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Cohomology of symmetric products

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This paper is devoted to a study of the cohomology groups of the n-fold symmetric product $SP_n(X)$ of a finite simplicial complex X, where $n=1,2,\cdots$. Our main success is as follows.

Since points of $SP_n(X)$ are represented by unordered sets $\{x_1, x_2, \cdots, x_n\}$ with $x_i \in X$, we shall define an into-homeomorphism $\iota_{m,n}: SP_m(X) \to SP_n(X)$ by $\iota_{m,n}\{x_1, x_2, \cdots, x_m\} = \{x_1\ x_2, \cdots, x_m, *, \cdots, *\}$, where $m \leqq n$ and $*$ is a base ponit of X. Let G be any coefficient group for cohomology groups, and consider for any q the homomorphism $\iota^*_{m,n}: H^q(SP_n(X); G) \to H^q(SP_m(X); G)$ induced by $\iota_{m,n}$. We have then the following which is an extension of the result due to S. D. Liao [6]: The homomorphism $\iota^*_{m,n}$ has the right inverse, to be denoted by $\mu_{m,n}$ (i.e. $\iota^*_{m,n}\ \mu_{m,n}$=the identity isomorphism of $H^q(SP_m(X), G)$), and therefore it follows that $\iota^*_{m,n}$ is an onto-homomorphism, and that $H^q(SP_n(X); G)$ has a subgroup isomorphic with $H^q(SP_m(X); G)$ for any $m \leqq n$. The construction of the homomorphism $\mu_{m,n}$ is based on a theorem stated as follows: Denote by $\overline{SP}_m(X)$ a space which is obtained from $SP_m(X)$ by identifying the image of $\iota_{m-1,m}$ to a point, then we have that $H^q(SP_m(X); G)$ is isomorphic with the direct sum $\sum_{l=1}^{m} H^q(\overline{SP}_l(X); G)$. We call $\overline{SP}_n(X)$ the reduced n-fold symmetric product of X.

Next, assuming that X is homologically $(r-1)$-connected*), we study the integral cohomology group of $\overline{SP}_n(X)$. For this purpose we utilize the Cartan-Leray spectral sequence of regular finite covering [2, 3]. As a result, we obtain that the homomorphism $\iota^*_{n-1,n}: H^q(SP_n(X)) \to H^q(SP_{n-1}(X))$ is isomorphic into for $q \leqq r+1$ if $n'=1$, and for $q \leqq \mathrm{Min}\ (r+n'-2, 2r+1)$ if $n'>1$, where $n=2^e n'$ ($e \geqq 0$, n': odd). This gives especially that $\iota^*_{1,n}: H^q(SP_n(X)) \approx H^q(X)$ for $q \leqq r+1$.

We have calculated in [7] the cohomology of the 2- or 3-fold symmetric product of an r-sphere S^r. At the present paper, some of the results stated there will be again proved by a different method from the preceding. We show also that $H^{r+2}(SP_n(S^r))=0$ for $r \geqq 3$ and $n \geqq 1$.

1. Special cohomology groups

Consider a connected, finite simplicial complex X. We denote by

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*) The space Y is said to be homologically $(r-1)$-connected if the integral homology groups $H_q(Y)=0$ for $0<q<r$ and the reduced homology group $\widetilde{H}_0(Y)=0$. If a simply connected space Y is homologically $(r-1)$-connected, then Y is $(r-1)$-connected in the usual sense.

$$P_n(X)^{1)} \qquad (n \geqq 1)$$

the n-fold cartesian product of X. As usual, points of P_n are represented by ordered sets

$$(x_1, x_2, \cdots, x_n) \qquad (x_i \in X).$$

Suppose now that X is ordered. Then a natural simplicial decomposition of P_n is introduced as follows [4, 6]: A point $w=(x_1, x_2, \cdots, x_n)$ is a vertex of the simplicial decomposition if and only if each x_i is a vertex v_i of X; Different $(q+1)$ vertices $w_0 = (v_{01}, v_{02}, \cdots, v_{0n})$, $w_1 = (v_{11}, v_{12}, \cdots, v_{1n}), \cdots, w_q = (v_{q1}, v_{q2}, \cdots, v_{qn})$ form a q-dimensional simplex if and only if, for each $k=1, 2, \cdots, n$, $(q+1)$ vertices $v_{0k}, v_{1k}, \cdots, v_{qk}$ are contained in a simplex of X and it holds that $v_{0k} \leqq v_{1k} \leqq \cdots \leqq v_{qk}$ with respect to the order $<$ in X. Throughout this paper, P_n will be always considered with this decomposition.

Denote by

$$\mathfrak{S}_n$$

the symmetric group of the letter $1, 2, \cdots, n$. Each $\alpha \in \mathfrak{S}_n$ yields a transformation

$$\alpha : P_n \to P_n$$

defined by

$$\alpha(x_1, x_2, \cdots, x_n) = (x_{\alpha(1)}, x_{\alpha(2)}, \cdots, x_{\alpha(n)}).$$

Therefore $\mathfrak{S}_n$ may be regarded as a transformation group acting on P_n. The orbit space $O(P_n; \mathfrak{S}_n)$ ove: P_n relative to $\mathfrak{S}_n$[2] is called the *n-fold symmetric product* of X, and is denoted by

$$SP_n(X)^{1)}$$

in the present paper. Write

$$\varphi_n : P_n \to SP_n$$

for the identification map, and put

$$\{x_1, x_2, \cdots, x_n\} = \varphi_n(x_1, x_2, \cdots, x_n).$$

Then points of SP_n are represented by unordered sets $\{x_1, x_2, \cdots, x_n\}$ with $x_i \in X$.

As is noted in [6], every transformation $\alpha : P_n \to P_n (\alpha \in \mathfrak{S}_n)$ is simplicial, and if a simplex of P_n is mapped onto itself by α then it remains point-wise fixed under α. Therefore it follows easily that the identification map φ_n carries the simplicial decomposition of P_n naturally to a cellular decomposition of SP_n.[3]

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1) For the sake of brevity, X in this notation will be usually omitted if there is no confusion.

2) Let Y be a Hausdorff space on which a group Γ acts. Then the orbit space $O(Y; \Gamma)$ over Y relative to Γ is defined as a space obtained from Y by identifying each point $y \in Y$ with its image $\gamma(y) (\gamma \in \Gamma)$.

3) As is shown by simple examples, φ_n does not necessarily produce a simplicial decomposition of SP_n. However, if we consider the first barycentric subdivision of P_n, this is carried by φ_n to a simplicial decomposition of SP_n, being a subdivision of the cellular decomposition of SP_n mentioned above.

In the following, we consider always SP_n as such a cell complex. Obviously φ_n is both proper and cellular[4]. Therefore φ_n defines a unique cochain homomorphism

$$\varphi_n^{*} : C^q(SP_n;\ G) \rightarrow C^q(P_n;\ G)$$

for each $q=0, 1, 2, \cdots$, and for any coefficient group G.

For each $\alpha\in\mathfrak{S}_n$, the transformation $\alpha : P_n\rightarrow P_n$ is simplicial, so that the cochain homomorphism

$$\alpha^{*} : C^q(P_n;\ G) \rightarrow C^q(P_n\ ;G)$$

can be defined. Therefore it follows that $C^q(P_n;\ G)$ may be regarded as an $\mathfrak{S}_n$-group by defining $\alpha(c)=\alpha^{*}(c)$ $(\alpha\in\mathfrak{S}_n,\ c\in C^q(P_n;\ G))$. Consider the subgroup

$$C^q(P_n;\ G)^{\mathfrak{S}_n}\text{[5]}$$

of $C^q(P_n;\ G)$. Then it is easily verified that the coboundary homomorphism δ maps $C^q(P_n;\ G)^{\mathfrak{S}_n}$ into $C^{q+1}(P_n;\ G)^{\mathfrak{S}_n}$. Thus we have a cochain complex

$$\cdots\cdots\xrightarrow{\delta} C^q(P_n;\ G)^{\mathfrak{S}_n}\xrightarrow{\delta} C^{q+1}(P_n;\ G)^{\mathfrak{S}_n}\xrightarrow{\delta}\cdots\cdots,$$

whose cohomology group is denoted by

$$H^q(P_n|\mathfrak{S}_n;\ G).$$

Since it is easily seen that the cochain map φ_n^{*} yields an isomorphism of $C^q(SP_n;\ G)$ onto $C^q(P_n;\ G)^{\mathfrak{S}_n}$, the following is obvious.

