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On some homogeneous spaces of classical Lie groups

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We shall, in this paper, give cellular decompositions of homogeneous spaces V_n, W_n and X_n of classical Lie groups $O(n)$, $U(n)$ and $Sp(n)$ by their diagonal subgroups respectively.

1. We denote by F one of three fields of real numbers R, complex numbers C or quaternion numbers Q, and by $d=d(F)$ the dimension of F over R; $d(R)=1$, $d(C)=2$ and $d(Q)=4$. Let F^n be a right vector space of dimension n over F and e_i $(i=1, \cdots, n)$ be the element of F^n whose i-th component is 1 and the others are 0. F^{n-1} is embedded in F^n as a vector subspace whose last component is 0.

Denote by $G(n)$ one of three classical Lie groups $O(n)$ (orthogonal group), $U(n)$ (unitary group) and $Sp(n)$ (symplectic group). $G(n)$ operates on F^n in the natural sense. $G(n-1)$ may be regarded as a subgroup of $G(n)$ by extending a point A of $G(n-1)$ to $G(n)$ by requirement that $Ae_n=e_n$.

The diagonal subgroup $D(n)$[1] of $G(n)$ is isomorphic to the product group $S^{d-1}\times \cdots\times S^{d-1}$ (n-fold), where S^{d-1} is the unit sphere in F which is a group. We define K_n to be $G(n)/D(n)$. Then we have $G(n-1)/D(n-1)=G(n-1)\times D(1)/D(n)\subset G(n)/D(n)$. Thus we have a sequence

$$K_1\subset K_2\subset \cdots \subset K_n\subset \cdots .$$

In the natural sense $G(n)$ operates on K_n, i.e. for $g\in G(n)$ and $a\in K_n$, we have $ga\in K_n$.

K_n is denoted by V_n, W_n or X_n, according as the field F is real, complex or quaternionic respectively.

Let Ω_{n-1} be the $d(n-1)$-dimensional projective space over F. If a point x of Ω_{n-1} has a representative $x=[x_1, \cdots, x_n]$, where $x_1, \cdots, x_n$ are, not all zero, in F, then the other representatives are $x=[x_1a, \cdots, x_na]$, where a is any non zero element of F. Hence we can choose a representative $x=[x_1, \cdots, x_n]$ such that $|x_1|^2+\cdots+|x_n|^2=1$. Now, if we define a mapping

$$\iota:\ \Omega_{n-1}\longrightarrow G(n)$$

by the formula

$$\iota([x_1, \cdots, x_n])=(\delta_{ij}-2x_i\bar{x}_j)\,, \qquad i, j=1, \cdots, n,$$

1) In the case $G(n)=U(n)$, $D(n)$ is a maximal torus of $U(n)$.

then ι is homeomorphic into, (see [2]). Hence we may consider Ω_{n-1} is a subset of $G(n)$. Ω_{n-1} is embedded in Ω_n as a subspace whose last component is 0.

The basic tools used here are indicated by the commutative diagram

$$\begin{tikzcd} \Omega_{n-1} \arrow[r, "\iota"] & G(n) \arrow[r, "r"] \arrow[d, "p"'] \arrow[dr, "\mu"] & S_F^{dn-1} \arrow[d, "\lambda"] \\ & K_n \arrow[r, "q"'] & \Omega_{n-1} \end{tikzcd}$$

where S_F^{dn-1} is the unit sphere in F^n and $(G(n), r, S_F^{dn-1}; G(n-1))$[2] $(G(n), p, K_n; D(n))$, $(S_F^{dn-1}, \lambda, \Omega_{n-1}; S^{d-1})$, $(K_n, q, \Omega_{n-1}; K_{n-1})$, and $(G(n), \mu, \Omega_{n-1}; G(n-1)\times S^{d-1})$ are the familiar fibre spaces.

