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SYMPLECTIC BIRATIONAL TRANSFORMATIONS OF THE PLANE

JÉRÉMY BLANC

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Abstract

We study the group of symplectic birational transformations of the plane. It is proved that this group is generated by $\mathrm{SL}(2, \mathbb{Z})$, the torus and a special map of order 5, as it was conjectured by A. Usnich.

Then we consider a special subgroup H, of finite type, defined over any field which admits a surjective morphism to the Thompson group of piecewise linear automorphisms of $\mathbb{Z}^2$. We prove that the presentation for this group conjectured by Usnich is correct.

1. Introduction

1.1. The group *Symp*. Recall that a rational map $f\colon \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$—or a *rational transformation* of $\mathbb{C}^2$—is given by

$$(x, y) \dashrightarrow (f_1(x, y),\ f_2(x, y)),$$

where f_1, f_2 are two rational functions (quotients of polynomials) in two variables. The map f is said to be *birational* if it admits a inverse of the same type, which is equivalent to say that f is locally bijective, or that f induces an isomorphism between two open dense subsets of $\mathbb{C}^2$. The group of birational maps of $\mathbb{C}^2$ is the famous *Cremona group*.

Following [4], we define *Symp* as the group of symplectic birational transformations of the plane, which is the group of birational transformations of $\mathbb{C}^2$ which preserve the differential form

$$\omega_0 = \frac{dx \wedge dy}{xy}.$$

In [4], a natural surjective morphism from *Symp* to the Thompson group of piecewise linear automorphisms of $\mathbb{Z}^2$ is constructed (see also [3]) although the Thompson group is not embedded in the Cremona group. The group *Symp*, related to other topics of mathematics, is also an interesting subgroup of the Cremona group, from the geometric point of view. The base-points of its elements are poles of the differential form

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ω_0, but its elements can contract curves which are not poles of ω_0. In this article, we describe the geometry of elements of *Symp*, and give proofs to two conjectures of [4] (Theorems 1 and 2 below).

1.2. The results. The two groups $\mathrm{SL}(2,\mathbb{Z})$ and $(\mathbb{C}^*)^2$ naturally are embedded into *Symp*; the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ corresponds to the map $(x,y) \dashrightarrow (x^a y^b, x^c y^d)$, and the pair $(\alpha,\beta) \in (\mathbb{C}^*)^2$ corresponds to $(x,y) \dashrightarrow (\alpha x, \beta y)$. Moreover, the map $P\colon (x,y) \dashrightarrow (y,(y+1)/x)$, of order 5, is also an element of *Symp*. Our first main result consists of proving the following result, conjectured in [4]:

Theorem 1. *The group Symp is generated by* $\mathrm{SL}(2,\mathbb{Z})$, $(\mathbb{C}^*)^2$ *and* P.

The map P is a well-known linearisable map ([2]), and the group $\langle \mathrm{SL}(2,\mathbb{Z}), (\mathbb{C}^*)^2 \rangle$ is a toric well-understood group. The mix of this group with P provides all the complexity to *Symp*. In the proof, the reader can see that all non-toric base-points come from P, but in fact, there are many relations in *Symp*, and we can have complicated elements with many non-toric base-points.

However, the natural subgroup $H \subset$ *Symp* generated by $\mathrm{SL}(2,\mathbb{Z})$ and P is easier to understand. It is an interesting subgroup of finite type of the Cremona group, which is moreover defined over $\mathbb{Q}$ or over any field. We write C, I the elements $C = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$. The presentation

$$\mathrm{SL}(2,\mathbb{Z}) = \langle I, C \mid I^4 = C^3 = [C, I^2] = 1 \rangle$$

is classical. We will prove the following result on the relations of H, conjectured in [4]:

Theorem 2. *The following is a presentation of the group* H:

$$H = \langle I, C, P \mid I^4 = C^3 = [C, I^2] = P^5 = 1, PCP = I \rangle.$$

The author thanks S. Galkin for asking him these questions in the Workshop on the Cremona group organised by I. Cheltsov in Edinburgh in the beginning of 2010.

2. Some reminders on birational transformations

Recall that any birational transformation of $\mathbb{C}^2$ extends to a unique birational transformation of the projective complex plane $\mathbb{P}^2$ (written also $\mathbb{P}^2_{\mathbb{C}}$ or $\mathbb{C}\mathbb{P}^2$) via the embedding $(x,y) \mapsto (x:y:1)$. We will take X, Y, Z as homogeneous coordinates on $\mathbb{P}^2$, so that the affine coordinates x, y on $\mathbb{C}^2$ correspond to $x = X/Z$ and $y = Y/Z$. Any birational transformation φ of $\mathbb{P}^2$ can be written as

$$\varphi\colon (X:Y:Z) \dashrightarrow (P_1(X,Y,Z):P_2(X,Y,Z):P_3(X,Y,Z)),$$

where the P_i are homogeneous polynomials of the same degree without common factor. The degree of the map is the degree of the P_i. If this one is >1, then there is a finite number of points of $\mathbb{P}^2$ where φ is not defined, which corresponds to the set of common zeros of P_1, P_2, P_3.

More generally, the base-points of φ are the points where all curves of the linear system $\sum \lambda_i P_i$, $\lambda_i \in \mathbb{C}$ pass through. Note that these points are not necessarily on $\mathbb{P}^2$, but maybe in some blow-up, and correspond thus to some tangent directions. See for example [1] for more details.

3. Normal cubic forms and geometric descriptions

Recall that the divisor of a differential form on $\mathbb{P}^2$ is a divisor of degree -3. In particular, the divisor corresponding to ω_0 on $\mathbb{P}^2$ is $-(X)-(Y)-(Z)$.

DEFINITION 3.1. We say that a differential form ω on $\mathbb{P}^2$ is a *normal cubic form* if $-\mathrm{div}(\omega)$ is the divisor of a (possibly reducible) singular cubic, whose singular points are ordinary double points (in particular we ask that $-\mathrm{div}(\omega)$ is effective and reduced).

Note that in the above definition, $-\mathrm{div}(\omega)$ can be either (i) the union of three lines with exactly three double points, (ii) the union of a smooth conic and a line intersecting into two distinct points, (iii) an irreducible cubic curve having a unique ordinary double point. The form ω_0 is a normal cubic form of type (i).

Before using the above definition, we remind the reader the following simple result, already observed in [4].

Lemma 3.2. *Let ω be a differential form on a smooth algebraic surface S and let $\eta\colon \hat{S} \to S$ be the blow-up of $q \in S$. We write $D = \mathrm{div}(\omega)$ the divisor of ω, $\tilde{D}$ its strict transform on $\hat{S}$, and E the exceptional curve contracted by η.*

Then $\mathrm{div}(\eta^*(\omega)) = \tilde{D} + (m+1)E$, where $m \in \mathbb{Z}$ is the multiplicity of D at q. In particular,

(1) *E is a zero of $\mathrm{div}(\eta^*(\omega)) \Leftrightarrow D$ has multiplicity ≥ 0 at q.*
(2) *E is a pole of $\mathrm{div}(\eta^*(\omega)) \Leftrightarrow D$ has multiplicity ≤ -2 at q;*
(3) *E is a pole of multiplicity one of $\mathrm{div}(\eta^*(\omega)) \Leftrightarrow D$ has multiplicity -2 at q.*

Proof. Let us take some local coordinates u, v on S at q so that this point corresponds to $u = v = 0$. The form ω locally corresponds to $\varphi(u,v)\cdot du \wedge dv$, where φ is a rational function in two variables, and D corresponds to $(\varphi(u,v))$.

