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ON CHARTS WITH TWO CROSSINGS II

Dedicated to Professor Akio Kawauchi for his 60th birthday

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Abstract

Let Γ be a chart with at most two crossings. In this paper, we show that if Γ is a 2-minimal generalized n-chart with $n \geq 5$, then Γ contains at least $4n - 10$ black vertices. And we show that if the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart. Hence the closure is a ribbon surface.

1. Introduction

S. Kamada introduced *charts* which correspond to surface braids [4], [5]. Charts are oriented labeled graphs in a disk with three kinds of vertices called black vertices, crossings, and white vertices. Kamada also introduced *C-moves* which are local modifications of charts in a disk. A C-move between two charts induces an ambient isotopy between the closures of the corresponding two surface braids. Two charts are said to be *C-move equivalent* if there exists a finite sequence of C-moves which modifies one of the two charts to the other.

A surface in $\mathbb{R}^4$ is called a *ribbon surface* if it is the boundary of an immersed handlebody with singularities which are mutually disjoint disks such that the preimage of each disk is a union of a proper disk of the domain and a disk in the interior of the domain, a handlebody. In the words of charts, a ribbon surface is the closure of a surface braid which corresponds to a *ribbon chart* where a ribbon chart is a chart which is C-move equivalent to a chart without white vertices [4].

Kamada showed that any 3-chart is a ribbon chart [4]. Nagase and Hirota extended Kamada's result: Any 4-chart with at most one crossing is a ribbon chart [7]. We showed that any n-chart with at most one crossing is a ribbon chart [11].

For a set X in a space, let $Cl(X)$ be the closure of the set X.

Let Γ be a chart. Let e_1 and e_2 be edges of Γ which connect two white vertices w_1 and w_2 where possibly $w_1 = w_2$. Suppose that the union $e_1 \cup e_2$ bounds an open disk E. Then $Cl(E)$ is called a *bigon* provided that any edge containing w_1 or w_2 does

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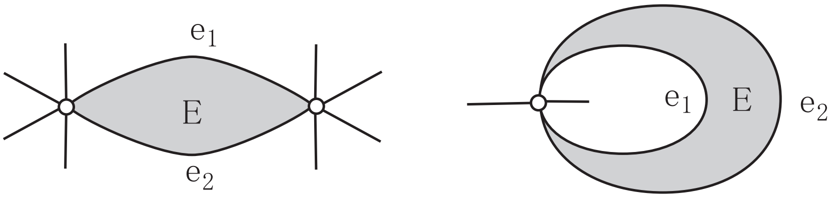

Fig. 1. The edges e_1 and e_2 do not contain crossings.

not intersect the open disk E (see Fig. 1). Since e_1 and e_2 are edges of Γ, they do not contain any crossings.

Let Γ be a chart. Let $w(\Gamma)$, $f(\Gamma)$ and $b(\Gamma)$ be the number of white vertices, the number of free edges and the number of bigons in Γ respectively. Let $C(\Gamma) = (w(\Gamma), -f(\Gamma), -b(\Gamma))$. The triplet $C(\Gamma)$ is called an *extended complexity* of the chart Γ (see [4] for complexities of charts).

For each non-negative integer k, let $c(\Gamma)$ be the number of crossings in a chart Γ and $C_k = \{\Gamma \mid c(\Gamma) \leq k\}$. A chart Γ in C_k is said to be *k-minimal* if its extended complexity is minimal among the charts in C_k which are C-move equivalent to the chart Γ with respect to the lexicographical order of the triad of the integers [11].

We showed that if a 2-minimal 4-chart contains exactly two crossings, then it contains at least eight black vertices [9]. It is well known that if the closure of the surface braid represented by a 4-chart is one sphere, then the chart contains exactly six black vertices. Hence we showed that any 4-chart with at most two crossings is a ribbon chart if the chart corresponds to a surface braid whose closure is one sphere [9]. We give another proof of this theorem [13].

Let Γ be a chart. For each label m, we denote by Γ_m the subgraph of Γ consisting of edges of label m and their vertices. In this paper,

> *crossings are vertices of Γ but we do not consider crossings as vertices of the subgraph Γ_m.*

A chart Γ with a white vertex is called a *generalized n-chart* if there exist two non-negative integers $p < q$ with $n = q - p$ such that

(i) Γ_i does not have a white vertex except for $p < i < q$, and

(ii) the both Γ_{p+1} and Γ_{q-1} have white vertices.

In this paper the following are main results:

Theorem 1.1. *Let Γ be a 2-minimal generalized n-chart. If $n \geq 5$, then Γ contains at least $4n - 10$ black vertices.*

Theorem 1.2. *Let Γ be a chart with at most two crossings. If the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart. Hence the closure is a ribbon surface.*

The 2-twist spun trefoil is represented by a chart with six white vertices and three crossings. It is well known that the 2-knot is not a ribbon surface. By Theorem 1.2, the chart representing the 2-knot must possess at least three crossings.

On the other hand, Hasegawa showed that if a chart representing a 2-knot is minimal, then the chart must possess at least six white vertices [2], where a minimal chart means its complexity $(w(\Gamma), -f(\Gamma))$ is minimal among the charts C-move equivalent to the chart with respect to the lexicographic order of pairs of integers. We know that there does not exist a minimal chart with one, two nor three white vertices. We show that there does not exist a minimal chart with five white vertices [8]. We show that the minimal chart with four white vertices is a ribbon chart, or a disjoint union of free edges, hoops and a chart representing a "turned T^2-link of Hopf link" [3] and [14].

Using the result in this paper, we get the following [15]: If Γ is a chart with at most three crossings and if the closure of the surface braid represented by Γ is a disjoint union of spheres, then Γ is a ribbon chart, or a disjoint union of free edges, hoops and a chart representing a 2-twist spun trefoil. The chart with six white vertices and three crossings representing a 2-twist spun trefoil is "primitive" k-minimal chart in some sense for $k \geq 3$. We study the properties of k-minimal charts and such primitive charts.

2. Preliminaries

Let n be a positive integer. An *n-chart* is an oriented labeled graph in a disk, which may be empty or have closed edges without vertices, called *hoops*, satisfying the following four conditions:

(i) Every vertex has degree 1, 4, or 6.

(ii) The labels of edges are in $\{1, 2, \dots, n-1\}$.

(iii) In a small neighborhood of each vertex of degree 6, there are six short arcs, three consecutive arcs are oriented inward and the other three are outward, and these six are labeled i and $i+1$ alternately for some i, where the orientation and label of each arc are inherited from the edge containing the arc.

(iv) For each vertex of degree 4, diagonal edges have the same label and are oriented coherently, and the labels i and j of the diagonals satisfy $|i-j|>1$.

A vertex of degree 1, 4, and 6 is called a *black vertex*, a *crossing*, and a *white vertex* respectively (see Fig. 2). Among six short arcs in a small neighborhood of a white vertex, a central arc of each three consecutive arcs oriented inward or outward is called a *middle arc* at the white vertex (see Fig. 2 (c)). There are two middle arcs in a small neighborhood of each white vertex.

C-moves are local modifications of charts in a disk (see [1], [6] for the precise definition). Kamada originally defined CI-moves as follows (A C-I-M2 move and a C-I-R2 move as shown in Fig. 3 are special cases of CI-moves): A chart Γ is obtained from a chart Γ' by a *CI-move*, if there exists a disk D such that

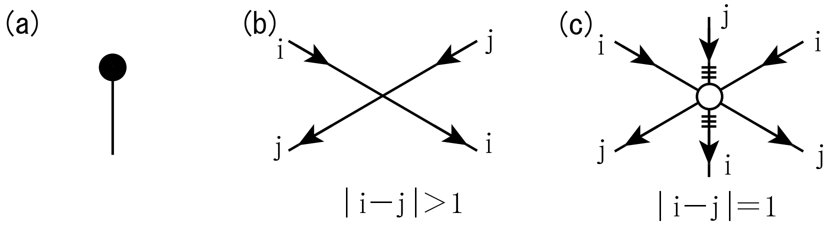

Fig. 2. (a) a black vertex, (b) a crossing, (c) a white vertex. Each arc with three transversal short arcs is a middle arc.

