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The structure of rings whose quotient rings are primitive rings with minimal one sided ideals

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Recently A. W. Goldie [2] has proved that the quotient ring of a prime ring with some ascending chain condition is a simple ring with minimal condition. In this note we shall show that we can obtain the properties of a ring whose quotient ring is a primitive ring with minimal one sided ideals (P.M.I.), which are analogous to those of a prime ring in [2]. The following example shows that there exists such a ring.

Let I be the ring of rational integers. Let R_n be a sub-ring of matrix ring with infinite degree over the ring of rational numbers such that

$$\begin{pmatrix} (a_{ij}) & & & \\ & 2m_1 & & \\ & & 2m_2 & \\ & & & \ddots \end{pmatrix} \quad m_i \in I, \quad (a_{ij}) \in I_n .$$

Let $R = \bigcup_n R_n$, then if an element a of R is not zero divisor, a is the following form:

$$a = \begin{pmatrix} (a_{ij}) & & & \\ & 2m_1 & & \\ & & 2m_2 & \\ & & & \ddots \end{pmatrix} \quad |a_{ij}| \neq 0, \quad m_i \neq 0 .$$

Hence the right (and left) quotient ring of R is $Q = \bigcup Q_n$:

$$Q_n = \begin{pmatrix} (a_{ij}) & & & \\ & m_1 & & \\ & & m_2 & \\ & & & \ddots \end{pmatrix} \quad (a_{ij}) \in Q_n \quad \text{and} \quad m_i \in Q' ,$$

where Q' is the ring of rational numbers, and Q is P.M.I..

In this note there are many statements which overlap [2], but we shall repeat those for the sake of completeness.

1. Preliminaries.

Let R be a ring with the right and left quotient ring Q and we shall call non zero divisor elements regular elements. We shall denote one sided ideals of R by Roman and ones of Q by German.

We have the following statements.

(1) *If* $c_1, c_2, \cdots, c_n$ *are regular elements of* R, *then there exist regular elements* $d_1, d_2, \cdots, d_n$ *and* c *such that*

$$c_i^{-1} = d_i c^{-1}.$$

We can prove this by the induction with respect to n, cf. Asano [1], and [2] Lemma 4. 2.

(2) *If* A *is a right, left and two sided ideal respectively, then* AQ, QA *and* QAQ *consist of* ac^{-1}, $c^{-1}a$ *and* $d^{-1}ac^{-1}$, $a \in A$ *and* $c, d \in R$, *respectively.*

Cf. [2] Lemma 4. 3.

(3) *Let* $\mathfrak{r}$ *be a non zero right ideal of* Q, *then* $\mathfrak{r} \cap R \neq (0)$.

Let S be a sub-set of Q. We shall define the following annihilators.

$$S_r = \{x \mid \in R,\ \ Sx = (0)\},$$
$$S_r{*} = \{x \mid \in Q,\ \ Sx = (0)\}\ \ \textit{and}$$
$$\tilde{S} = \{a \mid \in R,\ \textit{there exists a regular element } b \textit{ in } R \textit{ such that } b^{-1}a \in S\} \cup (S \cap R).$$

(4) *Let* $\mathfrak{r}$ *be a right ideal of* Q, *then*

$$\mathfrak{r} = (\mathfrak{r} \cap R)Q.$$

It is clear $\mathfrak{r} \supseteqq (\mathfrak{r} \cap R)Q$. If $x \in \mathfrak{r}$ then $x = ac^{-1}$ a, $c \in R$ and $a = xc \in \mathfrak{r} \cap R$ Hence $x \in (\mathfrak{r} \cap R)Q$.

(5) $$S_r = S_r{*} \cap R \quad \textit{and} \quad S_r{*} = S_r Q.$$

It is clear that $S_r{*} \supseteqq S_r Q$. If $a \in S_r{*}$ and $a = bc^{-1}$, b, $c \in R$, then $(0) = Sa = Sbc^{-1}$ hence $b \in S_r$.

We have clearly

(6) $$S_r{*} = (\tilde{S})_r{*} \quad \textit{and} \quad (\tilde{S})_r = S_r{*} \cap R.$$

Let $\mathfrak{l}$ *be a left ideal of* Q, *then*

(7) $$(\mathfrak{l} \cap R)_r = \mathfrak{l}_r{*} \cap R.$$

By the definition $\tilde{\mathfrak{l}} = \mathfrak{l} \cap R$ and by (6) we have $(\mathfrak{l} \cap R)_r = \tilde{\mathfrak{l}}_r = \mathfrak{l}_r{*} \cap R$.