The blow-up can be viewed locally as $(u,v)\mapsto(uv,v)$, and $\eta^*(\omega)$ becomes $\varphi(uv,v)\cdot d(uv)\wedge dv=\varphi(uv,v)v\cdot du\wedge dv$. In these coordinates, v is the equation of the divisor E and $\varphi(uv,v)$ corresponds to $\eta^*(\mathrm{div}(\omega))$. Moreover $\varphi(uv,v)=v^m\cdot\psi(u,v)$, where $m\in\mathbb{Z}$ is the multiplicity of D at q (which is the multiplicity of φ at $(0,0)$), and where $\psi(0,0)\in\mathbb{C}^*$. Observing that $\psi(u,v)$ corresponds to $\tilde{D}$, we obtain the result. □

DEFINITION 3.3. Let S be a smooth surface, let ω be a differential form on S and let $p\in S$. We define the multiplicity of ω at p to be the multiplicity of $\mathrm{div}(\omega)$ at p. If this multiplicity is negative, we say that p is a pole of ω.

We can now relate the base points of birational maps to the image of normal cubic forms. The following proposition deals with base-points of a birational map φ of $\mathbb{P}^2$, which belong to $\mathbb{P}^2$ or to blow-ups of $\mathbb{P}^2$. Saying that the points are pole of ω corresponds to use the above definition with the lift of the differential form on the corresponding blow-up of $\mathbb{P}^2$.

Proposition 3.4. *Let $\varphi\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^2$ be a birational map, and let ω be a normal cubic form. The following are equivalent*:

(1) *All base-points of φ are poles of the transform of ω*;

(2) *The form $\varphi_*(\omega)$ is a normal cubic form.*

Proof. If φ has no base-point, both assertions are trivially true, so we may assume that φ has at least one base-point.

We denote by $\eta\colon S\to\mathbb{P}^2$ the blow-up of all base-points of φ, and by $\epsilon\colon S\to\mathbb{P}^2$ the morphism $\varphi\eta$, which is the blow-up of all base-points of φ^{-1}.

Suppose first that at least one base-point q of φ (which may be infinitely near to $\mathbb{P}^2$) is not a pole of ω. By Lemma 3.2, the exceptional curve of this point, and of all infinitely near points, are zeros of $\eta^*(\omega)$. Since q is a base-point, at least one of these curves is not contracted by η, and thus $\varphi_*(\omega)=\epsilon_*(\eta^*(\omega))$ has zeros; it is therefore not a normal cubic form.

Suppose now that all base-points of φ are poles of ω. If $-\mathrm{div}(\omega)$ is an irreducible cubic curve, it has a unique ordinary double point, we assume that η blows-up this point, by replacing η by its composition with the blow-up if needed, obtaining another (non-minimal) resolution of φ. We now prove the following assertion:

The divisor $D_S=-\mathrm{div}(\eta^*(\omega))$ is linearly equivalent to $-K_S$ and is an effective reduced divisor consisting of a loop of smooth rational curves (*i.e. a finite number of smooth rational curves where each one intersect exactly two others, and each intersection is transversal*).

Firstly, since $\mathrm{div}(\omega)$ is linearly equivalent to $K_{\mathbb{P}^2}$, by definition of the canonical divisor. Secondly, we recall that $-\mathrm{div}(\omega)$ is an effective divisor, and that it is either a loop of smooth rational curves or an irreducible nodal cubic curve. In this latter case, writing $\mu\colon\mathbb{F}_1\to\mathbb{P}^2$ the blow-up of the singular point, $-\mathrm{div}(\mu^*(\omega))$ is the union of the exceptional curve with the strict transform of the cubic, and is thus a loop of smooth rational curves. We proceed then by induction on the number of points blown-up by η, applying Lemma 3.2 at each step; blowing-up a smooth point on a loop does not change the structure of the loop, and blowing-up a singular point only adds one component. The assertion is now clear.

The fact that D_S is an effective divisor linearly equivalent to $-K_S$ implies that $D = -\mathrm{div}(\varphi_*(\omega)) = -\mathrm{div}(\epsilon_*(\eta^*(\omega))) = \epsilon_*(D_S)$ is an effective divisor linearly equivalent to $-K_{\mathbb{P}^2}$, and is thus a cubic curve. All components of D_S being rational, D cannot be smooth. It remains to see that all singular points of D are ordinary double points. Writing $\omega' = \varphi_*(\omega)$, if $D = -\mathrm{div}(\omega')$ had one other singularities, we can check using Lemma 3.2 that $D_S = -\mathrm{div}(\epsilon^*(\omega'))$ would not be a loop. □

4. Decomposition into quadratic maps

It is well known that any birational transformation of the plane decomposes into quadratic maps. Using Proposition 3.4, we can deduce the same for elements which send a normal cubic form on another one (Lemma 4.1), and then with a more careful study to elements which preserve the divisor of ω_0 (Proposition 4.2).

Lemma 4.1. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map of degree $d > 1$, and let ω be a normal cubic form. If $\varphi_*(\omega)$ is a normal cubic form, there exist quadratic transformations $\phi_1, \dots, \phi_n$ such that*
(1) $\varphi = \phi_n \circ \cdots \circ \phi_1$;
(2) *for $i = 1, \dots, n$, $(\phi_i \circ \cdots \circ \phi_1)_*(\omega)$ is a normal cubic form.*

Proof. We start as in the classical proof of Noether–Castelnuovo theorem, by taking a de Jonquières transformation ψ (a birational map of $\mathbb{P}^2$ which preserves a pencil of lines) such that each base-point of ψ is a base-point of φ and $\varphi\psi^{-1}$ has degree $< d$. The existence of such a ψ can be checked for example in Chapter 8 of [1] (see in particular the proof of Theorem 8.3.4).

Since all base-points of φ are poles of ω (Proposition 3.4), the same is true for ψ, so $\psi_*(\omega)$ is a normal cubic form.

It remains thus to prove the lemma in the case where φ is a de Jonquières transformation of degree $d > 1$, which preserves the pencil of lines passing through $s \in \mathbb{P}^2$. We prove the result by induction on d, the case $d = 2$ being clear. We follow the classical proof of the theorem of Noether–Castelnuovo.

The linear system of φ (which is the pull-back by φ of the system of lines of the plane) consists of curves of degree d passing through s with multiplicity $d-1$ and through $2d-2$ other points $t_1, \dots, t_{2d-2}$ with multiplicity one.

If at least one of the t_i's is a proper point of $\mathbb{P}^2$, say t_1, there exists another t_j, say t_2, and a quadratic de Jonquières transformation ϕ_1 with base-points s, t_1, t_2. The linear systems of ϕ_1 and φ intersecting into $d-1$ free points, the map $\varphi \circ (\phi_1)^{-1}$ is a de Jonquières transformation of degree $d-1$. Since $(\phi_1)_*(\omega)$ is a normal cubic form, the result follows from the induction hypothesis.

If no one of the t_i's is a proper point of the plane, there exists at least one of these, say t_1, which corresponds to a tangent direction of s, and another point t_j, say t_2, which is infinitely near to t_1. We choose a proper point u in $\mathbb{P}^2$ which is a pole of ω and which is not aligned with s and t_1. There exists a quadratic de Jonquières transformation ϕ_1 with base-points s, t_1, u. The linear systems of ϕ_1 and φ intersecting into d free points, the map $\theta = \varphi \circ (\phi_1)^{-1}$ is a de Jonquières transformation of degree d. The linear system of θ is the image by ϕ_1 of the linear system of φ; it has one proper base-point distinct from q, which corresponds to the "image" of t_2 by ϕ_1 (in the decomposition of ϕ_1 into blow-ups and blow-downs, the exceptional curve associated to t_1 is sent onto two a line of $\mathbb{P}^2$ and t_2 is sent onto a general point of this line). Since $(\phi_1)_*(\omega)$ is a normal cubic form, we can apply the preceding case to θ. □

Proposition 4.2. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map of degree > 1, and assume that*

$$\operatorname{div}(\varphi_*(\omega_0)) = \operatorname{div}(\omega_0)$$

(*where ω_0 is the differential form $dx \wedge dy/(xy)$*). *Then, there exist quadratic transformations $\phi_1, \dots, \phi_n$ such that*
(1) $\varphi = \phi_n \circ \cdots \circ \phi_1$;
(2) *for* $i = 1, \dots, n$, $\operatorname{div}((\phi_i \circ \cdots \circ \phi_1)_*(\omega_0)) = \operatorname{div}(\omega_0)$.