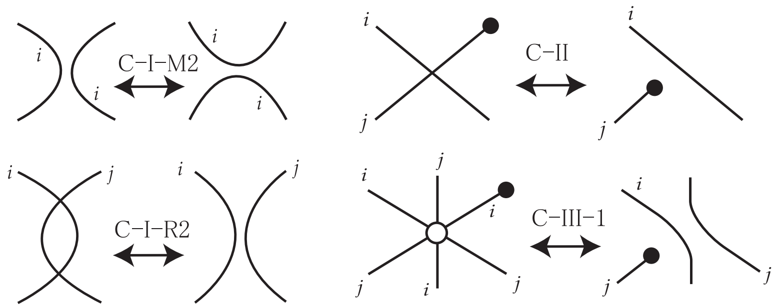

Fig. 3. For the C-III-1 move, the edge containing the black vertex does not contain a middle arc in the left figure.

(i) the two charts Γ and Γ' intersect the boundary of D transversely or do not intersect the boundary of D,
(ii) $\Gamma \cap D^c = \Gamma' \cap D^c$, and
(iii) neither of $\Gamma \cap D$ nor $\Gamma' \cap D$ contains a black vertex,
where $(\cdots)^c$ is the complement of $(\cdots)$.

Let Γ be a chart. An *edge* of Γ is the closure of a connected component of the set obtained by taking out all white vertices and crossings from Γ. On the other hand, an *edge* of Γ_m is the closure of a connected component of the set obtained by taking out all white vertices from Γ_m. A closed edge of Γ_m is called *a ring* if it contains a crossing but does not contain a white vertex nor a black vertex. A *hoop* is a closed edge of Γ without vertices (hence without crossings, neither). An edge of Γ or Γ_m is called a *free edge* if it has two black vertices. An edge of Γ or Γ_m is called a *terminal edge* if it has a white vertex and a black vertex. Note that free edges and terminal edges may contain crossings of Γ.

To make the argument simple, we assume that the charts lie on the 2-sphere instead of the disk. In this paper,

all charts are contained in the 2-*sphere* S^2.

We have the special point in the 2-sphere S^2, called *the point at infinity*, denoted by ∞. In this paper, all charts are contained in a disk which does not contain the point at infinity ∞.

A hoop is said to be *simple* if one of the complementary domain of the hoop does not contain any white vertices.

We can assume that any k-minimal charts Γ satisfy the following five assumptions (cf. [10] and [11]):

ASSUMPTION 1. Any terminal edge of Γ_m does not contain a crossing. Hence any terminal edge of Γ_m is a terminal edge of Γ and any terminal edge of Γ_m contains a middle arc.

ASSUMPTION 2. Any free edge of Γ_m does not contain a crossing. Hence any free edge of Γ_m is a free edge of Γ.

ASSUMPTION 3. All free edges and simple hoops in Γ are moved into a small neighborhood U_∞ of the point at infinity ∞.

ASSUMPTION 4. Each complementary domain of any ring must contain at least one white vertex.

ASSUMPTION 5. Hence we can assume that the subgraph obtained from Γ by omitting free edges and simple hoops does not meet the set U_∞. And also we can assume that Γ does not contain free edges nor simple hoops, otherwise mentioned. Therefore we can assume that if an edge of Γ_m contains a black vertex, then it is a terminal edge and that each complementary domain of any hoops and rings of Γ contains a white vertex, otherwise mentioned.

Furthermore as shown in [10], we can also assume the following assumption:

ASSUMPTION 6. The point at infinity ∞ is moved in any complementary domain of Γ.

For a set X in a space, let $Int(X)$, $\partial(X)$ be the interior, the boundary of the set X respectively.

3. Tangles

For each graph G in S^2, let (see Fig. 4)

$M(G)$ = the maximal subgraph of G without vertices of degree 1,

$Out(G)$ = the complementary domain of $M(G)$ containing the point at infinity ∞,

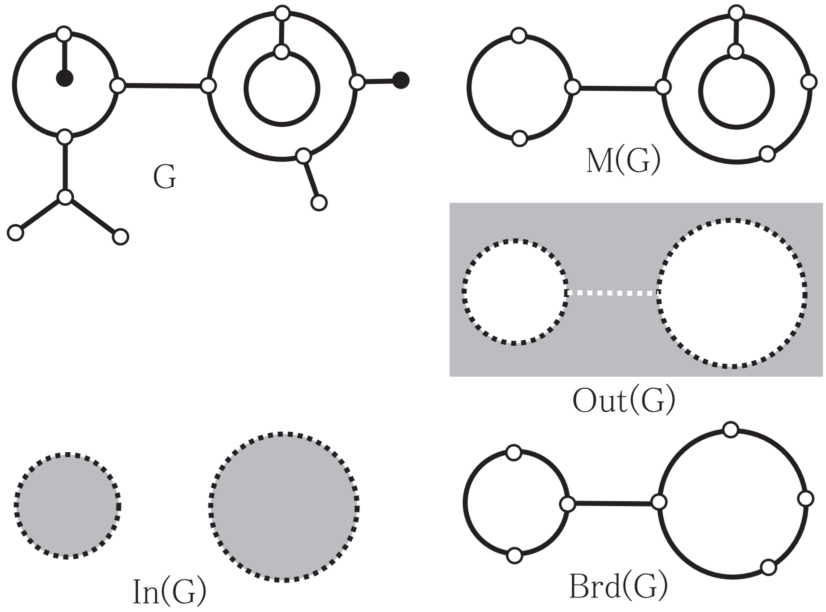

Fig. 4. *Out*(G) and *In*(G) are shaded areas.

$In(G) = (Cl(Out(G)))^c$, and

$Brd(G) = M(G) \cap Cl(Out(G))$.

Lemma 3.1 ([11, Lemma 5.1]). *Let G be a connected graph in S^2. Let D be a disk containing G. Then the following hold*:
(1) *$Out(G)$ is an open disk.*
(2) *Each connected component of $In(G)$ is an open disk whose closure is a disk.*
(3) *A regular neighbourhood of $In(G) \cup G$ in S^2 is a disk, and so is a regular neighbourhood of $In(G) \cup G$ in D.*

Let Γ be a chart. For a subset X in Γ, let

$$w(X) = \text{the number of white vertices in } X.$$

Let Γ be a chart and D a disk. The pair $(D \cap \Gamma, D)$ is called a *tangle* if it satisfies the following two conditions:
(1) ∂D does not contain any white vertices, black vertices nor crossings of the chart Γ, and
(2) ∂D transversely intersects edges of Γ.

Let Γ be a chart, $(D \cap \Gamma, D)$ a tangle and $G_i = D \cap \Gamma_i$ $(i = 1, 2, \dots)$. The tangle $(D \cap \Gamma, D)$ is called a *T-tangle* of label n (tangle with at most three labels) if

it satisfies the following two conditions:
(i) $G_i = \emptyset$ except for $n-1 \leq i \leq n+1$.
(ii) $w(D \cap \Gamma) \geq 1$ but D does not contain any crossing.
If $In(G_n) = \emptyset$ then we say that the T-tangle is *linear*. If $Cl(In(G_n))$ is a disk then we say that the T-tangle is *cellular*.

Let $(D \cap \Gamma, D)$ be a T-tangle of label n. If an edge e of Γ_n intersects ∂D, then $e \cap D$ is called an *exceptional arc* of the T-tangle.

