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CLASS NUMBER PARITY OF A QUADRATIC TWIST OF A CYCLOTOMIC FIELD OF PRIME POWER CONDUCTOR

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Abstract

Let p be a fixed odd prime number and K_n the p^{n+1}-st cyclotomic field. For a fixed integer $d \in \boldsymbol{Z}$ with $\sqrt{d} \notin K_0$, denote by L_n the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Let h_n^* and h_n^- be the relative class numbers of K_n and L_n, respectively. We give an explicit constant n_d depending on p and d such that (i) for any integer $n \geq n_d$, the ratio h_n^-/h_{n-1}^- is odd if and only if h_n^*/h_{n-1}^* is odd and (ii) for $1 \leq n < n_d$, h_n^-/h_{n-1}^- is even.

1. Introduction

Let p be a fixed odd prime number. Let $K_n = \boldsymbol{Q}(\zeta_{p^{n+1}})$ be the p^{n+1}-st cyclotomic field for an integer $n \geq 0$, and $K_\infty = \bigcup_n K_n$. Let $d \in \boldsymbol{Z}$ be a fixed integer with $\sqrt{d} \notin K_0$. We denote by L_n the imaginary quadratic subextension of the biquadratic extension $K_n(\sqrt{d})/K_n^+$ with $L_n \neq K_n$. Here, K^+ denotes the maximal real subfield of an imaginary abelian field K. When $d < 0$, we have $L_n = K_n^+(\sqrt{d})$. We call L_n the quadratic twist of K_n associated to the integer d. The extension $L_\infty = \bigcup_n L_n$ is the cyclotomic $\boldsymbol{Z}_p$-extension over L_0 with the n-th layer L_n. We call L_∞/L_0 the quadratic twist of the cyclotomic $\boldsymbol{Z}_p$-extension K_∞/K_0 associated to d. Let h_n^* and h_n^- be the relative class numbers of K_n and L_n, respectively. It is known and easy to show that h_{n-1}^* (resp. h_{n-1}^-) divides h_n^* (resp. h_n^-) using class field theory. The parity of h_0^* behaves rather irregularly when p varies (see a table in Schoof [6]). However, it is recently shown that when $p \leq 509$, the ratio h_n^*/h_{n-1}^* is odd for all $n \geq 1$ ([3, Theorem 2]). And it might be possible that the ratio is odd for any prime p and any $n \geq 1$. The purpose of this paper is to study the parity of the ratio h_n^-/h_{n-1}^- of the quadratic twist L_n. We already know that h_n^-/h_{n-1}^- is odd for sufficiently large n by a theorem of Washington [8] on the non-p-part of the class number in a cyclotomic $\boldsymbol{Z}_p$-extension. Denote by $S = S_d$ the set of prime numbers $l \neq p$ which ramify in $\boldsymbol{Q}(\sqrt{d})/\boldsymbol{Q}$. The set S is non-empty as $\sqrt{d} \notin K_0$. We define an integer $n_d \geq 1$ by

$$n_d = \max\{\mathrm{ord}_p(l^{p-1} - 1) \mid l \in S\},$$

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where $\operatorname{ord}_p(*)$ is the normalized p-adic additive valuation. The following is the main theorem of this paper.

Theorem 1. *Under the above setting, the following assertions hold.*
(I) *When $n \geq n_d$, the ratio h_n^-/h_{n-1}^- is odd if and only if h_n^*/h_{n-1}^* is odd.*
(II) *When $n_d \geq 2$ and $1 \leq n < n_d$, the ratio h_n^-/h_{n-1}^- is even.*

From Theorem 1 and [3, Theorem 2], we immediately obtain the following:

Corollary 1. *Under the above setting, let p be an odd prime number with $p \leq$ 509. Then the ratio $h_n^-/h_{n_d-1}^-$ is odd for all $n \geq n_d$.*

This corollary, though given in a very special setting, is an explicit version of the above mentioned theorem of Washington. In [4], we showed Theorem 1 when $d = -1$ and $L_n = K_n^+(\sqrt{-1})$ using some results of cyclotomic Iwasawa theory. In this paper, we prove Theorem 1 by using a main theorem of Conner and Hurrelbrink [1, Theorem 2.3].

