

## BLOWDOWNS AND MCKAY CORRESPONDENCE ON FOUR DIMENSIONAL QUASITORIC ORBIFOLDS

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### Abstract

We prove the existence of torus invariant almost complex structure on any positively omnioriented four dimensional primitive quasitoric orbifold. We construct pseudo-holomorphic blowdown maps for such orbifolds. We prove a version of McKay correspondence when the blowdowns are crepant.

### 1. Introduction

Quasitoric orbifolds are generalizations or topological counterparts of simplicial projective toric varieties. They admit an action of the real torus of half dimension such that the orbit space has the combinatorial type of a simple convex polytope. Davis and Januskiewicz [4], who introduced the notion of quasitoric space, showed that the formula for the cohomology ring of a quasitoric manifold, and hence of any nonsingular projective toric variety, may be deduced by purely algebraic topology methods. This was generalized to the orbifold case in [10].

In general quasitoric manifolds do not have integrable or almost complex structure. However, they always have stable almost complex structure. Moreover, positively omnioriented quasitoric manifolds have been known to have an almost complex structure, see [2]. It was recently proved by Kustarev [6, 7] that any positively omnioriented quasitoric manifold has an almost complex structure which is torus invariant. We extend his result to four dimensional primitive quasitoric orbifolds, see Theorem 3.1. Note that for higher dimensional positively omnioriented quasitoric orbifolds the existence of an almost complex structure, torus invariant or otherwise, remains an open problem. We hope to address this in future.

Inspired by birational geometry of toric varieties, we introduce the notion of blowdown into the realm of quasitoric orbifolds. Our blowdown maps contract an embedded orbifold sphere (exceptional sphere) to a point. They admit a description in terms of a finite collection of coordinate charts, very much in the spirit of toric geometry. These maps are torus invariant, continuous and diffeomorphism of orbifolds away from the exceptional sphere, see Theorem 4.1. However they are not morphisms of orbifolds near the exceptional sphere. For a blowdown map between two primitive positively omnioriented

quasitoric orbifolds of dimension four (see Remark 2.1 and Subsection 2.7 for definitions), we can choose almost complex structures on them so that the blowdown is an analytic map near the exceptional sphere and an almost complex morphism of orbifolds away from it, see Theorem 4.2. We explain in Corollary 4.3 that the blowdown is a pseudo-holomorphic map in the sense that it pulls back invariant pseudo-holomorphic functions to similar functions.

We restrict our study to the case where the exceptional sphere has at most one orbifold singularity. Any singularity on a 4-dimensional primitive positively omnioriented quasitoric orbifold may be resolved by a sequence of blowdowns of this type. However as our method for studying pseudo-holomorphicity of blowdowns is not very amenable to induction, we leave the study of such a sequence of blowdowns for future work.

An immediate consequence of the existence of an almost complex structure is that Chen–Ruan orbifold cohomology [3] is defined. We define when an almost complex blowdown is crepant, see Definition 5.2. We prove that the Betti numbers of orbifold cohomology are preserved under a crepant blowdown. This is a form of McKay correspondence. Such correspondence has been widely studied for algebraic orbifolds.

Masuda [8] proved the existence of invariant almost complex structure on positively omnioriented four dimensional quasitoric manifolds, many of which are not toric varieties. This shows that our results are not redundant. We give explicit examples in the last section. In fact, using blowdown we can construct almost complex quasitoric orbifolds that are not toric varieties.

We refer the reader to [1] for relevant definitions and results regarding orbifolds.

## 2. Preliminaries

Many results of this section are part of folklore. However to set up our notation and due to lack of a good reference, we give an explicit description.

**2.1. Definition.** Let  $N$  be a free  $\mathbb{Z}$ -module of rank 2. Let  $T_N := (N \otimes_{\mathbb{Z}} \mathbb{R})/N \cong \mathbb{R}^2/N$  be the corresponding 2-dimensional torus. A primitive vector in  $N$ , modulo sign, corresponds to a circle subgroup of  $T_N$ . More generally, suppose  $M$  is a free submodule of  $N$  of rank  $m$ . Then  $T_M := (M \otimes_{\mathbb{Z}} \mathbb{R})/M$  is a torus of dimension  $m$ . Moreover there is a natural homomorphism of Lie groups  $\xi_M: T_M \rightarrow T_N$  with finite kernel, induced by the inclusion  $M \hookrightarrow N$ .

**DEFINITION 2.1.** Define  $T(M)$  to be the image of  $T_M$  under  $\xi_M$ . If  $M$  is generated by a vector  $\lambda \in N$ , denote  $T(M)$  by  $T(\lambda)$ .

**DEFINITION 2.2 ([10]).** A 4-dimensional quasitoric orbifold  $\mathbf{Y}$  is an orbifold whose underlying topological space  $Y$  has a  $T_N$  action, where  $N$  is a fixed free  $\mathbb{Z}$ -module of rank 2, such that the orbit space is (diffeomorphic to) a 2-dimensional

polytope  $P$ . Denote the projection map from  $Y$  to  $P$  by  $\pi: Y \rightarrow P$ . Furthermore every point  $x \in Y$  has

A1) a  $T_N$ -invariant neighborhood  $V$ ,

A2) an associated free  $\mathbb{Z}$ -module  $M$  of rank 2 with an isomorphism  $\theta: T_M \rightarrow U(1)^2$  and an injective module homomorphism  $i: M \rightarrow N$  which induces a surjective covering homomorphism  $\xi_M: T_M \rightarrow T_N$ ,

A3) an orbifold chart  $(\tilde{V}, G, \xi)$  over  $V$ , where  $\tilde{V}$  is  $\theta$ -equivariantly diffeomorphic to an open set in  $\mathbb{C}^2$ ,  $G = \ker \xi_M$  and  $\xi: \tilde{V} \rightarrow V$  is an equivariant map i.e.  $\xi(t \cdot y) = \xi_M(t) \cdot \xi(y)$  inducing a homeomorphism between  $\tilde{V}/G$  and  $V$ .

Fix  $N$ . Let  $P$  be a convex polytope in  $\mathbb{R}^2$  with edges  $E_i$ ,  $i \in \mathcal{E} := \{1, 2, \dots, m\}$ . Identify the set of edges with  $\mathcal{E}$ . A function  $\Lambda: \mathcal{E} \rightarrow N$  is called a characteristic function for  $P$  if  $\Lambda(i)$  and  $\Lambda(j)$  are linearly independent whenever  $E_i$  and  $E_j$  meet at a vertex of  $P$ . We write  $\lambda_i$  for  $\Lambda(i)$  and call it a characteristic vector.

REMARK 2.1. In this article we assume that all characteristic vectors are primitive. Corresponding quasitoric orbifolds have been termed primitive quasitoric orbifold in [10]. They are characterized by the codimension of singular set being greater than or equal to four.

Let  $\Lambda$  be a characteristic function for  $P$ . For any face  $F$  of  $P$ , let  $N(F)$  be the submodule of  $N$  generated by  $\{\lambda_i: F \subset E_i\}$ . For any point  $p \in P$ , denote by  $F(p)$  the face of  $P$  whose relative interior contains  $p$ . Define an equivalence relation  $\sim$  on the space  $P \times T_N$  by

$$(2.1) \quad (p, t) \sim (q, s) \quad \text{if and only if} \quad p = q \quad \text{and} \quad s^{-1}t \in T(N(F(p))).$$

The quotient space  $X := P \times T_N / \sim$  can be given the structure of a 4-dimensional quasitoric orbifold. Moreover any 4-dimensional primitive quasitoric orbifold may be obtained in this way, see [10]. We refer to the pair  $(P, \Lambda)$  as a model for the quasitoric orbifold.

The space  $X$  inherits an action of  $T_N$  with orbit space  $P$  from the natural action on  $P \times T_N$ . Let  $\pi: X \rightarrow P$  be the associated quotient map. The space  $X$  is a manifold if the characteristic vectors  $\lambda_i$  and  $\lambda_j$  form a  $\mathbb{Z}$ -basis of  $N$  whenever the edges  $E_i$  and  $E_j$  meet at a vertex.

The points  $\pi^{-1}(v) \in X$ , where  $v$  is any vertex of  $P$ , are fixed by the action of  $T_N$ . For simplicity we will denote the point  $\pi^{-1}(v)$  by  $v$  when there is no confusion.