REMARK 4.3. A differential form ω satisfies $\operatorname{div}(\omega) = \operatorname{div}(\omega_0)$ if and only if $\omega = \mu\omega_0$ for some $\mu \in \mathbb{C}^*$. In fact, as we can see from the proofs in Section 5, if a birational map $\varphi \in \operatorname{Bir}(\mathbb{P}^2)$ satisfies $\varphi_*(\omega_0) = \mu\omega_0$ for some $\mu \in \mathbb{C}^*$, then μ is ± 1, and both are possible (taking for example $\varphi\colon (x : y : z) \mapsto (y : x : z)$ we get $\mu = -1$).

Proof. Applying Lemma 4.1, we obtain a decomposition $\varphi = \phi_n \circ \cdots \circ \phi_1$ where $\omega_i := (\phi_i \circ \cdots \circ \phi_1)_*(\omega_0)$ is a normal cubic form for $i = 0, \dots, n$.

Denote by m the maximal degree of the irreducible components of $-\operatorname{div}(\omega_i)$ for $i = 0, \dots, n$, denote by r the minimal index where ω_r has a component of degree m. We now prove the result by induction on the pairs $(m, n-r)$, ordered lexicographically.

If $m = 1$, $-\operatorname{div}(\omega_i)$ is the union of three lines for each i. Composing the quadratic maps with an automorphism of $\mathbb{P}^2$ which sends $\operatorname{div}(\omega_i)$ onto $\operatorname{div}(\omega_0)$, we can assume that $\operatorname{div}(\omega_i) = \operatorname{div}(\omega_0)$ for each i and obtain the result.

Suppose now that $m = 2$, which implies that $0 < r < n$, since $\omega_0 = \omega_n \neq \omega_r$. The divisor $-\operatorname{div}(\omega_r)$ is the union of a line L and a conic Γ, and the divisor $-\operatorname{div}(\omega_{r-1})$ is the union of three lines. In particular, the curve $\Gamma_0 = (\phi_r^{-1})_*(\Gamma)$ is a line and $L_0 = (\phi_r^{-1})_*(L)$ is either a point or a line. This implies that the three base-points s_1, s_2, s_3 of ϕ_r^{-1} belong to Γ (as proper or infinitely near points) and that at least one of them lies on L. Up to renumbering, s_1 is one of the two points of $\Gamma \cap L$, and s_2 is either a proper point of Γ or the point infinitely near to s_1 corresponding to the tangent of Γ. The curve $\Gamma_5 = (\phi_{r+1})_*(\Gamma)$ is either a conic or a line, so at least two of the three base-points t_1, t_2, t_3 of ϕ_{r+1} belongs to Γ, and $L_5 = (\phi_{r+1})_*(L)$ can be a point, a line or a conic. Up to renumbering, t_1 is a proper point of Γ, and t_2 is either another proper point of Γ, or the point infinitely near to t_1 corresponding to the tangent direction of Γ. We can also assume that if t_2 belongs to $\Gamma \cap L$, so does t_1.

$$\begin{tikzcd}
(L_0, \Gamma_0) & (L, \Gamma) \arrow[l, dashed, "\phi_r^{-1}"', "{[s_1,s_2,s_3]}"] \arrow[r, dashed, "\phi_{r+1}", "{[t_1,t_2,t_3]}"'] & (L_5, \Gamma_5)
\end{tikzcd}$$

We now define two proper points a, b of Γ. If $t_1 \in \Gamma \cap L$, the point b is a general point of Γ (i.e. distinct from the s_i and t_i), and otherwise b is such that $L \cap \Gamma = \{s_1, b\}$. The point a is a general point of Γ (i.e. distinct from b and all s_i, t_i). We define four birational quadratic maps χ_1, χ_2, χ_3, χ_4 of $\mathbb{P}^2$, with base points $[s_1, s_2, a]$, $[s_1, a, b]$, $[t_1, a, b]$ and $[t_1, t_2, b]$ respectively. We moreover set $\chi_0 = \phi_r^{-1}$ and $\chi_5 = \phi_{r+1}$. By construction, we have the following: for $i = 0, \dots, 4$, χ_i has its three base-points on Γ and at least one of them belongs to L, so $\Gamma_i = (\chi_i)_*(\Gamma)$ is a line, and $L_i = (\chi_i)_*(L)$ is either a point or a line; moreover χ_i and χ_{i+1} have two common base-points, so $\theta_i = \chi_{i+1} \circ \chi_i^{-1}$ is a quadratic map. We obtain the following commutative diagram:

$$\begin{tikzcd}
(L_0, \Gamma_0) \arrow[dd, dashed, "\theta_0"] & & & & (L_5, \Gamma_5) \\
& & (L, \Gamma) \arrow[ull, dashed, "\chi_0=\phi_r^{-1}"', "{[s_1,s_2,s_3]}"] \arrow[urr, dashed, "\chi_5=\phi_{r+1}", "{[t_1,t_2,t_3]}"'] \arrow[dll, dashed, "\chi_1"', "{[s_1,s_2,a]}"] \arrow[drr, dashed, "\chi_4", "{[t_1,t_2,b]}"'] \arrow[ddl, dashed, "\chi_2"', "{[s_1,a,b]}"] \arrow[ddr, dashed, "\chi_3", "{[t_1,a,b]}"'] & & \\
(L_1, \Gamma_1) \arrow[dr, dashed, "\theta_1"] & & & & (L_4, \Gamma_4) \arrow[uu, dashed, "\theta_4"'] \\
& (L_2, \Gamma_2) \arrow[rr, dashed, "\theta_2"] & & (L_3, \Gamma_3) \arrow[ur, dashed, "\theta_3"] &
\end{tikzcd}$$

By construction, $-\mathrm{div}((\chi_i)_*(\omega_r))$ is the union of three lines for $i = 0, \dots, 4$; replacing $\phi_{r+1} \circ \phi_r$ by $\theta_4\theta_3\theta_2\theta_1\theta_0$, we reduce the pair $(m, n-r)$.

Suppose now that $m = 3$ (which implies that $1 < r < n-1$). The divisor $-\mathrm{div}(\omega_r)$ consists of a nodal cubic curve Γ. The curve $\Gamma_0 = (\phi_r^{-1})_*(\Gamma)$ is a conic, so all base-points s_1, s_2, s_3 of ϕ_r^{-1} belong to Γ, and one of them, say s_1, is the singular point of Γ. Up to reordering, we can assume that s_2 is either a proper point of Γ or the point infinitely near to s_1 corresponding to the tangent of Γ. We denote by t_1, t_2, t_3 the three base-points of ϕ_{r+1}. The curve $\Gamma_4 = (\phi_{r+1})_*(\Gamma)$ is either a cubic or a conic, which means that either all t_i's belong to Γ or that only two belong to Γ but one of these two is the singular point s_1. If s_1 is a base-point of ϕ_{r+1}, we can assume that $t_1 = s_1$, that t_2 belongs to Γ and that either t_2 is a proper point of $\mathbb{P}^2$ or is infinitely near to $t_1 = s_1$. If s_1 is not equal to any of the t_i, we can assume that t_2 is a proper point of Γ. We choose a general proper point a of Γ, not collinear with any two of the s_i, t_i and define two birational quadratic maps χ_1, χ_2 of $\mathbb{P}^2$, with base points $[s_1, s_2, a]$ and