PROPOSITION (1. 1). *We have*

$$\varphi_n^{\blacktriangle} : H^q(SP_n;\ G)\approx H^q(P_n|\mathfrak{S}_n;\ G),$$

where $\varphi_n^{\blacktriangle}$ *is the homomorphism induced by* φ_n^{*}.

Take from X a vertex $*$ which is used as the base point. Let m and n $(m\leqq n)$ be integers, and consider a continuous map

$$f_{m,n} : P_m \rightarrow P_n$$

defined by

$$f_{m,n}\ (x_1, x_2, \cdots\cdots, x_m) = (x_1, x_2, \cdots\cdots, x_m, *, \cdots\cdots, *).$$

Then $f_{m,n}$ is a simplicial map. Given $\alpha\in\mathfrak{S}_m$, define $\varepsilon_{m,n}(\alpha)\in\mathfrak{S}_n$ by

$$\varepsilon_{m,n}(\alpha)\ (1, 2, \cdots\cdots, n) = (\alpha(1), \alpha(2), \cdots\cdots, \alpha(m), m+1, \cdots\cdots, n).$$

Obviously we have

$$\varepsilon_{m,n}(\alpha) f_{m,n} = f_{m,n}\,\alpha,$$

and so

$$f_{m,n}^{*}\,\varepsilon_{m,n}(\alpha)^{*} = \alpha^{*} f_{m,n}^{*}$$

for the cochain homomorphism $f_{m,n}^{*} : C^q(P_n;\ G) \rightarrow C^q(P_m;\ G)$ induced by $f_{m,n}$. It follows from this that $f_{m,n}^{*}$ yields a cochain homomorphism

$$f'^{*}_{m,n} : C^q(P_n;\ G)^{\mathfrak{S}_n} \rightarrow C^q(P_m;\ G)^{\mathfrak{S}_m},$$

and so a homomorphism

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4) We use here the terminologies '*cellular decomposition, cell complex* and *proper map*' in the sense of the book of Steenrod [9].

5) Let B be a Λ-group, then we denote by B^{Λ} the subgroup of B which consists of all $b\in B$ for which $\lambda(b)=b$ for all $\lambda\in\Lambda$.

$$f^{\square}_{m,n} : H^q(P_n|\mathfrak{S}_n;\ G) \to H^q(P_m|\mathfrak{S}_m;\ G).$$

Define a continuous map

$$\iota_{m,n} : SP_m \to SP_n$$

by

$$\iota_{m,n}\{x_1, x_2, \cdots, x_m\} = \{x_1, x_2, \cdots, x_m, *, \cdots, *\}.$$

Then $\iota_{m,n}$ is both proper and cellular, and further

$$\varphi_n f_{m,n} = \iota_{m,n}\varphi_m.$$

Therefore we can consider the cochain homomorphism $\iota^{\divideontimes}_{m,n} : C^q(SP_n; G) \to C^q(SP_m; G)$, and it follows that

$$f^{\divideontimes}_{m,n}\varphi^{\divideontimes}_n = \varphi^{\divideontimes}_m \iota^{\divideontimes}_{m,n}.$$

Thus the commutativity holds in the diagram:

$$(1.\ 2)\qquad \begin{array}{ccc} H^q(SP_n;\ G) & \xrightarrow{\iota^*_{m,n}} & H^q(SP_m;\ G) \\ \downarrow \varphi^{\blacktriangle}_n & & \downarrow \varphi^{\blacktriangle}_m \\ H^q(P_n|\mathfrak{S}_n;\ G) & \xrightarrow{f^{\square}_{m,n}} & H^q(P_m|\mathfrak{S}_m;\ G) \end{array}$$

Write $(P_n(X), P'_n(X))$[1] for the n-fold cartesian product of $(X, *)$:

$$(1.\ 3)\qquad (P_n, P'_n) = (X, *)\times(X, *)\times \cdots \times(X, *).$$

Obviously P'_n is a subcomplex of P_n, consisting of all points $(x_1, x_2, \cdots\cdots, x_n)$ such that $x_i = *$ for some $i=1, 2, \cdots\cdots, n$. Therefore P'_n is a subcomplex invariant under $\mathfrak{S}_n$, so that, for each $\alpha\in\mathfrak{S}_n$, the cochain homomorphism $\alpha^{\divideontimes}$ may be regarded as also a cochain homomorphism of $C^q(P_n, P'_n;\ G)$ onto itself. Thus, by the way similar to defining $H^q(P_n|\mathfrak{S}_n; G)$, we can define the cohomology group

$$H^q((P_n, P'_n)|\mathfrak{S}_n;\ G)$$

from the cochain complex

$$\cdots\cdots \xrightarrow{\delta} C^q(P_n, P'_n; G)^{\mathfrak{S}_n} \xrightarrow{\delta} C^{q+1}(P_n, P'_n;\ G)^{\mathfrak{S}_n} \xrightarrow{\delta} \cdots\cdots$$

Write

$$SP'_n(X)^{1)}$$

for the image of $P'_n(X)$ by φ_n. Then it follows that SP'_n is a subcomplex of SP_n, and is the orbit space $O(P'_n;\ G_n)$.[2] Therefore the cochain homomorphism $\varphi^{\divideontimes}_n$ maps $C^q(SP_n, SP'_n;\ G)$ isomorphically onto $C^q(P_n, P'_n;\ G)^{\mathfrak{S}_n}$, so that we have the proposition similar to (1. 1):

PROPOSITION (1. 4). *It holds that*

$$\varphi^{\blacktriangle}_n : H^q(SP_n, SP'_n;\ G) \approx H^q((P_n, P'_n)|\mathfrak{S}_n;\ G)$$

for the homomorphism $\varphi^{\blacktriangle}_n$ induced by $\varphi^{\divideontimes}_n$.

The following is trivial.

LEMMA (1. 5). *The map $\iota_{n-1,n}$ $(n\geqq 1)$ gives a homeomorphism of SP_{n-1} onto SP'_n, where we make a convention: $SP_0=*$.*

Let $j^{\divideontimes}_n : C^q(P_n, P'_n;\ G) \to C^q(P_n;\ G)$ be the cochain homomorphism induced by the inclusion map $j_n : P_n \to (P_n, P'_n)$. Then $j^{\divideontimes}_n$ maps $C^q(P, P'_n;\ G)^{\mathfrak{S}_n}$ into $C^q(P_n;$

$G)^{\mathfrak{S}_n}$, and hence it defines a homomorphism

$$j_n^{\square} : H^q((P_n, P_n')|\mathfrak{S}_n;\ G) \to H^q(P_n|\mathfrak{S}_n;\ G).$$

It is obvious that the commutativity holds in the diagram:

$$(1.\ 6)\qquad \begin{array}{ccc} H^q(SP_n, SP_n';\ G) & \xrightarrow{j_n^*} & H^q(SP_n;\ G) \\ \downarrow \varphi_n^{\blacktriangle} & & \downarrow \varphi_n^{\blacktriangle} \\ H^q((P_n, P_n')|\mathfrak{S}_n;\ G) & \xrightarrow{j_n^{\square}} & H^q(P_n|\mathfrak{S}_n;\ G) \end{array}$$

where j_n^* is the homomorphism induced by the inclusion map $j_n : SP_n \to (SP_n, SP_n')$.

2. The right inverse of $\iota_{m,n}^*$

Let m and n be positive integers such that $m \leqq n$. We call then $(n;\ m)$-*array* each ordered set $(i_1, i_2, \cdots, i_m)$ of mutually distinct m integers $\leqq n$. If an $(n; m)$-array $(i_1, i_2, \cdots, i_m)$ satisfies a condition: $i_1 < i_2 < \cdots < i_m$, it is said to be *normal*.