**2.2. Differentiable structure.** Consider open neighborhoods  $U_v \subset P$  of the vertices  $v$  such that  $U_v$  is the complement in  $P$  of all edges that do not contain  $v$ . Let

$$(2.2) \quad X_v := \pi^{-1}(U_v) = U_v \times T_N / \sim.$$

For a face  $F$  of  $P$  containing  $v$  there is a natural inclusion of  $N(F)$  in  $N(v)$ . It induces an injective homomorphism  $T_{N(F)} \rightarrow T_{N(v)}$  since a basis of  $N(F)$  extends to a basis of  $N(v)$ . We will regard  $T_{N(F)}$  as a subgroup of  $T_{N(v)}$  without confusion.

Define an equivalence relation  $\sim_v$  on  $U_v \times T_{N(v)}$  by  $(p, t) \sim_v (q, s)$  if  $p = q$  and  $s^{-1}t \in T_{N(F)}$  where  $F$  is the face whose relative interior contains  $p$ . Then the space

$$(2.3) \quad \tilde{X}_v := U_v \times T_{N(v)} / \sim_v$$

is  $\theta$ -equivariantly diffeomorphic to an open ball in  $\mathbb{C}^2$ , where  $\theta: T_{N(v)} \rightarrow U(1)^2$  is an isomorphism, see [4]. This is also evident from the discussion on local models below.

The map  $\xi_{N(v)}: T_{N(v)} \rightarrow T_N$  induces a map  $\xi_v: \tilde{X}_v \rightarrow X_v$  defined by  $\xi_v([(p, t)]^{\sim_v}) = [(p, \xi_{N(v)}(t))]^{\sim}$  on equivalence classes. The kernel of  $\xi_{N(v)}$ ,  $G_v = N/N(v)$ , is a finite subgroup of  $T_{N(v)}$  and therefore has a natural smooth, free action on  $T_{N(v)}$  induced by the group operation. This induces smooth action of  $G_v$  on  $\tilde{X}_v$ . This action is not free in general. Since  $T_N \cong T_{N(v)}/G_v$ ,  $X_v$  is homeomorphic to the quotient space  $\tilde{X}_v/G_v$ . An orbifold chart (or uniformizing system) on  $X_v$  is given by  $(\tilde{X}_v, G_v, \xi_v)$ .

Up to homeomorphism we may regard the set  $U_v$  as a cone  $\sigma(v)$  with the same edges as  $U_v$ . The neighborhood  $X_v$  is then homeomorphic to  $\sigma(v) \times T_N/\sim$ . We say that a local model for  $X$  near  $v$  consists of a cone  $\sigma$  and characteristic vectors, say,  $\lambda_1, \lambda_2$  along its edges  $E_1$  and  $E_2$ .

Let  $p_1, p_2$  denote the standard coordinates on  $\mathbb{R}^2 \supset P$ . Let  $q_1, q_2$  be the coordinates on  $N \otimes \mathbb{R}$  with respect to the standard basis of  $N$ . They correspond to standard angular coordinates on  $T_N$ . The local model where  $\sigma = \mathbb{R}_{\geq}^2 := \{(p_1, p_2) \in \mathbb{R}^2: p_i \geq 0\}$  and the characteristic vectors are  $(1, 0)$  along  $p_1 = 0$  and  $(0, 1)$  along  $p_2 = 0$  is called standard. In this case there is a homeomorphism from the  $\mathbb{R}_{\geq}^2 \times T_N/\sim$  to  $\mathbb{C}^2 = \mathbb{R}^4$  given by

$$(2.4) \quad x_i = \sqrt{p_i} \cos(2\pi q_i), y_i = \sqrt{p_i} \sin(2\pi q_i) \quad \text{where } i = 1, 2.$$

For any cone  $\sigma(v) \subset \mathbb{R}^2$  with arbitrary characteristic vectors we will define a canonical homeomorphism  $\phi(v): \tilde{X}_v \rightarrow \mathbb{R}^4$  as follows. Order the edges  $E_1, E_2$  of  $\sigma(v)$  so that the clockwise angle from  $E_1$  to  $E_2$  is less than  $180^\circ$ . Denote the coordinates of the vertex  $v$  by  $(\alpha, \beta)$ . Let the equations of the edge  $E_i$  be  $a_i(p_1 - \alpha) + b_i(p_2 - \beta) = 0$ . Assume that the interior of  $\sigma(v)$  is contained in the half-plane  $a_i(p_1 - \alpha) + b_i(p_2 - \beta) \geq 0$ . Suppose  $\lambda_1 = (c_{11}, c_{21})$  and  $\lambda_2 = (c_{12}, c_{22})$  be the characteristic vectors assigned to  $E_1$  and  $E_2$  respectively. If  $q_1(v), q_2(v)$  are angular coordinates of an element of  $T_N$  with respect to the basis  $\lambda_1, \lambda_2$  of  $N \otimes \mathbb{R}$ , then the standard coordinates  $q_1, q_2$  may be expressed as

$$(2.5) \quad \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{pmatrix} q_1(v) \\ q_2(v) \end{pmatrix}.$$

Then define the homeomorphism  $\phi(v): \tilde{X}_v \rightarrow \mathbb{R}^4$  by

$$(2.6) \quad \begin{aligned} x_i &= x_i(v) := \sqrt{p_i(v)} \cos(2\pi q_i(v)), \\ y_i &= y_i(v) := \sqrt{p_i(v)} \sin(2\pi q_i(v)) \quad \text{for } i = 1, 2 \end{aligned}$$

where

$$(2.7) \quad p_i(v) = a_i(p_1 - \alpha) + b_i(p_2 - \beta), \quad i = 1, 2.$$

Similar homeomorphism has been used in [11].

Now consider the action of  $G_v = N/N(v)$  on  $\tilde{X}_v$ . An element of  $G_v$  is represented by a vector  $g = a_1\lambda_1 + a_2\lambda_2$  in  $N$  where  $a_1, a_2 \in \mathbb{Q}$ . The action of  $g$  transforms the coordinates  $(q_1(v), q_2(v))$  to  $(q_1(v) + a_1, q_2(v) + a_2)$ . If we write  $z_i = x_i + \sqrt{-1}y_i$ , then

$$(2.8) \quad g \cdot (z_1, z_2) = (e^{2\pi\sqrt{-1}a_2}z_1, e^{2\pi\sqrt{-1}a_1}z_2).$$

Since  $\lambda_1$  and  $\lambda_2$  are both primitive, neither of  $a_1, a_2$  is an integer. Therefore the only orbifold singularity on  $X_v$  is at the point with coordinates  $z_1 = z_2 = 0$ , namely the vertex  $v$ .

We show the compatibility of the charts  $(\tilde{X}_v, G_v, \xi_v)$ . Let  $v_1$  and  $v_2$  be two adjacent vertices. Assume that edges  $E_1, E_2$  meet at  $v_1$  and edges  $E_2, E_3$  meet at  $v_2$ . Let  $\lambda_i$  be the characteristic vector corresponding to  $E_i$ . Since all characteristic vectors are primitive, we may assume without loss of generality that  $\lambda_1 = (1, 0)$ . Suppose  $\lambda_2 = (a, b)$  and  $\lambda_3 = (c, d)$ . Let  $\Delta = ad - bc$ .

Up to choice of coordinates we may assume that the edge  $E_1$  has equation  $p_1 = 0$ , the edge  $E_3$  has equation  $p_2 = 0$ , and the edge  $E_2$  has equation  $\hat{p} = 0$ . Here  $\hat{p} = p_2 + sp_1 - t$  where  $s$  and  $t$  are positive reals. We assume that the quantities  $p_1, p_2$  and  $\hat{p}$  are positive in the interior of the polytope.