2. Let E_F^{dn} be the unit cell in F^n of $x=(x_1, \cdots, x_n)$ such that $|x|=\sqrt{|x_1|^2+\cdots+|x_n|^2}\leqq 1$ and $\mathcal{E}_F^{dn}$ be $E_F^{dn}-(E_F^{dn})^{\bullet}$. Let D_F^{dn} be the subset of E_F^{dn} of x such that $1/\sqrt{2}\leqq |x|=\sqrt{|x_1|^2+\cdots+|x_n|^2}\leqq 1$ and S_+^{dn-1}, S_-^{dn-1} be the subsets of D_F^{dn} consisting of x such that $|x|=1/\sqrt{2}$, $|x|=1$ respectively. Then D_F^{dn} is homeomorphic to $S^{dn-1}\times I$.[3] And define $\mathfrak{D}_F^{dn}$ by the set $D_F^{dn}-(S_+^{dn-1}\cup S_-^{dn-1})$.

Now, we define mappings

$$\tilde{f}_{n-1,n}:\ E_F^{d(n-1)} \longrightarrow G(n)$$

and

$$f_{n-1,n}:\ E_F^{d(n-1)} \longrightarrow K_n$$

by setting

$$\tilde{f}_{n-1,n}(x) = (\delta_{ij}-2x_i\bar{x}_j)\,, \qquad i, j=1, \cdots, n,$$

where $x=(x_1, \cdots, x_{n-1})\in E_F^{d(n-1)}$ and $x_n=\sqrt{1-|x|^2}$,
and

$$f_{n-1,n} = p\circ \tilde{f}_{n-1,n}\,.$$

If a mapping $\xi_F:\ D_F^{d(n-1)} \longrightarrow \Omega_{n-1}$ is defined by

$$\xi_F = q\circ f_{n-1,n}\,,$$

then we have the following

LEMMA 2.1. *ξ_F maps $\mathfrak{D}_F^{d(n-1)}$ homeomorphically onto $\Omega_{n-1}-(\Omega_{n-2}\cup\omega_0)$ and maps $S_-^{d(n-1)-1}$ to Ω_{n-2}, $S_+^{d(n-1)-1}$ to ω_0 respectively, where ω_0 is a point $[0, \cdots, 0, 1]$ of Ω_{n-1}.*

Proof. In the formula

$$\xi_F(x) = \xi_F((x_1, \cdots, x_n)) = [-2x_1x_n\,, \cdots, -2x_{n-1}x_n, 1-2x_n^2]\,,$$

2) $(E, p, B; F)$ indicates a fibre space with total space E, base space B, fibre F and projection p.

3) S^{dn-1} is a $(dn-1)$-sphere and I is $[0, 1]$ interval.

we have $0<1-2x_n^2<1$ for $x\in\mathfrak{D}_F^{d(n-1)}$. Hence for a point a of $\Omega_{n-1}-(\Omega_{n-2}\cup\omega_0)$, we can determine $x_1,\cdots,x_{n-1}$ from $\xi_F(x)=a$ uniquely and continuously with respect to a. The last two assertions are obvious, since $1-2x_n^2=0$ for $x\in S_-^{d(n-1)-1}$ and $x_n=0$ for $x\in S_+^{d(n-1)-1}$.

From this lemma, we see that $\tilde{f}_{l-1,l}$ (resp. $f_{l-1,l}$) maps $\mathfrak{D}_F^{d(l-1)}$ homeomorphically into $G(l)\subset G(n)$ (resp. $K_l\subset K_n$) for $n\geqq l\geqq 3$, Put

$$\varepsilon_{l-1,l}^{d(l-1)}=f_{l-1,l}(\mathfrak{D}_F^{d(l-1)}),$$

and

$$e_{1.2}^d=f_{1.2}(\mathcal{E}_F^d).$$

We shall call $\varepsilon_{l-1,l}^{d(l-1)}$ $(n\geqq l\geqq 3)$ the (quasi)-primitive cell and $e_{1.2}^d$ the primitive cell of K_n.

3. Remember that Ω_{n-2} is a cell complex composed of cells u^0, u^d, $\cdots$, $u^{d(n-2)}$, where u^{dk} is given as the image of $\mathcal{E}_F^{dk}$ by the characteristic mapping φ_k for u^{dk}

$$\varphi_k:\ E_F^{dk}\longrightarrow\Omega_{n-2},$$
$$\varphi_k(x)=[x_1,\cdots,x_{k+1},0,\cdots,0],$$

where $x=(x_1,\cdots,x_k)\in E_F^{dk}$ and $x_{k+1}=\sqrt{1-|x|^2}$.

