{i=1}^{6} J_i. \end{aligned}$$

After integrating by parts we have

$$
\begin{aligned}
J_1 &= \frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1+x)B}u'(1) - e^{-xB}u'(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}(e^{-(1+x)B}u(1) - e^{-xB}u(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(x+s)B}u(s)\,ds, \\
J_2 &= \frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(3-x)B}u'(1) - e^{-(2-x)B}u'(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}(e^{-(3-x)B}u(1) - e^{-(2-x)B}u(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2-x+s)B}u(s)\,ds, \\
J_3 &= \frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1+x)B}u'(1) - e^{-(2+x)B}u'(0)) \\
&\quad - \frac{1}{2}(I+Z)^{-1}(e^{-(1+x)B}u(1) - e^{-(2+x)B}u(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2+x-s)B}u(s)\,ds, \\
J_4 &= -\frac{1}{2}(I+Z)^{-1}B^{-1}(e^{-(1-x)B}u'(1) - e^{-(2-x)B}u'(0)) \\
&\quad + \frac{1}{2}(I+Z)^{-1}(e^{-(1-x)B}u(1) - e^{-(2-x)B}u(0)) \\
&\quad - \frac{1}{2}(I+Z)^{-1}\int_0^1 Be^{-(2-x-s)B}u(s)\,ds, \\
J_5 &= -\frac{1}{2}B^{-1}(u'(x) - e^{-xB}u'(0)) + \frac{1}{2}(u(x) - e^{-xB}u(0)) \\
&\quad - \frac{1}{2}\int_0^x Be^{-(x-s)B}u(s)\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
J_6 &= -\frac{1}{2}B^{-1}(e^{-(1-x)}u'(1) - u'(x)) - \frac{1}{2}(e^{-(1-x)}u(1) - u(x)) \\
&\quad - \frac{1}{2}\int_x^1 Be^{-(s-x)B}u(s)\,ds.
\end{aligned}
$$

The last integral is well defined since $u \in C^1([0, 1]; X)$.

We obtain that

$$\sum_{i=1}^{6} H_i + \sum_{i=1}^{6} J_i$$
$$= -(I+Z)^{-1}[(e^{-xB} + e^{-(2-x)B})d_0 + (e^{-(1-x)B} - e^{-(1+x)B})B^{-1}n_1] + u(x)$$

from which we deduce formula (8). We obtain that

$$u(1) = (I+Z)^{-1}[2e^{-B}d_0 + (I-Z)B^{-1}n_1] + (I+Z)^{-1}\left[\int_0^1 B^{-1}e^{-(1+s)B} f(s)\,ds - \int_0^1 B^{-1}e^{-(1-s)B} f(s)\,ds\right],$$

then

$$\begin{aligned} B^{-1}n_1 &= u(1) - 2(I+Z)^{-1}e^{-B}d_0 + (I+Z)^{-1}e^{-2B}B^{-1}n_1 \\ &\quad - (I+Z)^{-1}e^{-B}\int_0^1 B^{-1}e^{-sB} f(s)\,ds \\ &\quad + (I+Z)^{-1}\int_0^1 B^{-1}e^{-(1-s)B} f(s)\,ds \\ &= u(1) - 2e^{-B}(I+Z)^{-1}d_0 + e^{-2B}(I+Z)^{-1}B^{-1}n_1 \\ &\quad - e^{-B}(I+Z)^{-1}\int_0^1 B^{-1}e^{-sB} f(s)\,ds \\ &\quad + (I+Z)^{-1}\int_0^1 B^{-1}e^{-(1-s)B} f(s)\,ds \\ &= \sum_{i=1}^{5} a_i. \end{aligned} \tag{9}$$

It is clear that $a_1$, $a_2$, $a_3$, $a_4$ are in $D(A)$. In addition, from

$$\begin{aligned} a_5 &= (I+Z)^{-1}B^{-1}\int_0^1 e^{-(1-s)B} f(s)\,ds \\ &= -A^{-1}(I+Z)^{-1}\int_0^1 Be^{-(1-s)B}(f(s) - f(1))\,ds \\ &\quad + A^{-1}(I+Z)^{-1}(I - e^{-B})f(1) \end{aligned}$$

we have $a_5 \in D(A)$. Summing up we deduce that $n_1 \in D(B)$.