REMARK. When $p \equiv 1 \bmod 4$ (resp. $p \equiv 3 \bmod 4$), we can show that two integers d_1 and d_2 give the same twist L_∞/L_0 of K_∞/K_0 if and only if $d_2 = d_1 x^2$ or $d_2 = p d_1 x^2$ (resp. $d_2 = -p d_1 x^2$) for some $x \in \boldsymbol{Q}^\times$. Hence, the set S_d and the integer n_d depend only on the twist L_∞/L_0 and not on the choice of d.

2. Exact hexagon of Conner and Hurrelbrink

In this section, we recall the exact hexagon of Conner and Hurrelbrink. Let k be an imaginary abelian field with 2-power degree, and F a real abelian field with $2 \nmid [F : \boldsymbol{Q}]$. We put $K = kF$, and

$$G = \operatorname{Gal}(K/k) = \operatorname{Gal}(K^+/k^+) = \operatorname{Gal}(F/\boldsymbol{Q}).$$

For a number field N, let A_N be the 2-part of the ideal class group of N, $\mathcal{O}_N$ the ring of integers, and $E_N = \mathcal{O}_N^\times$ the group of units of N. The groups A_K and E_K are naturally regarded as modules over $\operatorname{Gal}(K/K^+)$ and at the same time as those over G. For a $\operatorname{Gal}(K/K^+)$-module X, denote by $H^i(X) = H^i(K/K^+; X)$ the Tate cohomology group with $i = 0, 1$. When $X = A_K$ or E_K, the group $H^i(X)$ is also regarded as G-modules. In [1, Theorem 2.3], Conner and Hurrelbrink introduced the following exact hexagon of G-modules to study the 2-part of the class number of a relative quadratic extension.

$$\begin{array}{ccccccc} & & H^1(A_K) & \longrightarrow & H^1(E_K) & & \\ & \nearrow & & & & \searrow & \\ R^0(K) & & & & & & R^1(K) \\ & \nwarrow {\scriptstyle i_0} & & & & \swarrow & \\ & & H^0(E_K) & \longleftarrow & H^0(A_K) & & \end{array}$$

Here, $R^i(K)$ is a certain G-module associated to K/K^+ defined in [1]. We describe the G-module structure of $R^i(K)$ following [1]. Let T_f be the set of prime ideals $\wp$ of k^+ for which a prime ideal $\mathfrak{P}$ of K^+ over $\wp$ ramifies in K. Let T_∞ be the set of infinite prime divisors of k^+. We put $T = T_f \cup T_\infty$. For each $v \in T$, let $G_v \subseteq G$ be the decomposition group of v at K^+/k^+. When v is an infinite prime, the group G_v is trivial. We define G-modules Ω_f and Ω_∞ by

$$\Omega_f = \bigoplus_{\wp \in T_f} \boldsymbol{F}_2[G/G_\wp] \quad \text{and} \quad \Omega_\infty = \bigoplus_{v \in T_\infty} \boldsymbol{F}_2[G/G_v] = \bigoplus_{v \in T_\infty} \boldsymbol{F}_2[G],$$

respectively, where $\boldsymbol{F}_2 = \boldsymbol{Z}/2\boldsymbol{Z}$ is the finite field with two elements. (When T_f is empty, $\Omega_f = \{0\}$ by definition.) For each prime divisor w of K^+ with the restriction $w_{|k^+} \in T$ and an element $x \in (K^+)^\times$, we put $\iota_w(x) = 0$ or 1 according as $x \in N(K_w^\times)$ or not. Here, K_w is the completion of K at the unique prime divisor of K over w and $N = N_{K/K^+}$ is the norm map. For $g \in G$ and $x \in (K^+)^\times$, we see that

$$\iota_{w^g}(x) = \iota_w(x^{g^{-1}}) \tag{1}$$

by local class field theory. For a prime ideal $\mathfrak{P}$ of K^+ with $\mathfrak{P} \cap k^+ \in T_f$, let $\tilde{\mathfrak{P}}$ be the unique prime ideal of K over $\mathfrak{P}$. For an ideal $\mathfrak{A}$ of K, writing $\mathfrak{A} = \tilde{\mathfrak{P}}^e \mathfrak{B}$ with an integer e and an ideal $\mathfrak{B}$ relatively prime to $\tilde{\mathfrak{P}}$, we put $\mathrm{ord}_{\mathfrak{P}}(\mathfrak{A}) = e$.