(8) *Let* I_r *be a maximal annihilator in* R, *then* $I_r Q$ *is so in* Q.
Let $\mathfrak{l}_r{*}$ *be a maximal annihilator in* Q, *then* $\mathfrak{l}_r{*} \cap R$ *is so in* R.

It is clear that $I_r Q$ is an annihilator. If there exists an annihilator $\mathfrak{l}_r{*}$ such that $\mathfrak{l}_r{*} \supseteqq I_r Q$, then $(\mathfrak{l} \cap R)_r = \mathfrak{l}_r{*} \cap R \supseteqq I_r Q \cap R \supseteqq I_r$. By (3) $\mathfrak{l} \cap R \neq (0)$, and $(\mathfrak{l} \cap R)_r \neq R$, hence $(\mathfrak{l} \cap R)_r = I_r$ and $\mathfrak{l}_r{*} = (\mathfrak{l}_r{*} \cap R)Q = (\mathfrak{l} \cap R)_r Q = I_r Q$. Conversely let $\mathfrak{r}$ be a maximal annihilator, then $\mathfrak{r} \cap R$ is an annihilator in R by (7). If $I_r \supseteqq \mathfrak{r} \cap R$, by (4) we have

$$\mathfrak{r} = (\mathfrak{r} \cap R)Q \subseteqq I_r Q = I_r{*}, \quad \text{hence} \quad \mathfrak{r} = I_r{*} \supseteqq I_r \quad \text{and} \quad \mathfrak{r} \cap R = I_r.$$

Let I $(\neq(0))$ be a right ideal in R. We shall call maximal right ideals J with $J\cap I=(0)$ *complements* of I (denoted by I^c, $I^{c'}$, $\cdots$).

Let I be a right ideal in R. For any complement I^c of I in R there exists a complement $(IQ)^{c'}$ of IQ such that

$$I^cQ=(IQ)^{c'}, \tag{9}$$

and conversely for any complement $(IQ)^{c'}$ of IQ there exists a complement I^c of I satisfying (9).

If $x\in(IQ\cap I^cQ)$, then $x=ic^{-1}=jd^{-1}$, $i\in I$, $j\in I^c$ and we have by (2) $c^{-1}=af^{-1}$, $d^{-1}=bf^{-1}$, hence $ia=jb\in I\cap I^c=(0)$ and $x=0$. If there exists a right ideal $\mathfrak{j}$ such that $I^cQ\subseteqq\mathfrak{j}$ and $\mathfrak{j}\cap IQ=(0)$, then $\mathfrak{j}\cap R\cap I\subseteqq\mathfrak{j}\cap IQ=(0)$, hence since $I^c\subseteqq\mathfrak{j}\cap R$, $I^c=\mathfrak{j}\cap R$ and $I^cQ=(\mathfrak{j}\cap R)Q=\mathfrak{j}$. Therefore I^cQ is a complement of IQ. Conversely let $(IQ)^{c'}$ be a complement, then from the fact $(IQ)^{c'}\cap R\cap I=(0)$, $(IQ)^{c'}\cap R\subseteqq I^c$ hence $(IQ)^{c'}=((IQ)^{c'}\cap R)Q\subseteqq I^cQ$. From the above $I^cQ=(IQ)^{c''}$, hence $(IQ)^{c'}=(IQ)^{c''}=I^cQ$.

Let $\mathfrak{i}$ be a right ideal in Q. For any complement $\mathfrak{i}^c$ of $\mathfrak{i}$ in Q there exists a complement $(\mathfrak{i}\cap R)^{c'}$ of $(\mathfrak{i}\cap R)$ in R such that

$$\mathfrak{i}^c\cap R=(\mathfrak{i}\cap R)^{c'} \tag{10}$$

and conversely for any complement $(\mathfrak{i}\cap R)^{c'}$ there exists a complement right ideal $\mathfrak{i}^c$ in Q satisfying (10).

From the fact $\mathfrak{i}\cap R\cap\mathfrak{i}^c\cap R=(0)$ we have $\mathfrak{i}^c\cap R\subseteqq(\mathfrak{i}\cap R)^{c'}$. $\mathfrak{i}^c=(\mathfrak{i}^c\cap R)Q\subseteqq(\mathfrak{i}\cap R)^{c'}Q=((\mathfrak{i}\cap R)Q)^{c''}=\mathfrak{i}^{c''}$ by (9). Hence $\mathfrak{i}^c=\mathfrak{i}^{c''}$ and $\mathfrak{i}^c\cap R=\mathfrak{i}^{c''}\cap R=(\mathfrak{i}\cap R)^{c'}Q\cap R\supseteqq(\mathfrak{i}\cap R)^{c'}$. Conversely $(\mathfrak{i}\cap R)^{c'}\cap(\mathfrak{i}\cap R)=(0)$, then $\mathfrak{i}\cap(\mathfrak{i}\cap R)^{c'}Q=(0)$. Hence $(\mathfrak{i}\cap R)^{c'}Q\subseteqq\mathfrak{i}^c$ for some complement I^c of I and $\mathfrak{i}^c\cap R\subseteqq(\mathfrak{i}\cap R)^{c'}$. By the above $\mathfrak{i}^c\cap R=(\mathfrak{i}\cap R)^{c''}\supseteqq(\mathfrak{i}\cap R)^{c'}$, hence $\mathfrak{i}^c\cap R=(\mathfrak{i}\cap R)^{c'}$.