We shall write down explicit coordinates on  $\tilde{X}_{v_i}$ . For this purpose it is convenient to express all angular coordinates in terms of  $(q_1, q_2)$  by inverting equation (2.5).

$$(2.9) \quad \begin{pmatrix} q_1(v) \\ q_2(v) \end{pmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

where the matrix  $(d_{ij})$  is the inverse of the matrix  $(c_{ij})$ . Then by equations (2.6), (2.7), (2.9) we have the following expressions for coordinates  $z_i(v_j) := x_i(v_j) + \sqrt{-1}y_i(v_j)$  on  $\tilde{X}_{v_j}$ .

$$(2.10) \quad \begin{aligned} z_1(v_1) &:= \sqrt{p_1}e^{2\pi\sqrt{-1}(q_1-(a/b)q_2)}, & z_2(v_1) &:= \sqrt{\hat{p}}e^{2\pi\sqrt{-1}(1/b)q_2}, \\ z_1(v_2) &:= \sqrt{\hat{p}}e^{2\pi\sqrt{-1}(dq_1-cq_2)/\Delta}, & z_2(v_2) &:= \sqrt{p_2}e^{2\pi\sqrt{-1}((-bq_1+aq_2)/\Delta)}. \end{aligned}$$

The coordinates on  $\tilde{X}_{v_2}$  are related to those on  $\tilde{X}_{v_1}$  over the intersection  $X_{v_1} \cap X_{v_2}$  as follows.

$$(2.11) \quad z_1(v_2) = z_1(v_1)^{d/\Delta} z_2(v_1) \sqrt{p_1}^{-d/\Delta}, \quad z_2(v_2) = z_1(v_1)^{-b/\Delta} \sqrt{p_2} \sqrt{p_1}^{b/\Delta}.$$

Take any point  $x \in X_{v_1} \cap X_{v_2}$ . Let  $\tilde{x}$  be a preimage of  $x$  with respect to  $\xi_{v_1}$ . Choose a small ball  $B(\tilde{x}, r)$  around  $\tilde{x}$  such that  $(g \cdot B(\tilde{x}, r)) \cap B(\tilde{x}, r)$  is empty for all nontrivial  $g \in G_{v_1}$ . Then  $(B(\tilde{x}, r), \{1\}, \xi_{v_1})$  is an orbifold chart around  $x$ . This chart admits natural embedding (or injection) into the chart  $(\tilde{X}_{v_1}, G_{v_1}, \xi_{v_1})$  given by inclusion. We show that for sufficiently small value of  $r$ , this chart embeds into  $(\tilde{X}_{v_2}, G_{v_2}, \xi_{v_2})$  as well. Choose a branch of  $z_1(v_1)^{1/\Delta}$  so that the branch cut does not intersect  $B(\tilde{x}, r)$ . Assume  $r$  to be small enough so that the functions  $z_1(v_1)^{d/\Delta}$  and  $z_1(v_1)^{\frac{b}{\Delta}}$  are one-to-one on  $B(\tilde{x}, r)$ . Then equation (2.11) defines a smooth embedding  $\psi$  of  $B(\tilde{x}, r)$  into  $\tilde{X}_{v_2}$ . Note that  $p_1$  and  $p_2$  are smooth nonvanishing functions on  $\xi_{v_1}^{-1}(X_{v_1} \cap X_{v_2})$ . Assume  $r$  to be small enough so that  $(h \cdot \psi(B(\tilde{x}, r))) \cap \psi(B(\tilde{x}, r))$  is empty for all  $h$  in  $G_{v_2}$ . Then  $(\psi, id): (B(\tilde{x}, r), \{1\}, \xi_{v_1}) \rightarrow (\tilde{X}_{v_2}, G_{v_2}, \xi_{v_2})$  is an embedding of orbifold charts.

**REMARK 2.2.** We will denote the topological space  $X$  endowed with the above orbifold structure by  $\mathbf{X}$ . In general we denote the underlying space of an orbifold  $\mathbf{Y}$  by  $Y$ . We denote the set of smooth points of  $Y$ , i.e. points having trivial local group, by  $Y_{\text{reg}}$ .

**REMARK 2.3.** The equivariant homeomorphism or diffeomorphism type of a quasitoric orbifold does not depend on the choice of signs of the characteristic vectors. However these signs do effect the local complex structure on the coordinate charts obtained via the pullback of the standard complex structure on  $\mathbb{C}^2/G_v$ . A theorem of Prill [9] proves that the analytic germ of singularity at  $v$  is characterized by the linearized action of  $G_v$ .

The following lemma shows that the orbifold structure on  $\mathbf{X}$  does not depend on the shape of the polytope  $P$ .

**Lemma 2.4.** *Suppose  $\mathbf{X}$  and  $\mathbf{Y}$  are four dimensional quasitoric orbifolds whose orbit spaces  $P$  and  $Q$  are diffeomorphic and the characteristic vector of any edge of  $P$  matches with the characteristic vector of the corresponding edge of  $Q$ . Then  $\mathbf{X}$  and  $\mathbf{Y}$  are equivariantly diffeomorphic.*

*Proof.* Pick any vertex  $v$  of  $P$ . For simplicity we will write  $p_i$  for  $p_i(v)$ , and  $q_i$  for  $q_i(v)$ . Suppose the diffeomorphism  $f: P_1 \rightarrow P_2$  is given near  $v$  by  $f(p_1, p_2) = (f_1, f_2)$ . It induces a map of local charts  $\tilde{X}_v \rightarrow \tilde{Y}_{f(v)}$  by

$$(2.12) \quad (\sqrt{p_i} \cos(q_i), \sqrt{p_i} \sin(q_i)) \mapsto (\sqrt{f_i} \cos(q_i), \sqrt{f_i} \sin(q_i)) \quad \text{for } i = 1, 2.$$

This is a smooth map if the functions  $\sqrt{f_i/p_i}$  are smooth functions of  $p_1, p_2$ . Without loss of generality let us consider the case of  $\sqrt{f_1/p_1}$ . We may write

$$(2.13) \quad f_1(p_1, p_2) = f_1(0, p_2) + p_1 \frac{\partial f_1}{\partial p_1}(0, p_2) + p_1^2 g(p_1, p_2)$$

where  $g$  is smooth, see section 8.14 of [5]. Note that  $f_1(0, p_2) = 0$  as  $f$  maps the edge  $p_1 = 0$  to the edge  $f_1 = 0$ . Then it follows from equation (2.13) that  $f_1/p_1$  is smooth. We have

$$(2.14) \quad \frac{f_1}{p_1} = \frac{\partial f_1}{\partial p_1}(0, p_2) + p_1 g(p_1, p_2).$$

Note that  $f_1/p_1$  is nonvanishing away from  $p_1 = 0$ . Moreover we have

$$(2.15) \quad \frac{f_1}{p_1} = \frac{\partial f_1}{\partial p_1}(0, p_2) \quad \text{when} \quad p_1 = 0.$$

Since  $f_1(0, p_2)$  is identically zero,  $(\partial f_1/\partial p_2)(0, p_2) = 0$ . As the Jacobian of  $f$  is non-singular we must have

$$(2.16) \quad \frac{\partial f_1}{\partial p_1}(0, p_2) \neq 0.$$

Thus  $f_1/p_1$  is nonvanishing even when  $p_1 = 0$ . Consequently  $\sqrt{f_1/p_1}$  is smooth. Therefore the map (2.12) is smooth and induces an isomorphism of orbifold charts.  $\square$

**2.3. Torus action.** An action of a group  $H$  on an orbifold  $\mathbf{Y}$  is an action of  $H$  on the underlying space  $Y$  with some extra conditions. In particular for every sufficiently small  $H$ -stable neighborhood  $U$  in  $Y$  with uniformizing system  $(\tilde{U}, G, \xi)$ , the action should lift to an action of  $H$  on  $\tilde{U}$  that commutes with the action of  $G$ . The  $T_N$ -action on the underlying topological space of a quasitoric orbifold does not lift to an action on the orbifold in general.

**2.4. Metric.** Any cover of  $X$  by  $T_N$ -stable open sets induces an open cover of  $P$ . Choose a smooth partition of unity on the polytope  $P$  subordinate to this induced cover. Composing with the projection map  $\pi: X \rightarrow P$  we obtain a partition of unity on  $X$  subordinate to the given cover, which is  $T_N$ -invariant. Such a partition of unity is smooth as the map  $\pi$  is smooth, being locally given by maps  $p_j = x_j^2 + y_j^2$ .

**DEFINITION 2.3.** By a torus invariant metric on  $\mathbf{X}$  we will mean a metric on  $\mathbf{X}$  which is  $T_{N(v)}$ -invariant in some neighborhood of each vertex  $v$  and  $T_N$ -invariant on  $X_{\text{reg}}$ .

For instance, choose a  $T_{N(v)}$ -invariant metric on each  $\tilde{X}_v$ . Then using a partition of unity as above we can define a metric on  $\mathbf{X}$ . Such a metric is  $T_N$ -invariant on  $X_{\text{reg}}$ . We use variants of this construction in what follows.