Now, for $n-2\geqq k\geqq 0$, we define mappings

$$\tilde{f}_{k,n}:\ E_F^{dk}\longrightarrow G(n)$$

and

$$f_{k,n}:\ E_F^{dk}\longrightarrow K_n$$

by setting

$$\tilde{f}_{k,n}(x)=\left(\begin{array}{c:c:c}\delta_{ij}-x_i\bar{x}_j & 0 & \begin{matrix}x_1\\ \vdots\\ x_{k+1}\end{matrix}\\ \hdashline 0 & \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} & 0\\ \hdashline \bar{x}_1,\cdots,\bar{x}_{k+1} & 0 & 0\end{array}\right)\quad i,j=1,\cdots,k+1,$$

where $x=(x_1,\cdots,x_k)\in E_F^{dk}$ and $x_{k+1}=\sqrt{1-|x|^2}$,

and

$$f_{k,n}=p\circ\tilde{f}_{k,n}.$$

Obviously we have the following

LEMMA 3.1. $$\varphi_k=q\circ f_{k,n}.$$

From this lemma, we see that $\tilde{f}_{k,l}$ (resp. $f_{k,l}$) maps $\mathcal{E}_F^{d_k}$ homeomorphically into $G(l)\subset G(n)$ (resp. $K_l\subset K_n$) for $l-2\geqq k\geqq 0$. Put

$$e_{k,l}^{dk}=f_{k,l}(\mathcal{E}_F^{dk}).$$

We shall call $e_{k,l}^{dk}$ $(n-2\geqq l-2\geqq k\geqq 1)$ the primitive cell of K_n.

4. For integers $k_1, \cdots, k_j$; $l_1, \cdots, l_j$ such that $n \geqq l_1 > \cdots > l_j \geqq 2$ and $l_i \geqq k_i \geqq 0$ $(i=1, \cdots, j)$, we shall define a mapping

$$f_{k_1, \cdots, k_j ; l_1, \cdots, l_j} : {}'E_F^{dk_1} \times \cdots \times {}'E_F^{dk_j} \longrightarrow K_n$$

by setting

$$f_{k_1, \cdots, k_j ; l_1, \cdots, l_j}(y_1, \cdots, y_j) = \tilde{f}_{k_1, l_1}(y_1) \cdots \tilde{f}_{k_{j-1}, l_{j-1}}(y_{j-1}) f_{k_j, l_j}(y_j)$$

where ${}'E_F^{dk_i}$ indicates one of either $D_F^{dk_i}$ or $E_F^{dk_i}$. Put

$$\varepsilon_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)} = f_{k_1, \cdots, k_j ; l_1, \cdots, l_j}({}'\mathcal{E}_F^{dk_1} \times \cdots \times {}'\mathcal{E}_F^{dk_j})^{4)}$$

and

$$e_{0,1}^0 = K_1 ,$$

where ${}'\mathcal{E}_F^{dk_i}$ indicates one of eithar $\mathfrak{D}_F^{dk_i}$ or $\mathcal{E}_F^{dk_i}$.

LEMMA 4.1. *K_n is the union of the subsets $e_{0,1}^0$ and $\varepsilon_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)}$ with $n \geqq l_1 > \cdots > l_j \geqq 2$ and $l_i \geqq k_i \geqq 0$ $(i=1, \cdots, j)$.*

Proof. Since $K_1 = e_{0,1}^0$ and K_2 (which is d-dim projective space $\Omega_1 = S^{d-1}) = e_{0,1}^0 \cup e_{1,2}^d$, we shall assume that the assertion is valid for K_{n-1}. Suppose that $a \in K_n$ but $a \notin K_{n-1}$ (i.e. $q(a) \neq \omega_0$). In the case of $q(a) \notin \Omega_{n-2}$, we can choose a point $y \in \mathfrak{D}_F^{d(n-1)}$ such that $\xi_F(y) = q(a)$ by Lemma 2.1. Put $U = \tilde{f}_{k,n}(y)$. In the case of $q(a) \in \Omega_{n-2}$, $q(a)$ belongs to some cell u^{dk} of Ω_{n-2}, hence we can choose a point $y \in \mathcal{E}_F^{dk}$ such that $\varphi_k(y) = q(a)$. Put $U = \tilde{f}_{k,n}(y)$. In either cases, $U^* a \in K_{n-1}$. By the assumption of the induction, $U^* a$ belongs to some subset $\varepsilon_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)}$ with $n-1 \geqq l_1 > \cdots > l_j \geqq 2$ and $l_i > k_i \geqq 0$ (or to $e_{0,1}^0$) of K_{n-1}. Therefore a belongs to $\varepsilon_{n-1, k_1, \cdots, k_j ; n, l_1, \cdots, l_j}^{d(n-1+k_1 + \cdots + k_j)}$ in the first case and to $\varepsilon_{k, k_1, \cdots, k_j ; n, l_1, \cdots, l_j}^{d(k+k_1 + \cdots + k_j)}$ in the second case.