2. Uniform right ideals.

We can classify the right ideals in R as follows;

$I\equiv J$ if and only if there exist regular elements d, d' in R such that for any elements $r\in I$, $r'\in J$, $rd\in J$ and $r'd'\in I$.

It is clear that

$$I\equiv J \text{ if and only if } IQ=JQ.$$

We shall denote the class containing I by $[I]$.

PROPOSITION 1. *The right ideals in Q are lattice isomorphic to* $\{[I]\}$.

Proof. From the definition and (3) it is clear that this correspondence is onto and that $(I_1\cap I_2)Q\subseteqq I_1Q\cap I_2Q$. If $x\in I_1Q\cap I_2Q$, $x=r_1q_1^{-1}=r_2q_2^{-1}$, $r_i\in I_i$ and by (1) we have $x=r_1p_1t^{-1}=r_2p_2t^{-1}$, hence $r_1p_1=r_2p_2\in I_1\cap I_2$ and $x\in(I_1\cap I_2)Q$. We have clearly $(I_1\cup I_2)Q=I_1Q\cup I_2Q$.

$[I]Q_\cap R$ is the unique maximal right ideal in [I]. Since Q is P.M.I. there exist minimal right ideals and we call a right ideal in R which corresponds to a minimal right ideal in Q an *uniform right ideal* and the unique maximal right ideal in this class *basic right ideal.*

PROPOSITION 2. *If U is a uniform right ideal, then for any non zero right ideals I, J ($\subseteqq U$) $I_\cap J \neq (0)$.*

Proof. Since U is uniform, UQ is irreducible, hence $IQ=JQ=UQ$. From Proposition 1 $I_\cap J\neq 0$.

LEMMA 1.[1] *Let Q be a P.M.I. ring. If a right ideal $\mathfrak{r}$ is not minimal, then it contains at least two minimal right ideals.*

Proof. Let $\mathfrak{r}$ contain only one minimal right ideal $\mathfrak{r}_0$. Then $\mathfrak{r}\mathfrak{z}\subset\mathfrak{r}_\cap\mathfrak{z}$ and $\mathfrak{r}\mathfrak{z}=\mathfrak{r}_0=eQ$ where $\mathfrak{z}$ is the socle of Q. Hence $\mathfrak{r}\mathfrak{z}=e\mathfrak{r}\mathfrak{z}$. For any elements $r\in\mathfrak{r}$, $z\in\mathfrak{z}$ we have $rz=erz$ i.e. $(er-r)z=0$. Therefore $er-r\in\mathfrak{z}_l=(0)$ and $er=r$. Hence $e\mathfrak{r}=\mathfrak{r}=eQ$.

PROPOSITION 3. *Let U be a right ideal in R. If for any non zero right ideals I, J in U $I_\cap J\neq(0)$, then U is uniform.*

Proof. If U is not uniform, there exist two minimal right ideals $\mathfrak{r}_1$, $\mathfrak{r}_2$ in UQ by Lemma 1. Since $\mathfrak{r}_{1\cap}U\neq(0)$, $\mathfrak{r}_{2\cap}U\neq(0)$ and $\mathfrak{r}_{1\cap}U_\cap\mathfrak{r}_{2\cap}U=(0)$, it is a contradiction.

PROPOSITION 4. *Let I be a right ideal in R. I is uniform if and only if there exist elements y_1, y_2 and regular elements y_1', y_2' in R such that for any elements x, $x'\in I$ $xy_1'=x'y_1$, $x'y_2'=xy_2$.*

Proof. Let xq^{-1} and $x'q'^{-1}$ be elements in IQ. Then by the hypothesis $x'y'=xy$ with regular element y'. Hence $x'q'^{-1}=xyy'^{-1}q'^{-1}=xq^{-1}qyy'^{-1}q'^{-1}\in xQ$, therefore IQ is irreducible. The converse is similar.