Lemma 3.2 ([12, Lemma 4.2]). *Any linear T-tangle in a k-minimal chart possesses at least two exceptional arcs.*

Lemma 3.3. *Let $(D \cap \Gamma, D)$ be a linear T-tangle of label n with exactly two exceptional arcs in a k-minimal chart Γ. Then we have*
(1) *each white vertex in D is contained in a terminal edge of label n, and*
(2) *there exists a unique arc in $D \cap \Gamma_n$ connecting the two points $\partial D \cap \Gamma_n$ such that all the white vertices in the arc are contained in terminal edges.*

Proof. For (1). Let G be a connected component of $D \cap \Gamma_n$. Since the T-tangle is linear, G is a tree. Then $\partial D \cap G$ consists of two points by Lemma 3.2. Now consider the two points $\partial D \cap \Gamma_n$ as vertices of G. Let B be the number of terminal edges in G which is equal to the number of black vertices in G, W the number of white vertices in G, and E the number of edges in G. Since each white vertex in G is of degree 3, we have $3W + (B + 2) = 2E$. Since G is a tree, we have the Euler characteristic $(W + B + 2) - E = 1$. Thus $3W + B + 2 = 2(W + B + 1)$. Namely $W = B$. Since the chart is k-minimal, each white vertex in G is contained in at most one terminal edges of label n by Assumption 1. Hence the equality $W = B$ implies that each white vertex in G is contained in a terminal edge of label n.

For (2). By taking all terminal edges off from G, we get a unique simple arc. □

4. Tiny cellular *T*-tangles

Lemma 4.1. *Let $(D \cap \Gamma, D)$ be a T-tangle of label n in a k-minimal chart Γ. Let G be the closure of a connected component of $(D \cap \Gamma_n) - Cl(In(D \cap \Gamma_n))$. If G is not a terminal edge, then it is a tree containing at least two points in $Brd(D \cap \Gamma_n) \cup \partial D$.*

Proof. If G is an arc, then G is either a terminal edge or an arc containing two points in $Brd(D \cap \Gamma_n) \cup \partial D$. Hence we can assume that G is a tree containing a white vertex.

Suppose that G contains at most one point in $Brd(D \cap \Gamma_n) \cup \partial D$. Let D' be a regular neighborhood of $Cl(In(D \cap \Gamma_n))$ in D, $G' = G \cap Cl(D - D')$, and N a regular neighborhood of G' in $Cl(D - D')$. Then $N \cap \Gamma_n = G'$ and $\partial N \cap \Gamma_n$ contains at most one point. Since G contains a white vertex, $w(N \cap \Gamma) \geq 1$. Since G' is a tree, N is a disk.

Since $(D \cap \Gamma, D)$ is a T-tangle of label n, $(N \cap \Gamma, N)$ is a T-tangle of label n with at most one exceptional arc. Since G is a tree, $(N \cap \Gamma, N)$ is linear. This contradicts Lemma 3.2. Hence G contains at least two points in $Brd(D \cap \Gamma_n) \cup \partial D$. □

A tangle $(D_1 \cap \Gamma, D_1)$ *contains* a tangle $(D_2 \cap \Gamma, D_2)$ provided that $D_1 \supset D_2$.

Let Γ be a chart, and $(D \cap \Gamma, D)$ a cellular T-tangle of label n. The tangle $(D \cap \Gamma, D)$ is *tiny* provided that the closure of each component of $(D - Cl(In(D \cap \Gamma_n))) \cap \Gamma$ is

(i) an arc connecting a point in ∂D and a point in $Brd(D \cap \Gamma_n)$, or

(ii) a terminal edge of label n.

NOTE. For any cellular T-tangle of label n, let X be the union of all the terminal edges of label n in D each of which intersects $Cl(In(D \cap \Gamma_n))$, and N a regular neighborhood of $Cl(In(D \cap \Gamma_n)) \cup X$ in D. Then $(N \cap \Gamma, N)$ is a tiny cellular T-tangle of label n.

Lemma 4.2. *Let $(D \cap \Gamma, D)$ be a non-linear T-tangle of label n with p exceptional arcs in a k-minimal chart Γ. Then $(D \cap \Gamma, D)$ contains a tiny cellular T-tangle with at most p exceptional arcs.*

Proof. Since $(D \cap \Gamma, D)$ is not linear, $In(D \cap \Gamma_n) \neq \emptyset$. Let Z be a connected component of $D \cap \Gamma_n$ such that $In(Z)$ contains a connected component of $In(D \cap \Gamma_n)$. Then $Z \cap \partial D$ consists of at most p points.

Let $D^* = Cl(In(Z))$ and Y the union of the closures of connected components of $Z - Cl(In(D \cap \Gamma_n))$ each of which is not a terminal edge (see Fig. 5). By Lemma 3.1 (2), $D^* = Cl(In(Z))$ consists of disjoint disks. And Y consists of disjoint trees.

Suppose $Y = \emptyset$. Then the closure of a connected component of $Z - Cl(In(D \cap \Gamma_n))$ is a terminal edge, and $Cl(In(Z))$ is a disk. Let N be a regular neighborhood of $In(Z) \cup Z$ in D. Then $(N \cap \Gamma, N)$ is a tiny cellular T-tangle without exceptional arcs. Hence we have a desired result. We can assume $Y \neq \emptyset$.

Let q be the number of points in $D^* \cap Y$. For $i = 1, 2, 3, \ldots$, let

$d_i =$ the number of connected components of D^* containing i points in $D^* \cap Y$,

$t_i =$ the number of trees in Y containing i points in $D^* \cap Y$.

Then we have

$$\sum_{i=1}^{\infty} i \times d_i = \sum_{i=1}^{\infty} i \times t_i = q. \tag{1}$$

Since $Y \neq \emptyset$, we have $q \geq 1$. Since $D^* \cup Y$ is contractible, by Euler formula we have

$$\sum_{i=1}^{\infty} d_i + \sum_{i=1}^{\infty} t_i - q = 1. \tag{2}$$

By using the equation (1) and the equation obtained by doubling each side of the equation (2), we have

$$\begin{aligned} 2 &= 2\sum_{i=1}^{\infty} d_i + 2\sum_{i=1}^{\infty} t_i - 2q \\ &= 2\sum_{i=1}^{\infty} d_i + 2\sum_{i=1}^{\infty} t_i - \left(\sum_{i=1}^{\infty} i \times d_i + \sum_{i=1}^{\infty} i \times t_i\right) \\ &= \sum_{i=1}^{\infty}(2-i)d_i + \sum_{i=1}^{\infty}(2-i)t_i = d_1 - \sum_{i=3}^{\infty}(i-2)d_i + t_1 - \sum_{i=3}^{\infty}(i-2)t_i. \end{aligned}$$

Thus we have

$$\sum_{i=3}^{\infty}(i-2)d_i + \sum_{i=3}^{\infty}(i-2)t_i = d_1 + t_1 - 2. \tag{3}$$

By Lemma 4.1, if the closure of a connected component of $(D \cap \Gamma_n) - Cl(In(D \cap \Gamma_n))$ is not a terminal edge, then it contains at least two points in $Brd(D \cap \Gamma_n) \cup \partial D$. This implies that for a connected component G of Y, if $D^* \cap G$ consists of one point, then G contains a point in ∂D. Thus each tree in Y contributing to t_1 must contain a point in ∂D. Since there are at most p connected components of Y intersecting ∂D, we have $t_1 \leq p$.

We shall show that there exists an integer $1 \leq j \leq p$ with $d_j \neq 0$.

If $p = 1$, then $t_1 \leq 1$. Since the left side of the equation (3) is non negative, we have $d_1 + t_1 - 2 \geq 0$. Hence $d_1 \geq 2 - t_1 \geq 2 - 1 = 1$. Therefore $d_1 \neq 0$. We can assume $p \geq 2$.