Furthermore

$$\begin{aligned}
&Au(1) - f(1)\\
&= 2(I+Z)^{-1}e^{-B}Ad_0 - (I+Z)^{-1}(I-Z)Bn_1\\
&\quad - (I+Z)^{-1}\left[e^{-B}\int_0^1 Be^{-sB}f(s)\,ds - \int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds\right]\\
&\quad + (I+Z)^{-1}(I-e^{-B})f(1) - (I+Z)^{-1}(I+Z)f(1)\\
&= (I+Z)^{-1}[2e^{-B}Ad_0 - (I+Z-2Z)Bn_1] - (I+Z)^{-1}e^{-B}\int_0^1 Be^{-sB}f(s)\,ds\\
&\quad + (I+Z)^{-1}\int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds - (I+Z)^{-1}(e^{-B}+e^{-2B})f(1)\\
&= Bn_1 + 2(I+Z)^{-1}[e^{-B}Ad_0 + e^{-2B}Bn_1] - (I+Z)^{-1}e^{-B}\int_0^1 Be^{-sB}f(s)\,ds\\
&\quad + (I+Z)^{-1}\int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds - (I+Z)^{-1}(e^{-B}+e^{-2B})f(1).
\end{aligned}$$

Then

$$\begin{aligned}
Bn_1 &= [Au(1)-f(1)] - 2(I+Z)^{-1}[e^{-B}Ad_0 + e^{-2B}Bn_1]\\
&\quad + (I+Z)^{-1}e^{-B}\int_0^1 Be^{-sB}f(s)\,ds\\
&\quad - (I+Z)^{-1}\int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds + (e^{-B}+e^{-2B})(I+Z)^{-1}f(1)\\
&= \sum_{i=1}^{6} b_i.
\end{aligned}$$

Since $b_1$, $b_2$, $b_3$, $b_4$, $b_5$ and $b_6$ are in $\overline{D(A)}$, it hods that $Bn_1 \in \overline{D(A)}$.

Conversely, we assume that

$$d_0 \in D(A), \quad n_1 \in D(B),$$
$$Ad_0 - f(0) \in \overline{D(A)} \quad \text{and} \quad Bn_1 \in \overline{D(A)}.$$

From (8) we obtain

$$\begin{aligned}
u''(x) &= -(I+Z)^{-1}(I+e^{-2(1-x)B})e^{-xB}Ad_0 + (I+Z)^{-1}(I-e^{-2xB})e^{-(1-x)B}Bn_1\\
&\quad + \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(x+s)B}f(s)\,ds + \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x+s)B}f(s)\,ds\\
&\quad + \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2+x-s)B}f(s)\,ds - \frac{1}{2}(I+Z)^{-1}B\int_0^1 e^{-(2-x-s)B}f(s)\,ds\\
&\quad - \frac{1}{2}B\int_0^x e^{-(x-s)B}f(s)\,ds - \frac{1}{2}B\int_x^1 e^{-(s-x))B}f(s)\,ds + f(x).
\end{aligned}$$

We write

$$u''(x) = N(x, B)n_1 + D(x, A)d_0 + F(x, B) + G(x, B) + H(x, B) + f(x), \tag{10}$$

where

$$N(x, B)n_1 = (I + Z)^{-1}e^{-(1-x)B}Bn_1 - (I + Z)^{-1}e^{-(1+x)B}Bn_1,$$

$$D(x, A)d_0 = -(I + Z)^{-1}e^{-xB}(Ad_0 - f(0)) - (I + Z)^{-1}e^{-(2-x)B}Ad_0,$$

$$\begin{aligned} F(x, \ B) = {} & \frac{1}{2}(I + Z)^{-1}e^{-xB}\int_0^1 Be^{-sB}(f(s) - f(0))\,ds \\ & + \frac{1}{2}(I + Z)^{-1}e^{-(2-x)B}\int_0^1 Be^{-sB}f(s)\,ds \\ & + \frac{1}{2}(I + Z)^{-1}e^{-(1+x)B}\int_0^1 Be^{-(1-s)B}f(s)\,ds \\ & - \frac{1}{2}(I + Z)^{-1}e^{-(1-x)B}\int_0^1 Be^{-(1-s)B}(f(s) - f(1))\,ds, \end{aligned}$$

$$G(x, \ B) = -\frac{1}{2}\int_0^x Be^{-(x-s)B}(f(s) - f(x))\,ds - \frac{1}{2}\int_x^1 Be^{-(s-x))B}(f(s) - f(x))\,ds$$

and

$$\begin{aligned} H(x, \ B) = {} & -\frac{1}{2}(I + Z)^{-1}e^{-(1+x)B}f(0) - \frac{1}{2}(I + Z)^{-1}e^{-(2+x)B}f(0) \\ & + \frac{1}{2}(I + Z)^{-1}[e^{-(2-x)B}f(1) + Ze^{-(1-x))B}f(1)] \\ & + \frac{1}{2}e^{-(1-x))B}(f(x) - f(1)) + \frac{1}{2}e^{-xB}(f(x) - f(0)). \end{aligned}$$