PROPRSITION 5. *There exist mutually isomorphic uniform right ideals in any two classes which contain basic right ideals.*

Proof. Let I_1 and I_2 be basic. Since Q is P.M.I. there exists a Q-isomorphism λ of I_1Q to I_2Q. Let $I_iQ=e_iQ$, $e_i=r_ix_i^{-1}$, $r_i\in I_i$, $x_i\in R$ and $\lambda(e_1)=e_2q$, $q\in Q$. Then $\lambda(r_1)=\lambda(e_1x_1)=e_2qx_1$. If we put $x_2qx_1=yz^{-1}$, $y, z\in R$, we have $0\neq\lambda(r_1z)=e_2qx_1z$ $=e_2x_2^{-1}x_2qx_1z=r_2y$. Since I_1Q and I_2Q are irreducible, $[r_1zR^1]$[2]$=[I_1]$ and $[r_2yR^1]$ $=[I_2]$. Hence λ sends r_1zR^1 isomorphically onto r_2yR^1.

If e is a primitive idempotent in R, then so is e in Q, hence eR is basic. But basic right ideals are not always principal even if R has the unit. For example, let K be a field and x be an independent over K and R_0 be the subring of elements in $K[x]$ without constant-term. If we put $R=EK+\cup(R_0)_n$ as in the first ex-

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1) Mr. Kanzaki kindly pointed out to me this proof.

2) aR^1 means the right ideal in R generated by a.

ample, then its quotient ring is $Q=EK+\bigcup K(x)_n$. Let $\mathfrak{r}=e_{11}Q$. If $\mathfrak{r}\cap R$ is principal: $\mathfrak{r}\cap R=\begin{pmatrix} f_1, f_2\cdots f_n \\ 0\cdots 0\cdots 0 \\ 0\cdots 0\cdots 0\end{pmatrix}R$, there exist $g_1,\cdots,g_n$ and $k\neq 0\in K$ such that $f_1(k+g_1)+\cdots+f_ng_n=x$, hence min. degree of $f_1=1$. On the other hand there exist $g'_1,\cdots,g'_n$ and $k'\neq 0\in K$ such that $f_1(k'_1+g'_1)+\cdots+f_ng'_n=0$, hence min. degree of $f_1(x)\geqq 2$. This is a contradiction. Next example shows that basic right ideals are not always mutually isomorphic. Let $R=(R_0)_n+e_{33}K+\cdots+e_{nn}K$. If an element x of R is not a zero-divisor in R then x is regular in $K(x)_n$, for the adjoint of x is in R. Let (x_{ij}), (y_{ij}) be elements of R, and suppose that (x_{ij}) is non zero-divisor. Then $(x_{ij})^{-1}(y_{ij})|x_{ij}|E=adj\ (x_{ij})\cdot(y_{ij})$ is in R, hence $(x_{ij})\,adj\ (x_{ij})\cdot(y_{ij})=(y_{ij})|x_{ij}|E$ and $|x_{ij}|E$ is a non zero divisor. Therefore R has the quotient ring $Q=K(x)_n$. $e_{11}Q\cap R$ is basic and not principal, because if $e_{11}Q\cap R=(e_{11}f_1+e_{12}f_2+\cdots+e_{1n}f_n)R$, $f_i\in R_0$ then $x=\sum_{i=1}^{n}f_ig_i$, $g_i\in R_0$ which is a contradiction. On the other hand $e_{33}Q\cap R=e_{33}R$ is basic and principal. Therefore $e_{11}Q\cap R$ is not isomorphic to $e_{33}R$.

PROPOSITION 6. *Any right ideal I in R contains a uniform right ideal in R.*

Proof. Since Q is P.M.I., IQ contains a minimal right ideal $\mathfrak{r}$ in Q, and further $(0)\neq I\cap\mathfrak{r}=I\cap\mathfrak{r}\cap R$ and $(I\cap\mathfrak{r}\cap R)Q=\mathfrak{r}$, hence $I\cap\mathfrak{r}\cap R$ is uniform.

PROPOSITION 7. *Let U be a uniform right ideal in R. Then*

$$U_l=\{x|\in R,\ x_r\cap U\neq(0)\}.$$

Proof. If $xu=0$ for any $u\in U$, then since UQ is irreducible, $UQ=uQ$, hence $xUQ=xuQ=(0)$. Therefore $x\in U_l$.

An element u in R is called right uniform if uR^1 is a uniform right ideal (equivalently if uR is uniform $(R_l=R_r=(0))$).