$[s_1, t_2, a]$ respectively. We moreover set $\chi_0 = \phi_r^{-1}$ and $\chi_3 = \phi_{r+1}$. By construction, we have the following: for $i = 0, \ldots, 2$, χ_i has its three base-points on Γ and at least one of them is s_1, so $\Gamma_i = (\chi_i)_*(\Gamma)$ is a conic; moreover χ_i and χ_{i+1} have two common base-points, so $\theta_i = \chi_{i+1} \circ \chi_i^{-1}$ is a quadratic map. We obtain the following commutative diagram:

$$\begin{tikzcd}
 & & \Gamma \arrow[dll, dashed, "\chi_0=\phi_r^{-1}" above, "{[s_1,s_2,s_3]}" below] \arrow[ddl, dashed, "\chi_1" left, "{[s_1,s_2,a]}" right] \arrow[ddr, dashed, "\chi_2" right, "{[s_1,t_2,a]}" left] \arrow[drr, dashed, "\chi_3=\phi_{r+1}" above, "{[t_1,t_2,t_3]}" below] & & \\
\Gamma_0 \arrow[dr, dashed, "\theta_0"] & & & & \Gamma_3 \\
 & \Gamma_1 \arrow[rr, dashed, "\theta_1"] & & \Gamma_2 \arrow[ur, dashed, "\theta_2"] &
\end{tikzcd}$$

By construction, $-\text{div}((\chi_i)_*(\omega_r))$ is the union of the conic Γ_i and a line for $i = 0, \ldots, 4$; replacing $\phi_{r+1} \circ \phi_r$ by $\theta_2\theta_1\theta_0$, we reduce the pair $(m, n - r)$. $\square$

5. Quadratic elements of *Symp* and the proof of Theorem 1

We now describe some of the main quadratic elements of *Symp*, useful in the generation of elements of *Symp* (see Proposition 4.2).

We fix notation for some points which are poles of ω_0. The points p_1, p_2, p_3 are the vertices of the triangle $XYZ = 0$, and q_1, q_2, q_3 are points on edges:

$$p_1 = (1:0:0), \quad p_2 = (0:1:0), \quad p_3 = (0:0:1),$$
$$q_1 = (0:1:-1), \quad q_2 = (1:0:-1), \quad q_3 = (1:-1:0).$$

Any quadratic birational transformation of $\mathbb{P}^2$ has three base-points. We describe now some quadratic transformations, by giving their description on $\mathbb{C}^2$, $\mathbb{P}^2$ (writing only the image of (x, y) and $(X : Y : Z)$ respectively) and by giving their base-points. Firstly, we describe the classical generators:

$$\begin{array}{llll}
I^2, & \left(\dfrac{1}{x}, \dfrac{1}{y}\right), & (YZ : XZ : XY), & p_1, p_2, p_3, \\
P, & \left(y, \dfrac{y+1}{x}\right), & (XY : (Y+Z)Z : XZ), & p_1, p_2, q_1, \\
P^2, & \left(\dfrac{y+1}{x}, \dfrac{x+y+1}{xy}\right), & (Y(Y+Z) : Z(X+Y+Z) : XY), & p_1, q_1, q_2, \\
P^3, & \left(\dfrac{x+y+1}{xy}, \dfrac{x+1}{y}\right), & ((X+Y+Z)Z : X(X+Z) : XY), & p_2, q_1, q_2, \\
P^4, & \left(\dfrac{x+1}{y}, x\right), & (Z(X+Z) : XY : YZ), & p_1, p_2, q_2.
\end{array}$$

Secondly, we construct more complicated elements. For any $\lambda \in \mathbb{C}^*$, we denote by $\rho_\lambda \in \text{Aut}(\mathbb{P}^2)$ the automorphism $(X : Y : Z) \mapsto (\lambda X : Y : Z)$. If $\lambda \neq -1$, the maps S_λ and

T_λ, respectively given by $S_\lambda = (P^2C)^{-1}\rho_{-\lambda}P^2C$ and $T_\lambda = P^2\rho_{-\lambda}CP^2$, are described in the following table:

$$\begin{array}{lll} S_\lambda, & (-\lambda X(X+Y+Z) : Y(X+Y-\lambda Z) : Z(-\lambda X+Y-\lambda Z)), & (0:\lambda:1), q_2, q_3, \\ T_\lambda, & (XY : (Y+Z)(\lambda Z-Y) : -\lambda XZ), & p_1, q_1, (0:\lambda:1). \end{array}$$

Recall that C is the automorphism $(X:Y:Z) \mapsto (Y:Z:X)$ of $\mathbb{P}^2$, which corresponds to the birational map $(x, y) \dashrightarrow (y/x, 1/x)$ of $\mathbb{C}^2$, and thus to the matrix $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ of $\mathrm{SL}(2,\mathbb{Z})$. We denote by $\mathrm{Sym}_{(X,Y,Z)} \subset \mathrm{Aut}(\mathbb{P}^2)$ the symmetric group of permutations of the variables, generated by C and $(X:Y:Z) \mapsto (Y:X:Z)$. We now describe linear and quadratic elements of *Symp*.

Lemma 5.1. *The group of automorphisms of $\mathbb{P}^2$ which preserve the triangle*

$$XYZ = 0$$

is $(\mathbb{C}^*)^2 \rtimes \mathrm{Sym}_{(X,Y,Z)}$, and its subgroup $(\mathbb{C}^*)^2 \rtimes \langle C\rangle$ is equal to the group of automorphisms of $\mathbb{P}^2$ which are symplectic.

Proof. Follows from a simple calculation. □

Lemma 5.2. *Let $\varphi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a birational map of degree 2 which has three proper base-points. The following condition are equivalent*:

(1) $\mathrm{div}(\varphi_*(\omega_0)) = \mathrm{div}(\omega_0)$.

(2) *$\varphi = \alpha Q\beta$, where $\alpha \in (\mathbb{C}^*)^2 \rtimes \mathrm{Sym}_{(X,Y,Z)}$, $\beta \in (\mathbb{C}^*)^2 \rtimes \langle C\rangle$ and $Q \in \{I^2, P, P^2, P^3, P^4\}$ or $Q \in \{S_\lambda, T_\lambda\}$ for some $\lambda \in \mathbb{C}^* \setminus \{-1\}$.*

Proof. The second assertion clearly implies the first one, since $Q \in$ *Symp* is a quadratic map with three proper base-points. It remains thus to prove the other direction.

Denote by $L_1, L_2, L_3 \subset \mathbb{P}^2$ the three lines of equation $X = 0$, $Y = 0$ and $Z = 0$. Each of the three lines L_i is a pole of ω_0 and its image by φ is thus either a point or a line. So for each i, one or two of the base-points of φ belong to L_i.

Denote by $k \in \{0, 1, 2, 3\}$ the number of base-points of φ which are vertices of the triangle $XYZ = 0$. Replacing φ by φC or φC^2 if needed, the k vertices are the k first points of the triple (p_1, p_2, p_3). We will find $Q \in \{I, P, P^2, P^3, P^4, S_\lambda, T_\lambda\}$ and $\beta \in (\mathbb{C}^*)^2 \rtimes \mathrm{Sym}_{(X,Y,Z)}$ such that φ and $Q\beta$ have the same base-points.

Before proving the existence of Q, β, let us prove how it yields the result. The fact that φ and $Q\beta$ have the same base-points implies that $\varphi = \alpha Q\beta$ for some $\alpha \in \mathrm{Aut}(\mathbb{P}^2)$. Since $\mathrm{div}(\omega_0) = \mathrm{div}(\varphi_*(\omega_0)) = \mathrm{div}(Q_*(\omega_0)) = \mathrm{div}(\beta_*(\omega_0))$, we also have $\mathrm{div}(\alpha_*(\omega_0)) = \mathrm{div}(\omega_0)$, which means that $\alpha \in (\mathbb{C}^*)^2 \rtimes \mathrm{Sym}_{(X,Y,Z)}$ (Lemma 5.1).