**2.5. Characteristic suborbifolds.** The  $T_N$ -invariant subset  $\pi^{-1}(E)$ , where  $E$  is any edge of  $P$ , is a suborbifold of  $\mathbf{X}$ . It is called a characteristic suborbifold. Topologically it is a sphere. It can have orbifold singularity only at the two vertices. If  $\lambda$  is the characteristic vector attached to  $E$ , then  $\pi^{-1}(E)$  is fixed by the circle subgroup  $T(\lambda)$  of  $T_N$ . A characteristic suborbifold is a quasitoric orbifold, see [10].

**2.6. Orientation.** Consider the manifold case first. Note that for any vertex  $v$   $dp_i(v) \wedge dq_i(v) = dx_i(v) \wedge dy_i(v)$ . Therefore  $\omega(v) := dp_1(v) \wedge dp_2(v) \wedge dq_1(v) \wedge dq_2(v)$  equals  $dx_1(v) \wedge dx_2(v) \wedge dy_1(v) \wedge dy_2(v)$ . The standard coordinates  $(p_1, p_2)$  are related to  $(p_1(v), p_2(v))$  by a diffeomorphism. Similarly for  $(q_1, q_2)$  and  $(q_1(v), q_2(v))$ . Therefore  $\omega := dp_1 \wedge dp_2 \wedge dq_1 \wedge dq_2$  is a nonzero multiple of each  $\omega(v)$  and defines a nonvanishing form on  $X$ . Thus a choice of orientations for  $P \subset \mathbb{R}^2$  and  $T_N$  induces an orientation for  $X$ .

In the orbifold case the action of  $G_v$  on  $\tilde{X}_v$ , see equation (2.8), preserves  $\omega(v)$  for each vertex  $v$  as  $dx_i(v) \wedge dy_i(v) = \frac{\sqrt{-1}}{2} dz_i(v) \wedge d\bar{z}_i(v)$ . Hence the same conclusion holds.

**2.7. Omniorientation.** An omniorientation is a choice of orientation for the orbifold as well as an orientation for each characteristic suborbifold. At any vertex  $v$ , the  $G_v$ -representation  $\mathcal{T}_0 \tilde{X}_v$  splits into the direct sum of two  $G_v$ -representations corresponding to the linear subspaces  $z_i(v) = 0$ . Thus we have a decomposition of the orbifold tangent space  $\mathcal{T}_v \mathbf{X}$  as a direct sum of tangent spaces of the two characteristic suborbifolds that meet at  $v$ . Given an omniorientation, we set the sign of a vertex  $v$  to be positive if the orientations of  $\mathcal{T}_v(\mathbf{X})$  determined by the orientation of  $\mathbf{X}$  and orientations of characteristic suborbifolds coincide. Otherwise we say that sign of  $v$  is negative. An omniorientation is then said to be positive if each vertex has positive sign.

Note that the normal bundle of any characteristic suborbifold is naturally oriented by the action of its isotropy circle. The action and hence the orientation depends on the sign of the characteristic vector. Thus for a fixed orientation on  $\mathbf{X}$  an omniorientation is determined by a choice of sign for each characteristic vector. Assume henceforth that  $\mathbf{X}$  is oriented via standard orientations on  $P$  and  $T_N$ . Then an omniorientation on  $\mathbf{X}$  is positive if the matrix of adjacent characteristic vectors, with clockwise ordering of adjacent edges, has positive determinant at each vertex.

### 3. Almost complex structure

Kustarev [6] showed that the obstruction to existence of a torus invariant almost complex structure on a quasitoric manifold, which is furthermore orthogonal with respect



to a torus invariant metric, reduces to the obstruction to its existence on a section of the orbit map. We use the same principle here for 4-dimensional quasitoric orbifolds. The obstruction theory for orbifolds in higher dimensions seems to be more complicated.

Let  $\mathbf{X}$  be a positively omnioriented 4-dimensional primitive quasitoric orbifold.

**DEFINITION 3.1.** We say that an almost complex structure on  $\mathbf{X}$  is torus invariant if it is  $T_{N(v)}$ -invariant in some neighborhood of each vertex  $v$  and  $T_N$ -invariant on  $X_{\text{reg}}$ .

Denote the set of all  $i$ -dimensional faces of  $P$  by  $sk_i(P)$ . We refer to  $\pi^{-1}(sk_i(P))$  as the  $i$ -th skeleton of  $X$ . We fix an embedding  $\iota: P \rightarrow X$  that satisfies

$$(3.1) \quad \begin{aligned} \pi \circ \iota &= id \quad \text{and} \\ \iota|_{\text{int}(G)} &\text{ is smooth for any face } G \subset P. \end{aligned}$$

A choice of  $\iota$  is given by the composition  $P \xrightarrow{i} P \times T_N \xrightarrow{j} X$  where  $i$  is the inclusion given by  $i(p_1, p_2) = (p_1, p_2, 1, 1)$  and  $j$  is the quotient map that defines  $X$ .

We also fix a torus invariant metric  $\mu$  on  $\mathbf{X}$  as follows. Choose an open cover of  $P$  such that each vertex of  $P$  has a neighborhood which is contained in exactly one open set of the cover. This induces a cover of  $X$ . On each open set  $W_v$  of this cover, corresponding to the vertex  $v$ , choose the standard metric with respect to the coordinates in (2.6). On the remaining open sets, choose any  $T_N$ -invariant metric. Then use a  $T_N$ -invariant partition of unity, subordinate to the cover, to glue these metrics and obtain  $\mu$ .

**Theorem 3.1.** *There exists a torus invariant almost complex structure  $J$  on  $\mathbf{X}$  such that  $J$  is orthogonal with respect to  $\mu$ .*

**Proof.** Choose small orbifold charts  $(\tilde{X}'_v, G_v, \xi_v)$  around each vertex  $v$  where  $X'_v \subset W_v$ . Choose coordinates  $x_i(v), y_i(v), i = 1, 2$  on  $\tilde{X}'_v$  according to (2.6). Declare  $z_i(v) = x_i(v) + \sqrt{-1}y_i(v)$ . Choose the standard complex structure  $J_v$  on  $\tilde{X}'_v$  with respect to these coordinates, i.e.  $z_1(v), z_2(v)$  are holomorphic coordinates under  $J_v$ . Since  $J_v$  commutes with action of  $T_{N(v)}$  and  $G_v$  is a subgroup of  $T_{N(v)}$ , we may regard  $J_v$  as a torus invariant complex structure on a neighborhood of  $v$  in the orbifold  $\mathbf{X}$ . Note that these local complex structures are orthogonal with respect to  $\mu$  near the vertices.

We will first construct an almost complex structure on the first skeleton of  $X$  that agrees with the above local complex structures. Let  $E$  be any edge of  $P$ . Suppose  $E$  joins the vertices  $u$  and  $v$ . Assume that  $\lambda$  is the characteristic vector attached to  $E$ . The characteristic suborbifold corresponding to  $E$  is an orbifold sphere which we denote by  $\mathbf{S}^2$ . Let  $\nu$  denote the orbifold normal bundle to  $\mathbf{S}^2$  in  $\mathbf{X}$ . This is an orbifold line bundle with action of  $T(\lambda)$ , see Definition 2.1.

Take any  $x \in \mathbf{S}^2$ . Observe, after Lemma 3.1 of [6], that  $v_x$  and  $\mathcal{T}_x \mathbf{S}^2$  are orthogonal with respect to  $\mu$ . This may be verified as follows. Take any nonzero vectors  $\eta_1 \in v_x$  and  $\eta_2 \in \mathcal{T}_x \mathbf{S}^2$ . Let  $\theta$  be an element of  $T(\lambda)$  that acts on  $v_x$  as multiplication by  $-1$ . Since  $T(\lambda)$  acts trivially on  $\mathcal{T}_x \mathbf{S}^2$ , we have

$$\mu(\eta_1, \eta_2) = \mu(\theta \cdot \eta_1, \eta_2) = \mu(-\eta_1, \eta_2) = 0.$$

Thus we may split the construction of an orthogonal  $J$  into constructions of orthogonal almost complex structures on  $v$  and  $\mathcal{T}\mathbf{S}^2$ .

Define the restriction of  $J$  to  $v$  as rotation by the angle  $\frac{\pi}{2}$  with respect to the metric  $\mu$  in the counterclockwise direction as specified by the orientation of  $v$  obtained from the  $T(\lambda)$  action. Then  $J|_v$  is torus invariant as  $\mu$  is preserved by torus action.