We can define similarly left uniform elements. But the left uniform elements coincide with the right uniform elements, because if u is left uniform, then $Qu=Qe$ is irreducible where e is a primitive idempotent, since Q is P.M.I., eQ is irreducible, hence $uQ=ueQ$ is also irreducible. Therefore u is right uniform, and the coverse is similar. Hence we may call right (left) uniform elements simply uniform elements.

PROPOSITION 8. *Let I be a right ideal in R. If there exists some uniform element u such that $u_r\cap I=(0)$, then I is uniform. Furthermore if R is prime, then the converse is true.*

Proof. If $u_r\cap I=(0)$, for any element $aq^{-1}\in u_r*\cap IQ$, $a\in I$ we have $ua=0$, hence $a\in I\cap u_r=(0)$ and so $u_r*\cap IQ=(0)$. Let θ be a mapping: $q\to uq$. Since $\theta^{-1}(0)\cap IQ=(0)$, we have a isomorphism $IQ\approx uQ$, hence I is uniform. Let R be prime and I be uniform. If $u_r\cap I\neq(0)$ for all element u in I, then $I^2=0$ by Proposition 7. This is a contradiction.

From the definition xU is uniform if U is so, hence the sum R_0 of all uniform right ideals is two sided ideal and R_0 is the sum of all uniform elements. Therefore R_0 coincides with the sum of all left uniform ideals. Furthermore R_0Q is the socle $\mathfrak{z}$ of Q. $R_0Q\subseteqq\mathfrak{z}$ and since $(\mathfrak{r}_i\cap R)Q=\mathfrak{r}_i$, for $x\in\mathfrak{z}$, $x\in\Sigma\mathfrak{r}_i$ and $x\in R_0Q$.

THEOREM 1. *The cardinal numbers of the maximal length of direct-sums of basic right ideals are equal. Further if Q is a sub-P.M.I. ring of $\mathfrak{L}_{\mathfrak{m}'}(\mathfrak{m})$ with Δ-dim$\mathfrak{m}=\Delta$-dim $\mathfrak{m}'$, then the cardinal numbers for basic left ideals coincide with ones for basic right ideals, where $\mathfrak{L}_{\mathfrak{m}'}(\mathfrak{m})$ is the ring of continuous endomorphisms of $\mathfrak{m}$, topologized by $\mathfrak{m}'$-topology, and Δ is the division ring of $\mathfrak{L}_{\mathfrak{m}'}(\mathfrak{m})$-endomorphisms of $\mathfrak{m}$.*

Proof. Let $B=\{B_\alpha\}$ be the set of basic right ideals. We can order direct-sums $S_j=\sum_{\alpha\in j}\oplus B_\alpha$ of elements B_α of B as follows:
$S_i>S_j$ if and only if $S_i=S_j\oplus\sum_{\alpha\in i-j}B_\alpha$. By the Zorn's Lemma there exists a maximal element S_0 in this order. Then S_0 meets all basic right ideals. If $S_0Q\subsetneqq\mathfrak{z}$ there exists a minimal right ideal $\mathfrak{r}_0$ such that $\mathfrak{r}_0\cap S_0Q=(0)$. Hence $(0)=R\cap\mathfrak{r}_0\cap S_0Q\supseteqq R\cap\mathfrak{r}_0\cap S_0$ and since $R\cap\mathfrak{r}_0$ is basic, it is a contradction. Therefore $S_0Q=\mathfrak{z}$. Since Q is P.M.I. the right dimension of $\mathfrak{z}$ is constant. It is also true for left basic ideals. Further if Q is as in Theorem, then the left dimension coincides with the right one.

THEOREM 2. *Let U be a uniform right ideal in R and $\varepsilon(U)$ be the R-endomorphism ring of U. Then non zero element of $\varepsilon(U)$ is non singular. $\varepsilon(U)$ has the right quotient division ring which is the Q-endomorphism ring of Q-irreducible module.*

Proof. If $\phi\in\varepsilon(U)$, then ϕ can be extended to a Q-endomorphism of UQ. Because if $uq^{-1}=u'q'^{-1}\in UQ$, then there exist p, s, d by (1) such that $q^{-1}=pd^{-1}$, $q'^{-1}=sd^{-1}$, hence $\phi(uq^{-1})=\phi(u)q^{-1}=\phi(u)pd^{-1}=\phi(up)d^{-1}=\phi(u's)d^{-1}=\phi(u')sd^{-1}=\phi(u')q'^{-1}$. Since UQ is irreducible, the Q-endomorphism ring of UQ is a division ring. Hence if ϕ is not zero, then ϕ is non singular. Let ψ be any Q-endomorphism of UQ. Then there exists y in UQ such that $\psi(y)=u\in U$; $y=u'x^{-1}$, $u'\in U$ and for any element w in U $\psi\lambda_{u'}w=\psi(u'w)=\psi(yxw)=uxw=\lambda_{ux}w$ where $\lambda_a: x\to ax$, $x\in R$. Hence $\psi=\lambda_{ux}\lambda_{u'}^{-1}$.