We denote by $I(K)$ the group of (fractional) ideals of K. Let X be the subgroup of $I(K)$ consisting of ideals $\mathfrak{A}$ with $\mathfrak{A}^J = \mathfrak{A}$. Here, J is the complex conjugation acting on several objects associated to K. Let X_0 be the subgroup of X consisting of ideals $\mathfrak{A} \in I(K)$ with $\mathfrak{A} = x\mathfrak{B}^{1+J}$ for some $x \in (K^+)^\times$ and $\mathfrak{B} \in I(K)$. The G-module $R^1(K)$ is isomorphic to the quotient X/X_0. For this, see the lines 1–2 from the bottom of p. 6 and Lemma 2.1 of [1]. For each prime ideal $\wp \in T_f$, we fix a prime ideal $\mathfrak{P}$ of K^+ over $\wp$. From the argument in [1, §5], we obtain the following isomorphism of G-modules:

$$R^1(K) \cong \Omega_f; \quad \mathfrak{A}X_0 \to \bigoplus_{\wp \in T_f} \left(\sum_{\bar{g}} \mathrm{ord}_{\mathfrak{P}^g}(\mathfrak{A})\bar{g} \right), \tag{2}$$

where $\bar{g}$ (with $g \in G$) runs over the quotient $G/G_\wp$.

Let Y be the subgroup of the multiplicative group $(K^+)^\times \times I(K)$ consisting of pairs $(x, \mathfrak{A})$ with $x\mathfrak{A}^{1+J} = \mathcal{O}_K$. Let Y_0 be the subgroup of Y consisting of pairs $(N(y), y^{-1}\mathfrak{B}^{1-J})$ with $y \in K^\times$ and $\mathfrak{B} \in I(K)$. By definition, $R^0(K) = Y/Y_0$. We denote by $[x, \mathfrak{A}] \in R^0(K)$ the class containing $(x, \mathfrak{A})$. The map i_0 in the hexagon is defined by

$$i_0 \colon H^0(E_K) = E_{K^+}/N(E_K) \to R^0(K); \quad [\epsilon] \to [\epsilon, \mathcal{O}_K]$$

with $\epsilon \in E_{K^+}$. For each $v \in T_\infty$, we fix a prime divisor $\tilde{v}$ of K^+ over v. Using (1), we observe that the homomorphisms

$$\alpha_\infty \colon (K^+)^\times \to \Omega_\infty; \quad x \to \bigoplus_{v \in T_\infty} \left(\sum_{g \in G} \iota_{\tilde{v}^g}(x) g \right)$$

and

$$\alpha_f \colon (K^+)^\times \to \Omega_f; \quad x \to \bigoplus_{\wp \in T_f} \left(\sum_{\bar{g}} \iota_{\mathfrak{P}^g}(x) \bar{g} \right)$$

are compatible with the action of G. Further, α_∞ is nothing but the "sign" map. From the argument in [1, §4], we obtain the following exact sequence of G-modules:

(3) $$\{0\} \to R^0(K) \xrightarrow{\alpha} \Omega_f \oplus \Omega_\infty \xrightarrow{\beta} \boldsymbol{F}_2 \to \{0\}.$$

Here, α is defined by $\alpha([x, \mathfrak{A}]) = (\alpha_f(x), \alpha_\infty(x))$, β is the argumentation map and G acts trivially on $\boldsymbol{F}_2$.