Given an $(n;\ m)$-array $(i_1, i_2, \cdots, i_m)$, it is associated with a continuous map

$$\psi_{i_1, i_2, \ldots, i_m} : P_n(X) \to P_m(X)$$

defined by

$$\psi_{i_1, i_2, \ldots, i_m}(x_1, x_2, \cdots, x_n) = (x_{i_1}, x_{i_2}, \cdots, x_{i_m}).$$

If $w_0, w_1, \cdots, w_q$ are vertices contained in a simplex of P_n, then $\psi_{i_1, i_2, \ldots, i_m}(w_0)$, $\psi_{i_1, i_2, \ldots, i_m}(w_1)$, $\cdots$, $\psi_{i_1, i_2, \ldots, i_m}(w_q)$ are obviously vertices contained in a single simplex of P_m. Therefore $\psi_{i_1, i_2, \ldots, i_m}$ is a simplicial map, so that it induces the cochain homomorphism

$$\psi_{i_1, i_2, \ldots, i_m}^{\divideontimes} : C^q(P_m;\ G) \to C^q(P_n;\ G).$$

Define now a cochain homomorphism

$$\Psi_{m,n} : C^q(P_m;\ G) \to C^q(P_n;\ G)$$

by

$$\Psi_{m,n} = \sum_{(n;\ m)} \psi_{i_1, i_2, \ldots, i_m}^{\divideontimes},$$

the sum being taken over all normal $(n;\ m)$-arraies $(i_1, i_2, \cdots, i_m)$. Since it is obvious that

$$\beta\, \psi_{i_1, i_2, \ldots, i_m} = \psi_{\beta(i_1), \beta(i_2), \ldots, \beta(i_m)} \quad (\beta \in \mathfrak{S}_m),$$

we have

$$\psi_{\beta(i_1), \beta(i_2), \ldots, \beta(i_m)}^{\divideontimes}(c) = \psi_{i_1, i_2, \ldots, i_m}^{\divideontimes}\beta^{\divideontimes}(c) = \psi_{i_1, i_2, \ldots, i_m}^{\divideontimes}(c)$$

for each cochain $c \in C^q(P_m;\ G)^{\mathfrak{S}_m}$. Thus it follows that

$$\Psi_{m,n}(c) = \sum_{(n;\ m)}{}' \frac{1}{m!} \psi_{i_1, i_2, \ldots, i_m}^{\divideontimes}(c) \quad \text{for } c \in C^r(P_m;\ G)^{\mathfrak{S}_m},$$

where $\sum'_{(n;\ m)}$ denotes the summation extended over all $(n;\ m)$-arraies $(i_1, i_2, \cdots, i_m)$. Since

$$\psi_{i_1, i_2, \ldots, i_m}\alpha = \psi_{\alpha(i_1), \alpha(i_2), \ldots, \alpha(i_m)} \quad (\alpha \in \mathfrak{S}_n),$$

it follows further that

$$\begin{aligned}\alpha^{\divideontimes}\Psi_{m,n}(c) &= \sum_{(n;m)}{}' \frac{1}{m!}\alpha^{\divideontimes}\psi^{\divideontimes}_{i_1,i_2,\ldots,i_m}(c)\\ &= \sum_{(n;m)}{}' \frac{1}{m!}\psi^{\divideontimes}_{\alpha(i_1),\alpha(i_2),\ldots,\alpha(i_m)}(c)\\ &= \sum_{(n;m)}{}' \frac{1}{m!}\psi^{\divideontimes}_{i_1,i_2,\ldots,i_m}(c)\\ &= \Psi_{m,n}(c).\end{aligned}$$

This means that $\Psi_{m,n}(c) \in C^q(P_n;\ G)^{\mathfrak{S}_n}$ if $c \in C^q(P_m;\ G)^{\mathfrak{S}_m}$. Thus $\Psi_{m,n}$ yields a cochain homomorphism

$$\Psi_{m,n} : C^q(P_m;\ G)^{\mathfrak{S}_m} \to C^q(P_n;\ G)^{\mathfrak{S}_n},$$

and hence a homomorphism

$$\Psi^{\square}_{m,n} : H^q(P_m|\mathfrak{S}_m;\ G) \to H^q(P_n|\mathfrak{S}_n;\ G).$$

Write

(2. 1) $$\Psi^{*}_{m,n} = \varphi^{\blacktriangle -1}_n \Psi^{\square}_{m,n}\varphi^{\blacktriangle}_m : H^q(SP_m;\ G) \to H^q(SP_n;\ G).$$

Trivially we obtain

LEMMA (2. 2). *$\Psi^{*}_{m,m}$ is the identity isomorphism.*

Consider the commutative diagram:

(2. 3) $$\begin{array}{ccc} H^q(SP'_n;\ G) & \xleftarrow{\ i^*\ } & H^q(SP_n;\ G)\\ & {\scriptstyle \iota'^*_{n-1,n}}\searrow\approx \quad \swarrow{\scriptstyle \iota^*_{n-1,n}} & \\ & H^q(SP_{n-1};\ G) & \end{array}$$

where $\iota'^*_{n-1,n}$ and i^* are the homomorphisms induced by $\iota_{n-1,n}$ and the inclusion map : $SP'_n \to SP_n$ respectively. It follows then from (1. 5) that $\iota'^*_{n-1,n}$ is an onto-isomorphism. Write

$$K^q(SP_n;\ G) = \text{the Kernel of } \iota^*_{n-1,n}.$$

Then the above diagram and the cohomology exact sequence for $(SP_n,\ SP'_n)$ yield the following:

LEMMA (2. 4). *$K^r(SP_n;\ G)$ is the image group of the homomorphism j^*_n : $H^q(SP_n,\ SP'_n;\ G) \to H^q(SP_n;\ G)$.*

We shall prove

LEMMA (2. 5). *Let $l \leqq m \leqq n$, and consider the diagram:*

$$\begin{array}{ccc} H^q(SP_n;\ G) & \xrightarrow{\ \iota^*_{m,n}\ } & H^q(SP_m;\ G)\\ & {\scriptstyle \Psi^*_{l,n}}\nwarrow \quad \nearrow{\scriptstyle \Psi^*_{l,m}} & \\ & H^q(SP_l;\ G) & \end{array}$$

Then it holds that $\iota^*_{m,n}\Psi^*_{l,n} = \Psi^*_{l,m}$ on $K^q(SP_l;\ G)$.

Proof. It follows from (2. 4) that our purpose is accomplished if we prove

$$\iota^*_{m,n}\Psi^*_{l,n}j^*_l = \Psi^*_{l,m}j^*_l.$$

Further, by (1. 6), (2. 1) and (1. 2), this is reduced to prove

$$f^{\square}_{m,n}\Psi^{\square}_{l,n}j^{\square}_l = \Psi^{\square}_{l,m}j^{\square}_l.$$

For this purpose, it is sufficient to prove that

(A) $$\sum_{(n;\,l)} f^{\divideontimes}_{m,n}\psi^{\divideontimes}_{i_1,i_2,\dots,i_l}(c)=\sum_{(m;\,l)}\psi^{\divideontimes}_{i_1,i_2,\dots,i_l}(c),\quad \big(c\in C^q(P_l,\ P_l';\ G)\big).$$

where $\sum_{(s;\,l)}$ $(s=n$ or $m)$ denotes the summation extended over all normal $(s;\,l)$-arraies $(i_1, i_2, \cdots, i_l)$.

Let w be any point of P_m. It is then obvious from the definitions that $\psi_{i_1,i_2,\dots,i_l}f_{m,n}(w)$ is $\psi_{i_1,i_2,\dots,i_l}(w)$ if the $(n;\,l)$-array $(i_1, i_2, \cdots, i_l)$ is an $(m;\,l)$-array, and is in P_l' otherwise. Therefore, for each simplex $\varDelta^q=(w_0, w_1, \cdots, w_q)$ in P_m, we have that

$$\big(f^{\divideontimes}_{m,n}\psi^{\divideontimes}_{i_1,i_2,\dots,i_l}(c)\big)(\varDelta^q)=\big(\psi^{\divideontimes}_{i_1,i_2,\dots,i_l}(c)\big)(\varDelta^q)\quad\text{or}\quad =0$$

according as the $(n;\,l)$-array $(i_1, i_2, \cdots, i_l)$ is an $(m;\,l)$-array or not. This proves (A), and the proof of the lemma is complete.