3. Complements and annihilators.

THEOREM 3. *Let B be basic then $B=B_{lr}$. A right ideal B in R is basic if and only if B is a minimal annihilator. A right ideal M in R is a maximal annihilator if and only if $M=u_r$ where u is a uniform element.*

Proof. Let B be basic, then $B=BQ\cap R$ and $BQ=eQ$, $e=e^2$. By (7) $B_{lr}=(BQ\cap R)_{lr}=BQ_{lr}\cap R=eQ_{lr}\cap R=eQ\cap R=B$. If $B\supseteqq L_r$ then $(QL)_r=L_rQ\subseteqq BQ$. Since BQ is irreducible $BQ=(QL)_r$. Hence $B=BQ\cap R=(QL)_r\cap R=L_r$. Therefore

B is a minimal annihilator. Let $I=L_r$ be a minimal annihilator. If $L_rQ\supseteqq L'^*_r$ for some subset L' in Q, then $L_r=L_rQ\cap R\supseteqq L'^*_r\cap R=\tilde{L}'_r$ by (6). Hence $L_r=\tilde{L}'_r$ and $L_rQ=\tilde{L}'_rQ=L'^*_r$. Therefore L_rQ is also a minimal annihilator. Let $\mathfrak{r}=eQ$ be an irreducible right ideal in Q contained in L_rQ. Then $eQ=(Q_{(1-e)})_r$ and since L_rQ is a minimal annihilator, $eQ=L_rQ$, hence $L_r=L_rQ\cap R$ is basic. Let M be a maximal annihilator. By (8) $MQ=\mathfrak{l}_r^*$ is so in Q. Let $\mathfrak{l}_0$ be an irreducible left ideal contained in $\mathfrak{l}$, then $Q\neq\mathfrak{l}_{0r}^*\supseteqq\mathfrak{l}_r^*$, hence $\mathfrak{l}_{0r}^*=\mathfrak{l}_r^*=MQ$. Therefore $M\subseteqq\mathfrak{l}_{0r}^*\cap R=(\mathfrak{l}_0\cap R)_r=B_r$ and B is basic. From Proposition 7 we have $B_r=u_r$ for any element u in B. Conversely if u is a uniform element, then Qu is irreducible, hence $(Qu)_r^*=u_r^*$ is a maximal right ideal. By (8) $u_r=u_r^*\cap R$ is a maximal annihilator.

THEOREM 4. *Let M be a right ideal in R. M is a maximal complement in R if and only if MQ is a maximal one of right ideals $\mathfrak{r}$ with $(\mathfrak{r}:Q)_r=(0)$ and $MQ\ \ R=M$ or if and only if $M=B^c$ where B is basic. Let M be a maximal complement in R. Then* (1) *for any basic right ideal B $M\supseteqq B$ or $M\cap B=(0)$,* (2) *M is minimal irreducible*[3], (3) *if M_0 is of the maximal length of direct-sum of basic right ideals contained in M, then there exists a basic right ideal B such that $M\oplus B$ is of the maximal length of direct-sum of basic right ideals in R and* (4) *M^c is basic. Maximal annihilators are maximal complements.*