3. Consequences

In this section, we derive some consequences of the exact hexagon and (2), (3). All of them are G-decomposed versions of the corresponding results in [1]. We work under the setting of Section 2. Denote by $\tilde{A}_{K^+}$ the 2-part of the narrow class group of K^+. Letting $K^+_{>0}$ be the group of totally positive elements of K^+, we have an exact sequence

(4) $$\{0\} \to (K^+)^\times/(K^+_{>0} E_{K^+}) \to \tilde{A}_{K^+} \to A_{K^+} \to \{0\}$$

of G-modules. We define the minus class group A_K^- to be the kernel of the norm map $A_K \to A_{K^+}$. Let χ be a $\bar{\boldsymbol{Q}}_2$-valued character of $G = \mathrm{Gal}(K/k) = \mathrm{Gal}(F/\boldsymbol{Q})$, which we also regard as a primitive Dirichlet character. For a module M over $\boldsymbol{Z}_2[G]$, we denote by $M(\chi)$ the χ-part of M. Here, $\boldsymbol{Z}_2$ is the ring of 2-adic integers and $\bar{\boldsymbol{Q}}_2$ is a fixed algebraic closure of the 2-adic rationals $\boldsymbol{Q}_2$. (For the definition of the χ-part and some of its properties, see Tsuji [7, §2].) Denote by S_K the set of prime numbers lying below some prime ideal in T_f. In all what follows, we assume that χ is a *nontrivial* character. The following is a version of [1, Theorem 13.8].

Theorem 2. *Under the above setting, the groups $H^i(K/K^+; A_K)(\chi)$ with $i = 0$ and 1 are trivial if and only if*
(i) $\chi(l) \neq 1$ *for all* $l \in S_K$ *and*
(ii) $|\tilde{A}_{K^+}(\chi)| = |A_{K^+}(\chi)|$.

The following corollary is a version of [1, Corollary 13.10] and Hasse [2, Satz 45].

Corollary 2. *Under the above setting, the group $A_K^-(\chi)$ is trivial if and only if*
(i) $\chi(l) \neq 1$ *for all* $l \in S_K$ *and*
(ii) $\tilde{A}_{K^+}(\chi)$ *is trivial.*

Let $\tilde{h}_M$ be the class number in the narrow sense of a number field M. When M is an imaginary abelian field, let h_M^- be the relative class number of M. We can easily show that h_k^- (resp. $\tilde{h}_{k^+}$) divides h_K^- (resp. $\tilde{h}_{K^+}$) using class field theory. The following is an immediate consequence of Corollary 2.

Corollary 3. *Under the above setting, the ratio h_K^-/h_k^- is odd if and only if*
(i) *no prime number l in S_K splits in F and*
(ii) $\tilde{h}_{K^+}/\tilde{h}_{k^+}$ *is odd.*

To prove these assertions, we prepare the following two lemmas. For a number field L, let $\mu(L)$ be the group of roots of unity in L and $\mu_2(L)$ the 2-part of $\mu(L)$.

Lemma 1. *The group $H^1(K/K^+; E_K)(\chi)$ is trivial.*

Proof. Let ${}_N E_K$ be the group of units $\epsilon \in E_K$ with $N(\epsilon) = \epsilon^{1+J} = 1$. We have $N(\epsilon) = 1$ if and only if $\epsilon \in \mu(K)$ by a theorem on units of a CM-field (cf. Washington [9, Theorem 4.12]). Since $\mu(K)^2 = \mu(K)^{1-J} \subseteq E_K^{1-J}$, we obtain a surjection

$$\mu(K)/\mu(K)^2 \to H^1(K/K^+; E_K) = {}_N E_K/E_K^{1-J}$$

of G-modules. However, as $[K : k]$ is odd, we have

$$\mu(K)/\mu(K)^2 = \mu_2(K)/\mu_2(K)^2 = \mu_2(k)/\mu_2(k)^2.$$

Since χ is nontrivial, the χ-part $(\mu_2(k)/\mu_2(k)^2)(\chi)$ is trivial. Hence, we obtain the assertion. □