In view of Lemmas 5 and 7, $N(\cdot, B)n_1$ and $D(\cdot, A)d_0$ are in $C([0, 1]; X)$ and $F(\cdot, B)$, $G(\cdot, B)$ and $H(\cdot, B)$ are continuous since $f \in C^\theta([0, 1]; X)$, from which we deduce that $u''$ is in $C([0, 1]; X)$. In the same way we prove that $Au$ is in $C([0, 1]; X)$.

Note that

$$\begin{aligned} Au(x) = {} & (I + Z)^{-1}[e^{-xB}Ad_0 + e^{-(2-x)B}Ad_0 - e^{-(1-x)B}Bn_1 + e^{-(1+x)B}Bn_1] \\ & - \frac{1}{2}(I + Z)^{-1}\left[e^{-xB}\int_0^1 Be^{-sB}f(s)\,ds + e^{-(2-x)B}\int_0^1 Be^{-sB}f(s)\,ds\right] \\ & - \frac{1}{2}(I + Z)^{-1}e^{-(1+x)B}\int_0^1 Be^{-(1-s)B}f(s)\,ds \\ & - \frac{1}{2}(I + Z)^{-1}e^{-(1-x)B}\int_0^1 Be^{-(1-s)B}f(s)\,ds \\ & + \frac{1}{2}\int_0^x Be^{-(x-s)B}f(s)\,ds + \frac{1}{2}\int_x^1 Be^{-(s-x))B}f(s)\,ds. \end{aligned}$$

Then

$$u''(x) + Au(x) = f(x). \qquad \square$$

Finally we obtain the following maximal regularity theorem:

**Theorem 9.** *Under* (3) *let* $f \in C^{\theta}([0,1];X)$, $0 < \theta < 1$. *Then the following assertions are equivalent.*

(1) *The unique solution* $u$ *of Problem* (1)–(2) *has the maximal regularity property*:

$$u'',\ Au \in C^{\theta}([0,1];X).$$

(2) *The elements* $d_0$ *and* $n_1$ *satisfy the conditions*

$$d_0 \in D(A),\ n_1 \in D(\sqrt{-A}),\ Ad_0 - f(0) \in (D(A),X)_{1-\theta/2,\infty}$$
$$\text{and} \quad \sqrt{-A}n_1 \in (D(A),X)_{1-\theta/2,\infty}.$$

Proof. Assume that there exists a strict solution $u$ of Problem (1)–(2) having the maximal regularity property. From the previous theorem, we have

$$d_0 \in D(A), \quad n_1 \in D(B).$$

Also the first and the second terms in formula (10) are in $C^{\theta}([0,1];X)$ and hence

$$e^{-B\cdot}(Ad_0 - f(0)) \in C^{\theta}([0,1];X),$$
$$e^{(1-\cdot)B}Bn_1 \in C^{\theta}([0,1];X).$$

Using [17, Remark], we have

$$Ad_0 - f(0) \in (D(B),X)_{1-\theta,\infty},$$
$$Bn_1 \in (D(B),X)_{1-\theta,\infty}.$$

We finish the proof of (1) $\Longrightarrow$ (2) if we note that

$$(D(B),X)_{1-\theta,\infty} = (D(A),X)_{1-\theta/2,\infty}.$$

Conversely assume that

$$d_0 \in D(A),\ n_1 \in D(B),\ Ad_0 - f(0) \in (D(A),X)_{1-\theta/2,\infty}$$
$$\text{and} \quad Bn_1 \in (D(A),X)_{1-\theta/2,\infty}.$$

Using [3, Theorem 1.4] we have

$$
\begin{aligned}
&e^{-\sqrt{-A}\cdot}(Ad_0 - f(0)) \in C^\theta([0,1];X),\\
&e^{(1-\cdot)\sqrt{-A}}Bn_1 \in C^\theta([0,1];X),\\
&\int_0^1 Be^{-sB}(f(s)-f(0))\,ds \in C^\theta([0,1];X),\\
&\int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds \in C^\theta([0,1];X),
\end{aligned}
$$

and thus

$$u'',\ Au \in C^\theta([0,1];X). \qquad \square$$

**Proposition 10.** *Under* (3) *let* $f \in C^\theta([0,1];X)$, $0<\theta<1$, *and assume that*

$$
\begin{aligned}
&d_0 \in D(A),\ n_1 \in D(\sqrt{-A}),\ Ad_0 - f(0) \in (D(A),X)_{1-\theta/2,\infty}\\
&\text{and}\quad \sqrt{-A}n_1 \in (D(A),X)_{1-\theta/2,\infty}.
\end{aligned}
$$