Proof. Let M be a maximal complement in $R; M=I^c$. By (9) $MQ=(IQ)^{c'}$. Let $MQ\subseteqq\mathfrak{i}^c$. Since $(\mathfrak{i}\cap R)^{c'}=\mathfrak{i}^c\cap R\supseteqq M$, $MQ=(\mathfrak{i}^c\cap R)Q=\mathfrak{i}^c$, hence MQ is a maximal complement in Q, and $MQ\cap R=M$. Let $\mathfrak{r}$ be a right ideal with $(\mathfrak{r}:Q)_r=(0)$ and $\mathfrak{r}\supseteqq MQ$. Then since $\mathfrak{r}\not\supset\mathfrak{z}$ there exists a minimal right ideal $\mathfrak{r}_0$ such that $\mathfrak{r}\cap\mathfrak{r}_0=(0)$. Hence $\mathfrak{r}$ is contained in a maximal complement. Therefore $\mathfrak{r}=MQ$. Conversely if MQ satisfies the property mentioned in Theorem, then $MQ\not\supset\mathfrak{z}$ and $MQ\cap\mathfrak{r}_0=(0)$; $\mathfrak{r}_0$ a minimal right ideal, and $MQ\subseteqq\mathfrak{r}_0^c$. Since $(\mathfrak{r}_0^c:Q)=(0)$, $MQ=\mathfrak{r}_0^c$. If $MQ\subset\mathfrak{r}^c$, then $(\mathfrak{r}^c:Q)_r=(0)$. By (10) $M=MQ\cap R=(\mathfrak{r}_0^c\cap R)=(\mathfrak{r}_0\cap R)^{c'}$. Further if $M\subseteqq I^c$ then $MQ\subseteqq I^cQ=(IQ)^{c'}$, hence $MQ=I^cQ$. Therefore $M=I^c$, and M is a maximal complement. Let M be a maximal complement in R, then MQ is so in Q. Hence there exists a minimal right ideal $\mathfrak{r}_0$ such that $\mathfrak{r}_0\cap MQ=(0)$ and $MQ=\mathfrak{r}_0^c$. $M=MQ\cap R$ $=\mathfrak{r}_0^c\cap R=(\mathfrak{r}_0\cap R)^{c'}$ by (10) and $\mathfrak{r}_0\cap R$ is basic. Conversely let $M=B^c$. $MQ=B^cQ$ $=(BQ)^{c'}$ and BQ is minimal. If $\mathfrak{z}=(BQ^c\cap\mathfrak{z})\oplus BQ\oplus\mathfrak{r}_1$, where $\mathfrak{z}$ is the socle of Q, then for $y(\in(BQ^{c'}\oplus\mathfrak{r}_1)\cap BQ)=x_1+x_2$, $x_1\in BQ^{c'}$, $x_2\in\mathfrak{r}_1$, we have $x_1=y-x_2\in(BQ\oplus\mathfrak{r}_1)\cap BQ^{c'}\subseteqq\mathfrak{z}\cap(BQ)^{c'}$. Hence $((BQ)^{c'}\oplus\mathfrak{r}_1)\cap BQ=(0)$ and $\mathfrak{r}_1=(0)$. If $MQ\subseteqq\mathfrak{r}^c$, then $r^c\not\supset\mathfrak{z}$ hence $\mathfrak{r}^c\cap BQ=(0)$ and $\mathfrak{r}^c=MQ$. Therefore MQ is a maximal complement in Q and further $B^c=M\subseteqq MQ\cap R=(BQ)^{c'}\cap R=(BQ\cap R)^{c''}=B^{c''}$ and we have $M=MQ\cap R$. 1). Let B be basic. Since BQ is a minimal right ideal, $BQ\subseteqq MQ$ or

3) From this theorem a right ideal I is called irreducible if $I=M\cap N$ implies $I=M$ or $I=N$.

$BQ_{\frown}MQ=(0)$. Hence $B\subseteqq MQ_{\frown}R=M$ or $M_{\frown}B=(0)$. 2). If $M\subsetneqq N$, $M\subsetneqq S$, and $M=N_{\frown}S$, then $MQ\subsetneqq SQ$ for $MQ=SQ$ implies $M=S$. Hence $SQ\supseteqq\mathfrak{z}$ and $NQ\supseteqq\mathfrak{z}$. Therefore $NQ_{\frown}SQ\supseteqq\mathfrak{z}$ and this is a contradiction. If $M_0\subsetneqq M$, then $M_0=(M_0\oplus M^c)_{\frown}M$, hence M is minimal irreducible. From the above argument and the fact that $MQ_{\frown}R=M$, M is a maximal complement. Let $MQ_{\frown}\mathfrak{r}_0=(0)$ for a minimal right ideal $\mathfrak{r}_0$. Since $MQ\oplus\mathfrak{r}_0\supsetneqq MQ$, $MQ\oplus\mathfrak{r}_0\supseteqq\mathfrak{z}$. We define the right ideal

$$\mathfrak{j} = \{j\mid\in MQ,\text{ there exists an element } z\in\mathfrak{z} \text{ such that } z=j+r,\ r\in\mathfrak{r}_0\}.$$