LEMMA 4.2. *The subsets in the preceeding lemma are disjoint to each other and $f_{k_1, \cdots, k_j ; l_1, \cdots, l_j}$ maps ${}'\mathcal{E}_F^{dk_1} \times \cdots \times {}'\mathcal{E}_F^{dk_j}$ homeomorphically onto $\varepsilon_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)}$.*

Proof. If $U_1 U_2 \cdots U_{s-1} a_s = V_1 V_2 \cdots V_{t-1} b_t$, where $U_m \in \tilde{f}_{k_m, l_m}({}'\mathcal{E}_F^{dk_m})$, $a_s \in f_{k_s, l_s}({}'\mathcal{E}_F^{dk_s})$ and if $m > m'$ then $l_m < l_{m'}$ and V_m, b_t are also similar ones, then $q(U_1 U_2 \cdots U_{s-1} a_s) = q(V_1 V_2 \cdots V_{t-1} b_t)$. This follows $\mu(U_1) = \mu(V_1)$. Since $\mu(U) = q \circ p \circ f(y) = q \circ f(y) = \xi(y)$ or $\varphi(y)$ for some $y \in {}'\mathcal{E}_F$ and ξ or φ is homeomorphic, it follows $U_1 = V_1$. Hence we have $U_2 \cdots U_{s-1} a_s = V_2 \cdots V_{t-1} b_t$. Analogously $U_2 = V_2$ and so on. Consequently we have $s = t$ and $a_s = b_t$. This proves that these subsets are disjoint and $f_{k_1, \cdots, k_j ; l_1, \cdots, l_j}$ is one-to-one. The fact that $f_{k_1, \cdots, k_j ; l_1, \cdots, l_j}$ is homeomorphism is obvious from Lemmas 2.1 and 3.1.

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4) If $l_i - 2 \geqq k_i \geqq 0$ $(i=1, \cdots, j)$, then $\varepsilon_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)}$ will be written by $e_{k_1, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_1 + \cdots + k_j)}$, and also by $e_{k_1, l_1}^{dk_1} \cdot e_{k_2, \cdots, k_j ; l_1, \cdots, l_j}^{d(k_2 + \cdots + k_j)}$.

Furthermore, it will be readily verified the following

LEMMA 4.3. *$f_{k_1,\cdots,k_j;l_1,\cdots,l_j}$ maps the boundary of $'E_F^{dk_1}\times\cdots\times'E_F^{dk_j}$ to the lower dimensional skelton of K_n than $d(k_1+\cdots+k_j)$.*

5. Since the quasi-primitive cell $\varepsilon_{l-1,l}^{d(l-1)}$ is not a cell in the natural sense, the above construction does not give a cell structure in the sense of J. H. C. Whitehead [1]. In order to correct this, we shall decompose $\varepsilon_{l-1,l}^{d(l-1)}$ into two cells $e_{l-1,l}^{d(l-1)}$ and $e_{l-1,l}^{1}$. The details are the following.

Let $\mathfrak{D}^1$ be the subset of $\mathfrak{D}_F^{dn}$ consisting of real numbers $x=(x,0,\cdots,0)$ such that $1/\sqrt{2}<x<1$ and $'\mathfrak{D}_F^{dn}$ be $\mathfrak{D}_F^{dn}-\mathfrak{D}^1$. Put

$$e_{l-1,l}^{d(n-1)}=f_{l-1,l}('\mathfrak{D}_F^{d(l-1)})$$

and

$$e_{l-1,l}^{1}=f_{l-1,l}(\mathfrak{D}^1)\,.$$

Then we have $\varepsilon_{l-1,l}^{d(l-1)}=e_{l-1,l}^{d(l-1)}\cup e_{l-1,l}^{1}$. $e_{l-1,l}^{d(l-1)}$ $(n\geqq l\geqq 3)$ will be also called the primitive cell of K_n.