Suppose that $d_i = 0$ for $i = 1, 2, \ldots, p$. By the equation (1), we have $\sum_{i=1}^{\infty} i \times d_i = q \geq 1$. Thus there exists an integer $j > p \geq 2$ with $d_j \neq 0$. Hence for the left side of the equation (3) we have

$$\sum_{i=3}^{\infty}(i-2)d_i + \sum_{i=3}^{\infty}(i-2)t_i \geq \sum_{i=j}^{\infty}(i-2)d_i \geq j - 2 > p - 2. \tag{4}$$

On the other hand, for the right side of the equation (3) we have

$$d_1 + t_1 - 2 = t_1 - 2.$$

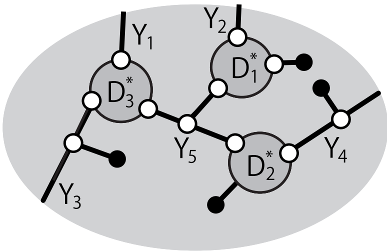

Fig. 5. $p = 4$, $q = 7$, D_1^* and D_2^* are disks in D^* containing two points in $D^* \cap Y$, D_3^* is a disk in D^* containing three points in $D^* \cap Y$, Y_1, Y_2, Y_3 and Y_4 are trees in Y containing one point in $D^* \cap Y$, Y_5 is a tree in Y containing three points in $D^* \cap Y$, $d_1 = 0$, $d_2 = 2$, $d_3 = 1$, $t_1 = 4$, $t_2 = 0$, $t_3 = 1$.

Since $t_1 \leq p$, we have

$$d_1 + t_1 - 2 \leq p - 2. \tag{5}$$

We have a contradiction comparing (4) and (5). Therefore there exists an integer $1 \leq j \leq p$ with $d_j \neq 0$.

Since $d_j \neq 0$ for some integer $1 \leq j \leq p$, there exists a connected component N of $Cl(In(D \cap \Gamma_n))$ such that N intersects at most p connected components in Y. By Lemma 3.1 (2), N is a disk. Let X be the union of terminal edges in $D \cap \Gamma_n$ intersecting N. Let N^* be a regular neighborhood of $N \cup X$. Then $(N^* \cap \Gamma, N^*)$ is a tiny cellular T-tangle with at most p exceptional arcs. □

5. T_2-tangles

Let Γ be a chart. A tangle $(D \cap \Gamma, D)$ is called an *NS-tangle of label m* (new significant tangle) if it satisfies the following two conditions:

(i) If $i \neq m$, then $\partial D \cap \Gamma_i$ is at most one point, and

(ii) $w(D \cap \Gamma) \geq 1$ and D contains at most one crossing.

Lemma 5.1 ([12, Theorem 3.5]). *There does not exist any NS-tangle in a k-minimal chart Γ.*

Let $(D\cap\Gamma, D)$ be a T-tangle of a chart Γ. If s is the number of labels in $\{i \mid \partial D \cap \Gamma_i \neq \emptyset\}$, then the T-tangle is called a *T_s-tangle*. Thus a T-tangle means a T_0-tangle, a T_1-tangle, a T_2-tangle or a T_3-tangle.

NOTE. Since T_0-tangles and T_1-tangles are NS-tangles, there do not exist any T_0-tangles nor T_1-tangles in a k-minimal chart by Lemma 5.1.

Lemma 5.2 ([12, Theorem 5.4]). *Let $(D \cap \Gamma, D)$ be a tiny cellular T_2-tangle of label n in a k-minimal chart Γ which possesses exceptional arcs.*
(1) *The tangle possesses at least two exceptional arcs.*
(2) *If the tangle possesses exactly two exceptional arcs, then D contains at least two terminal edges of label n.*

Let Γ be a chart, $X \subset \Gamma$. Let

$$\alpha(X) = \min\{i \mid \Gamma_i \cap X \neq \emptyset\},$$
$$\beta(X) = \max\{i \mid \Gamma_i \cap X \neq \emptyset\}.$$

Lemma 5.3 (Boundary condition lemma ([12, Lemma 4.1])). *Let $(D \cap \Gamma, D)$ be a tangle in a k-minimal chart Γ such that D does not contain any crossing. Let $a = \alpha(\partial D \cap \Gamma)$ and $b = \beta(\partial D \cap \Gamma)$. Then $D \cap \Gamma_i = \emptyset$ except for $a \leq i \leq b$.*

Lemma 5.4. *Let $(D \cap \Gamma, D)$ be a non-linear T_2-tangle of label n in a k-minimal chart Γ. If the T_2-tangle possesses exactly two exceptional arcs, then the tangle possesses at least two terminal edges of label n.*

Proof. Since the T_2-tangle possesses an exceptional arc, there exists an integer $\varepsilon \in \{+1, -1\}$ with $\partial D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\varepsilon}$. Thus we have $D \cap \Gamma \subset \Gamma_n \cup \Gamma_{n+\varepsilon}$ by the Boundary condition lemma (Lemma 5.3). Hence the tangle $(D \cap \Gamma, D)$ contains a tiny cellular T_2-tangle $(D' \cap \Gamma, D')$ with at most two exceptional arcs by Lemma 4.2.

By Lemma 5.2 (1), the tangle $(D' \cap \Gamma, D')$ possesses exactly two exceptional arcs. By Lemma 5.2 (2), there exist at least two terminal edges of label n in D. □

By Lemmas 3.3 and 5.4, we have the following corollary:

Corollary 5.5. *Let $(D \cap \Gamma, D)$ be a T_2-tangle of label n with exactly two exceptional arcs in a k-minimal chart Γ. Then the following hold*:
(1) *The disk D contains at least one terminal edge of label n.*
(2) *If D contains exactly one terminal edge of label n, then $(D \cap \Gamma, D)$ is linear.*

6. Charts with at most three crossings

Let Γ be a chart, D a disk. Let m be a label with $D \cap \Gamma_m \neq \emptyset$. A connected component G of $D \cap \Gamma_m$ is a *two-color component* of label m in D provided that
(i) $G \cap \partial D$ consists of at most one point,
(ii) there exists an integer $\delta \in \{+1, -1\}$ such that all the white vertices in G are contained in $\Gamma_{m+\delta}$, and
(iii) G is not an arc contained in a terminal edge.
Note that a two-color component may contain a crossing.

Lemma 6.1 ([12, Lemma 3.6]). *Let Γ be a k-minimal chart and D a disk. Then for any two-color component G in D, $G \cup In(G)$ contains at least two crossings.*

Let G be a graph. Then an edge e in G is called a *cut edge* of G provided that $G - e$ is disconnected.

Lemma 6.2. *Let Γ be a k-minimal chart and G a two-color component of label m in a disk D such that*
(1) *$G \cap \partial D = \emptyset$, and*
(2) *G contains a cut edge.*
Then Γ contains at least four crossings.

Proof. Let e be a cut edge of G. Since by Assumption 6 we can move the point at infinity ∞ to any complementary domain of Γ, we can assume $e \subset Cl(Out(G))$. Since e is a cut edge of G, $Cl(G - e)$ consists of two connected components, say G_1 and G_2. For $i = 1, 2$ let N_i be a regular neighbourhood of $G_i \cup In(G_i)$ and $G'_i = N_i \cap G$. Then N_i is a disk by Lemma 3.1 (3). Thus G'_i is a two-color component in N_i. Hence by Lemma 6.1, each of $G'_1 \cup In(G'_1)$ and $G'_2 \cup In(G'_2)$ contains at least two crossings. Now $e \subset Cl(Out(G))$ implies $N_1 \cap N_2 = \emptyset$. Therefore Γ contains at least four crossings. □

Lemma 6.3. *Let Γ be a k-minimal chart with at most three crossings. Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$. Then*
(1) *each of Γ_α and Γ_β is connected,*
(2) *each of $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ is a simple closed curve, and*
(3) *$Brd(\Gamma_\alpha) \cap Brd(\Gamma_\beta)$ consists of two crossings.*

Proof. Let G_α be a connected component of Γ_α. Let N be a regular neighbourhood of $G_\alpha \cup In(G_\alpha)$. Since G_α is connected, N is a disk by Lemma 3.1 (3). Let $D^* = Cl(S^2 - N)$ where S^2 is the 2-sphere. Then D^* is a disk, too.

Now $\alpha = \alpha(\Gamma)$ implies that any white vertices in G_α are contained in $\Gamma_\alpha \cap \Gamma_{\alpha+1}$. Thus G_α is a two-color component of label α in the disk N.