Then $\mathfrak{j}\subseteqq\mathfrak{z}_{\frown}MQ$ and $\mathfrak{z}=\mathfrak{j}+\mathfrak{r}_0$, $\mathfrak{j}=\Sigma\oplus\mathfrak{r}_i$, $\mathfrak{r}_i$'s are minimal ideals. $M=MQ_{\frown}R\supseteqq\mathfrak{z}_{\frown}MQ_{\frown}R\supseteqq\mathfrak{j}_{\frown}R=\Sigma\mathfrak{r}_{i\frown}R$. If $(MQ)^c$ is not minimal, then it contains two minimal right ideals, $\mathfrak{r}_1$, $\mathfrak{r}_2$ by Lemma 1. Hence $MQ_{\frown}(\mathfrak{r}_1\oplus\mathfrak{r}_2)=(0)$ and $(MQ\oplus\mathfrak{r}_1)_{\frown}\mathfrak{r}_2=(0)$. Therefore since $(MQ)^c$ is minimal and $(MQ)^c=M^{c'}Q$, $M^{c'}$ is uniform and by (1) $M^{c'}$ is basic. Let M' be a maximal annihilator. By (8) $M'Q=\mathfrak{l}_r{}^*$ is so in Q. If $\mathfrak{l}_0=Qe$ is a minimal left ideal in $\mathfrak{l}$, then $\mathfrak{l}_r{}^*=\mathfrak{l}_{0r}{}^*=(1-e)Q$ and $\mathfrak{l}_r{}^*{}_{\frown}eQ=(0)$. Since $\mathfrak{l}_r{}^*$ is maximal, $\mathfrak{l}_r{}^*$ is a maximal complement.

The following example with field $Q/\mathfrak{z}$ analogous to the first one in this note shows that a maximal complement is not always a maximal annihilator. Let $\mathfrak{r}$ be the right ideal generated by elements $e_{11}+e_{21}$, $e_{22}+e_{32},\cdots$. Since $(\mathfrak{m}/\mathfrak{r}\mathfrak{m}:\Delta)=1$, $\mathfrak{r}$ is a maximal right ideal contained in $\mathfrak{z}$, where $\mathfrak{m}$ is an irreducible Q-module and Δ is its Q-endomorphism ring. If $\mathfrak{r}^*\supsetneqq\mathfrak{r}$ then an element x of $\mathfrak{r}^*-\mathfrak{r}$ is of the following from

$$x = x_1+\alpha E\,,\quad \alpha\in\Delta \quad\text{and}\quad x_1\in\mathfrak{z}\,.$$

If $\alpha\neq0$, then $xe_{ii}=\alpha e_{ii}\in\mathfrak{r}^*$ for a sufficiently large i. Hence $\mathfrak{r}^*\supseteqq\mathfrak{z}$. If $\alpha=0$, then $x\in\mathfrak{z}$. Therefore $\mathfrak{r}^*\supseteqq\mathfrak{z}$. From Theorem 4 $R_{\frown}\mathfrak{r}$ is a maximal complement but not a maximal annihilator since $\mathfrak{r}$ is not maximal. Furthermore in this ring R if a right ideal M is minimal irreducible and $M=MQ_{\frown}R$, then M is maximal complement. Because if M is minimal irreducible then MQ is so in Q. Since $\mathfrak{r}$ is minimal irreducible $MQ\not\supset\mathfrak{z}$, hence $MQ\subseteq\mathfrak{r}_0^c$ for some minimal right ideal $\mathfrak{r}_0$. If $\mathfrak{r}_0^c\supsetneqq MQ$ then $MQ=\mathfrak{r}_0^c{}_{\frown}(MQ\oplus\mathfrak{r}_0)$ is not irreducible. Hence $MQ=\mathfrak{r}_0^c$ and by the first mention in the proof MQ is a maximal complent.

THEOREM 5. *If Q satisfies the minimal conditions, then the complement right ideals coincide with the annihilator right ideals. A right ideal M is a maximal complement if and only if M is minimal irreducible and M conatins no regular elements.*

Proof. Let $I=J^c$ be a complement right ideal. $IQ=J^cQ=(JQ)^{c'}=(eQ)^{c'}=(1-e)Q=(Qe)_r$ where $e^2=e$, $JQ=eQ$, because Q is a simple ring with minimal conditions. On the other hand if $IQ_{\frown}R=I'\supsetneqq I$ then $I'_{\frown}J\neq(0)$, hence $(0)\neq I'Q_{\frown}JQ$ which is a contradiction. Therefore $I=J^c=J^cQ_{\frown}R=(Qe)_{r\frown}R=(Qe_{\frown}R)_r$. Conversely if $I=J_r$ then $J_r=J_r{}^*{}_{\frown}R=(eQ)^c{}_{\frown}R$ where $J_r{}^*=(1-e)Q$. By (10)

$J_r = J_r^* \cap R = (eQ)^c \cap R = (eQ \cap R)^c$. Let M be a minimal irreducible right ideal with $MQ \neq Q$. Then there exists a maximal right ideal $\mathfrak{r}$ which contains MQ, $\mathfrak{r} \cap R \supseteqq M$ and since from Theorem 4 $\mathfrak{r} \cap R$ is minimal irreducible, $M = \mathfrak{r} \cap R$ is a maximal complement.

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