*Then the unique strict solution* $u$ *of Problem* (1)–(2) *with the maximal regularity property*:

$$u'',\ Au \in C^\theta([0,1];X)$$

*has also the cross-regularity*

$$Au(\cdot) - f(\cdot) \in B([0,1];(D(A),X)_{1-\theta/2,\infty}).$$

Proof. We recall that

$$
\begin{aligned}
&Au(x) - f(x)\\
&= (I+Z)^{-1}[e^{-xB}(Ad_0 - f(0)) + e^{-(1-x)B}Bn_1 + e^{-(2-x)B}Ad_0 - e^{-(1+x)B}Bn_1]\\
&\quad - \frac{1}{2}(I+Z)^{-1}\left[e^{-xB}\int_0^1 Be^{-sB}(f(s)-f(0))\,ds + e^{-(2+x)B}f(0) - e^{-(1+x)B}f(0)\right]\\
&\quad - \frac{1}{2}(I+Z)^{-1}\left[e^{-(2-x)B}\int_0^1 Be^{-sB}f(s)\,ds + e^{-(1+x)B}\int_0^1 Be^{-(1-s)B}f(s)\,ds\right]\\
&\quad + \frac{1}{2}(I+Z)^{-1}\left[e^{-(1-x)B}\int_0^1 Be^{-(1-s)B}(f(s)-f(1))\,ds - e^{-(2-x)B}f(1)\right]\\
&\quad + \frac{1}{2}(I+Z)^{-1}Ze^{-(1-x))B}f(1) + \frac{1}{2}\int_0^x Be^{-(x-s)B}(f(s)-f(x))\,ds\\
&\quad - \frac{1}{2}e^{-xB}(f(x)-f(0)) + \frac{1}{2}\int_x^1 Be^{-(s-x))B}(f(s)-f(x))\,ds\\
&\quad - \frac{1}{2}e^{-(1-x))B}(f(x)-f(1))\\
&= \sum_{i=1}^{16} k_i(x).
\end{aligned}
$$

Note that

$$(D(A), X)_{1-\theta/2,\infty} = (D(B), X)_{1-\theta,\infty}.$$

So in order to prove that $(Au(x) - f(x)) \in (D(A), X)_{1-\theta/2,\infty}$ it suffices to show that

$$\sup_{t>0} \|t^{-\theta}(e^{-tB} - I)[Au(x) - f(x)]\|_X \leq K.$$

As $Ad_0 - f(0) \in (D(A), X)_{1-\theta/2,\infty}$ and $\sqrt{-A}n_1 \in (D(A), X)_{1-\theta/2,\infty}$, it follows that $k_1(x)$ and $k_2(x)$ are in $(D(A), X)_{1-\theta/2,\infty}$.

It is clear that for $i = 3, 4, 6, 7, 8, 9, 11$ and $12$, $k_i(x)$ are in $D(B)$ and hence in $(D(B), X)_{1-\theta,\infty}$.

Concerning $k_{13}$, we have

$$\begin{aligned}
&\|e^{-tB}k_{13}(x) - k_{13}(x)\|_X \\
&= \left\| \frac{1}{2}\left[ e^{-tB} \int_0^x Be^{-(x-s)B}(f(s) - f(x))\, ds - \int_0^x Be^{-(x-s)B}(f(s) - f(x))\, ds \right] \right\|_X \\
&= \left\| \frac{1}{2} \int_0^x [Be^{-(x+t-s)B} - Be^{-(x-s)B}](f(s) - f(x))\, ds \right\|_X \\
&= \left\| \frac{1}{2} \int_0^x \int_{x-s}^{x+t-s} B^2 e^{-\sigma B}(f(s) - f(x))\, d\sigma\, ds \right\|_X \\
&\leq C \int_0^x \int_{x-s}^{x+t-s} \sigma^{-2} \|f(s) - f(x)\|_X\, d\sigma\, ds \\
&\leq C \int_0^x \int_{x-s}^{x+t-s} \sigma^{-2}(x-s)^\theta \|f\|_{C^\theta(X)}\, d\sigma\, ds \\
&\leq C \int_0^x \int_y^{y+t} \sigma^{-2} y^\theta \|f\|_{C^\theta(X)}\, d\sigma\, dy \leq C \int_0^x y^\theta \left[ \frac{1}{y} - \frac{1}{y+t} \right] \|f\|_{C^\theta(X)}\, dy \\
&\leq C \int_0^{x/t} (ut)^\theta \left[ \frac{1}{ut} - \frac{1}{t(u+1)} \right] \|f\|_{C^\theta(X)} t\, du \\
&\leq C \int_0^{x/t} u^\theta t^\theta \left[ \frac{1}{u} - \frac{1}{u+1} \right] \|f\|_{C^\theta(X)}\, du \leq Ct^\theta \|f\|_{C^\theta(X)}.
\end{aligned}$$