Write $\varPsi^0_{m,n}$ for the homomorphism $\varPsi^*_{m,n}: H^q(SP_m;\ G)\to H^q(SP_n;\ G)$ restricted on $K^q(SP_m;\ G)$. Then we have

PROPOSITION (2. 6). *For any $q>0$, the homomorphisms*

$$\varPsi^0_{m,n}: K^q(SP_m;\ G)\to H^q(SP_n;\ G)\qquad (1\leqq m\leqq n)$$

yield an injective representation of $H^q(SP_n;\ G)$ as a direct sum[6], *i.e. for each $a\in H^q(SP_n;\ G)$, there exists a unique system of n elements $\{a_m\}$ $(m=1,2,\cdots,n)$ with $a_m\in K^q(SP_m;\ G)$ satisfying $a=\sum_{m=1}^{n}\varPsi^0_{m,n}(a_m)$.*

Proof. This is trivial for $n=1$. Assume, inductively, the validity for $n-1$. Consider the homomorphism

$$\iota^*_{n-1,n}: H^q(SP_n;\ G)\to H^q(SP_{n-1};\ G).$$

By the assumption of inducion, we have

$$\iota^*_{n-1,n}(a)=\sum_{m=1}^{n-1}\varPsi^0_{m,n-1}(a_m),\qquad \big(a_m\in K^q(SP_m;\ G)\big).$$

Put

$$a_n=a-\sum_{m=1}^{n-1}\varPsi^0_{m,n}(a_m).$$

Then we have by (2. 5)

$$\begin{aligned}\iota^*_{n-1,n}(a_n)&=\iota^*_{n-1,n}(a)-\sum_{m=1}^{n-1}\iota^*_{n-1,n}\varPsi^0_{m,n}(a_m)\\&=\iota^*_{n-1,n}(a)-\sum_{m=1}^{n-1}\varPsi^0_{m,n-1}(a_m)=0,\end{aligned}$$

and so $a_n\in K^q(SP_n;\ G)$. Since $a_n=\varPsi^0_{n,n}(a_n)$ from (2. 2), it follows that

$$a=a_n+\sum_{m=1}^{n-1}\varPsi^0_{m,n}(a_m)=\sum_{m=1}^{n}\varPsi^0_{m,n}(a_m).$$

Namely it was proved that a can be expressed as $\sum_{m=1}^{n}\varPsi^0_{m,n}(a_m)$ with $a_m\in K^q(SP_m;\ G)$.

Next we shall prove the uniqueness of such a expression. Suppose that we

6) See p. 8 in [4] for the definition of this terminology.

have two expressions:

$$a = \sum_{m=1}^{n} \Psi^0_{m,n}(a_m) = \sum_{m=1}^{n} \Psi^0_{m,n}(a'_m),$$

where a_m, $a'_m \in K^q(SP_m;\ G)$. Then we have by (2. 2)

$$0 = \sum_{m=1}^{n} \Psi^0_{m,n}(a_m - a'_m)$$

$$= a_n - a'_n + \sum_{m=1}^{n-1} \Psi^0_{m,n}(a_m - a'_m).$$

Apply $\iota^*_{n-1,n}$ to this equation. Since $a_m - a'_m \in K^q(SP_m;\ G)$, it follows from (2. 5) that

$$0 = \sum_{m=1}^{n-1} \iota^*_{n-1,n} \Psi^0_{m,n}(a_m - a'_m) = \Psi^0_{m,n-1}(a_m - a'_m).$$

By the assumption of induction, this implies that $a_m - a'_m = 0$ $(m=1, 2, \cdots, n-1)$. Therefore we have also $a_n = a'_n$. This proves the uniqueness, and the proof of the proposition is accomplished.

Owing to (2. 6), we can define a homomorphism

$$\mu_{m,n} : H^q(SP_m;\ G) \to H^q(SP_n;\ G) \qquad (q > 0,\ m \leqq n)$$

as follows: Let $a \in H^q(SP_m;\ G)$ be an element, and

$$a = \sum_{l=1}^{m} \Psi^0_{l,m}(a_l), \qquad (a_l \in K^q(SP_l;\ G)).$$

Then $\mu_{m,n}(a)$ is given by

$$\mu_{m,n}(a) = \sum_{l=1}^{m} \Psi^0_{l,n}(a_l).$$

It is obvious that $\mu_{m,n}$ is a homomorphism. We shall now prove

THEOREM (2. 7). *It holds that $\iota^*_{m,n}\mu_{m,n} =$ the identity isomorphism of $H^q(SP_m;\ G)$ for any $q>0$.*

Proof. It follows from the definition and (2. 5) that

$$\iota^*_{m,n}\mu_{m,n}(a) = \sum_{l=1}^{m} \iota^*_{m,n} \Psi^0_{l,n}(a_l)$$

$$= \sum_{l=1}^{m} \Psi^0_{l,m}(a_l)$$

$$= a.$$

This proves the theorem.

As a trivial consequence of (2. 7), we have

COROLLARY (2. 8). *The homomorphism $\iota^*_{m,n} : H^q(SP_n;\ G) \to H^q(SP_m;\ G)$ is onto, and the homomorphism $\mu_{m,n} : H^q(SP_m;\ G) \to H^q(SP_n;\ G)$ is isomorphic into.*

Consider the cohomology exact sequence for (SP_m, SP'_m). Then it follows from (2. 3) and (2. 8) that $i^* : H^q(SP_m;\ G) \to H^q(SP'_m;\ G)$ is onto for each q. Therefore we obtain that $j^* : H^q(SP_m, SP'_m;\ G) \to H^q(SP_m;\ G)$ is isomorphic into for any q. Thus the following is concluded by (2. 4) and (2. 6).

THEOREM (2. 9). *We have the direct sum relation*:

$$H^q(SP_n;\ G)\approx\sum_{m=1}^{n}H^q(SP_m,\ SP'_m;\ G)\qquad(q>0).$$

3. Reduced symmetric products

Let

$$\bar{P}_n(X)\qquad(\text{resp. } \overline{SP}_n(X))^{1)}$$

denote a finite CW-complex obtained from the complex $P_n(X)$ (resp. $SP_n(X)$) by shrinking the subcomplex $P'_n(X)$ (resp. $SP'_n(X)$) to the point $(*_n)=(*, *, \cdots, *)$ (resp. $\{*_n\}=\{*, *, \cdots, *\}$). We refer to $\bar{P}_n(X)$ (resp. $\overline{SP}_n(X)$) as the *n-fold reduced cartesian* (resp. *symmetric*) *product* of X. It is obvious that the shrinking maps

$$t_n: (P_n,\ P'_n)\to(\bar{P}_n,\ (*_n)),$$

$$\tau_n: (SP_n,\ SP'_n)\to(\overline{SP}_n,\ \{*_n\})$$

are relative homeomorphisms.[7] Therefore we have

PROPOSITION (3. 1). *For any $q>0$, it holds that*

$$t_n^*: H^q(\bar{P}_n;\ G)\approx H^q(P_n,\ P'_n;\ G),$$

$$\tau_n^*: H^q(\overline{SP}_n;\ G)\approx H^q(SP_n,\ SP'_n;\ G),$$

where t_n^* and τ_n^* are the homomorphisms induced by t_n and τ_n respectively.

By this and (2. 9), we can find at once the cohomology groups of the symmetric product from those of the reduced symmetric product. The latter will be studied in §§ 4 and 6. We make some preparations for the study in the remainder of this section.

For each $\alpha\in\mathfrak{S}_n$, the transformation $\alpha: P_n\to P_n$ obviously determines a transformation $\bar{\alpha}: \bar{P}_n\to\bar{P}_n$ such that $\bar{\alpha}\ t_n=t_n\ \alpha$. Therefore $\bar{P}_n$ may be regarded as a space on which $\mathfrak{S}_n$ acts. Consider a continuous map

$$\bar{\varphi}_n: \bar{P}_n\to\overline{SP}_n$$

defined by

$$\bar{\varphi}_n\ t_n=\tau_n\ \varphi_n.$$

It is then easily verified that $\overline{SP}_n$ is the orbit space $O(\bar{P}_n;\ \mathfrak{S}_n)$ whose identification map is $\bar{\varphi}_n$.

Define a space $F_n(X)^{1)}\subset P_n(X)$ $(n\geqq 1)$ as follows: F_n $(n>1)$ consists of all points $(x_1, x_2, \cdots, x_n)$ such that $x_i=x_j$ for some i and j $(i\neq j)$ and $F_1=*$. Define further a space

$$\bar{F}_n(X)^{1)}\subset\bar{P}_n(X)$$

as the image of $F_n(X)$ by the map t_n. Then it follows easily that F_n is a subcomplex of $\bar{P}_n$ which is invariant under $\mathfrak{S}_n$, and consists of all points which are

7) For the definition, see p 266 in [4].

fixed under some transformations (except the identity) of $\mathfrak{S}_n$. Therefore $\bar{P}_n - \bar{F}_n$ is a locally compact space on which $\mathfrak{S}_n$ acts without fixed points, (i.e. on which any transformation $\alpha \neq$ the identity ($\alpha \in \mathfrak{S}_n$) admits no fixed point). Define a space

$$\overline{SF}_n(X)^{1)} \subset \overline{SP}_n(X)$$

as the image of $\bar{F}_n(X)$ by the map $\bar{\varphi}_n$. Then it is obvious that $\overline{SP}_n - \overline{SF}_n$ is the orbit space $O(\bar{P}_n - \bar{F}_n; \mathfrak{S}_n)$ whose identification map is $\bar{\varphi}_n$. Thus it follows that the space $\bar{P}_n - \bar{F}_n$ is a locally compact principal fiber space of structure group $\mathfrak{S}_n$ over the space $\overline{SP}_n - \overline{SF}_n$. Therefore we can apply the Cartan-Leray theory [2, 3].[8] Before we state the result, we shall make some remarks.