Since there are at most three crossings, G_α does not contain a cut edge by Lemma 6.2. By Assumption 5, G_α is not a free edge. Thus G_α is not a tree. Now $Cl(In(G_\alpha))$ consists of disks by Lemma 3.1 (2). Since G_α does not contain a cut edge, $Cl(In(G_\alpha))$ consists of only one disk. Hence we have

(i) $\quad Brd(G_\alpha)$ is a simple closed curve.

Suppose that $Brd(G_\alpha)$ contains at most one crossing. Since G_α does not contain a cut edge, $\partial D^* \cap (\Gamma - \Gamma_{\alpha+1})$ is at most one point (see Fig. 6 (a)).

By Lemma 6.1, $G_\alpha \cup In(G_\alpha)$ contains at least two crossings. Since there are at most three crossings, D^* contains at most one crossing.

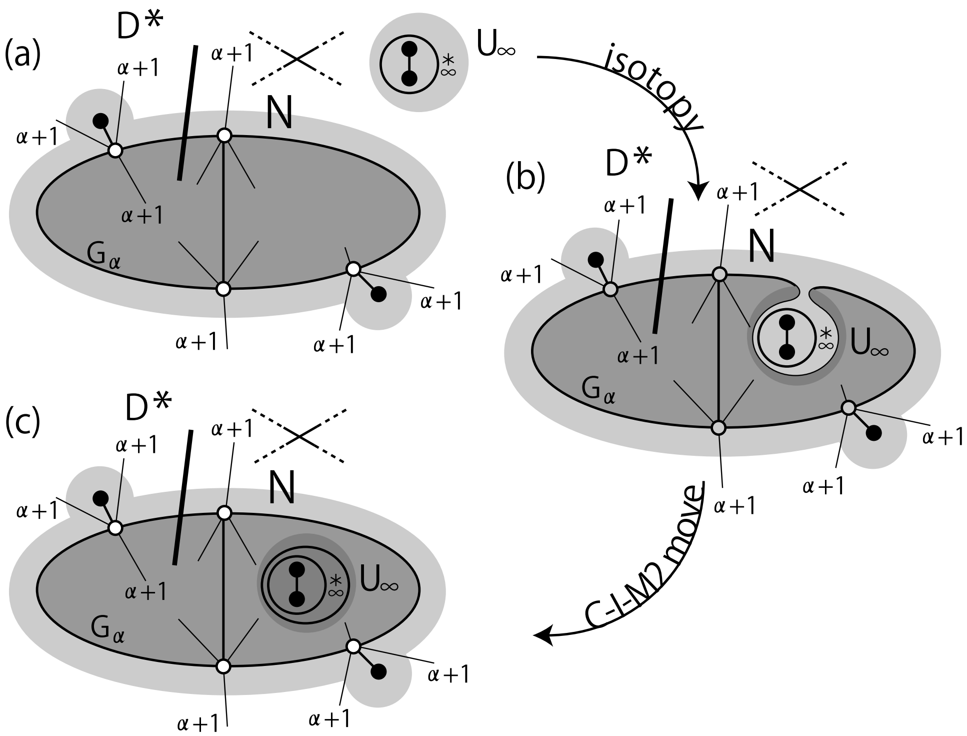

Fig. 6.

As mentioned in Assumption 6, by applying C-I-M2 moves we can push the neighbourhood U_∞ out from D^* without increasing the complexity of the chart (see Fig. 6 (c)). Then $(D^* \cap \Gamma, D^*)$ is an NS-tangle. This contradicts Lemma 5.1. Therefore

(ii) $\quad Brd(G_\alpha)$ contains at least two crossings.

Since each connected component of Γ_α contains at least two crossings and since there are at most three crossings, there exists only one connected component in Γ_α. Thus $G_\alpha = \Gamma_\alpha$. Thus Γ_α is connected.

Since Γ_α is connected and since Γ_α satisfies (i), Γ_α satisfies (2).

Similarly we can show that Γ_β is connected, satisfies (2) and $Brd(\Gamma_\beta)$ contains at least two crossings.

Since there are at most three crossings, $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ must intersect. Since $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ are simple closed curves, $Brd(\Gamma_\alpha) \cap Brd(\Gamma_\beta)$ consists of an even number of points. Since there are at most three crossings, $Brd(\Gamma_\alpha) \cap Brd(\Gamma_\beta)$ consists of two points. □

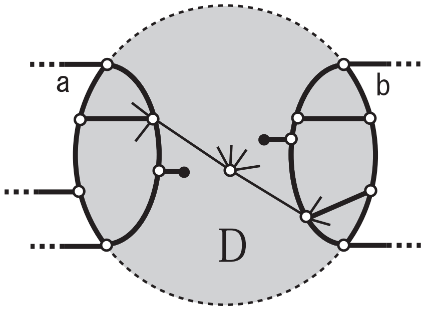

Fig. 7. The shaded area is the disk D.

7. 2-minimal charts

Lemma 7.1 ([11, Theorem 2]). *Any n-chart with at most one crossing is a ribbon chart.*

Let Γ be a chart and D a disk. Let $a = \alpha(D \cap \Gamma)$ and $b = \beta(D \cap \Gamma)$. The disk D is called an *N-rectangle* if it satisfies the following four conditions (see Fig. 7):
(i) D does not contain any crossing,
(ii) both of $\partial D \cap \Gamma_a$ and $\partial D \cap \Gamma_b$ are connected,
(iii) $\partial D \cap \Gamma \subset \Gamma_a \cup \Gamma_b$, and
(iv) there exists an arc in $D \cap \Gamma$ connecting a point in $D \cap \Gamma_a$ and a point in $D \cap \Gamma_b$.

From now on throughout this section, we assume that
(i) Γ is a 2-minimal chart with exactly two crossings,
(ii) Γ is not a ribbon chart, and
(iii) $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$.

By Lemma 6.3, each of $Brd(\Gamma_\alpha)$ and $Brd(\Gamma_\beta)$ is a simple closed curve containing the two crossings. Let

$\Delta_\alpha =$ the closure of the complementary domain of the simple closed curve $Brd(\Gamma_\alpha)$ such that Δ_α does not contain the point at infinity ∞,

$\Delta_\beta =$ the closure of the complementary domain of the simple closed curve $Brd(\Gamma_\beta)$ such that Δ_β does not contain the point at infinity ∞.

Let v_1 and v_2 be the crossings in Γ. Let $N_1 = N(v_1)$ and $N_2 = N(v_2)$ be regular neighborhoods of v_1 and v_2 respectively, and $N = N_1 \cup N_2$ (see Fig. 8). Let

$$P_1 = (\Gamma_\alpha - Int(N)) \cap \Delta_\beta, \qquad P_3 = (\Gamma_\alpha - Int(N)) \cap Cl(\Delta_\beta^c),$$
$$P_2 = (\Gamma_\beta - Int(N)) \cap \Delta_\alpha, \qquad P_4 = (\Gamma_\beta - Int(N)) \cap Cl(\Delta_\alpha^c),$$
$$Q_1 = (\Delta_\alpha \cap \Delta_\beta) - Int(N), \qquad Q_3 = (Cl(\Delta_\alpha^c) \cap Cl(\Delta_\beta^c)) - Int(N),$$
$$Q_2 = (\Delta_\alpha \cap Cl(\Delta_\beta^c)) - Int(N), \qquad Q_4 = (Cl(\Delta_\alpha^c) \cap \Delta_\beta) - Int(N).$$

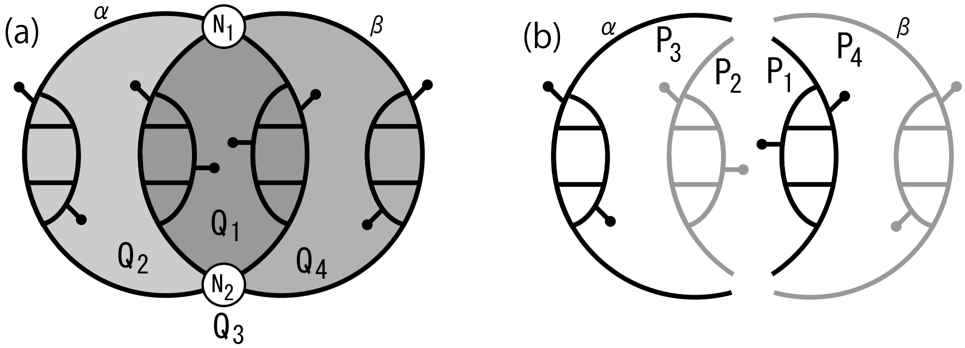

Fig. 8.