For $k_{14}$ we have

$$\begin{aligned}
&\|e^{-tB}k_{14}(x) - k_{14}(x)\|_X \\
&= \left\| \frac{1}{2} \int_x^{x+t} Be^{-\sigma B}(f(x) - f(0))\, d\sigma \right\|_X \\
&\leq C \int_x^{x+t} \sigma^{-1} \|f(x) - f(0)\|_X\, d\sigma \leq C \int_x^{x+t} \sigma^{-1} x^\theta \|f\|_{C^\theta(X)}\, d\sigma \\
&\leq C \int_x^{x+t} \sigma^{-1} \sigma^\theta \|f\|_{C^\theta(X)}\, d\sigma \leq Ct^\theta \|f\|_{C^\theta(X)}.
\end{aligned}$$

For $k_5(x)$ we can write

$$B\int_0^1 e^{-(x+s)B} f(s)\,ds = B\int_0^x e^{-(x-s)B}e^{-2sB} f(s)\,ds + e^{-2xB}B\int_x^1 e^{-(s-x)B} f(s)\,ds.$$

In this way all the terms are seen to be in $(D(B), X)_{1-\theta,\infty}$. □

## 5. A priori estimates

**Proposition 11.** *Under* (3) *let* $f \in C^\theta([0,1]; X)$, $0 < \theta < 1$, *and assume that*

$$d_0 \in D(A),\ n_1 \in D(\sqrt{-A}),\ \ Ad_0 - f(0) \in (D(A), X)_{1-\theta/2,\infty}$$
$$\textit{and}\quad \sqrt{-A}n_1 \in (D(A), X)_{1-\theta/2,\infty}.$$

*Then* $\exists C > 0$:

$$\|u''\|_{C(X)} + \|Au\|_{C(X)} \le C[\|f\|_{C^\theta(X)} + \|Ad_0 - f(0)\|_X + \|\sqrt{-A}n_1\|_X]$$

*and*

$$\|u''\|_{C^\theta(X)} + \|Au\|_{C^\theta(X)}$$
$$\le C[\|f\|_{C^\theta(X)} + \|Ad_0 - f(0)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|\sqrt{-A}n_1\|_{(D(A),X)_{1-\theta/2,\infty}}].$$

Proof. Writing $u''$ as in formula (10) we get

$$\max_{0\le x\le 1}\|u''(x)\|_X \le C[\|f\|_{C^\theta(X)} + \|Ad_0 - f(0)\|_X + \|Bn_1\|_X].$$

This gives the proof of statement 1. For statement 2 it suffices to prove that

$$\max_{\substack{0\le x,t\le 1\\ x\ne t}}\|u''(x) - u''(t)\|_X$$
$$\le C|x-t|^\theta[\|f\|_{C^\theta(X)} + \|Ad_0 - f(0)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|\sqrt{-A}n_1\|_{(D(A),X)_{1-\theta/2,\infty}}].$$

In formula (10) we define

$$u''(x) = \sum_{i=1}^{17} h_i(x).$$

Then

$$\begin{aligned}\|h_1(x) - h_1(t)\|_X &= \|-(I+Z)^{-1}e^{-xB}(I - e^{-(t-x)B})(Ad_0 - f(0))\|_X\\ &\le \|e^{-xB}(I - e^{-(t-x)B})(Ad_0 - f(0))\|_X\\ &\le C|t-x|^\theta\|Ad_0 - f(0)\|_{(D(A),X)_{1-\theta/2,\infty}}.\end{aligned}$$

Similarly

$$\|h_2(x) - h_2(t)\|_X \le C|t-x|^\theta \|Bn_1\|_{(D(A),X)_{1-\theta/2,\infty}}.$$