We find now β and Q, by studying the possibilities for k.

If $k=3$, the base-points of φ are p_1, p_2, p_3 and it suffices to choose $Q=I^2$ and $\beta=1$.

If $k=2$, the base-points are p_1, p_2, u, where $u\in(L_1\cup L_2)\setminus L_3$. We choose $\beta\in(\mathbb{C}^*)^2$ which sends u onto q_1 or q_2, and choose respectively $Q=P$ or $Q=P^4$.

If $k=1$, the base-points are p_1,u,v, where $u\in L_1\setminus(L_2\cup L_3)$. If $v\in L_2$ we choose $\beta\in(\mathbb{C}^*)^2$ which sends u onto q_1 and v onto q_2, and choose then $Q=P^2$. If $v\in L_3$, we choose $\beta=\beta' C^{-1}$, where $\beta'\in(\mathbb{C}^*)^2$, such that β sends respectively p_1,u,v onto p_2,q_2,q_1, and choose $Q=P^3$. If $v\in L_1$, we choose $\beta\in(\mathbb{C}^*)^2$ which sends u onto $q_1=(0:-1:1)$; the point v is sent onto $(0:\lambda:1)$ for some $\lambda\in\mathbb{C}^*\setminus\{-1\}$. We can thus choose $Q=T_\lambda$.

If $k=0$, the base-points are u,v,w, which belong respectively to L_1, L_2, L_3. We choose $\beta\in(\mathbb{C}^*)^2$ which sends v onto q_2 and w onto q_3. The point u is sent onto $(0:\lambda:1)$, for some $\lambda\in\mathbb{C}^*\setminus\{-1\}$ (λ is not -1 because u,v,w are not collinear). We choose $Q=S_\lambda$. □

Now, using all above results, we can prove Theorem 1, which is a direct consequence of the following proposition.

Proposition 5.3. *The group Symp is generated by* $(\mathbb{C}^*)^2$, C *and* P.

Proof. Let f be an element of *Symp*. If its degree is 1, it is an automorphism of $\mathbb{P}^2$, which is thus generated by C and P (Lemma 5.1).

Otherwise, we write $f=\theta_n\circ\cdots\circ\theta_1$ using Proposition 4.2, and denote by m the number of θ_i which have at least one base-point which is not a proper point of $\mathbb{P}^2$. We prove the result by induction on the pairs (m,n), ordered lexicographically, the case $m=n=0$ being induced by Lemma 5.1.

Suppose first that the three base-points of θ_1 are proper points of $\mathbb{P}^2$. In this case, we apply Lemma 5.2 and write $\theta_1=\alpha Q\beta$, where $\alpha\in(\mathbb{C}^*)^2\rtimes\mathrm{Sym}_{(X,Y,Z)}$, and Q, β are generated by $(\mathbb{C}^*)^2$, C and P. Replacing f with $f(Q\beta)^{-1}$, we replace the pair (m,n) with $(m,n-1)$.

Suppose now that at least one base-point of θ_1, say a, is not a proper point of $\mathbb{P}^2$. Denote by $L_1, L_2, L_3\subset\mathbb{P}^2$ the three lines of equation $X=0$, $Y=0$ and $Z=0$. Each of the three lines L_i is a pole of ω_0 and its image by θ_1 is thus either a point or a line. This means that there is at least one base-point on each of the three lines L_1, L_2, L_3, and thus that the two other base-points of θ_1 are proper points $b,c\in\mathbb{P}^2$, and at least one of the two points b,c belongs to L_i for $i=1,\dots,3$. We choose some proper point d of the triangle, not aligned with any two of the points a,b,c. There exists a quadratic transformation Q of $\mathbb{P}^2$ with base-points b,c,d. The map Q sends ω_0 onto a normal cubic form by Proposition 3.4. Moreover, the image of any of the lines L_1, L_2, L_3 is a line or a point, so Q sends ω_0 onto a normal cubic form corresponding to a triangle. Replacing Q with its composition with an automorphism of $\mathbb{P}^2$, we may assume that the triangle is $XYZ=0$. Because a is not aligned with any two of the points b, c, d, the linear system of conics passing through a, b, c is sent by Q onto a system of conics with three proper base-points. In consequence, $\theta_1 Q^{-1}$ is a quadratic transformation with three proper base-points. Replacing θ_1 with $(\theta_1 Q^{-1}) \circ Q$, we replace (m, n) with $(m-1, n+1)$. □

6. The group $H = \langle \mathrm{SL}(2, \mathbb{Z}), P \rangle$

Let us now focus ourselves on the group H of finite type generated by $\mathrm{SL}(2, \mathbb{Z})$ and P, or simply by C, I, P (and in fact only by P and C since $I = PCP$). Recall that C is the automorphism $(X : Y : Z) \mapsto (Y : Z : X)$ of order 3 of $\mathbb{P}^2$ and that I and P have respectively order 4 and 5.

Recall the following notation for the points $p_1, p_2, p_3, q_1, q_2, q_3$.

$$p_1 = (1:0:0), \quad p_2 = (0:1:0), \quad p_3 = (0:0:1),$$
$$q_1 = (0:1:-1), \quad q_2 = (1:0:-1), \quad q_3 = (1:-1:0).$$

We moreover denote by p_1^Y the point in the first neighbourhood of p_1 which corresponds to the tangent $Y = 0$, and do the same for p_1^Z, p_1^{Y+Z}, p_2^X, p_2^Z, q_1^X and so on.

We now define twelve quadratic maps contained in H, whose three base-points belong to the set $\{p_1, p_1^Y, p_1^Z, p_1^{Y+Z}, q_1, q_1^X\}$ or to its orbit by C.

$Q_1 = I,$	$\left(\frac{1}{y}, x\right),$	$(Z^2 : XY : YZ),$	$p_1, p_2, p_1^Y,$
$Q_2 = I^3,$	$\left(y, \frac{1}{x}\right),$	$(XY : Z^2 : XZ),$	$p_1, p_2, p_2^X,$
$Q_3 = I^2,$	$\left(\frac{1}{x}, \frac{1}{y}\right),$	$(YZ : XZ : XY),$	$p_1, p_2, p_3,$
$Q_4 = P,$	$\left(y, \frac{y+1}{x}\right),$	$(XY : (Y+Z)Z : XZ),$	$p_1, p_2, q_1,$
$Q_5 = P^{-1},$	$\left(\frac{x+1}{y}, x\right),$	$(Z(X+Z) : XY : YZ),$	$p_1, p_2, q_2,$
$Q_6 = PI^2,$	$\left(\frac{1}{y}, x\frac{(y+1)}{y}\right),$	$(Z^2 : (Y+Z)X : YZ),$	$p_1, p_2, p_1^{Y+Z},$
$Q_7 = P^{-1}I^2,$	$\left(y\frac{x+1}{x}, \frac{1}{x}\right),$	$((X+Z)Y : Z^2 : XZ),$	$p_1, p_2, p_2^{X+Z},$
$Q_8 = I^2P,$	$\left(\frac{1}{y}, \frac{x}{y+1}\right),$	$(Z(Y+Z) : XY : Y(Y+Z)),$	$p_1, p_1^Y, q_1,$
$Q_9 = IP,$	$\left(\frac{x}{y+1}, y\right),$	$(XZ : Y(Y+Z) : Z(Y+Z)),$	$p_1, p_1^Z, q_1,$
$Q_{10} = P^2,$	$\left(\frac{y+1}{x}, \frac{x+y+1}{xy}\right),$	$(Y(Y+Z) : Z(X+Y+Z) : XY),$	$p_1, q_1, q_2,$

$$Q_{11} = P^3C^{-1}, \left(\frac{x+y+1}{xy}, \frac{x+1}{y}\right), ((X+Y+Z)Y : Z(Y+Z) : XZ), p_1, q_1, q_3,$$
$$Q_{12} = PIP, \quad \left(y, \frac{y+1)^2}{x}\right), \qquad (XY : (Y+Z)^2 : XZ), \qquad p_1, q_1, q_1^X.$$

Any element of H can be written as a word written with the letters C, I, P. We will say that a *linear word* is a word of type C^a with $a \in \{0, 1, 2\}$. Similarly, we will say that a *quadratic word* is a word of type $C^aQ_iC^b$, where $1 \leq i \leq 12$, $a, b \in \{0, 1, 2\}$. Note that a linear word corresponds to a linear automorphism of $\mathbb{P}^2$ and that a quadratic word corresponds to a quadratic birational transformation of $\mathbb{P}^2$.