Now, each time when $\varepsilon_{k_1,\cdots,k_j;l_1,\cdots,l_j}^{d(k_1+\cdots+k_j)}$ contains the suffix such that $k_i=l_i-1$, $l_i\geqq 3$, we decompose it into two disjoint subset $'\varepsilon_{k_1,\cdots,k_i,\cdots,k_j;l_1,\cdots,l_j}^{d(k_1+\cdots+k_j)}$ and $\varepsilon_{k_1,\cdots,\dot{k}_i,\cdots,k_j;l_1,\cdots,l_i,\cdots,l_j}^{d(k_1+\cdots+\dot{k}_i+\cdots+k_j)+1}$. This process decomposes the subset ε into the union of disjoint cells. Thus we have a cellular decomposition of K_n.

6. Let K_n^0 be the abstract cell complex which is composed of $e_{0,1}^0$ and $e_{k_1,\cdots,k_j;l_1,\cdots,l_j}^{d(k_1+\cdots+k_j)}$ which is the product of primitive cells of K_n. Then we have

LEMMA. 6.1. *The injection $i: K\to K_n$ induces an isomorphism*

$$i_*:\ H(K_n^0;\,Z)\longrightarrow H(K_n;\,Z)^{5)}\,.$$

Proof. It will be readily verified that the boundary of the chain $e_{k_1,\cdots,k_j;l_1,\cdots,l_j}^{d(k_1+\cdots+k_j)}$ of K_n is independent of e_0^0 and $e_{l-1,l}^1$ $(n\geqq 3)$. We shall orient the cell $e_{l-1,l}^1$ such that $\partial e_{l-1,l}^1=e_{0,l}^0-e_{0,1}^0$. Now we define a chain map $\rho: K_n\to K_n^0$ by

$$\begin{cases}\rho(e_{0,l}^0)=e_{0,1}^0 & \text{for }\ n\geqq l\geqq 3\,,\\ \rho(e_{l-1,l}^1)=0 & \text{for }\ n\geqq l\geqq 3\,,\\ \rho(e_{0,1}^0)=e_{0,1}^0\,, & \\ \rho(e_{k,l}^{dk})=e_{k,l}^{dk} & \text{for the primitive cell } e_{k,l}^{dk}\,,\\ \rho(e_{k_1,\cdots,k_j;l_1,\cdots,l_j}^{d(k_1+\cdots+k_j)})=\rho(e_{k_1,l_1}^{ek_1})\cdot\rho(e_{k_2,\cdots,k_j;l_2,\cdots,l_j}^{d(k_2+\cdots+k_j)})\,. & \end{cases}$$

To see that i_* is an isomorphism, we shall construct a chain homotopy D by setting

5) Z is a free cyclic group with one generator.

$$\begin{cases} D(e^0_{0,l}) = -e^1_{l-1,l} & \text{for } n \geqq l \geqq 3, \\ D(e^0_{0,1}) = D(e^1_{l-1,l}) = 0 & \text{for } n \geqq l \geqq 3, \\ D(e^{dk}_{k,l}) = 0 & \text{for the primitive cell } e^{dk}_{k,l}, \\ D(e^{d(k_1+\cdots+k_j)}_{k_1,\cdots,k_j;\, l_1,\cdots,l_j}) = De^{dk_1}_{k_1,l_1}\cdot e^{d(k_2+\cdots+k_j)}_{k_2,\cdots,k_j;\, l_2,\cdots,l_j} & \\ \qquad\qquad +(-1)^{dk_1} i\circ\rho(e^{dk_1}_{k_1,l_1})\cdot D(e^{d(k_2+\cdots+k_j)}_{k_2,\cdots,k_j;\, l_2,\cdots,l_j}). & \end{cases}$$

Then, we have directly

$$\partial D + D\partial = i\circ\rho - 1.$$

Thus the lemma is completed.