For $h_3$ we write

$$\begin{aligned} h_3(x) &= -(I+Z)^{-1}e^{-(2-x)B}Ad_0 \\ &= -(I+Z)^{-1}e^{-(2-x)B}(Ad_0 - f(0)) - (I+Z)^{-1}e^{-(2-x)B}f(0). \end{aligned}$$

As above, the first term is seen to be in $C^\theta([0,1];X)$ and the second is the same. As for $h_{13}$ we have

$$\begin{aligned} &h_{13}(x) - h_{13}(t) \\ &= \frac{1}{2}\int_t^x Be^{-(x-s)B}(f(s)-f(x))\,ds + \frac{1}{2}\int_0^t \int_{t-s}^{x-s} B^2 e^{-\sigma B}(f(s)-f(t))\,d\sigma\,ds \\ &\quad + \frac{1}{2}(e^{-xB} - e^{-(x-t)B})(f(t)-f(x)). \end{aligned}$$

The first and last terms are clearly estimated by $C|t-x|^\theta\|f\|_{C^\theta(X)}$. For the second term, we estimate

$$\begin{aligned} &\left\| \frac{1}{2}\int_0^t \int_{t-s}^{x-s} B^2 e^{-\sigma B}(f(s)-f(t))\,d\sigma\,ds \right\|_X \\ &\le C\|f\|_{C^\theta(X)} \int_0^t |s-t|^\theta \int_{t-s}^{x-s} \sigma^{-2}\,d\sigma\,ds \\ &\le C\|f\|_{C^\theta(X)}(x-t)\int_0^t (t-s)^{\theta-1}(x-s)^{-1}\,ds \\ &\le C\|f\|_{C^\theta(X)}(x-t)^\theta \int_0^{t/x-t} \nu^{\theta-1}(1+\nu)^{-1}\,d\nu \le C|x-t|^\theta\|f\|_{C^\theta(X)}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} &\|h_{14}(x) - h_{14}(t)\|_X \\ &= \left\| \frac{1}{2}e^{-xB}(f(x)-f(t)) + \frac{1}{2}\int_t^x Be^{-sB}(f(t)-f(0))\,ds \right\|_X \\ &\le C|t-x|^\theta\|f\|_{C^\theta(X)} + \int_t^x \frac{C}{s}t^\theta\,ds\|f\|_{C^\theta(X)} \\ &\le C|t-x|^\theta\|f\|_{C^\theta(X)} + \int_t^x \frac{C}{s}s^\theta\,ds\|f\|_{C^\theta(X)} \le C|t-x|^\theta\|f\|_{C^\theta(X)}. \end{aligned}$$

In this way, we finally prove that

$$\begin{aligned} \|u''(x) - u''(t)\|_X \le C|t-x|^\theta[\|f\|_{C^\theta(X)} + \|Ad_0 - f(0)\|_{(D(A),X)_{1-\theta/2,\infty}} \\ + \|\sqrt{-A}n_1\|_{(D(A),X)_{1-\theta/2,\infty}}]. \end{aligned}$$

Hence we complete the proof. □

## 6. Concrete applications

**6.1. An anisotropic interpolation result.** Set

$$\begin{cases} \mathcal{E}_\theta = \{u \in C^{2+\theta}([0,1];X) \cap C^\theta([0,1];D(A)) \colon u''(0) \in (D(A),X)_{1-\theta/2,\infty}\}, \\ \mathcal{T}_\theta = D(A) \times D_{\sqrt{-A}}(\theta+1,\infty), \end{cases}$$

where

$$D_{\sqrt{-A}}(\theta+1,\infty) = \{\xi \in D_{\sqrt{-A}} \colon \sqrt{-A}\xi \in (D(A),X)_{1-\theta/2,\infty}\}.$$

**Proposition 12.** *The mapping* $J$

$$J \colon \mathcal{E}_\theta \to \mathcal{T}_\theta,$$
$$u \mapsto (u(0), u'(1))$$

*is well-defined, linear, continuous and bijective.*

Proof. The mapping $J$ is obviously linear. Let $u \in \mathcal{E}_\theta$, then

$$u'', Au \in C^\theta([0,1];X)$$

and hence

$$\begin{cases} u''(x) + Au(x) = u''(x) + Au(x) := g(x), & \\ u(0) = d_0 & \text{in} \quad D(A), \\ u'(1) = n_1 & \text{in} \quad X, \end{cases}$$

furthermore

$$u \in C^{2+\theta}([0,1];X) \cap C^\theta([0,1];D(A)).$$

From Theorem 9, we obtain that

$$d_0 \in D(A), n_1 \in D(\sqrt{-A})$$

and

$$Ad_0 - g(0) \in (D(A),X)_{1-\theta/2,\infty}, \quad \sqrt{-A}n_1 \in (D(A),X)_{1-\theta/2,\infty},$$