Recall that the space of all orthogonal complex structures on the oriented vector space  $\mathbb{R}^2$  is parametrized by  $SO(2)/U(1)$  which is a single point. We orient  $\mathcal{T}\mathbf{S}^2$  according to the given omniorientation. Consider the path  $\iota(E) \in \mathbf{S}^2$  given by the embedding  $\iota$  in (3.1). Since this path is contractible, the restriction of  $\mathcal{T}\mathbf{S}^2$  on it is trivial. Thus there is a canonical choice of an orthogonal almost complex structure on  $\mathcal{T}\mathbf{S}^2|_{\iota(E)}$ . We want this structure to agree with the complex structures  $J_u|_{\mathcal{T}\mathbf{S}^2}$  and  $J_v|_{\mathcal{T}\mathbf{S}^2}$  near the vertices, chosen earlier. This is possible since the omniorientation is positive. Then we use the torus action to define  $J|_{\mathcal{T}\mathbf{S}^2}$  on each point  $x$  in  $\mathbf{S}^2$ . Find  $y$  in  $\iota(E)$  and  $\theta \in T_N/T(\lambda)$  such that  $x = \theta \cdot y$ . Then define  $J(x) = d\theta \circ J(y) \circ d\theta^{-1}$ . This completes the construction of a torus invariant orthogonal  $J$  on the 1-skeleton of  $\mathbf{X}$ .

Choose a simple loop  $\gamma$  in  $P$  that goes along the edges for the most part but avoids the vertices. By the previous step of our construction,  $J$  is given on  $\iota(\gamma)$ . Let  $D$  be the disk in  $P$  bounded by  $\gamma$ . The set  $X_0 := \pi^{-1}(D) \subset X$  is a smooth manifold with boundary. The restriction of  $\mathcal{T}X_0$  to  $\iota(D)$  is a trivial vector bundle. Fix a trivialization. Recall that the space of all orthogonal complex structures on oriented vector space  $\mathbb{R}^4$ , up to isomorphism, is homeomorphic to  $SO(4)/U(2)$ . This is a simply connected space. Thus  $J$  may be extended from  $\iota(\gamma)$  to  $\iota(D)$ . Then we produce a  $T_N$ -invariant orthogonal  $J$  on  $\mathcal{T}X_0$  by using the  $T_N$  action as in the last paragraph. This completes the proof.  $\square$

#### 4. Blowdown

Our blowdown is analogous to partial resolution of singularity in complex geometry. Topologically it is an inverse to the operation of connect sum with a complex 2-dimensional weighted projective space. At the combinatorial level, it corresponds to deletion of an edge and its characteristic vector. To be precise, the polytope is modified by removing one edge and extending its neighboring edges till they meet at a new vertex, using Lemma 2.4 if necessary.

Suppose the orbifold  $\mathbf{X}$  corresponds to the model  $(P, \Lambda)$ . Suppose the edge  $E_2$  of  $P$  is deleted and the neighboring edges  $E_1$  and  $E_3$  are extended produce a new polytope  $\hat{P}$ . Denote the characteristic vector attached to the edge  $E_i$  by  $\lambda_i$ . Assume that  $\lambda_1$  and  $\lambda_3$  are linearly independent. Then  $\hat{P}$  inherits a characteristic function  $\hat{\Lambda}$  from  $\Lambda$ . Let  $\mathbf{Y}$  be the orbifold corresponding to the model  $(\hat{P}, \hat{\Lambda})$ . We only consider the case where at least one of the vertices of  $E_2$  correspond to a smooth point of  $\mathbf{X}$ . Then there exists a continuous  $T_N$ -invariant map of underlying topological spaces  $\rho: X \rightarrow Y$  called a blowdown. This map contracts the sphere  $\pi^{-1}(E_2) \subset X$  to a point in the image and it is a diffeomorphism away from this sphere. The sphere  $\pi^{-1}(E_2)$  is called the exceptional set.

The blowdown is not an orbifold morphism near the exceptional set as it cannot be lifted locally to a continuous equivariant map on orbifold charts. This is not surprising as it also happens for resolution of quotient singularities in algebraic geometry. However we can give a neighborhood of the exceptional set an analytic structure such that the blowdown is analytic in this neighborhood. We can extend these local complex structures on  $\mathbf{X}$  and  $\mathbf{Y}$  to almost complex structures so that the blowdown map is an almost complex diffeomorphism of orbifolds away from the exceptional set. Moreover the blowdown is a  $J$ -holomorphic map in a natural sense described in Corollary 4.3.

**Theorem 4.1.** *If  $\det[\lambda_1, \lambda_2] = 1$  and  $0 < \det[\lambda_2, \lambda_3] \leq \det[\lambda_1, \lambda_3]$ , then there exists a continuous map  $\rho: X \rightarrow Y$  which is a diffeomorphism away from the set  $\pi^{-1}(E_2) \subset X$ .*

*Proof.* Let  $v_1$  and  $v_2$  be the vertices of  $P$  corresponding to  $E_1 \cap E_2$  and  $E_2 \cap E_3$  respectively. Let  $w$  be the vertex of  $\hat{P}$  where the extended edges  $E_1$  and  $E_3$  meet.

Up to change of basis of  $N$  we may assume that  $\lambda_1 = (1, 0)$ ,  $\lambda_2 = (0, 1)$ , and  $\lambda_3 = (-k, m)$  where  $0 < k \leq m$  are positive integers. Then  $v_1$  is a smooth point in  $\mathbf{X}$ , but  $v_2$  is a possibly singular point in  $\mathbf{X}$  with local group  $\mathbb{Z}_k$ . The point  $w$  in  $\mathbf{Y}$  has local group  $\mathbb{Z}_m$ .

Up to choice of coordinates, the equations of the sides  $E_1$  and  $E_3$  may be assumed to be  $p_1 = 0$  and  $p_2 = 0$  respectively. Suppose the equation of  $E_2$  is  $\hat{p} := p_2 + sp_1 - t = 0$  where  $s$  and  $t$  are positive constants.

Choose small positive numbers  $\epsilon_1 < \epsilon_2 < 1$  and a non-decreasing function  $\delta: [0, \infty) \rightarrow \mathbb{R}$  which is smooth away from 0 such that

$$(4.1) \quad \delta(t) = \begin{cases} t^{1/m} & \text{if } t < \epsilon_1, \\ 1 & \text{if } t > \epsilon_2. \end{cases}$$

The blow down map  $\rho: (P \times T_N / \sim) \rightarrow (\hat{P} \times T_N / \sim)$  may be defined by

$$(4.2) \quad \rho(p_1, p_2, q_1, q_2) = (\delta(\hat{p})^k p_1, \delta(\hat{p}) p_2, q_1, q_2).$$

It is enough to study the map  $\rho$  in the open sets  $X_{v_1}$  and  $X_{v_2}$ , as it is identity elsewhere. The coordinates on  $\tilde{X}_{v_1}$ ,  $\tilde{X}_{v_2}$  and  $\tilde{Y}_w$  are

$$(4.3) \quad \begin{aligned} z_1(v_1) &:= \sqrt{p_1} e^{2\pi \sqrt{-1}(q_1)}, & z_2(v_1) &:= \sqrt{\hat{p}} e^{2\pi \sqrt{-1}q_2}, \\ z_1(v_2) &:= \sqrt{\hat{p}} e^{2\pi \sqrt{-1}(mq_1+kq_2)/k}, & z_2(v_2) &:= \sqrt{p_2} e^{2\pi \sqrt{-1}(-q_1/k)}, \\ z_1(w) &:= \sqrt{r_1} e^{2\pi \sqrt{-1}(mq_1+kq_2)/m}, & z_2(w) &:= \sqrt{r_2} e^{2\pi \sqrt{-1}q_2/m}. \end{aligned}$$

For questions related to smoothness, it is convenient to describe (lift of)  $\rho$  in terms of these coordinates. The formulas that follow make sense on suitable (small) open sets in the complement of the exceptional sphere or its image (and with choice of branch of roots), although we may sometimes neglect to explicitly say so for the sake of brevity.