Lemma 2. *The natural map $A_{K^+}(\chi) \to A_K(\chi)$ is injective.*

Proof. Denote the natural map $A_{K^+} \to A_K$ by ι. Let $\mathfrak{A}$ be an ideal of K^+ with the class $[\mathfrak{A}] \in \ker\iota$. Then $\mathfrak{A}\mathcal{O}_K = x\mathcal{O}_K$ for some $x \in K^\times$. We see that $\epsilon = x^{1-J}$ is a unit of K with $N(\epsilon) = 1$. It is known that the map

$$\ker\iota \to H^1(K/K^+; E_K); \, [\mathfrak{A}] \to x^{1-J} E_K^{1-J}$$

is an injective G-homomorphism ([1, Theorem 7.1]). Then, from Lemma 1, we see that the χ-part $(\ker\iota)(\chi)$ is trivial, from which we obtain the assertion. □

Proof of Theorem 2. Let $\wp$ be a prime ideal in T_f, and $l = \wp \cap \boldsymbol{Q} \in S_K$. We see that the χ-part $\boldsymbol{F}_2[G/G_\wp](\chi) \neq \{0\}$ if and only if χ factors through $G/G_\wp$, which is equivalent to $\chi(G_\wp) = \{1\}$. Since $[k^+ : \boldsymbol{Q}]$ is a 2-power and $[F : \boldsymbol{Q}]$ is odd, we have $\chi(G_\wp) = \{1\}$ if and only if $\chi(l) = 1$. Hence, we have shown that the condition (i) in Theorem 2 is equivalent to the condition $\Omega_f(\chi) = \{0\}$. By the hexagon and Lemma 1, we see that $H^0(A_K)(\chi)$ and $H^1(A_K)(\chi)$ are trivial if and only if (iii) $R^1(K)(\chi) = \{0\}$ and (iv) the map

$$i_0 \colon H^0(E_K)(\chi) = (E_{K^+}/N(E_K))(\chi) \to R^0(K)(\chi)$$

is an isomorphism. By (2) and the above, the condition (iii) is equivalent to (i). Under the condition (i), we see that $R^0(K)(\chi) = \Omega_\infty(\chi)$ from the exact sequence (3), and that for each class $[\epsilon] \in H^0(E_K)(\chi)$ with $\epsilon \in E_{K^+}$, we have $i_0([\epsilon]) = \alpha_\infty(\epsilon)$ from the definitions of the maps i_0 and α. Further, the 2-rank of $\Omega_\infty(\chi)$ is larger than or equal to that of $H^0(E_K)(\chi)$ by a theorem of Minkowski on units of a Galois extension (cf. Narkiewicz [5, Theorem 3.26]). Therefore, under (i), we observe that the condition (iv) holds if and only if $\alpha_\infty(E_{K^+})(\chi) = \Omega_\infty(\chi)$. We see that the last condition is equivalent to the condition (ii) in Theorem 2 because of the exact sequence (4) and $\alpha_\infty((K^+)^\times)(\chi) = \Omega_\infty(\chi)$. Therefore, we obtain Theorem 2. □

Proof of Corollary 2. First, we show the "only if" part assuming that $A_K^-(\chi)$ is trivial. By Lemma 2, we can regard $A_{K^+}(\chi)$ as a subgroup of $A_K(\chi)$. Assume that $A_{K^+}(\chi)$ is nontrivial. Then there exists a class $c \in A_{K^+}(\chi)$ of order 2. We have $c^J = c = c^{-1}$, and hence $c \in A_K^-(\chi)$. It follows that $A_K^-(\chi)$ is nontrivial, a contradiction. Hence, $A_{K^+}(\chi) = \{0\}$. It follows that $A_K(\chi)$ is trivial by the exact sequence

$$\{0\} \to A_K^-(\chi) \to A_K(\chi) \xrightarrow{1+J} A_{K^+}(\chi) \to \{0\}.$$