Let $H^q(\bar{P}_n - \bar{F}_n; G)$ (resp. $H^q(\overline{SP}_n - \overline{SF}_n; G)$) denote the Čech cohomology group (with compact supports) of the locally compact space $\bar{P}_n - \bar{F}_n$ (resp. $\overline{SP}_n - \overline{SF}_n$). It follows then from the well-known general property of cohomology that $H^q(\bar{P}_n - \bar{F}_n; G)$ (resp. $H^q(\overline{SP}_n - \overline{SF}_n; G)$) is canonically isomorphic with the relative cohomology group $H^q(\bar{P}_n, \bar{F}_n; G)$ (resp. $H^q(\overline{SP}_n, \overline{SF}_n; G)$) of the cellular pair $(\bar{P}_n, \bar{F}_n)$ (resp. $(\overline{SP}_n, \overline{SF}_n)$).

Given $\alpha \in \mathfrak{S}_n$ and $a \in H^q(\bar{P}_n, \bar{F}_n; G)$, define $\alpha(a) \in H^q(\bar{P}_n, \bar{F}_n; G)$ by

$$(3.\ 2) \qquad \alpha(a) = \bar{\alpha}^{*-1}(a).$$

Then it follows that $H^q(\bar{P}_n, \bar{F}_n; G)$ is an $\mathfrak{S}_n$-group by this operation.

We shall utilize the usual notations (as is seen in [8]) with respect to the spectral sequence. Then the general theory of Čartan-Leray gives

PROPOSITION (3. 3). *There exists a cohomology spectral sequence* (E_r) *in which the term* $E_2^{p,q}$ *is isomorphic with the cohomology group* $H^p(\mathfrak{S}_n; H^q(\bar{P}_n, \bar{F}_n; G))$ *of the group* $\mathfrak{S}_n$ *with coefficients in the* $\mathfrak{S}_n$*-group* $H^q(\bar{P}_n, \bar{F}_n; G)$, *and* $E_\infty^{p,q}$ *is isomorphic with the graduated group* $D^{p,q}/D^{p+1,q-1}$ *associated with a certain filtration* $(D^{s,t})$ $(s+t=p+q)$ *of* $H^{p+q}(\overline{SP}_n, \overline{SF}_n; G)$.

The cohomology groups of $\overline{SP}_n$ and of $(\overline{SP}_n, \overline{SF}_n)$ are related to each other by the exact sequence. This relation is explained to some extent by evaluating the cohomology groups of $\overline{SF}_n$. To do this, we define a space

$$\overline{SF}{}_n^t(X) \subset \overline{SF}_n(X)^{1)}$$

for each integer $t \geqq 1$ as follows: Write $SF_n^t(X)$ $(1 \leqq t \leqq n/2)$ for a subset of $SF_n(X)$ which consists of all points $\{x_1, x_1, x_2, x_2, \cdots, x_t, x_t, x_{t+1}, x_{t+2}, \cdots, x_{n-t}\} \in SP_n(X)$, and put $SF_n^t(X) = \{*_n\}$ for $t > n/2$. Then $\overline{SF}{}_n^t(X)$ is defined as the image of $SF_n^t(X)$ by the map τ_n. It is easily seen that $\overline{SF}{}_n^t$ are subcomplexes of $\overline{SP}_n$, and satisfy a condition:

$$(3.\ 4) \qquad \overline{SP}_n \supset \overline{SF}_n = \overline{SF}{}_n^1 \supset \cdots\cdots \supset \overline{SF}{}_n^t \supset \overline{SF}{}_n^{t+1} \supset \cdots\cdots .$$

We shall prove

8) See also A. Borel [1].

LEMMA (3. 5). *For each t $(1 \leqq t \leqq n/2)$, there is a relative homeomorphism*

$$\bar{\omega}_n^t : (\overline{SP}_t, \{*_t\}) \times (\overline{SP}_{n-2t}, \overline{SF}_{n-2t}) \rightarrow (\overline{SF}_n^t, \overline{SF}_n^{t+1})$$

with the following conventions: $\overline{SP}_0 = *$, $\overline{SF}_0 =$ *empty set.*

Proof. Define a continuous map

$$\omega_n^t : SP_t \times SP_{n-2t} \rightarrow SF_n^t$$

by

$$\omega_n^t(\{x_1, x_2, \cdots, x_t\} \times \{x_1', x_2' \cdots, x_{n-2t}'\})$$
$$= \{x_1, x_1, x_2, x_2, \cdots, x_t, x_t, x_1', x_2', \cdots, x_{n-2t}'\} \qquad (x_i, x_j' \in X).$$

Then ω_n^t maps $(SP_t' \times SP_{n-2t}) \cup (SP_t \times SP_{n-2t}')$ in $(SF_n^t \cup SP_n')$. Therefore ω_n^t can be regarded as a continuous map of $(SP_t, SP_t') \times (SP_{n-2t}, SP_{n-2t}')$ into $(SF_n^t, SF_n^t \cup SP_n')$, so that ω_n^t defines a continuous map

$$\bar{\omega}_n^t : \overline{SP}_t \times \overline{SP}_{n-2t} \rightarrow \overline{SF}_n^t.$$

Since $\bar{\omega}_n^t$ maps $\{*_t\} \times \overline{SP}_{n-2t}$ in $\{*_n\}$ and $\overline{SP}_t \times \overline{SF}_{n-2t}$ in $\overline{SF}_n^{t+1}$, $\bar{\omega}_n^t$ can be regarded as a continuous map of $(\overline{SP}_t, \{*_t\}) \times (\overline{SP}_{n-2t}, \overline{SF}_{n-2t})$ into $(\overline{SF}_n^t, \overline{SF}_n^{t+1})$. Notice here that each point of $\overline{SF}_n^t - \overline{SF}_n^{t+1}$ can be represented uniquely as $\{x_1, x_1, x_2, x_2, \cdots, x_t, x_t, x_{t+1}, x_{t+2}, \cdots, x_{n-t}\}$ with $x_i \neq *$ $(1 \leqq i \leqq n-t)$ and $x_j \neq x_k$ $(t+1 \leqq j < k \leqq n-t)$. It follows then that $\bar{\omega}_n^t$ is a one-to-one correspondence of $\overline{SP}_t \times \overline{SP}_{n-2t} - (\{*_t\} \times \overline{SP}_{n-2t}) \cup (\overline{SP}_t \times \overline{SF}_{n-2t})$ onto $\overline{SF}_n^t - \overline{SF}_n^{t+1}$. Thus $\bar{\omega}_n^t$ is a relative homeomorphism. This completes the proof.

4. Symmetric products of homologically (r−1)-connected complexes

Assuming that X is homologically $(r-1)$-connected $(r \geqq 2)$, we calculate in this section some integral cohomology groups of the symmetric product of X.

We have

PROPOSITION (4. 1). *If X is homologically $(r-1)$-connected, $\bar{F}_n$ is homologically $(r+n-3)$-connected.*

The proof needs some preparation, and hence will be given in the next section.

From this, we have

LEMMA (4. 2). *Let X be homologically $(r-1)$-connected, then the integral homology group $H_q(\bar{P}_n, \bar{F}_n) = 0$ for $q \leqq r+n-2$.*

Proof. It follows from (1. 3) and (3. 1) by the Künneth formula that $H_q(\bar{P}_n) = 0$ for $0 < q \leqq nr-1$. Therefore, if we consider the homology exact sequeuce for $(\bar{P}_n, \bar{F}_n)$, the lemma follows at once from (4. 1).

We shall now prove

PROPOSITION (4. 3). *Let X be homologically $(r-1)$-connected, then the integral homology group $H_q(\overline{SP}_n, \overline{SF}_n) = 0$ for $q \leqq r+n-2$.*

Proof. It follows from (4. 2) by the universal coefficient theorem that the integral cohomology group $H^q(\bar{P}_n, \bar{F}_n) = 0$ for $q \leqq r+n-2$, and $H^{r+n-1}(\bar{P}_n, \bar{F}_n)$

is free abelian. Therefore, if we consider the spectral sequence in (3. 3), we have that if $q\leqq r+n-2$ then $E_2^{p,q}=0$ and hence $E_\infty^{p,q}=0$. Thus $H^q(\overline{SP}_n, \overline{SF}_n) = D^{0,q} = D^{1,q-1} = \cdots\cdot = D^{-1,q+1}=0$ for $q\leqq r+n-2$. Further we have $H^{r+n-1}(\overline{SP}_n, \overline{SF}_n)= D^{0,r+n-1}=E_\infty^{0,r+n-1}=E_2^{0,r+n-1}=H^0(\mathfrak{S}_n; H^{r+n-1}(\overline{P}_n, \overline{F}_n))=H^{r+n-1}(\overline{P}_n, \overline{F}_n)^{\mathfrak{S}_n}$,[5] by the well-known fact [2]. Therefore $H^{r+n-1}(\overline{SP}_n, \overline{SF}_n)$ is a subgroup of a free abelian group $H^{r+n-1}(\overline{P}_n, \overline{F}_n)$, so that $H^{r+n-1}(\overline{SP}_n, \overline{SF}_n)$ itself is free abelian. The proposition is now clear by the universal coefficient theorem.