If we take, as a coefficient group, Z_2[6] for V_n and Z for W_n and X_n, then it will be readily verified that the boundary homomorphism is trivial for the chain which is represented as the product of primitive cells. Therefore by Lemma 6.1, we have

THEOREM 6.1. *V_n is a cell complex having $\frac{n(n-1)}{2}$ primitive cells $e^k_{k,l}$ with $n \geqq l > k \geqq 1$. And its Poincaré polynomial mod 2 is*

$$_2P_{V_n}(t) = (1+t)(1+t+t^2)\cdots(1+t+t^2+\cdots+t^{n-1}).$$

THEOREM 6.2. *W_n (resp. X_n) is a cell complex having $\frac{n(n-1)}{2}$ primitive cells $e^{2k}_{k,l}$ (resp. $e^{4k}_{k,l}$) with $n \geqq l > k \geqq 1$. And W_n (resp. X_n) has no torsion and its Poincaré polynomial is*

$$P_{W_n}(t) = (1+t^2)(1+t^2+t^4)\cdots(1+t^2+t^4+\cdots+t^{2n-2})$$
$$(\textit{resp. } P_{X_n}(t) = (1+t^4)(1+t^4+t^8)\cdots(1+t^2+t^8+\cdots+t^{4n-4})).$$

7. In the case V_n, we shall compute the boundary formula the more details. We orient each cell $u^k (n-1 \geqq k \geqq 0)$ of the $(n-1)$-dimensional real projective space P_{n-1} such that

$$\partial u^k = \begin{cases} 0 & \text{for } k \text{ is odd,} \\ 2u^{k-1} & \text{for } k \text{ is even.} \end{cases}$$

Next, we orient the cells $e^k_{k,l}\ (n-1 \geqq l-1 > k \geqq 0)$ and $e^{l-1}_{l-1,l}\ (n \geqq l \geqq 2)$ such that q preserves the orientations.

LEMMA 7.1.

$$\partial e^k_{k,l} = \begin{cases} 0 & \text{for } k \text{ is odd and } k < l-1, \\ 2e^{k-1}_{k-1,l} & \text{for } k \text{ is oven and } k < l-1, \end{cases}$$

$$\partial e^l_{l,l+1} = \begin{cases} 0 & \text{for } l \text{ is odd,} \\ 2e^{l-1}_{l-1,l+1} \pm 4e^{l-1}_{l-1,l} & \text{for } l \text{ is even,} \end{cases}$$

and

6) Z_2 is a cyclic group of order 2.

$$\partial e^{k_1+\cdots+k_j}_{k_1,\cdots,k_j;l_1,\cdots,l_j} = \sum_{i=1}^{j} (1+(-1)^{k_i})(e^{k_1\cdots+k_i-1+\cdots+k_j}_{k_1,\cdots,k_i-1,\cdots,k_j;l_1,\cdots,l_j} \pm 2\varepsilon_i e^{k_1+\cdots+k_i-1+\cdots k_j}_{k_1,\cdots,k_i-1,\cdots,k_j;l_1,\cdots,l_i-1,\cdots,l_j})$$

where $$\varepsilon_i = \begin{cases} 0 & \text{for } k_i < l_i - 1\,, \\ 1 & \text{for } k_i = l_i - 1\,. \end{cases}$$

Proof. The mapping degree of $r\circ\tilde{f}_{l-1,l}: S^{l-1}_{+} \to S^{l-1}_{R}$

$$r\circ\tilde{f}_{l-1,l}(x_1,\cdots,x_l) = (-2x_1x_l,\cdots,-2x_1x_l,1-2x_l^2)$$

is 0 if l is odd and ± 2 if l is even. Furthermore the mapping degree $\lambda: S^{l-1}_{R} \to P_{l-1}$

$$\lambda(y_1,\cdots,y_l) = [y_1,\cdots,y_l]$$

is also 0 if l is odd and ± 2 if l is even. Hence the mapping degree of $\lambda\circ r\circ\tilde{f}_{l-1,l} = q\circ f_{l-1,l}$ is 0 if l is odd and ± 4 if l is even. Therefore the cell $e^{l}_{l,l+1}$ is attached to $e^{l-1}_{l-1,l}$ by the degree 0 or 4. The rest of the lemma will be obvious.

THEOREM 7.1. *V_n has only torsion of order* 2.

References

[1] T. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc., 55 (1949) 1–28.
[2] I. Yokota, *On the homology of classical Lie groups*, Jour. Inst. Poly. Osaka City Univ., vol. 8 (1957) 93–120.