Therefore, the "only if" part follows from Theorem 2. Next, assume that the conditions (i) and (ii) in Corollary 2 are satisfied. Then, $A_{K^+}(\chi) = \{0\}$, and the groups $H^i(A_K)(\chi)$ $(i = 0, 1)$ are trivial by Theorem 2. As the cohomology groups are trivial, we obtain an exact sequence

$$\{0\} \to A_{K^+}(\chi) \hookrightarrow A_K(\chi) \xrightarrow{1-J} A_K^{1-J}(\chi) = A_K^-(\chi) \to \{0\}.$$

Since $A_{K^+}(\chi) = \{0\}$, we see that $A_K(\chi) = A_K^-(\chi)$, and

$$A_K^-(\chi) = A_K^-(\chi)^{1-J} = A_K^-(\chi)^2$$

from the above exact sequence. Therefore, $A_K^-(\chi)$ is trivial. □

4. Proof of Theorem 1

We use the same notation as in Section 1. In particular, $d \in \boldsymbol{Z}$ is a fixed integer with $\sqrt{d} \notin K_0$ and L_n is the quadratic twist of K_n associated to d. We have $L_n^+ = K_n^+$. Let k (resp. k_d) be the maximal intermediate field of $K_0/\boldsymbol{Q}$ (resp. $L_0/\boldsymbol{Q}$) of 2-power degree, and let F_0 be the maximal subfield of $K_0^+ = L_0^+$ of odd degree over $\boldsymbol{Q}$. Then k and k_d are imaginary abelian fields with $k^+ = k_d^+$. Let $\boldsymbol{B}_n/\boldsymbol{Q}$ be the real abelian field with conductor p^{n+1} and $[\boldsymbol{B}_n : \boldsymbol{Q}] = p^n$. We put $F_n = F_0\boldsymbol{B}_n$. Then $L_n = k_d F_n$ and $K_n = kF_n$. The triples (k_d, F_n, L_n) and (k, F_n, K_n) correspond to (k, F, K) in Sections 2 and 3. We see that

$$S_{L_n} = S_d \quad \text{or} \quad S_d \cup \{p\} \tag{5}$$

and $S_{K_n} = \{p\}$. We put

$$G_n = \mathrm{Gal}(F_n/\boldsymbol{Q}) = \mathrm{Gal}(L_n/k_d) = \mathrm{Gal}(K_n/k),$$

and

$$\Delta = \mathrm{Gal}(F_0/\boldsymbol{Q}), \quad \Gamma_n = \mathrm{Gal}(F_n/F_0) = \mathrm{Gal}(\boldsymbol{B}_n/\boldsymbol{Q}).$$

Then we have a natural decomposition $G_n = \Delta \times \Gamma_n$. For characters φ and ψ of Δ and Γ_n respectively, we regard $\varphi\psi = \varphi \times \psi$ as a character of G_n. Further, we regard φ, ψ and $\varphi\psi$ also as primitive Dirichlet characters. The class groups $A_{L_n}^-$, $A_{K_n}^-$ and $\tilde{A}_{K_n^+}$ are modules over G_n. We can naturally regard $A_{L_{n-1}}^-$ as a subgroup of $A_{L_n}^-$ since L_n/L_{n-1} is a cyclic extension of degree $p \neq 2$ and $A_{L_{n-1}}^-$ is the 2-part of the class group. Actually, it is a direct summand of $A_{L_n}^-$ (cf. [9, Lemma 16.15]). We see that

$$A_{L_n}^-/A_{L_{n-1}}^- = \bigoplus_{\varphi,\psi_n} A_{L_n}^-(\varphi\psi_n) \tag{6}$$

where φ (resp. ψ_n) runs over a complete set of representatives of the $\boldsymbol{Q}_2$-conjugacy classes of the $\bar{\boldsymbol{Q}}_2$-valued characters of Δ (resp. Γ_n of order p^n). Regarding $A_{K_{n-1}}^-$ as a subgroup of $A_{K_n}^-$, we have a similar decomposition for $A_{K_n}^-/A_{K_{n-1}}^-$. As $S_{K_n} = \{p\}$ and $(\varphi\psi_n)(p) = 0$, we obtain the following assertion from Corollary 2 for the triple (k, F_n, K_n).