We shall prove the following result with respect to the reduced symmetric product.

PROPOSITION (4. 4). *Let X be homologically $(r-1)$-connected $(r\geqq 2)$, then it holds that $H^q(\overline{SP}_n)=0$ for $0<q\leqq r+1$ and $n\geqq 2$.*

First we give a

Proof of (4. 4) *for* $n=2$. Since $\overline{SF}_2$ is obviously homeomorphic with $\overline{F}_2$ and hence with X, it follows that $H_q(\overline{SF}_2)=0$ for $0<q\leqq r-1$. On the other hand, it follows from (4. 3) that $H_q(\overline{SP}_2, \overline{SF}_2)=0$ for $q\leqq r$. Therefore it is concluded by the homology exact sequence for $(\overline{SP}_2, \overline{SF}_2)$ that $H_q(\overline{SP}_2)=0$ for $0<q\leqq r-1$. This implies that $H^q(\overline{SP}_2)=0$ for $0<q\leqq r-1$. Thus it remains to prove $H^q(\overline{SP}_2)=0$ for $q=r$ and $r+1$.

Since $r\geqq 2$, it follows that $H_q(\overline{P}_2)=0$ if $0<q\leqq r+1$. Hence $H^q(\overline{P}_2)=0$ for $0<q\leqq r+1$. This implies

(A) The coboundary homomorphism $\delta^*: H^r(\overline{F}_2)\rightarrow H^{r+1}(\overline{P}_2, \overline{F}_2)$ is isomorphic onto.

Let $\alpha\in\mathfrak{S}_2$, then the transformation α is the identity on $\overline{F}_2$. Therefore the homomorphism $\alpha^*: H^r(\overline{F}_2)\rightarrow H^r(\overline{F}_2)$ is the identity isomorphism. This, together with (A) and (3. 2), proves that $\mathfrak{S}_2$ operates on $H^{r+1}(\overline{P}_2, \overline{F}_2)$ trivially. Thus, by some usual arguments in spectral sequence [8], it follows that the homomorphism $\overline{\varphi}^*: H^{r+1}(\overline{SP}_2, \overline{SF}_2)\rightarrow H^{r+1}(\overline{P}_2, \overline{F}_2)$ can be written as follows:

$$\begin{aligned}\overline{\varphi}^*: H^{r+1}(\overline{SP}_2, \overline{SF}_2) &= D^{0,r+1}\rightarrow E_\infty^{0,r+1}\subset E_2^{0,r+1}\\ &= H^0(\mathfrak{S}_2; H^{r+1}(\overline{P}_2, \overline{F}_2)) = H^{r+1}(\overline{P}_2, \overline{F}_2).\end{aligned}$$

However, as was seen in the proof of (4. 3), $D^{0,r+1}=E_\infty^{0,r+1}=E_2^{0,r+1}$. Therefore we obtain

(B) $\overline{\varphi}^*: H^{r+1}(\overline{SP}_2, \overline{SF}_2)\rightarrow H^{r+1}(\overline{P}_2, \overline{F}_2)$ is an onto-isomorphism.

Consider the commutative diagram

$$\begin{array}{ccc} H^r(\overline{SF}_2) & \xrightarrow{\delta^*} & H^{r+1}(\overline{SP}_2, \overline{SF}_2) \\ \downarrow{\overline{\varphi}^*} & & \downarrow{\overline{\varphi}^*} \\ H^r(\overline{F}_2) & \xrightarrow{\delta^*} & H^{r+1}(\overline{P}_2, \overline{F}_2) \end{array}$$

Since $\overline{\varphi}$ gives a homeomorphism of $\overline{F}_2$ onto $\overline{SF}_2$, the left $\overline{\varphi}^*$ is isomorphic onto. Therefore it follows from (A) and (B) that the upper δ^* is also isomorphic onto. By considering the exact sequence for $(\overline{SP}_2, \overline{SF}_2)$, we have

(C) The homomorphism $j^*: H^r(\overline{SP}_2, \overline{SF}_2)\rightarrow H^r(\overline{SP}_2)$ is onto, and the homomorphism $i^*: H^{r+1}(\overline{SP}_2)\rightarrow H^{r+1}(\overline{SF}_2)$ is isomorphic into.

Since it follows from (4. 3) that $H^r(\overline{SP}_2,\overline{SF}_2)=0$, we have $H^r(\overline{SP}_2)=0$ by (C).

Consider the commutative diagram

$$\begin{array}{ccc} H^{r+1}(\overline{SP}_2) & \xrightarrow{i^*} & H^{r+1}(\overline{SF}_2) \\ \downarrow{\bar{\varphi}^*} & & \downarrow{\bar{\varphi}^*} \\ H^{r+1}(\bar{P}_2) & \xrightarrow{i^*} & H^{r+1}(\bar{F}_2) \end{array}$$

Then the upper i^* is isomorphic into by (C), and the right $\bar{\varphi}^*$ is obviously isomorphic onto. Therefore $\bar{\varphi}^*i^*$ is isomorphic into. On the other hand, $H^{r+1}(\bar{P}_2)=0$ as was seen above. Hence $\bar{\varphi}^*i^*=i^*\bar{\varphi}^*$ is trivial. Thus we must have $H^{r+1}(\overline{SP}_2)=0$. This completes the proof of (4. 4) for $n=2$.

Proof of (4. 4). We proceed by induction on n. Assume the validity of (4. 4) for $n=2,3,\cdots,p-1$, and we shall prove (4. 4) for $n=p$.

Consider the sequence

$$\Theta_p: H^q(\overline{SP}_p)\xrightarrow{\theta_0^*} H^q(\overline{SF}_p^1)\xrightarrow{\theta_1^*}\cdots\cdots\longrightarrow H^q(\overline{SF}_p^t)\xrightarrow{\theta_t^*} H^q(\overline{SF}_p^{t+1})\longrightarrow\cdots\cdots,$$

which terminates in $H^q(\overline{SF}_p^{p/2})$ or $H^q(\overline{SF}_p^{(p-1)/2})$ according as p is even or odd, where θ_i^* $(0\leqq i\leqq p/2-1)$ denotes the homomorphism induced by the inclusion (3. 4). Then we assert

(D) Θ_p is isomorphic into for $q\leqq\mathrm{Min}\,(r+p-2,\,2r+1)$.

For the proof, it is sufficient to show that $H^q(\overline{SP}_p,\,\overline{SF}_p)$ and $H^q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})$ $(1\leqq t\leqq p/2-1)$ are trivial for $q\leqq\mathrm{Min}\,(r+p-2,\,2r+1)$. This is obvious by (4. 3) as for $H^q(\overline{SP}_p,\overline{SF}_p)$. To prove the result for $H^q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})$, consider the homology group $H_q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})$. It follows then from (3. 5) by the excision property of homology that $H_q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})$ is isomorphic with $H_q\big((\overline{SP}_t,\,\{*_t\})\times(\overline{SP}_{p-2t},\,\overline{SF}_{p-2t})\big)$. Therefore it follows from (4. 3) and the assumption of induction by the Künneth formula that $H_q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})$ $(2\leqq t\leqq p/2-1)$ is trivial for $q\leqq 2r-2t+p-1$ and $H_q(\overline{SF}_p^1,\,\overline{SF}_p^2)$ is trivial for $q\leqq 2r+p-4$. Hence the cohomology group $H^q(\overline{SF}_p^t,\,\overline{SF}_p^{t+1})=0$ for $q\leqq\underset{2\leq t\leq p/2-1}{\mathrm{Min}}\,\mathrm{Min}\,(2r-2t+p-1,2r+p-4)=\mathrm{Min}\,(2r+1,2r+p-4)$. Since $r+p-2\leqq 2r+p-4$ we have the desired result. Thus we obtain (D).