thus $(d_0, n_1) \in \mathcal{T}_\theta$. In order to prove the surjection, let $(d_0, n_1) \in \mathcal{T}_\theta$. Then Problem (1)–(2) has a unique strict solution $u$ such that

$$u \in C^{2+\theta}([0,1];X) \cap C^\theta([0,1];D(A)),$$

where

$$u''(0) = -Ad_0 + g(0) \in (D(A),X)_{1-\theta/2,\infty}$$

so $\exists! u \in \mathcal{E}_\theta$ such that $J(u) = (d_0, n_1)$, then $J$ is bijective. Let $(d_0, n_1) \in \mathcal{T}_\theta$, then

$$\begin{aligned}\|J(u)\|_{\mathcal{T}_\theta} &= \|(d_0, n_1)\|_{D(A)\times D_{\sqrt{-A}}(\theta+1,\infty)} = \sup\{\|d_0\|_{D(A)}, \|n_1\|_{D_{\sqrt{-A}}(\theta+1,\infty)}\} \\ &\le \sup\{\|d_0\|_{D(A)}, \|\sqrt{-A}n_1\|_{(D(A),X)_{1-\theta/2,\infty}}\}.\end{aligned}$$

We have on one hand

$$\begin{aligned}\|d_0\|_{D(A)} &\le \|Ad_0\|_X \\ &\le \|Ad_0 - u''(0) + u''(0)\|_X \\ &\le \|u''(0)\|_X + \|Ad_0 - u''(0)\|_X \\ &\le c\left[\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}} + \sup_{x\in[0,1]}\|Au(x)\|_X + \sup_{x\in[0,1]}\|u''(x)\|_X\right] \\ &\le c[\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|Au\|_{C([0,1],X)} + \|u''\|_{C([0,1],X)}] \\ &\le c[\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|Au\|_{C^\theta([0,1],X)} + \|u''\|_{C^\theta([0,1],X)}] \\ &\le c\|u\|_{\mathcal{E}_\theta}.\end{aligned}$$

On the other hand, from formula (9) we have

$$\begin{aligned}&\|n_1\|_{D_{\sqrt{-A}}(\theta+1,\infty)} \\ &= \|\sqrt{-A}n_1\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &= \|(I-Z)^{-1}(I+Z)Au(1) + 2(I-Z)^{-1}e^{-\sqrt{-A}}Ad_0\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &\quad + \left\|(I-Z)^{-1}\sqrt{-A}\int_0^1 e^{-(1+s)\sqrt{-A}}g(s)\,ds\right\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &\quad + \left\|(I-Z)^{-1}\sqrt{-A}\int_0^1 e^{-(1-s)\sqrt{-A}}g(s)\,ds\right\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &\le \|(I-Z)^{-1}(I+Z)Au(1)\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &\quad + \|2(I-Z)^{-1}e^{-\sqrt{-A}}Ad_0\|_{(D(A),X)_{1-\theta/2,\infty}} + c\|g\|_{C^\theta([0,1];X)} \\ &\le c[\|Au(1)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|Ad_0\|_{(D(A),X)_{1-\theta/2,\infty}}] \\ &\quad + c[\|Au\|_{C^\theta([0,1],X)} + \|u''\|_{C^\theta([0,1],X)}] \\ &\le c[\|Au(1)\|_{(D(A),X)_{1-\theta/2,\infty}} + \|Ad_0 + u''(0) - u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}} \\ &\qquad + \|Au\|_{C^\theta([0,1],X)} + \|u''\|_{C^\theta([0,1],X)}].\end{aligned}$$

Then

$$
\begin{aligned}
&\|n_1\|_{D_{\sqrt{-A}}(\theta+1,\infty)}\\
&\leq c\Big[\sup_{x>0}\|x^{-\theta}(e^{-x\sqrt{-A}}-1)Au(1)\|_X+\sup_{x>0}\|x^{-\theta}(e^{-x\sqrt{-A}}-1)(Ad_0+u''(0))\|_X\\
&\qquad+\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}}+\|Au\|_{C^\theta([0,1],X)}+\|u''\|_{C^\theta([0,1],X)}\Big]\\
&\leq c\Big[\sup_{x>0}\|Au(x)\|_X+\sup_{x>0}\|u''(x)\|_X+\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}}\Big]\\
&\quad+c\|Au\|_{C^\theta([0,1],X)}+\|u''\|_{C^\theta([0,1],X)}\\
&\leq c[\|Au\|_{C^\theta([0,1],X)}+\|u''\|_{C^\theta([0,1],X)}+\|u''(0)\|_{(D(A),X)_{1-\theta/2,\infty}}]\\
&\leq c\|u\|_{\mathcal{E}_\theta}.
\end{aligned}
$$