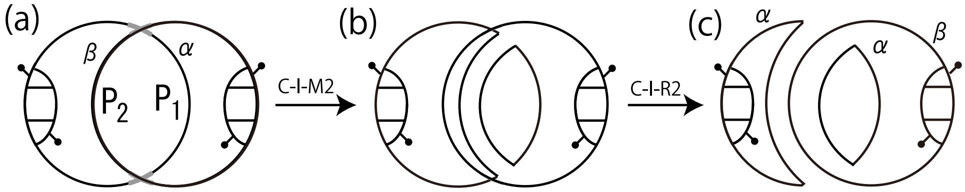

Fig. 9.

Lemma 7.2. *There are two N-rectangles among Q_1, Q_2, Q_3 and Q_4. Moreover among P_1, P_2, P_3 and P_4, three of them contain white vertices.*

Proof. We show our lemma by three steps.

STEP 1. We claim that P_1 or P_2 contains a white vertex. For, suppose that neither P_1 nor P_2 contains a white vertex. Apply a C-I-M2 move between two points in P_1 along the arc P_2 further apply a C-I-R2 move so that we can eliminate the crossings v_1 and v_2 (see Fig. 9 (c)). Hence Γ can be modified to a chart without crossings by C-moves. By Lemma 7.1, Γ is a ribbon chart. This contradicts the assumption (ii) of this section: Γ is not a ribbon chart. Hence one of P_1 and P_2 contains a white vertex. Without loss of generality we can assume that P_1 contains a white vertex.

STEP 2. We claim that Q_1 or Q_4 is an N-rectangle. For, suppose that neither Q_1 nor Q_4 is an N-rectangle. Then for $i = 1, 4$, there exists a simple arc l_i in Q_i connecting a point in ∂N_1 and a point in ∂N_2 with $l_i \cap \Gamma = \emptyset$ (see Fig. 10). Let D be the closure of the connected component of $\Delta_\beta - (l_1 \cup l_4 \cup N)$ containing P_1. Then $\partial D \cap \Gamma \subset \Gamma_\alpha$. Since $P_1 \subset D$ and since P_1 contains a white vertex, we have $w(D \cap \Gamma) \geq 1$. Since D does not contain any crossing, $(D \cap \Gamma, D)$ is an NS-tangle of label α. This contradicts Lemma 5.1. Hence one of Q_1 and Q_4 is an N-rectangle. Without loss of generality we can assume that Q_1 is an N-rectangle.

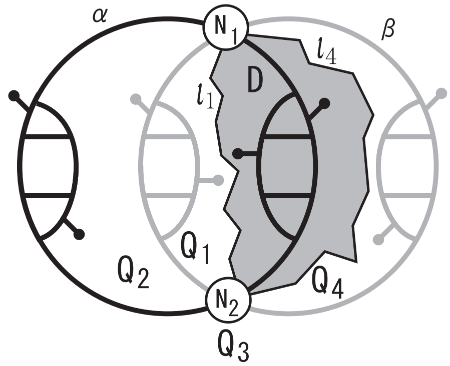

Fig. 10.

STEP 3. Hence both of P_1 and P_2 contain white vertices. We can show that one of P_3 and P_4 contains a white vertex by the same way as the one in Step 1. Hence among P_1, P_2, P_3 and P_4, three of them contain white vertices.

If P_3 contains a white vertex, then we can show that one of Q_2 and Q_3 is an N-rectangle in the same way as the one in Step 2. If P_4 contains a white vertex, then we can show that one of Q_3 and Q_4 is an N-rectangle in the same way as the one in Step 2. Therefore two of Q_1, Q_2, Q_3 and Q_4 are N-rectangles. $\square$

Lemma 7.3. *Both of Δ_α and Δ_α^c contain white vertices of Γ_i for any label i $(\alpha + 2 \leq i \leq \beta - 2)$, or both of Δ_β and Δ_β^c contain white vertices of Γ_i for any label i $(\alpha + 2 \leq i \leq \beta - 2)$.*

Proof. By Lemma 7.2, two of Q_1, Q_2, Q_3 and Q_4 are N-rectangles. Without loss of generality we can assume that Q_1 is an N-rectangle. There exists an integer j in $\{2, 3, 4\}$ such that Q_j is an N-rectangle.

For the case $j = 2$, we have $Q_1 \subset \Delta_\beta$ and $Q_2 \subset Cl(\Delta_\beta^c)$. Since Q_1 is an N-rectangle, by the condition (iii) of N-rectangles, $\partial Q_1 \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_\beta$. By the condition (iv) of N-rectangles, there exists an arc γ in $Q_1 \cap \Gamma$ connecting a point in $\partial Q_1 \cap \Gamma_\alpha$ and a point in $\partial Q_1 \cap \Gamma_\beta$. Hence for each label i $(\alpha + 2 \leq i \leq \beta - 2)$ there exists a white vertex in $\Gamma_i \cap \gamma$. Since $\partial Q_1 \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_\beta$, the white vertex of Γ_i is contained in $Int(\Delta_\beta)$. Since Q_2 is an N-rectangle, in a similar way as the one above we can show that there exists a white vertex of Γ_i $(\alpha + 2 \leq i \leq \beta - 2)$ in Δ_β^c.

For the case $j = 3$ or 4, we have $Q_1 \subset \Delta_\alpha$ and $Q_j \subset Cl(\Delta_\alpha^c)$. Similarly we can show that there exist white vertices of Γ_i for any label i $(\alpha + 2 \leq i \leq \beta - 2)$ in Δ_α and Δ_α^c respectively. $\square$

A connected component G' of a graph G is called a *small component* of G if it satisfies $(In(G') - G') \cap G = \emptyset$. In Fig. 11, X is a small component of $X \cup Y$, but Y is not a small component of $X \cup Y$.

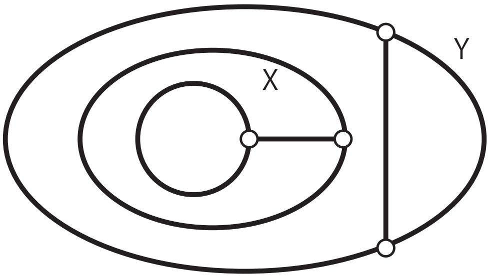

Fig. 11.

Lemma 7.4 ([12, Theorem 4.8]). *Let Γ be a k-minimal chart. Let G be a small component of Γ_n such that $G \cup In(G)$ does not contain any crossing. Then G contains at least two terminal edges of label n.*

Proposition 7.5. (1) *For any label i $(\alpha+2 \leq i \leq \beta-2)$ the subgraph Γ_i contains at least four black vertices.*
(2) *If $\alpha + 2 < \beta$, then $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices.*

Proof. (1) By Lemma 7.3 we can assume that both of Δ_α and Δ_α^c contain white vertices of Γ_i for any label i $(\alpha + 2 \leq i \leq \beta - 2)$.

Let i be a label with $\alpha + 2 \leq i \leq \beta - 2$. Now $\partial\Delta_\alpha \subset \Gamma_\alpha$, $\alpha \neq i$ and $\alpha + 1 \neq i$ imply $\partial\Delta_\alpha \cap \Gamma_i = \emptyset$. Let G_i be a small component of $\Delta_\alpha \cap \Gamma_i$. Then G_i is a small component of Γ_i in $Int(\Delta_\alpha)$. Since $Int(\Delta_\alpha)$ does not contain any crossing, neither does $G_i \cup In(G_i)$. By Lemma 7.4, G_i contains at least two terminal edges of label i. Hence $Int(\Delta_\alpha)$ contains at least two terminal edges of label i.