We would like to prove that the relations

$$R = \{I^4 = C^3 = [C, I^2] = P^5 = 1, PCP = I\}$$

(which can easily be verified) generate all the others in $H = \langle I, C, P\rangle$. To do this (in Proposition 6.6), we need to prove some technical simple lemmas (Lemmas 6.1, 6.2 and 6.5) and one key proposition (Proposition 6.3).

Lemma 6.1. *If Q is a quadratic word, then Q^{-1} and $\tau Q\tau^{-1}$ are quadratic words, for any $\tau \in \mathrm{Sym}_{(X,Y,Z)} \subset \mathrm{Aut}(\mathbb{P}^2)$ (permutation of the coordinates).*

Proof. If $\tau = C$, then $\tau Q\tau^{-1}$ is a quadratic word by definition. We can thus assume that τ is the map $(X : Y : Z) \mapsto (Y : X : Z)$ (or $(x,y) \mapsto (y,x)$), which conjugates P, I, C to respectively P^{-1}, I^{-1}, C^{-1}. If Q is a power of I or of P, it is clear that $Q^{-1} = \tau Q\tau^{-1}$ is a quadratic word. It remains to study the case when $Q = Q_i$ with $i \in \{6, 7, 8, 9, 12\}$.

First, we do the case of inverses. Since $PCP = I$, we have $(Q_6)^{-1} = I^2P^{-1} = IPC = Q_9C$, and thus $(Q_9)^{-1} = CQ_6$. Moreover, $(Q_7)^{-1} = I^2P = Q_8$. Finally, using $I^4 = 1$ and $I = PCP$, we have

$$(Q_{12})^{-1} = P^{-1}I^{-1}P^{-1} = P^{-1}(PCP)I(PCP)P^{-1} = CPIPC = CQ_{12}C.$$

Now, the conjugation. We have $\tau Q_6\tau^{-1} = Q_7$ and $\tau Q_8\tau^{-1} = I^2P^{-1} = I(PCP)P^{-1} = IPC = Q_9C$. This implies that $\tau Q_9\tau^{-1} = Q_8C$. Finally, $\tau Q_{12}\tau^{-1} = (Q_{12})^{-1}$ is a quadratic word, as we just proved. □

Lemma 6.2. *The words*

$$I^aP^{\pm 1}, P^{\pm 1}I^a, P^{\pm 1}I^aP^{\pm 1},$$

where $a \in \mathbb{N}$, are equivalent, up to relations in R, to quadratic words.

Proof. From the list, we see that any non-trivial power of I or P is a quadratic word. In particular, the case $a = 0$ is trivial. Using $PCP = I$, we find the following table:

a	I^aP	PI^aP	PI^aP^{-1}
1	$IP = Q_9$	$PIP = Q_{12}$	$PIP^{-1} = P^2C$
2	$I^2P = Q_8$	$C^{-1}P^{-1}C^{-1}$	$PIPC = Q_{12}C$
3	$P^{-1}C^{-1}$	$PI^{-1}P = C^{-1}$	$PI^{-1}P^{-1} = C^{-1}P^3$

the result is now clear for I^aP, PI^aP and PI^aP^{-1}.

For any a, the word I^aP^{-1} is equal to $I^{a-1}(PCP)P^{-1} = I^{a-1}PC$, and is thus also a quadratic word. The words $P^{\pm 1}I^a$ being the inverses of $I^{-a}P^{\pm 1}$, these are also quadratic words (Lemma 6.1). The same holds for $P^{-1}I^aP^{-1} = (PI^{-a}P)^{-1}$. It remains to see that $P^{-1}I^aP$ is a quadratic word for each a. Since $I^a = PCPI^aP^{-1}C^{-1}P^{-1}$, we find $P^{-1}I^aP = CPI^aP^{-1}C^{-1}$, which is quadratic word since PI^aP^{-1} is one. □

Proposition 6.3. *Let f and g be two quadratic words in H. If the two quadratic maps associated have two* (*respectively three*) *common base-points, then fg^{-1} is equal to a quadratic* (*respectively linear*) *word, modulo the relations R.*

Proof. The list of the twelve quadratic words above give the possible base-points of f and g: the base-points of Q_i and CQ_i are the same, and the base-points of Q_iC are the image by C^{-1} of the base-points of h.

A quick look at the list shows that if f and g have the same three base-points, then $f = C^ig$, for some integer i. In particular, fg^{-1} is equal to a linear word. We have thus only to study the case when exactly two of the three base-points of f and g are common.

In the sequel, we will use the following observations:

(i) we can exchange the role of f and g since fg^{-1} is a a quadratic word if and only if its inverse gf^{-1} is (Lemma 6.1);

(ii) we can replace f and g with C^if and C^jg since this only multiplies fg^{-1} by some power of C;

(iii) we can replace both f and g with their conjugates under any permutation of (X, Y, Z), using Lemma 6.1.

Using (iii), we can "rotate" the two common points by acting with C, which acts as $p_3 \mapsto p_2 \mapsto p_1$, $q_3 \mapsto q_2 \mapsto q_1$, $p_3^Y \mapsto p_2^X \mapsto p_1^Z$ and so on. The possibilities for the two common base-points can thus be reduced to $\{p_1, p_2\}$, $\{q_1, q_2\}$, $\{q_1, q_1^X\}$ or $\{p_1, u\}$, where $u \in \{q_1, q_2, q_3, p_1^Y, p_1^Z, p_1^{Y+Z}\}$. Conjugating by $(X : Y : Z) \mapsto (X : Z : Y)$ if needed, u may be chosen in $\{q_1, q_2, p_1^Y, p_{Y+Z}^1\}$ only.

We study each case separately.

(a) Case $\{p_1, p_2\}$—Using (ii) and reading the list, we can choose that $f, g \in \{P^{\pm 1}, P^{\pm 1}I^2, I^*\}$ (here the star means any power of I). If both f and g are powers of I, or both are powers of P, so is the product fg^{-1}, and we are done. If f is

a power of I and g is a power of P, then fg^{-1} is equal to $I^iP^{\pm1}$ and the result follows from Lemma 6.2. If $g = P^{\pm1}I^2$ and f is a power of I, then fg^{-1} is again equal to $I^iP^{\pm1}$ for some integer i. If $g = P^{\pm1}I^2$ and $f = P^{\pm1}$, then $fg^{-1} = P^{\pm1}I^2P^{\pm1}$, which is a quadratic word by Lemma 6.2.
(b) Case $\{q_1, q_2\}$—The only possibilities for f, g are P^2 or P^3, and $fg^{-1} = P^{\pm1}$.
(c) Case $\{q_1, q_1^X\}$—The only possibility is $f = g = Q_{12} = PIP$, a contradiction.
(d) Case $\{p_1, q_1\}$—The third base-point can be respectively p_2, p_3, q_2, q_3, p_1^Y, p_1^Z or q_1^X, and this corresponds respectively to P, $P^4C^{-1} = I^3P$, P^2, $P^3C^{-1} = P^{-1}I^{-1}P$, I^2P, IP and PIP. In particular, f and g are equal to $f'P$ and $g'P$ where $f', g' \in \{P^{\pm1}, P^{\pm1}I, I^\star\}$. Here, fg^{-1} (or its inverse) belongs to $\{P^*, I^*, P^{\pm1}I^*, P^{\pm1}IP^{\pm1}\}$ and we are done by Lemma 6.2.
(e) Case $\{p_1, q_2\}$—The only possibilities for f, g are P^{-1} or P^2, and $fg^{-1} = P^{\pm3}$.
(f) Case $\{p_1, p_1^Y\}$—Here $f, g \in \{I, I^2P\}$ and $fg^{-1} = (I^2PI^{-1})^{\pm1}$, a quadratic word by Lemma 6.2.
(g) Case $\{p_1, p_1^{X+Y}\}$—Here $f, g \in \{PI^2, P^{-1}I^2C^{-1} = P^{-1}C^{-1}I^2\}$ and $fg^{-1} = (PCP)^{\pm1} = I^{\pm1}$. □

Corollary 6.4. *Let W_1, W_2 be two quadratic words. If W_2W_1 corresponds to a birational map of degree* 1 (*respectively* 2), *then W_2W_1 is equal, modulo the relations R, to a linear word* (*respectively to a quadratic word*).