Finally we deduce that

$$\exists c>0/\forall u\in\mathcal{E}_\theta,\quad \|J(u)\|_{\mathcal{T}_\theta}\leq c\|u\|_{\mathcal{E}_\theta}.$$ □

**Corollary 13.** *Let $u\in\mathcal{E}_\theta$ and $f\in C^\theta([0,1];X)$ then*

$$Au(\cdot)-f(\cdot)\in B([0,1];(D(A),X)_{1-\theta/2,\infty}).$$

Proof. The proof is a direct consequence of Propositions 10 and 12. □

**6.2. Example 1.** Let $X=L^2(\mathbb{R})$ and let

$$\begin{cases} D(A)=H^2(\mathbb{R}),\\ Au=u''.\end{cases}$$

It is well-known that $D(A)$ is dense in $X$. Set

$$\begin{cases}\mathcal{G}_\theta=\{u\in C^{2+\theta}([0,1];L^2(\mathbb{R}))\cap C^\theta([0,1];H^2(\mathbb{R}))\colon u''(0)\in(H^2(\mathbb{R}),L^2(\mathbb{R}))_{1-\theta/2,\infty}\},\\ \mathcal{N}_\theta=H^2(\mathbb{R})\times\mathcal{H}_\theta,\end{cases}$$

and

$$\mathcal{H}_\theta=\{\xi\in D(\sqrt{-A})\colon \sqrt{-A}\xi\in(H^2(\mathbb{R}),L^2(\mathbb{R}))_{1-\theta/2,\infty}=B^\theta_{2,\infty}(\mathbb{R})\}.$$

The following result is a consequence of Proposition 12:

**Proposition 14.** *The mapping $M$*

$$\begin{aligned} M\colon \mathcal{G}_\theta&\to\mathcal{N}_\theta,\\ u&\mapsto(u(0),u'(1))\end{aligned}$$

*is well-defined, linear, continuous and bijective.*

**6.3. Example 2.** Let $X = C([0,1])$ and let

$$\begin{cases} D(A) = \{u \in C^2([0,1]) : u(0) = u(1) = 0\}, \\ Au = u''. \end{cases}$$

Here $D(A)$ is not dense in $X$ since

$$\overline{D(A)} = \{u \in C([0,1]) : u(0) = u(1) = 0\}.$$

The characterization of $D(\sqrt{-A})$ is difficult, however we know that

$$D(\sqrt{-A}) \subset (D(A), X)_{1/2,\infty} = \{u \in C^1_*([0,1]) : u(0) = u(1) = 0\},$$

where

$$C^1_*([0,1]) = \left\{ u \in C([0,1] : \sup_{x,y,(x+y)/2 \in [0,1]} \left( \frac{|u(x) + u(y) - 2u((x+y)/2)|}{|x-y|} \right) < \infty \right\}.$$

Note that

$$\begin{aligned} &(D(A), C([0,1]))_{1-\theta/2,\infty} \\ &= \{u \in (C^2([0,1]), C([0,1]))_{1-\theta/2,\infty} : u(0) = u(1) = 0\} \\ &= \{u \in C^\theta([0,1]) : u(0) = u(1) = 0\}, \end{aligned}$$

see [19, 2.7.2]. Let

$$\begin{cases} \mathcal{C}_\theta = \{u \in C^{2+\theta}([0,1]; C([0,1])) \cap C^\theta([0,1]; D(A)) : \\ \qquad u''(0) \in (D(A), C([0,1]))_{1-\theta/2,\infty}\}, \\ \mathcal{T}_\theta = D(A) \times \mathcal{K}_\theta \end{cases}$$

where

$$\mathcal{K}_\theta = \{\xi \in H^1(\mathbb{R}) : \sqrt{-A}\xi \in (D(A), C([0,1]))_{1-\theta/2,\infty}\}.$$

We can apply Proposition 12 to obtain

**Proposition 15.** *The mapping* $N$

$$\begin{aligned} N : \mathcal{C}_\theta &\to \mathcal{T}_\theta, \\ u &\mapsto (u(0), u'(1)) \end{aligned}$$

*is well-defined, linear, continuous and bijective.*

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