Similarly we can show that Δ_α^c contains at least two terminal edges of label i. Hence Γ_i contains at least four black vertices.

(2) Since $\alpha + 2 < \beta$, we have $\alpha + 1 \neq \beta - 1$. By Lemma 7.2, three of P_1, P_2, P_3 and P_4 contain white vertices. Without loss of generality we can assume that all of P_1, P_2 and P_3 contain white vertices.

Since P_2 contains a white vertex and $P_2 \subset \Delta_\alpha \cap \Gamma_\beta$, the disk Δ_α contains a white vertex of $\Gamma_{\beta-1}$. Since $\partial\Delta_\alpha \subset \Gamma_\alpha$, $\alpha \neq \beta-1$ and $\alpha+1 \neq \beta-1$, we have $\partial\Delta_\alpha \cap \Gamma_{\beta-1} = \emptyset$. In a similar way to (1) we can show that $Int(\Delta_\alpha)$ contains at least two terminal edges of label $\beta - 1$.

Since P_1 contains a white vertex and $P_1 \subset \Delta_\beta \cap \Gamma_\alpha$, the disk Δ_β contains a white vertex of $\Gamma_{\alpha+1}$. Similarly we can show that $Int(\Delta_\beta)$ contains at least two terminal edges of label $\alpha + 1$.

Since P_3 contains a white vertex and $P_3 \subset Cl(\Delta_\beta^c) \cap \Gamma_\alpha$, the open disk Δ_β^c contains a white vertex of $\Gamma_{\alpha+1}$. Similarly we can show that Δ_β^c contains at least two terminal edges of label $\alpha + 1$.

Therefore $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices. $\square$

Proposition 7.6. *Both of Γ_α and Γ_β contain at least two black vertices.*

Proof. Let D_i be a regular neighborhood of $P_i \cup In(P_i)$ in $S^2 - Int(N_1 \cup N_2)$ ($i = 1, 2, 3, 4$). By Lemma 3.1 (3), D_i is a disk. By Lemma 7.2, three of P_1, P_2, P_3 and P_4 contain white vertices. Without loss of generality we can assume that all of P_1, P_2 and P_3 contain white vertices.

For $i = 1, 3$, we have $\partial D_i \cap \Gamma \subset \Gamma_\alpha \cup \Gamma_{\alpha+1}$. By the Boundary condition lemma (Lemma 5.3), $D_i \cap \Gamma_j = \emptyset$ except for $j \in \{\alpha, \alpha+1\}$. Similarly for $i = 2, 4$, we have $D_i \cap \Gamma_j = \emptyset$ except for $j \in \{\beta, \beta - 1\}$.

Since P_i ($i = 1, 3$) contains a white vertex, $(D_i \cap \Gamma, D_i)$ is a T_2-tangle of label α with two exceptional arcs. By Corollary 5.5 (1), the disk D_i ($i = 1, 3$) contains at least one terminal edge of label α. Hence Γ_α contains at least two black vertices.

Since P_2 contains a white vertex, $(D_2 \cap \Gamma, D_2)$ is a T_2-tangle of label β with two exceptional arcs. By Corollary 5.5 (1), the disk D_2 contains at least one terminal edge of label β.

Suppose that the disk D_2 contains exactly one terminal edge of label β. By Corollary 5.5 (2), $(D_2 \cap \Gamma, D_2)$ is linear. Let e_1 and e_2 be the two exceptional arcs of $(D_2 \cap \Gamma, D_2)$. By Lemma 3.3, $D_2 \cap \Gamma_\beta$ consists of the two arcs e_1, e_2 and the terminal edge. Let w be the white vertex in the terminal edge. Since the terminal edge contains a middle arc at w by Assumption 1, both of e_1 and e_2 contain inward arcs at w or outward arcs at w. Hence P_4 contains a white vertex. Hence $(D_4 \cap \Gamma, D_4)$ is a T_2-tangle of label β with two exceptional arcs. By Corollary 5.5 (1), the disk D_4 contains at least one terminal edge of label β. Hence Γ_β contains at least two black vertices. □

8. Proof of Theorems 1.1 and 1.2

Lemma 8.1. *Let C be a hoop or a ring in a k-minimal chart Γ. Suppose that C contains exactly s crossings with $s \leq 3$. Then Γ contains at least $s + 4$ crossings.*

Proof. Let U_1 and U_2 be the connected components of $S^2 - C$. Then each of U_1 and U_2 contains a white vertex by Assumptions 3 and 4.

Suppose that U_i ($i = 1, 2$) contains at most one crossing. There are at most three edges transversely intersecting C. Let N_i be a disk in U_i such that $U_i - N_i$ is a very thin open annulus. Then we can assume that N_i contains a white vertex and that $\partial N_i \cap \Gamma$ consists of at most three points. Then for the edges intersecting ∂N_i there are two cases:

(1) the labels of the edges are different each other, and

(2) at least two labels of the edges are same.

In each case, $(N_i \cap \Gamma, N_i)$ is an NS-tangle. This contradicts Lemma 5.1. Hence U_i contains at least two crossings. Hence Γ contains at least $s + 4$ crossings. □

The following corollary is a direct result of the above lemma.

Corollary 8.2. *Let Γ be a k-minimal chart with at most three crossings. Then Γ contains neither hoop nor ring.*

Proof of Theorem 1.1. Since Γ is a generalized n-chart, $w(\Gamma) \geq 1$. Since Γ is a 2-minimal chart, Γ contains at most two crossings and Γ is not a ribbon chart. By Lemma 7.1, Γ contains exactly two crossings. By Assumption 5, Γ does not contain any free edge. By Corollary 8.2, Γ contains neither hoop nor ring. Let $\alpha = \alpha(\Gamma)$, $\beta = \beta(\Gamma)$. Then $w(\Gamma_\alpha) \geq 1$, $w(\Gamma_\beta) \geq 1$. Since Γ is a generalized n-chart, we have $\beta - \alpha = n - 2$.

By Proposition 7.6, $\Gamma_\alpha \cup \Gamma_\beta$ contains at least four black vertices. Since $\beta - (\alpha + 2) = n - 4 \geq 5 - 4 = 1 > 0$, we have $\alpha + 2 < \beta$. By Proposition 7.5 (2), $\Gamma_{\alpha+1} \cup \Gamma_{\beta-1}$ contains at least six black vertices. By Proposition 7.5 (1), for any label i $(\alpha + 2 \leq i \leq \beta - 2)$ Γ_i contains at least four black vertices. We have that $\Gamma_{\alpha+2} \cup \Gamma_{\alpha+3} \cup \cdots \cup \Gamma_{\beta-2}$ contains at least $4((\beta-2)-(\alpha+2)+1)$ black vertices. Since $4((\beta-2)-(\alpha+2)+1) = 4(\beta-\alpha-3) = 4(n-5)$, we have the number of black vertices of $\Gamma \geq 4(n-5)+4+6 = 4n - 10$. $\square$

By [4, Remarks 8 (2)] we have the statement (1) in the following lemma.

Lemma 8.3. *Let Γ be an n-chart, and $\hat{S}_\Gamma$ the closure of the surface braid obtained from Γ.*

(1) *Let b be the number of black vertices of Γ. Then $\chi(\hat{S}_\Gamma) = 2n - b$.*

(2) *Let $\iota_p^q(\Gamma)$ be the $(n + p + q)$-chart obtained from Γ by shifting all labels i to $i + p$. Then the closure of the surface braid obtained from $\iota_p^q(\Gamma)$ contains at least $p + q + 1$ components.*

(3) *Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$. Then $\hat{S}_\Gamma$ contains at least $n - \beta + \alpha - 1$ components.*

Proof. We shall show the statement (2). Let $\hat{S}$ be the closure of a surface braid obtained from $\iota_p^q(\Gamma)$. Then the surface $\hat{S}$ is $\hat{S}_\Gamma$ with p parallel spheres in front of $\hat{S}_\Gamma$ and q parallel spheres behind $\hat{S}_\Gamma$ (cf. [6, p. 183]). Therefore $\hat{S}$ contains of at least $p + q + 1$ components.