As for the range of Θ_p, we have

(E) $H^q(\overline{SF}_p^{p/2})=0$ for $q\leqq r+1$ (p : even); $H^q(\overline{SF}_p^{(p-1)/2})=0$ for $q\leqq 2r+1$ if $p>3$, and for $q\leqq 2r-1$ if $p=3$ (p : odd).

In fact, it follows from (3. 5) that $H^q(\overline{SF}_p^{p/2})$ and $H^q(\overline{SF}_p^{(p-1)/2})$ are isomorphic with $H^q(\overline{SP}_{p/2})$ and $H^q\big((\overline{SP}_{(p-1)/2},\{*_{(p-1)/2}\})\times(X,\,*)\big)$ respectively. Therefore (E) follows easily from (4. 3) and the assumption of induction, by the same arguments as in the proof of (D).

As a direct consequence of (D) and (E), we have (4. 4) for $n=p$. This completes the proof of (4. 4).

In the above proof, we have proved simultaneouly the following : (See (D) and (E))

PROPOSITION (4. 5). *Let X be homologically $(r-1)$-connected $(r\geqq 2)$.* (i) *If n is odd then $H^q(\overline{SP}_n)=0$ for $q\leqq Min\ (r+n-2, 2r+1)$.* (ii) *If n is even, the homomorphism $\Theta'_n=\omega_n^{n/2*}\ \Theta_n : H^q(\overline{SP}_n)\rightarrow H^q(\overline{SP}_{n\ 2})$ is isomorphic into for $q\leqq Min\ (r+n-2, 2r+1)$.*

We shall prove

THEOREM (4. 6). *Let X be homologically $(r-1)$-connected $(r\geqq 2)$, and $n=2^e n'$, where $e\geqq 0$ and n' is odd. Then the homomorphism $\iota^*_{n-1,n} : H^q(SP_n)\rightarrow H^q(SP_{n-1})$ is isomorphic onto for $q\leqq r+1$ if $n'=1$, and for $q\leqq Min\ (r+n'-1, 2r+1)$ if $n'>1$.*

Proof. By (2. 4), (2. 8) and (3. 1), it is sufficient for this purpose to prove that

$$\text{(F)}\qquad H^q(\overline{SP}_n)=0\ \begin{cases}\text{for } q\leqq r+1 \text{ if } n'=1,\\ \text{for } q\leqq \text{Min } (r+n'-2, 2r+1) \text{ if } n'>1.\end{cases}$$

If $n'=1$, (F) is obvious from (4. 4). Let $n'>1$. Then it follows from (ii) of (4. 5) that $\Theta'_{2n'}\ \Theta'_{4n'}\cdots\ \Theta'_n : H^q(\overline{SP}_n)\rightarrow H^q(\overline{SP}_{n'})$ is isomorphic into for $q\leqq$ Min $(r+2n'-2, 2r+1)$, and that $H^q(\overline{SP}_{n'})=0$ for $q\leqq$ Min $(r+n'-2, 2r+1)$. This proves (F) for $n'>1$. Thus we have proved the theorem.

Especially we have

COROLLARY (4. 7). *Let X be homologically $(r-1)$-connected $(r\geqq 2)$. Then, for any $n\geqq 1$, SP_n is homologically $(r-1)$-connected*[9] *and it holds that*

$$\iota^*_{1,n} : H^q(SP_n)\approx H^q(X)\ \textit{for}\ q=r,\ r+1.$$

5. Homology of $\overline{F}_n(X)$ —— Proof of (4. 1)

Before we proceed to the proof of (4. 1), we make some algebraic preparations.

Let

$$\Pi[n]$$

consist of all partitions[10] $u, v, \cdots$ of the set $[n]$ of the integers $1, 2, \cdots, n$. Define in $\Pi[n]$ a partial order $>$ as follows: If u is a refinement of v, then $u\geqq v$. With this order, $\Pi[n]$ is a lattice.

If integers $i, j\in[n]$ are contained in the same subclass of a partition $u\in\Pi[n]$, we shall write $i\equiv j\ (u)$. The following result on the meet $u\cap v$ of two partitions $u, v\in\Pi[n]$ will be obvious: $i\equiv j\ (u\cap v)$ if and only if there is either $k\in[n]$ such that $i\equiv k\ (u)$ and $k\equiv j\ (v)$, or $l\in[n]$ such that $i\equiv l\ (v)$ and $l\equiv j\ (u)$.

Let $U_1, U_2, \cdots, U_h$ be the disjoint subclasses into which a partition u divides $[n]$. Then h is called the *height* of u, and is denoted by $h(u)$. Obviously, u with $h(u)=1$ is the minimal element of $\Pi[n]$, and u with $h(u)=n$ is the maximal element of $\Pi[n]$.

Given a partition $u=\{U_1 U_2, \cdots, U_h\}\in\Pi[n]$, define $Su\in\Pi[n+1]$ by

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9) S. D. Liao gives in [6] a proof of that if X is $(r-1)$-connected then so is SP_n.

10) By a partition of a set M, we mean a division of M into non-overlapping subclasses.

$$Su = \{U_1, U_2, \cdots, U_h, U_{h+1}\},$$

where U_{h+1} denotes the set $\{n+1\}$ of the single element $n+1$. This yields a correspondence

$$S: \Pi[n] \to \Pi[n+1].$$

The following is immediate.

LEMMA (5. 1). $S(u \cap v) = Su \cap Sv$. $h(Su) = h(u)+1$. *If* $u \neq v$, *then* $Su \neq Sv$.

Let $a \in [n]$ be any integer. Then we define a correspondence

$$^aS: \Pi[n] \to \Pi[n+1]$$

as follows: Let $u = \{U_1, U_2, \cdots, U_h\} \in \Pi[n]$, and $U_{\alpha(a)}$ denote the subclass containing a. Then ^aSu is given by

$$^aSu = \{U_1, \cdots, U_{\alpha(a)-1}, U'_{\alpha(a)}, U_{\alpha(a)+1}, \cdots, U_h\},$$

where $U'_{\alpha(a)}$ denotes the union of the sets $U_{\alpha(a)}$ and $\{n+1\}$. Immediately we have

LEMMA (5. 2). $^aS(u \cap v) = {}^aSu \cap {}^aSv$. $h(^aSu) = h(u)$. *If* $u \neq v$, *then* $^aSu \neq {}^aSv$.

Each sequence

$$\Phi = (u_1, u_2, \cdots, u_l)$$

of elements u_i $(i=1,2,\cdots,l)$ of $\Pi[n]$ is called the *sequeuce in* $\Pi[n]$. If $u_1, u_2, \cdots, u_l$ are mutually distinct, the sequence Φ is said to be *proper*. Let $l \geqq 2$, and $\Phi = (u_1, u_2, \cdots, u_l)$ a proper sequence in $\Pi[n]$. Then we define, for each i $(2 \leqq i \leqq l)$, a new proper sequence $D_i\Phi$ in $\Pi[n]$ as follows:

$$D_i\Phi = (u_1 \cap u_i, u_2 \cap u_i, \cdots, u_{i-1} \cap u_i)$$

with a convention: $u_k \cap u_i$ is omitted whenever $u_k \cap u_i = u_j \cap u_i$ for some $j<k$. A sequence $\Phi = (u_1, u_2, \cdots, u_l)$ with $h(u_1) = h(u_2) = \cdots = h(u_l)$ is said to be *homogeneous*. By the height $h(\Phi)$ of a homogeneous sequence Φ is meant the height of each element of Φ.

By making use of D_i, we define now the terminology *'regular sequence'* recursively as follows: Every (proper, homogeneous) sequence of a single element is regular. Especially a proper sequence of height 1 is regular. From here proceed by induction, and assume that it has been defined that a proper homogeneous sequence of height $h-1$ is regular. Let Φ be a proper homogeneous sequence of height h. Then Φ is said to be regular if each $D_i\Phi$ is a homogeneous sequence of height $h-1$ and is regular.

Since the meet of two distinct partitions of height 2 is always the minimal element, we have obviously

LEMMA (5. 3). *Every proper homogeneous sequence of height* 2 *is regular.*

REMARK. As is easily seen, a proper homogenecus sequence of height 3 is not necessarily regular.