Proof. The map corresponding to W_2W_1 has degree 1 (respectively 2) if and only if the maps corresponding to W_2 and $(W_1)^{-1}$ have 3 (respectively 3) common base-points. The result follows then from Proposition 6.3. □

Lemma 6.5. *Let a_1, a_2, a_3 be three non-collinear distinct points, such that*
(Q) *for $i = 1, 2, 3$, a_i is a base-point of a quadratic word*;
(P) *for $i = 1, 2, 3$, if a_i is not a proper point of the plane, it is infinitely near to a point a_j, $j \neq i$*;
($\diamond$) *for any line L of the triangle $XYZ = 0$ in $\mathbb{P}^2$, there exists an a_i which belongs to L.*
Then, there exists a quadratic word Q having a_1, a_2, a_3 as base-points.

Proof. Let us write $r = \#\{a_1, a_2, a_3\} \cap \{p_1, p_2, p_3\} \in \{0, 1, 2, 3\}$.

If $r \geq 2$, we can assume that $a_1 = p_1$, $a_2 = p_2$ (up to renumbering and multiplying by C or C^2). The last point a_3 being not collinear to a_1 and a_2, and being a base-point of a quadratic word, it belongs to $\{p_1^Y, p_2^X, p_3, q_1, q_2, p_1^{y+z}, p_2^{X+Z}\}$. We can choose $Q = Q_i$ for $i \in \{1, \dots, 7\}$.

If $r = 1$, we can assume that $a_1 = p_1$. Condition ($\diamond$) implies that q_1 is equal to a_2 or a_3. The possibilities for the remaining point are $\{p_1^Y, p_1^Z, q_2, q_3, q_1^X\}$, and we can choose $Q = Q_i$ for $i \in \{8, \dots, 12\}$.

The case $r = 0$ is not possible. Otherwise we would have

$$\{a_1, a_2, a_3\} \subset \{q_1, q_2, q_3, q_1^X, q_2^Y, q_3^Z\},$$

which is impossible since q_1, q_2, q_3 are collinear. □

Proposition 6.6. *Let W be a word in I, P, C. If W corresponds to a birational map of degree* 1 *or* 2, *it is equal, up to relations R, to a linear or quadratic word. In particular, if W corresponds to the identity of* $\mathrm{Bir}(\mathbb{P}^2)$, *it is equal to* 1 *modulo R.*

Proof. If W is a power of C, the result is obvious, so we can write $W = W_k \cdots W_2 W_1$ where each W_i is a quadratic word. Note that many such writings exist. We call Λ_0 the linear system of lines of $\mathbb{P}^2$. For $i = 1, \dots, k$, we denote by Λ_i the linear system of $W_i \cdots W_2 W_1(\Lambda_0)$ (identifying here the word with the corresponding quadratic map of $\mathbb{P}^2$), and by d_i its degree. Note that $d_k \in \{1, 2\}$ is the degree of (the birational map corresponding to) W. We write $D = \max\{d_i \mid i = 1, \dots, k\}$ and $n = \max\{i \mid d_i = D\}$.

Suppose first that $D = 2$. If $k > 1$, the map $W_2 W_1$ has degree 2 or 1 and we can replace it with a single quadratic or linear word (Corollary 6.4). Continuing in this way, we show that W is equivalent, modulo R, to a linear or a quadratic word.

We suppose now that $D > 2$, which implies that $1 < n < k$. We order the pairs (D, n) using lexicographical order, and proceed by induction. Proving that (D, n) can be decreased, we will reduce to the case $D = 2$ studied before.

If $r = \deg(W_{n+1} W_n) \in \{1, 2\}$, we can replace $W_{n+1} W_n$ with a single quadratic or linear word (Corollary 6.4), and this decreases (D, n). We can thus assume that $r = \deg(W_{n+1} W_n) \in \{3, 4\}$.

We are looking for a quadratic word Q satisfying the following property:

$$(\star) \qquad \left[\begin{array}{l} \deg(Q(\Lambda_n)) < d_n = \deg(\Lambda_n), \\ \{\deg(QW_n), \deg(W_{n+1}Q^{-1})\} = \begin{cases} \{2, 2\} & \text{if} \quad r = \deg(W_{n+1}W_n) = 3, \\ \{2, 3\} & \text{if} \quad r = \deg(W_{n+1}W_n) = 4. \end{cases} \end{array}\right.$$

We first show that such a Q gives us a way to decrease (D, n), before proving that Q exists.

If $r = 3$, both QW_n and $W_{n+1}Q^{-1}$ have degree 2 so are equivalent to, up to relations in R, to quadratic words σ_1 and σ_2 (Corollary 6.4). Replacing $W_{n+1} W_n = (W_{n+1} Q^{-1})(QW_n)$ by $\sigma_2 \sigma_1$, we decrease the pair (D, n). The replacement is described in the following commutative diagram.

$$\begin{tikzcd}
 & \Lambda_n \arrow[rd, dotted, "W_{n+1}"] \arrow[d, dotted, "Q"'] & \\
\Lambda_{n-1} \arrow[ru, dotted, "W_n"] \arrow[rd, "QW_n", "\sigma_1"'] & & \Lambda_{n+1} \\
 & Q(\Lambda_n) \arrow[ru, "W_{n+1}Q^{-1}", "\sigma_2"'] &
\end{tikzcd}$$

$$\deg(QW_n) = \deg(W_{n+1}Q^{-1}) = 2$$

If $r = 4$ and $\deg(QW_n) = 2$, QW_n is equivalent to a quadratic word σ_0. Moreover, since $\deg(W_{n+1}Q^{-1}) = 3$ and $\deg(Q(\Lambda_n)) < D$, we can use the case $r = 3$ described before to write $W_{n+1}Q^{-1}$ as a product of two quadratic words $\sigma_2\sigma_1$ satisfying $\deg(\sigma_1 Q(\Lambda)) < D$. The replacement of $W_{n+1}W_n$ with $\sigma_2\sigma_1\sigma_0$, described below, decreases the pair (D, n).

$$\begin{tikzcd}
 & \Lambda_n \arrow[rd, dotted, "W_{n+1}"] & \\
\Lambda_{n-1} \arrow[ru, dotted, "W_n"] \arrow[rd, "QW_n", "\sigma_0"'] & & \Lambda_{n+1} \\
 & Q(\Lambda_n) \arrow[uu, dotted, "Q^{-1}"] \arrow[ru, dotted, "W_{n+1}Q^{-1}"] \arrow[r, "\sigma_1"] & \sigma_1 Q(\Lambda) \arrow[u, "\sigma_2"]
\end{tikzcd}$$

$$\deg(QW_n) = 2,\ \deg(W_{n+1}Q^{-1}) = 3$$

If $r = 4$ and $\deg(W_{n+1}Q^{-1}) = 2$, $W_{n+1}Q^{-1}$ is equivalent to a quadratic word σ_0. We again apply case $r = 3$ (since $\deg(Q(\Lambda_n)) < D$) to replace QW_n with a product of two quadratic words $\sigma_2\sigma_1$ with $\deg(\sigma_1(\Lambda_{n-1})) < D$. The replacement of $W_{n+1}W_n$ with $\sigma_0\sigma_2\sigma_1$, described below, decreases the pair (D, n).