Lemma 3. *Let $n \geq 1$ be an integer, and the characters φ and ψ_n be as in* (6). *Then $A_{K_n}^-(\varphi\psi_n) = \{0\}$ if and only if $\tilde{A}_{K_n^+}(\varphi\psi_n) = \{0\}$.*

Proof of Theorem 1 (I). Let φ and ψ_n be as in (6). As the orders of φ and ψ_n are relatively prime to each other, we have $(\varphi\psi_n)(l) = 1$ if and only if $\varphi(l) = \psi_n(l) = 1$ for a prime number l. Let n be an integer with $n \geq n_d$. Then we have $\psi_n(l) \neq 1$ and hence $(\varphi\psi_n)(l) \neq 1$ for all prime numbers $l \in S = S_d$. Further, we have $(\varphi\psi_n)(p) = 0$. Hence, by (5), the condition (i) in Corollary 2 for the triple (k_d, F_n, L_n) is satisfied. It follows that the condition $A^-_{L_n}(\varphi\psi_n) = \{0\}$ is equivalent to $\tilde{A}_{K_n^+}(\varphi\psi_n) = \{0\}$. (Note that $L_n^+ = K_n^+$.) Therefore, we obtain Theorem 1(I) from Lemma 3. □

To show Theorem 1 (II), assume that $n_d \geq 2$ and let n be an integer with $1 \leq n < n_d$. We put

$$S^{(n)} = \{l \in S = S_d \mid \mathrm{ord}_p(l^{p-1} - 1) \geq n + 1\}.$$

From the definition, we see that

$$S \supseteq S^{(1)} \supseteq S^{(2)} \supseteq \cdots \supseteq S^{(n_d-1)}$$

and that each $S^{(n)}$ is non-empty. Let φ (resp. ψ_n) be a $\bar{\boldsymbol{Q}}_2$-valued character of Δ (resp. of Γ_n of order p^n). Denote by φ_0 the trivial character of Δ. Theorem 1 (II) is a consequence of the following assertion.

Proposition 1. *Under the above setting, the following hold.*
(I) *The class group $A^-_{L_n}(\varphi\psi_n)$ is nontrivial if $\varphi(l) = 1$ for some $l \in S^{(n)}$. In particular, $A^-_{L_n}(\varphi_0\psi_n)$ is nontrivial.*
(II) *If $A^-_{K_n}(\varphi\psi_n) = \{0\}$, the converse of the first assertion of (I) holds.*

Proof. Applying Corollary 2 for the triple (k_d, F_n, L_n), we see from Lemma 3 that $A^-_{L_n}(\varphi\psi_n) = \{0\}$ if and only if (i) $(\varphi\psi_n)(l) \neq 1$ for all $l \in S = S_d$ and (ii) $A^-_{K_n}(\varphi\psi_n) = \{0\}$. We have $\psi_n(l) = 1$ for $l \in S^{(n)}$, and $\psi_n(l) \neq 1$ for $l \in S \setminus S^{(n)}$. Therefore, we see that the condition (i) is satisfied if and only if $\varphi(l) \neq 1$ for all $l \in S^{(n)}$ noting that the orders of φ and ψ_n are relatively prime. From this, we obtain the proposition. □

We put $M_n = K_n(\sqrt{d}) = K_n L_n$. On the relative class number $h^-_{M_n}$ of M_n, the following assertion holds.

Proposition 2. (I) *When $n \geq n_d$, the ratio $h^-_{M_n}/h^-_{M_{n-1}}$ is odd if and only if h^*_n/h^*_{n-1} is odd.*
(II) *When $n_d \geq 2$ and $1 \leq n < n_d$, $h^-_{M_n}/h^-_{M_{n-1}}$ is even.*

To prove this proposition, we need to show the following lemma. For an imaginary abelian field N, we put

$$\mathcal{E}_N = E_N/\mu(N)E_{N^+}.$$

It is well known that the unit index $Q_N = |\mathcal{E}_N|$ is 1 or 2 ([9, Theorem 4.12]).