The restriction  $\rho: X_{v_1} \rightarrow Y_w$  is given by

$$(4.4) \quad z_1(w) = z_1(v_1)z_2(v_1)^{k/m} \sqrt{\frac{\delta(\hat{p})^k}{\hat{p}^{k/m}}}, \quad z_2(w) = z_2(v_1)^{1/m} \sqrt{\frac{\delta(\hat{p})p_2}{\hat{p}^{1/m}}}.$$

The function  $p_2$  is nonzero on  $X_{v_1}$  whereas the functions  $\delta(\hat{p})$  and  $\hat{p}$  and  $z_2(v_1)$  are smooth and nonvanishing away from  $\hat{p} = 0$ . Hence  $\rho$  is a smooth map on  $X_{v_1} \cap \pi^{-1}(E_2)^c$ . To show that  $\rho$  is a diffeomorphism away from the exceptional set we exhibit a suitable inverse map  $\rho^{-1}$ . The map  $\rho^{-1}: \rho(X_{v_1}) \cap \{(0, 0)\}^c \rightarrow X_{v_1}$  is given by

$$(4.5) \quad z_1(v_1) = \frac{z_1(w)}{z_2(w)^k} \sqrt{\frac{r_2^k}{(\delta(\hat{p}))^k}}, \quad z_2(v_1) = z_2(w)^m \sqrt{\frac{\hat{p}}{r_2^m}}.$$

Note that  $r_2$  is a smooth function of  $x_2(w)$  and  $y_2(w)$ . Moreover  $r_2 = p_2\delta(\hat{p})$  does not vanish on  $\rho(X_{v_1}) - \{(0, 0)\}$ . Hence  $z_2(w)$  does not vanish on this set either. Same holds for the functions  $\hat{p}$  and  $\delta(\hat{p})$ . It only remains to show that  $\hat{p}$  is a smooth function of  $x_i(w)$ s and  $y_j(w)$ s.

The map  $f: P \rightarrow \hat{P}$  given by  $(r_1, r_2) = (\delta(\hat{p})^k p_1, \delta(\hat{p})p_2)$  has Jacobian

$$(4.6) \quad J(f) = \begin{bmatrix} \delta(\hat{p})^k + skp_1\delta(\hat{p})^{k-1}\delta'(\hat{p}) & kp_1\delta(\hat{p})^{k-1}\delta'(\hat{p}) \\ sp_2\delta'(\hat{p}) & \delta(\hat{p}) + p_2\delta'(\hat{p}) \end{bmatrix}.$$

Since  $\det(J(f)) = \delta(\hat{p})^{k+1} + \delta(\hat{p})^k\delta'(\hat{p})(p_2 + skp_1) > 0$ ,  $f$  is a diffeomorphism away from  $\hat{p} = 0$ . Thus away from  $(r_1, r_2) = (0, 0)$ ,  $p_1$  and  $p_2$  are smooth functions of  $(r_1, r_2)$ . Hence  $\hat{p} = p_2 + sp_1 - t$  is also a smooth function of  $r_1$  and  $r_2$  away from the origin. Consequently  $\hat{p}$  is a smooth function of  $x_i(w)$ s and  $y_j(w)$ s away from the origin.

Similarly the blowdown map restriction  $\rho: X_{v_2} \rightarrow Y_w$  given by

$$(4.7) \quad z_1(w) = z_1(v_2)^{k/m} \sqrt{\frac{(\delta(\hat{p}))^k p_1}{\hat{p}^{k/m}}}, \quad z_2(w) = z_1(v_2)^{1/m} z_2(v_2) \sqrt{\frac{\delta(\hat{p})}{\hat{p}^{1/m}}},$$

is smooth on  $X_{v_2} \cap \pi^{-1}(E_2)^c$  as the functions  $p_1$ ,  $\hat{p}$  and  $z_1(v_2)$  are nonvanishing there.

The map  $\rho^{-1}: \rho(X_{v_2}) \cap \{(0, 0)\}^c \rightarrow X_{v_2}$  is given by

$$(4.8) \quad z_1(v_2) = \frac{z_2(w)}{z_1(w)^{1/k}} \sqrt{\frac{r_1^{1/k}}{\delta(\hat{p})}}, \quad z_1(v_2) = z_1(w)^{m/k} \sqrt{\frac{\hat{p}}{r_1^{m/k}}}.$$

The diffeomorphism argument is same as in the case of the vertex  $v_1$ . This completes the proof.  $\square$

**Theorem 4.2.** *Suppose  $\mathbf{X}$  and  $\mathbf{Y}$  are positively omnioriented quasitoric orbifolds and  $\rho: X \rightarrow Y$  a blowdown map as constructed above. Then we may choose torus invariant almost complex structures  $J_1$  on  $\mathbf{X}$  and  $J_2$  on  $\mathbf{Y}$  with respect to which  $\rho$  is analytic near the exceptional set and almost complex orbifold morphism away from the exceptional set.*

*Proof.* We use the notation of Theorem 4.1. It is convenient to make the following changes of coordinates. On the chart  $\tilde{X}_{v_1}$ , define (compare to equation (4.3)),

$$(4.9) \quad z'_1(v_1) = z_1(v_1) \sqrt{\frac{1}{p_2^k}}, \quad z'_2(v_1) = z_2(v_1) \sqrt{\frac{\delta(\hat{p})^m p_2^m}{\hat{p}}}.$$

This is a valid change of coordinates since  $\delta(\hat{p})^m/\hat{p}$  and  $p_2$  are nonzero. In these coordinates the map  $\rho: X_{v_1} \rightarrow Y_w$  takes the form

$$(4.10) \quad z_1(w) = z'_1(v_1) z'_2(v_1)^{k/m}, \quad z_2(w) = z'_2(v_1)^{1/m}.$$

Similarly on the chart  $\tilde{X}_{v_2}$  we choose new coordinates as follows,

$$(4.11) \quad z'_1(v_2) = z_1(v_2) \sqrt{\frac{\delta(\hat{p})^m p_1^{m/k}}{\hat{p}}}, \quad z'_2(v_2) = z_2(v_2) \sqrt{p_1^{-1/k}}.$$

In these coordinates the map  $\rho: X_{v_2} \rightarrow Y_w$  takes the form

$$(4.12) \quad z_1(w) = z'_1(v_2)^{k/m}, \quad z_2(w) = z'_1(v_2)^{1/m} z'_2(v_2).$$

Let  $U_2$  be a small tubular neighborhood of the edge  $E_2$  in  $P$ . Choose complex structures on  $\pi^{-1}(U_2) \cap X_{v_1}$  and  $\pi^{-1}(U_2) \cap X_{v_2}$  by declaring the coordinates  $z'_1(v_1)$ ,  $z'_2(v_1)$  and respectively  $z'_1(v_2)$ ,  $z'_2(v_2)$  to be holomorphic. These complex structures agree on the intersection and define a complex structure  $J_1$  on  $\pi^{-1}(U_2)$  since the coordinates are related as follows.

$$(4.13) \quad z'_1(v_2) = z'_1(v_1)^{m/k} z'_2(v_1), \quad z'_2(v_2) = z'_1(v_1)^{-1/k}.$$

Consider the neighborhood  $V := f(U_2)$  in  $\hat{P}$ . On  $\pi^{-1}(V) \subset Y_w$  choose a complex structure  $J_2$  by declaring the coordinates  $z_1(w), z_2(w)$  to be holomorphic. Consequently by equations (4.10) and (4.12), the blowdown map  $\rho$  is analytic on  $\pi^{-1}(U_2)$ .

We will extend  $J_1$  to an almost complex structure on  $\mathbf{X}$ . For this purpose choose standard metrics  $\mu_j$ , with respect to the coordinates  $(z'_1(v_j), z'_2(v_j))$ , on  $\pi^{-1}(U_2) \cap X_{v_j}$  for  $j = 1, 2$ . Let  $\{W_1, W_2\}$  be an open cover of  $U_2$  such that  $v_j \in W_j$  and  $W_j^c$  contains a neighborhood of  $v_k$  for  $j, k = 1, 2$  and  $j \neq k$ . Glue  $\mu_1$  and  $\mu_2$  by a torus invariant partition of unity subordinate to the cover  $\{\pi^{-1}(W_1), \pi^{-1}(W_2)\}$ . This produces a metric  $\mu'$  on  $\pi^{-1}(U_2)$  such that  $J_1$  is orthogonal with respect to  $\mu'$ . Moreover  $\mu'$  is standard with respect to the given coordinates near  $v_1$  and  $v_2$ . Let  $U \subset U_2$  be a smaller tubular neighborhood of  $E_2$  in  $P$ . Using suitable partition of unity, extend  $\mu'|_U$  to a torus invariant metric  $\mu$  on  $\mathbf{X}$  such that  $\mu$  is standard with respect to our choice of coordinates near each vertex. Then extend  $J_1|_U$  first to the union of the 1-skeleton of  $X$  and  $\pi^{-1}(U)$ , and then to entire  $\mathbf{X}$  as a torus invariant orthogonal almost complex structure. This can be done in a way similar to the proof of Theorem 3.