We shall show the statement (3). Let Γ' be the $(\beta - \alpha + 2)$-chart obtained from Γ by shifting all labels i to $i - \alpha + 1$. Then edges of Γ_α and edges of Γ_β change edges of Γ'_1 and edges of $\Gamma'_{\beta-\alpha+1}$ respectively. Hence $\alpha(\Gamma') = 1$ and $\beta(\Gamma') = \beta - \alpha + 1$.

Let $\Gamma'' = \iota_{\alpha-1}^{n-\beta-1}(\Gamma')$. Since $(\beta-\alpha+2)+(\alpha-1)+(n-\beta-1) = n$, the chart Γ'' is an n-chart. Since edges of Γ'_1 and edges of $\Gamma'_{\beta-\alpha+1}$ change edges of Γ''_α and edges of Γ''_β respectively, the chart Γ'' is the same as the chart Γ. Since $(\alpha - 1) + (n - \beta - 1) + 1 = n - \beta + \alpha - 1$, $\hat{S}_\Gamma$ contains of at least $n - \beta + \alpha - 1$ components by the statement (2) in this lemma. $\square$

Lemma 8.4. *Let Γ' be an n-chart and Γ'' the n-chart obtained from Γ' by omitting all the free edges. Let $\hat{S}_{\Gamma'}$ and $\hat{S}_{\Gamma''}$ be the closures of the surface braids obtained from Γ' and Γ'' respectively. If $\hat{S}_{\Gamma'}$ is a disjoint union of spheres, then so is $\hat{S}_{\Gamma''}$.*

Proof. Since the chart Γ' is obtained by adding free edges to the chart Γ'', the surface $\hat{S}_{\Gamma'}$ is obtained by attaching 1-handles from the surface $\hat{S}_{\Gamma''}$. Since $\hat{S}_{\Gamma'}$ is a disjoint union of spheres, so is $\hat{S}_{\Gamma''}$. □

Kamada showed that for $n = 1, 2, 3$ any n-chart is a ribbon chart [4]. We showed that if a 2-minimal 4-chart contains exactly two crossings, then it contains eight black vertices [9]. By the similar argument as above, we have the following remark:

REMARK 8.5. Let Γ be a k-minimal chart. Let $\alpha = \alpha(\Gamma)$ and $\beta = \beta(\Gamma)$.
(1) If $\beta - \alpha \leq 1$, then Γ is a ribbon chart.
(2) If $\beta - \alpha = 2$ and if Γ is a 2-minimal chart with exactly two crossings, then it contains eight black vertices.

Proof of Theorem 1.2. Let n be the integer such that Γ is an n-chart. Let Γ' be a 2-minimal generalized n'-chart C-move equivalent to Γ. If Γ' contains at most one crossing, then by Lemma 7.1 Γ' is a ribbon chart, so is Γ.

Suppose that Γ' contains exactly two crossings. By Corollary 8.2, Γ' contains neither hoop nor ring. Let Γ'' be the n-chart obtained from Γ' by omitting all the free edges. Since Γ' is a 2-minimal generalized n'-chart, Γ'' is a 2-minimal generalized n'-chart.

Let $\alpha = \alpha(\Gamma'')$ and $\beta = \beta(\Gamma'')$. Since Γ'' contains neither hoops, rings nor free edges, both of Γ''_α and Γ''_β contain white vertices. Since Γ'' is a generalized n'-chart, we have $n' = \beta - \alpha + 2$.

Let $\hat{S}_\Gamma$, $\hat{S}_{\Gamma'}$ and $\hat{S}_{\Gamma''}$ be the closures of surface braids obtained from Γ, Γ' and Γ'' respectively. Since Γ is C-move equivalent to Γ', $\hat{S}_\Gamma$ is ambient isotopic to $\hat{S}_{\Gamma'}$ (cf. [6, Theorem 18.20]). The closure $\hat{S}_\Gamma$ is a disjoint union of spheres, and so is $\hat{S}_{\Gamma'}$. By Lemma 8.4, $\hat{S}_{\Gamma''}$ is a disjoint union of spheres.

Since Γ'' is an n-chart, by Lemma 8.3 (3) $\hat{S}_{\Gamma''}$ contains at least $n-\beta+\alpha-1$ spheres. Since $n' = \beta - \alpha + 2$, $\hat{S}_{\Gamma''}$ contains at least $n - n' + 1$ spheres. By Lemma 8.3 (1) $2(n - n' + 1) \leq \chi(\hat{S}_{\Gamma''}) = 2n -$ (the number of black vertices of Γ''). Hence Γ'' contains at most $2n' - 2$ black vertices.

Suppose $n' = 4$. Since $n' = \beta - \alpha + 2$, we have $\beta - \alpha = 2$. By Remark 8.5 (2), the chart contains at least eight black vertices. Hence Γ'' contains at least eight black vertices. On the other hand since $2n' - 2 = 2 \times 4 - 2 = 6$, the chart Γ'' contains at most six black vertices. This is a contradiction.

Suppose $n' \geq 5$. By Theorem 1.1, the chart Γ'' contains at least $4n' - 10$ black vertices. On the other hand the chart Γ'' contains at most $2n' - 2$ black vertices. Hence $4n' - 10 \leq 2n' - 2$. Hence $n' \leq 4$. This is a contradiction.

Therefore $n' \leq 3$. Since $n' = \beta - \alpha + 2$, we have $\beta - \alpha \leq 1$. By Remark 8.5 (1), the chart is a ribbon chart. Hence Γ'' is a ribbon chart, so is Γ. $\square$

References

[1] J.S. Carter and M. Saito: Knotted Surfaces and Their Diagrams, Mathematical Surveys and Monographs **55**, Amer. Math. Soc., Providence, RI, 1998.

[2] I. Hasegawa: *The lower bound of the w-indices of non-ribbon surface-links*, Osaka J. Math. **41** (2004), 891–909.

[3] S. Ishida, T. Nagase and A. Shima: *Minimal n-charts with four white vertices*, J. Knot Theory Ramifications **20** (2011), 689–711.

[4] S. Kamada: *Surfaces in* $\mathbf{R}^4$ *of braid index three are ribbon*, J. Knot Theory Ramifications **1** (1992), 137–160.

[5] S. Kamada: 2*-dimensional braids and chart descriptions*; in Topics in Knot Theory (Erzurum, 1992), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **399**, Kluwer Acad. Publ., Dordrecht, 1993, 277–287.

[6] S. Kamada: Braid and Knot Theory in Dimension Four, Mathematical Surveys and Monographs **95**, Amer. Math. Soc., Providence, RI, 2002.

[7] T. Nagase and A. Hirota: *The closure of a surface braid represented by a* 4*-chart with at most one crossing is a ribbon surface*, Osaka J. Math. **43** (2006), 413–430.

[8] M. Ochiai, T. Nagase and A. Shima: *There exists no minimal n-chart with five white vertices*, Proc. Sch. Sci. Tokai Univ. **40** (2005), 1–18.

[9] T. Nagase and A. Shima: *On surface braids of index four with at most two crossings*, Fund. Math. **188** (2005), 167–193.

[10] T. Nagase and A. Shima: *Properties of minimal charts and their applications*, I, J. Math. Sci. Univ. Tokyo **14** (2007), 69–97.

[11] T. Nagase and A. Shima: *Any chart with at most one crossing is a ribbon chart*, Topology Appl. **157** (2010), 1703–1720.

[12] T. Nagase and A. Shima: *On charts with two crossings* I: *there exist no NS-tangles in a minimal chart*, J. Math. Sci. Univ. Tokyo **17** (2010), 217–241.

[13] T. Nagase and A. Shima: *On charts with two crossings* III, preprint.

[14] T. Nagase, A. Shima and H. Tsuji: *The closures of surface braids obtained from minimal n-charts with four white vertices*, preprint.

[15] T. Nagase and A. Shima: *On charts with three crossings representing* 2*-knots*, in preparation.

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