Lemma 4. *Let T and N be imaginary abelian fields with $N \subseteq T$. If the degree $[T : N]$ is odd, then $Q_T = Q_N$.*

Proof. We first show that the inclusion map $N \to T$ induces an injection $\mathcal{E}_N \hookrightarrow \mathcal{E}_T$. For a unit ϵ of N, assume that $\epsilon = \zeta\eta$ for some $\zeta \in \mu(T)$ and $\eta \in E_{T^+}$. Let ρ be a nontrivial element of the Galois group $G = \mathrm{Gal}(T/N)$. Then, as $\epsilon = \epsilon^\rho$, we see that $\zeta^{1-\rho} = \eta^{\rho-1} \in \mu(T) \cap E_{T^+}$. Hence, $\zeta^{1-\rho} = \pm 1$. However, as $N_{T/N}(\zeta^{1-\rho}) = 1$ and $[T : N]$ is odd, the case $\zeta^{1-\rho} = -1$ does not happen. Hence, $\zeta^{1-\rho} = 1$ for all $\rho \in G$. It follows that $\zeta \in \mu(N)$ and hence $\eta \in E_{N^+}$. Therefore, we can regard $\mathcal{E}_N$ as a subgroup of $\mathcal{E}_T$. In particular, Q_N divides Q_T.

Assume that $Q_N \neq Q_T$. Then we have $|\mathcal{E}_T| = |\mathcal{E}_T/\mathcal{E}_N| = 2$. Regarding $\mathcal{E}_T$ as a module over G, we have a canonical decomposition

$$\mathcal{E}_T = \mathcal{E}_T/\mathcal{E}_N = \bigoplus_{\chi} \mathcal{E}_T(\chi)$$

where χ runs over a complete set of representatives of the $\boldsymbol{Q}_2$-conjugacy classes of the *nontrivial* $\bar{\boldsymbol{Q}}_2$-valued characters of G. Hence, $|\mathcal{E}_T(\chi)| = 2$ for some such χ. Let $\boldsymbol{Z}_2[\chi]$ be the subring of $\bar{\boldsymbol{Q}}_2$ generated by the values of χ over $\boldsymbol{Z}_2$. The group $\mathcal{E}_T(\chi)$ is naturally regarded as a module over the principal ideal domain $\boldsymbol{Z}_2[\chi]$. Since the order of χ is odd and ≥ 3, we observe that $\boldsymbol{Z}_2[\chi] \cong \boldsymbol{Z}_2^d$ as $\boldsymbol{Z}_2$-modules for some $d \geq 2$. Hence, $|\mathcal{E}_n(\chi)|$ is a multiple of 2^d, which contradicts $|\mathcal{E}_n(\chi)| = 2$. Therefore, we obtain $Q_N = Q_T$. □

Proof of Proposition 2. By Lemma 4, we have $Q_{M_n} = Q_{M_{n-1}}$ and $Q_{L_n} = Q_{L_{n-1}}$ for all $n \geq 1$. Therefore, using the class number formula [9, Theorem 4.17], we see that

$$h^-_{M_n}/h^-_{M_{n-1}} = p \prod_{\varpi} \prod_{\psi_n} \left(-\frac{1}{2} B_{1,\varpi\psi_n} \right)$$

where ϖ runs over the odd Dirichlet characters associated to M_0, and ψ_n over the even characters of conductor p^{n+1} and order p^n. Further, $B_{1,\varpi\psi_n}$ denotes the generalized Bernoulli number. We easily see that ϖ equals an odd Dirichlet character associated to K_0 or L_0 since M_0/K_0^+ is an imaginary biquadratic extension with the imaginary quadratic subextensions K_0 and L_0. Hence, using the class number formulas for L_n, K_n and $Q_{L_n} = Q_{L_{n-1}}$, we obtain

$$h^-_{M_n}/h^-_{M_{n-1}} = h^*_n/h^*_{n-1} \times h^-_n/h^-_{n-1}.$$

Therefore, the assertion follows from Theorem 1. □

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