Now  $\mathbf{X} \cap \pi^{-1}(E_2)^c$  is diffeomorphic to  $\mathbf{Y} \cap \{w\}^c$  via the blowdown map. Thus  $J'_2 := d\rho \circ J_1 \circ d\rho^{-1}$  is an almost complex structure on  $\mathbf{Y} \cap \{w\}^c$ . We have the following equalities for a point in the intersection  $(\mathbf{Y} \cap \{w\}^c) \cap \pi^{-1}(f(U))$ .

$$(4.14) \quad J_2 = (d\rho)(d\rho)^{-1}J_2 = (d\rho)J_1(d\rho)^{-1} = J'_2.$$

The second equality is due to complex analyticity of  $\rho^{-1}$  on  $\pi^{-1}(f(U)) \cap \{w\}^c$ . So the two structures are the same and we obtain a torus invariant almost complex structure on  $\mathbf{Y}$ , which we denote again by  $J_2$  without confusion. The blowdown  $\rho$  is an almost complex diffeomorphism of orbifolds away from the exceptional set  $\pi^{-1}(E_2) \subset \mathbf{X}$  and the point  $w \in \mathbf{Y}$  with respect to  $J_1$  and  $J_2$ . It is an analytic map of complex analytic spaces near  $\pi^{-1}(E_2)$  and  $w$ .  $\square$

**DEFINITION 4.1.** A complex valued continuous function  $f: X \rightarrow \mathbb{C}$  on an orbifold  $\mathbf{X}$  is called smooth if  $f \circ \xi$  is smooth for any orbifold chart  $(\tilde{U}, G, \xi)$ . We denote the sheaf of smooth functions on  $\mathbf{X}$  by  $\mathcal{S}_X$ . This is a sheaf on the underlying space  $X$  but depends on the orbifold structure. A smooth function  $f$  on an almost complex orbifold  $(\mathbf{X}, J)$  is said to be  $J$ -holomorphic if the differential  $d(f \circ \xi)$  commutes with  $J$  for every chart  $(\tilde{U}, G, \xi)$ . We denote the sheaf of  $J$ -holomorphic functions on  $\mathbf{X}$  by  $\Omega_{J,X}^0$ . A continuous map  $\rho: X \rightarrow Y$  between almost complex orbifolds  $(\mathbf{X}, J_1)$  and  $(\mathbf{Y}, J_2)$  is said to be pseudo-holomorphic if  $f \circ \rho \in \Omega_{J_1,X}^0(\rho^{-1}(U))$  for every  $f \in \Omega_{J_2,Y}^0(U)$  for any open set  $U \subset Y$ ; that is,  $\rho$  pulls back pseudo-holomorphic functions to pseudo-holomorphic functions.

**Corollary 4.3.** *The blowdown map  $\rho$  of Theorem 4.1 is pseudo-holomorphic with respect to the almost complex structures given in Theorem 4.2.*

Proof. Since  $\rho$  is an almost complex diffeomorphism of orbifolds away from the exceptional sphere  $\pi^{-1}(E_2)$ , it suffices to check the statement on the open sets  $X_{v_1}$  and  $X_{v_2}$ . The ring  $\Omega_{J_1, X}^0(X_{v_1})$  is the ring of convergent power series in variables  $z'_i(v_1)$  as  $v_1$  is a smooth point. The ring  $\Omega_{J_2, Y}^0(Y_w)$  is the  $\mathbb{Z}_m$ -invariant subring of convergent power series in variables  $z'_i(w)$ . Let  $\eta$  denote a primitive  $m$ -th root of unity. Then the action of  $\mathbb{Z}_m$  on  $\tilde{Y}_w$  is given by  $\eta \cdot (z_1(w), z_2(w)) = (\eta^k z_1(w), \eta z_2(w))$ , see (2.8). Therefore the invariant subring is generated by  $\{z_1(w)^i z_2(w)^j : m \mid (ik + j)\}$ . Using (4.10), we have

$$(4.15) \quad z_1(w)^i z_2(w)^j \circ \rho = z'_1(v_1)^i z'_2(v_1)^{(ik+j)/m}.$$

When  $m$  divides  $(ik + j)$ ,  $z_1(w)^i z_2(w)^j \circ \rho$  belongs to  $\Omega_{J_1, X}^0(X_{v_1})$ .

Similarly the  $\mathbb{Z}_k$  action on  $\tilde{X}_{v_2}$  is given by  $\zeta \cdot (z'_1(v_2), z'_2(v_2)) = (\zeta^{m/k} z'_1(v_2), \zeta^{-1/k} z'_2(v_2))$  where  $\zeta$  is a primitive  $k$ -th root of unity. The ring  $\Omega_{J_1, X}^0(X_{v_2})$  is generated by  $\{z'_1(v_2)^a z'_2(v_2)^b : k \mid (am - b)\}$ . Using (4.12) we have,

$$(4.16) \quad z_1(w)^i z_2(w)^j \circ \rho = z'_1(v_2)^{(ik+j)/m} z'_2(v_2)^j.$$

Since  $(ik + j)m/m - j = ik$  is divisible by  $k$ ,  $z_1(w)^i z_2(w)^j \circ \rho$  belongs to  $\Omega_{J_1, X}^0(X_{v_2})$  if  $z_1(w)^i z_2(w)^j \in \Omega_{J_2, Y}^0(Y_w)$ .  $\square$

REMARK 4.4. The blowdown map  $\rho$  does not pull back a smooth function to a smooth function in general.

## 5. McKay correspondence

DEFINITION 5.1. Given an almost complex  $2n$ -dimensional orbifold  $(\mathbf{X}, J)$ , we define the canonical sheaf  $K_X$  to be the sheaf of continuous  $(n, 0)$ -forms on  $X$ ; that is, for any orbifold chart  $(\tilde{U}, G, \xi)$  over an open set  $U \subset X$ ,  $K_X(U) = \Gamma(\bigwedge^n \mathcal{T}^{1,0}(\tilde{U})^*)^G$  where  $\Gamma$  is the functor that takes continuous sections.

An orbifold singularity  $\mathbb{C}^n/G$  is said to be *SL* if  $G$  is a finite subgroup of  $SL(n, \mathbb{C})$ . For an *SL*-orbifold  $\mathbf{X}$ , i.e. one whose singularities are all *SL*, the canonical sheaf is a complex line bundle over  $X$ .

DEFINITION 5.2. A pseudo-holomorphic blowdown map  $\rho: \mathbf{X} \rightarrow \mathbf{Y}$  between two four dimensional primitive positively omnioriented quasitoric *SL* orbifolds is said to be crepant if  $\rho^* K_Y = K_X$ .

**Lemma 5.1.** *The orbifold singularities corresponding to the characteristic vectors  $\lambda_1 = (1, 0)$ ,  $\lambda_2 = (-k, m)$  and  $\lambda_1 = (0, 1)$ ,  $\lambda_2 = (-k, m)$  are *SL* if  $k + 1 = m$ .*

Proof. Consider the case  $\lambda_1 = (1, 0)$ ,  $\lambda_2 = (1 - m, m)$ . Refer to the description of the action of the local group in equation (2.8). Any element of this group is represented by an integral vector  $a_1\lambda_1 + a_2\lambda_2$ . Integrality of such a vector implies that  $a_1 + a_2 - ma_2$  and  $ma_2$  are integers. Hence  $a_1 + a_2$  is also an integer. This implies that the group acts as a subgroup of  $SL(2, \mathbb{C})$ . The other case is similar.  $\square$

**Lemma 5.2.** *Suppose  $\mathbf{X}$  and  $\mathbf{Y}$  are two 4-dimensional primitive, positively omni-oriented, quasitoric  $SL$  orbifolds and  $\rho: \mathbf{X} \rightarrow \mathbf{Y}$  is a pseudo-holomorphic blowdown as constructed in Theorems 4.1 and 4.2. Then  $\rho$  is crepant if and only if  $k + 1 = m$ .*

Proof. We consider the canonical sheaf as a sheaf of modules over the sheaf of continuous functions  $\mathcal{C}_X^0$ . Since  $\rho$  is an almost complex diffeomorphism away from the exceptional set it suffices to check the equality of the  $\rho^*K_Y$  and  $K_X$  on the neighborhood  $\pi^{-1}(U) \subset X$  of the exceptional set, defined in proof of Theorem 4.2.