$$\begin{tikzcd}
 & \Lambda_n \arrow[rd, dotted, "W_{n+1}"] \arrow[dd, dotted, "Q"'] & \\
\Lambda_{n-1} \arrow[ru, dotted, "W_n"] \arrow[rd, dotted, "QW_n"] \arrow[d, "\sigma_1"] & & \Lambda_{n+1} \\
\sigma_1(\Lambda_{n-1}) \arrow[r, "\sigma_2"] & Q(\Lambda_n) \arrow[ru, "W_{n+1}Q^{-1}", "\sigma_0"'] &
\end{tikzcd}$$

$$\deg(QW_n) = 3,\ \deg(W_{n+1}Q^{-1}) = 2$$

It remains to prove the existence of Q satisfying the property $(\star)$.

We have $D = d_n = \deg(\Lambda_n)$. The system $\Lambda_{n+1} = W_{n+1}(\Lambda_n)$ has degree $d_{n+1} < D$, and $\Lambda_{n-1} = (W_n)^{-1}\Lambda_n$ has degree $d_{n-1} \leq D$. Denote respectively by $S = \{s_1, s_2, s_3\}$ and $T = \{t_1, t_2, t_3\}$ the base-points of W_{n+1} and $(W_n)^{-1}$. For any point p, we will write $m(p)$ the multiplicity of Λ_n at p. The fact that W_{n+1} is a quadratic map with base-points s_1, s_2, s_3 implies that $d_{n+1} = \deg(W_{n+1}(\Lambda_n)) = 2D - \sum_{i=1}^{3} m(s_i)$. In particular $\sum_{i=1}^{3} m(s_i) > D$. Similarly, $d_{n-1} = 2D - \sum_{i=1}^{3} m(t_i)$ and $\sum_{i=1}^{3} m(t_i) \geq D$.

In order to find Q, we will find its three base-points. We are looking for three distinct points $a_1, a_2, a_3 \in S \cup T$ which satisfy the following conditions:

$$(\star\star) \quad \left[\begin{array}{l} \displaystyle\sum_{i=1}^{3} m(a_i) > D, \\ \{\{a_1, a_2, a_3\} \cap S, \{a_1, a_2, a_3\} \cap T\} = \begin{cases} \{2, 2\} & \text{if} \quad r = \deg(W_{n+1} W_n) = 3, \\ \{1, 2\} & \text{if} \quad r = \deg(W_{n+1} W_n) = 4. \end{cases} \end{array}\right.$$

$$(\diamond) \quad \left[\text{for each line } L \text{ of the triangle } XYZ = 0 \text{ one } a_i \text{ belongs to } L.\right.$$

The condition $\sum_{i=1}^{3} m(a_i) > D$ implies that the three points are not collinear (because Λ_n has no fixed component). Replacing a point a_i by a_i' if a_i is infinitely near to a_i' and if $a_i' \notin \{a_1, a_2, a_3\}$, and then applying condition $(\diamond)$, we get a quadratic word Q having a_1, a_2, a_3 as its base-points (Lemma 6.5). Condition $(\star\star)$ implies then $(\star)$.

It remains to find three points a_1, a_2, a_3 satisfying $(\star\star)$ and $(\diamond)$. This is now done separately in the cases $r = 3$ and $r = 4$.

Suppose that $r = 3$, which means that $S \cap T = \{u\}$, for some proper point u of the plane. We order the points of S and T such that $S = \{u, s_1, s_2\}$, $T = \{u, t_1, t_2\}$, with $m(s_1) \geq m(s_2)$ and $m(t_1) \geq m(t_2)$. We observe that at least one of the inequalities $m(u)+m(t_1)+m(s_2) > D$, $m(u)+m(s_1)+m(t_2) > D$ is satisfied. Indeed, otherwise the sum would give $\sum_{i=1}^{3} m(s_i) + \sum_{i=1}^{3} m(t_i) \leq 2D$, which is impossible. We assume first that $m(u)+m(s_1)+m(t_2) > D$, and write $A_1 = \{u, s_1, t_1\}$, $A_2 = \{u, s_1, t_2\}$. For $i = 1, 2$, we have $\sum_{p \in A_i} m(p) \geq m(u) + m(s_1) + m(t_2) > D$, and thus the three points of A_i satisfy condition $(\star\star)$ and in particular are not collinear. We claim now that at least one of the two sets A_1, A_2 satisfies condition $(\diamond)$. Suppose the converse for contradiction. This means that for $i = 1, 2$, there exists a line L_i in the standard triangle $XYZ = 0$ such that $L_i \cap A_i = \emptyset$. Since $T = \{u, t_1, t_2\}$ satisfies condition $(\diamond)$, we see that $t_1 \in L_2 \setminus L_1$ and $t_2 \in L_1 \setminus L_2$, in particular $L_1 \neq L_2$. Denoting by L_3 the last line of the triangle, we have $u, s_1 \in L_3 \setminus (L_1 \cup L_2)$. Since t_1 and t_2 are not collinear with u and s_1, both do not belong to L_3. This implies that $T = \{u, t_1, t_2\} = \{q_1, q_2, q_3\}$, which is impossible since q_1, q_2, q_3 are collinear (they belong to the line $X + Y + Z = 0$). The case $m(u) + m(t_1) + m(s_2) > D$ is the same, by just exchanging S and T in the proof.

Suppose that $r = 4$, which means that $S \cap T = \emptyset$. We order the points s_i and t_i such that $m(s_1) \geq m(s_2) \geq m(s_3)$ and $m(t_1) \geq m(t_2) \geq m(t_3)$. We observe that at least one of the inequalities $m(s_1) + m(t_2) + m(t_3) > D$, $m(t_1) + m(s_2) + m(s_3) > D$ is satisfied. Indeed, otherwise the sum would give $\sum_{i=1}^{3} m(s_i) + \sum_{i=1}^{3} m(t_i) \leq 2D$, which is impossible. We assume first that $m(s_1)+m(t_2)+m(t_3) > D$, and write $A_1 = \{s_1,t_2,t_3\}$, $A_2 = \{s_1, t_1, t_3\}$, $A_3 = \{s_1, t_1, t_2\}$. For $i = 1, 2, 3$, we have $\sum_{p \in A_i} m(p) \geq m(s_1) + m(t_2) + m(t_3) > D$, and thus the three points of A_i satisfy condition $(\star\star)$. We claim now that at least one of the three sets A_i satisfies condition $(\diamond)$. Suppose the converse for contradiction. This means that for $i = 1, 2, 3$, there exists a line L_i in the standard triangle such that $L_i \cap A_i = \emptyset$. Since $T \cap L_i \neq \emptyset$, we have $t_i \in L_i$ for each i and $t_i \notin L_j$

for $i \neq j$. This implies that the three points t_i are contained in $\{q_1, q_2, q_3, q_1^X, q_2^Y, q_3^Z\}$, which is impossible because $T = \{t_1, t_2, t_3\}$ is the set of base-points of a quadratic word (we can see this on the list of base-points of quadratic words, or simply observe that q_1, q_2, q_3 are collinear). The case $m(t_1) + m(s_2) + m(s_3) > D$ is the same, by just exchanging S and T in the proof. □

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Universität Basel
Mathematisches Institut
Rheinsprung 21, CH-4051 Basel
Switzerland
e-mail: Jeremy.Blanc@unibas.ch