On  $X_{v_1} \cap \pi^{-1}(U)$ , the sheaf  $K_X$  is generated over the sheaf  $\mathcal{C}_X^0$  by the form  $dz'_1(v_1) \wedge dz'_2(v_1)$ . On the other hand  $\rho^*(K_Y)$  is generated on this neighborhood over  $\mathcal{C}_X^0$  by

$$\begin{aligned}
 & \rho^* dz_1(w) \wedge dz_2(w) \\
 &= d(z'_1(v_1)z'_2(v_1)^{k/m}) \wedge d(z'_2(v_1)^{1/m}) \\
 (5.1) \quad &= z'_2(v_1)^{k/m} dz'_1(v_1) \wedge \frac{1}{m} (z'_2(v_1))^{1/m-1} dz'_2(v_1) \\
 &= \frac{1}{m} (z'_2)^{(k+1)/m-1} dz'_1(v_1) \wedge dz'_2(v_1).
 \end{aligned}$$

Thus  $\rho^*K_Y = K_X$  on  $X_{v_1} \cap \pi^{-1}(U)$  if and only if  $k + 1 = m$ . Similarly on  $X_{v_2} \cap \pi^{-1}(U)$ , the sheaf  $K_X$  is generated over  $\mathcal{C}_X^0$  by the form  $dz'_1(v_2) \wedge dz'_2(v_2)$ . On the other hand  $\rho^*(K_Y)$  is generated on this neighborhood over  $\mathcal{C}_X^0$  by

$$\begin{aligned}
 & \rho^* dz_1(w) \wedge dz_2(w) \\
 &= d(z'_1(v_2)^{k/m}) \wedge d(z'_1(v_2)^{1/m} z'_2(v_2)) \\
 (5.2) \quad &= \frac{k}{m} (z'_1(v_2))^{k/m-1} dz'_1(v_2) \wedge z'_1(v_2)^{1/m} dz'_2(v_2) \\
 &= \frac{k}{m} (z'_1(v_2))^{(k+1)/m-1} dz'_1(v_2) \wedge dz'_2(v_2).
 \end{aligned}$$

Again  $\rho^*K_Y = K_X$  on  $X_{v_2} \cap \pi^{-1}(U)$  if and only if  $k + 1 = m$ .  $\square$

It is easy to construct examples of a positively omni-oriented quasitoric  $SL$  manifold or orbifold which is not a toric variety and admits a crepant blowdown. For instance, let  $\mathbf{X}$  be the quasitoric manifold over a 7-gon with characteristic vectors  $(1, 0)$ ,  $(0, 1)$ ,  $(-1, 2)$ ,  $(-2, 3)$ ,  $(1, -2)$ ,  $(0, 1)$  and  $(-1, -1)$ . Then  $\mathbf{X}$  is positively omni-oriented, primitive and  $SL$ . However it is not a toric variety as its Todd genus is 2. The orbifold



$\mathbf{Y}$  over a 6-gon with characteristic vectors  $(1, 0)$ ,  $(-1, 2)$ ,  $(-2, 3)$ ,  $(1, -2)$ ,  $(0, 1)$  and  $(-1, -1)$  is a crepant blowdown of  $\mathbf{X}$ . Same holds for the orbifold  $\mathbf{Z}$  over a 6-gon with characteristic vectors  $(1, 0)$ ,  $(0, 1)$ ,  $(-2, 3)$ ,  $(1, -2)$ ,  $(0, 1)$  and  $(-1, -1)$ . It may be argued that  $\mathbf{Y}$  and  $\mathbf{Z}$  are not toric varieties as otherwise the blowup  $\mathbf{X}$  would be a toric variety.

The singular and Chen–Ruan cohomology groups (see [3]) of an almost complex quasitoric orbifold was calculated in [10]. For a four dimensional primitive positively omnioriented quasitoric orbifold  $\mathbf{X}$ , the Chen–Ruan cohomology groups are

$$(5.3) \quad H_{\text{CR}}^d(\mathbf{X}, \mathbb{Q}) = \begin{cases} H^0(X, \mathbb{Q}) & \text{if } d = 0, \\ H^d(X, \mathbb{Q}) \oplus \bigoplus_{\text{age}(g)=d} \mathbb{Q}(v, g) & \text{if } d > 0. \end{cases}$$

Here  $v$  varies over vertices of  $P$ ,  $g$  varies over nontrivial elements of the local group  $G_v$  and  $\text{age}(g)$  is the degree shifting number  $\iota(g)$  of [3].

The singular cohomology groups of  $X$  are

$$(5.4) \quad H^d(X, \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } d = 0, 4, \\ \bigoplus_{m-2} \mathbb{Q} & \text{if } d = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where  $m$  denotes the number of edges of  $P$ .

**Theorem 5.3.** *Suppose  $\rho: \mathbf{X} \rightarrow \mathbf{Y}$  be a pseudo-holomorphic blowdown between four dimensional primitive positively omnioriented quasitoric manifolds, as constructed in Theorems 4.1 and 4.2. If  $\rho$  is crepant then*

$$\dim(H_{\text{CR}}^d(\mathbf{X}, \mathbb{Q})) = \dim(H_{\text{CR}}^d(\mathbf{Y}, \mathbb{Q})).$$

**Proof.** By formulas (5.3) and (5.4), it suffices to compare the contributions of the edge  $E_2$  and vertices  $v_1, v_2$  of  $P$  to  $H_{\text{CR}}^d(\mathbf{X}, \mathbb{Q})$  with the contribution of the vertex  $w$  of  $\hat{P}$  to  $H_{\text{CR}}^d(\mathbf{Y}, \mathbb{Q})$ .

The edge  $E_2$  contributes a single generator to  $H_{\text{CR}}^2(\mathbf{X}, \mathbb{Q})$ . The vertex  $v_1$  has no contribution as it is a smooth point and  $G_{v_1}$  is trivial. The group  $G_{v_2}$  is isomorphic to  $\mathbb{Z}_k$ . Assume  $\rho$  is crepant. Then by Lemma 5.2  $m = k + 1$  and the characteristic vector  $\lambda_3 = (-k, k + 1)$ . Recall that  $\lambda_2 = (0, 1)$ . The elements of  $G_{v_2}$  are

$$(5.5) \quad g^p := (-p, 0) = \frac{-(k+1)p}{k} \lambda_2 + \frac{p}{k} \lambda_3 \quad \text{where } 0 \leq p \leq k-1.$$

By equations (2.8) and (4.11) the action of  $g^p$  is given by

$$(5.6) \quad g^p(z'_1(v_2), z'_2(v_2)) = (e^{2\pi \sqrt{-1}(-p-p/k)} z'_1(v_2), e^{2\pi \sqrt{-1}(p/k)} z'_2(v_2)).$$

The degree shifting number

$$(5.7) \quad \text{age}(g^p) = \left(1 - \frac{p}{k}\right) + \frac{p}{k} = 1.$$

Thus each  $g^p$ ,  $1 \leq p \leq k-1$ , contributes a generator  $(v_2, g^p)$  to  $H_{\text{CR}}^2(\mathbf{X}, \mathbb{Q})$ .

The characteristic vectors at  $w$  are  $\lambda_1 = (1, 0)$  and  $\lambda_3 = (-k, k+1)$ . The group  $G_w \cong \mathbb{Z}_{k+1}$  has elements

$$(5.8) \quad h^q := (0, q) = \frac{qk}{k+1}\lambda_1 + \frac{q}{k+1}\lambda_3 \quad \text{where } 0 \leq q \leq k.$$

By equations (2.8) the action of  $h^q$  is given by

$$(5.9) \quad h^q(z_1(w), z_2(w)) = (e^{2\pi\sqrt{-1}(q-q/(k+1))}z_1(w), e^{2\pi\sqrt{-1}q/(k+1)}z_2(w)).$$

The degree shifting number

$$(5.10) \quad \text{age}(h^q) = \left(1 - \frac{q}{k+1}\right) + \frac{q}{k+1} = 1.$$

Thus each  $h^q$ ,  $1 \leq q \leq k$ , contributes a generator  $(w, h^q)$  to  $H_{\text{CR}}^2(\mathbf{Y}, \mathbb{Q})$ .

Hence the contributions of the edge  $E_2$  and vertices  $v_1, v_2$  of  $P$  to dimension of  $H_{\text{CR}}^d(\mathbf{X}, \mathbb{Q})$  match the contribution of the vertex  $w$  of  $\hat{P}$  to dimension of  $H_{\text{CR}}^d(\mathbf{Y}, \mathbb{Q})$ .  $\square$

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