Given a proper homogeneous sequence $\Phi = (u_1, u_2, \cdots, u_l)$ in $\Pi[n]$, define sequences $S\Phi$ and $^aS\Phi$ as follows:

$$S\Phi = (Su_1, Su_2, \cdots, Su_l),$$
$$^aS\Phi = (^aSu_1, {}^aSu_2, \cdots, {}^aSu_l).$$

Then it follows at once from (5. 1) and (5. 2) that both $S\Phi$ and ${}^aS\Phi$ are proper homogeneous sequence in $\Pi[n+1]$, and $h(S\Phi)=h(\Phi)+1$, $h({}^aS\Phi)=h(\Phi)$. We shall prove

LEMMA (5. 4). *If Φ is regular, so are $S\Phi$ and ${}^aS\Phi$.*

Proof. The proof is done by induction on the height h of Φ. If $h=1$, the lemma is clear. Assume, inductively, that the lemma has been proved for every regular sequence of height $h-1$, and let $\Phi=(u_1, u_2, \cdots, u_l)$ be a regular sequence of height h. We may assume $l\geqq 2$. Then it follows from the definition that each $D_i\Phi$ $(2\leqq i\leqq l)$ is a regular sequence of height $h-1$. Therefore, by the assumption of induction, $SD_i\Phi$ (resp. ${}^aSD_i\Phi$) is the regular sequence of height h (resp. $h-1$). However it follows from (5. 1) that

$$\begin{aligned} SD_i\Phi &= \big(S(u_1\cap u_i), S(u_2\cap u_i), \cdots, S(u_{i-1}\cap u_i)\big) \\ &= (Su_1\cap Su_i, Su_2\cap Su_i, \cdots, Su_{i-1}\cap Su_i) \\ &= D_iS\Phi, \end{aligned}$$

and similarly from (5. 2) that

$${}^aSD_i\Phi = D_i\,{}^aS\Phi.$$

Thus each $D_iS\Phi$ (resp. $D_i{}^aS\Phi$) is the regular sequence of height h (resp. $h-1$), so that $S\Phi$ (resp. ${}^aS\Phi$) is regular by the definition. This completes the proof.

Let

$$\beta(s, t) = (t-2)(t-1)/2+s.$$

Then it is easily seen that, for a given integer $i>0$, there is a unique system (s, t) of integers such that $\beta(s, t)=i$ and $0<s<t$.[11] Define a partition $w_i^n\in\Pi[n]$ $(i=1,2,\cdots,\beta(n-1,n)$ and $n\geqq 2)$ as follows:

(5. 5) $$w_i^n = \big\{\{1\}, \cdots, \{s-1\}, \{s, t\}, \{s+1\}, \cdots, \{t-1\}, \{t+1\}, \cdots, \{n\}\big\},$$

where $i=\beta(s,t)$ and $s<t$. Then $h(w_i^n)=n-1$. Conversely, it is obvious that every $u\in\Pi[n]$ of height $n-1$ has such a form. As a direct consequence of the definitions, we have

LEMMA (5. 6). *Let $1\leqq s<t\leqq n$ and $1\leqq a\leqq n$, then $w^{n+1}_{\beta(s,t)} = Sw^n_{\beta(s,t)}$, $w^{n+1}_{\beta(s,t)}\cap w^{n+1}_{\beta(a,n+1)} = {}^aSw^n_{\beta(s,t)}$.*

Define a sequence $Q[n]$ by putting

(5. 7) $$Q[n] = (w_1^n, w_2^n, \cdots, w_i^n, \cdots, w^n_{\beta(n-1,n)}).$$

Then $Q[n]$ is both proper and homogeneous, and $h(Q[n])=n-1$. We shall prove

LEMMA (5. 8). *$Q[n]$ is regular.*

Proof. The proof is done by induction on n. The lemma for $n=2$ is trivial. Assuming the validity of (5. 8) for $n=k$, we shall prove that $Q[k+1]$ is regular. For this purpose, it is sufficient to prove that

$$D_iQ[k+1] = (w_1^{k+1}\cap w_i^{k+1}, w_2^{k+1}\cap w_i^{k+1}, \cdots, w_{i-1}^{k+1}\cap w_i^{k+1})$$

$(2\leqq i\leqq\beta(k, k+1))$ is a regular sequence of height $k-1$. Let $i=\beta(s, t)$, where $s<t$.

11) $\beta(1, 2)=1$, $\beta(1, 3)=2$, $\beta(2, 3)=3$, $\beta(1, 4)=4$, $\beta(2, 4)=5$, $\beta(3, 4)=6$, $\cdots$.

Case I: $t \leqq k$.

It follows from (5. 6) that $w_j^{k+1} = Sw_j^k$ for any $j \leqq i$. Therefore we have by (5. 1)

$$\begin{aligned} D_i Q[k+1] &= (Sw_1^k \cap Sw_i^k, Sw_2^k \cap Sw_i^k, \cdots, Sw_{i-1}^k \cap Sw_i^k) \\ &= \left(S(w_1^k \cap w_i^k), S(w_2^k \cap w_i^k), \cdots, S(w_{i-1}^k \cap w_i^k)\right) \\ &= SD_i Q[k]. \end{aligned}$$

Since $Q[k]$ is regular by the assumption of induction, each $D_i Q[k]$ is a regular sequence of height $k-2$. Therefore it follows from (5. 4) that $D_i Q[k+1] = SD_i Q[k]$ is a regular sequence of height $k-1$.

Case II: $t = k+1$.

If $j < k+1$ and $j < s$, then

$$w_{\beta(j,k+1)}^{k+1} \cap w_{\beta(s,k+1)}^{k+1} = w_{\beta(j,s)}^{k+1} \cap w_{\beta(s,k+1)}^{k+1}.$$

Therefore it follows from the definition of D_i that

$$D_i Q[k+1] = (w_1^{k+1} \cap w_{\beta(s,k+1)}^{k+1}, w_2^{k+1} \cap w_{\beta(s,k+1)}^{k+1}, \cdots, w_{\beta(k-1,k)}^{k+1} \cap w_{\beta(s,k+1)}^{k+1}).$$

Further it follows from (5. 6) that

$$\begin{aligned} D_i Q[k+1] &= ({}^sSw_1^k, {}^sSw_2^k, \cdots, {}^sSw_{\beta(k-1,k)}^k) \\ &= {}^sSQ[k]. \end{aligned}$$

By the assumption of induction, $Q[k]$ is a regular sequence of height $k-1$. Therefore we obtain by (5. 4) that $D_i Q[k+1] = {}^sSQ[k]$ is a regular sequence of height $k-1$. This completes the proof.

We return here to a topological consideration. We retain the usage of the notations in the above sections.

Given $u \in \Pi[n]$, define a subset $M(u) \subset P_n(X)$ by putting

$$M(u) = \{(x_1, x_2, \cdots, x_n) \in P_n \mid x_i = x_j \text{ if } i \equiv j\ (u)\},$$

and write

$$\overline{M}(u)$$

for the image of $M(u)$ by t_n. It is easily verified that $\overline{M}(u)$ is a subcomplex of $\overline{P}_n(X)$. Let $u = \{U_1, U_2, \cdots, U_h\}$, and let l_j $(1 \leqq j \leqq h)$ denote the least integer in U_j. Without loss of generality, we may assume that $l_1 < l_2 < \cdots < l_h$.

Define a continuous map

$$\rho : M(u) \to P_h$$

by

$$\rho(x_1, x_2, \cdots, x_n) = (x_{l_1}, x_{l_2}, \cdots, x_{l_h}).$$

Obviously ρ is a homeomorphism of $M(u)$ onto P_h, and ρ maps $M(u) \cap P_n'$ onto P_h'. Therefore $\rho : (M(u), M(u) \cap P_n') \to (P_h, P_h')$ is a relative homeomorphism, and hence so is the map $\overline{\rho} : (\overline{M}(u), (*_n)) \to (\overline{P}_h, (*_h))$ defined by ρ. This, together with (1. 3) and (3. 1), implies

LEMMA (5. 9). *Let X be homologically $(r-1)$-connected, and $u \in \Pi[n]$. Then $\overline{M}(u)$ is homologically $(hr-1)$-connected, where $h = h(u)$.*

As a direct consequence of the definition, we have

LEMMA (5. 10). *For any $u, v \in \Pi[n]$, it holds that $\bar{M}(u \cap v) = \bar{M}(u) \cap \bar{M}(v)$. where $\cap$ in the right stands for the intersection of the sets.*

Extend the definition of $\bar{M}$ to every sequence $\Phi = (u_1, u_2, \cdots u_l)$ in $\Pi[n]$ by setting

$$\bar{M}(\Phi) = \bar{M}(u_1) \cup \bar{M}(u_2) \cup \cdots \cup \bar{M}(u_l),$$

where $\cup$ denotes the union of the sets. Using the Mayer-Vietoris sequence of homology groups [4], we shall prove the following:

LEMMA (5. 11). *Let X be homologically $(r-1)$-connected, and Φ a regular sequence in $\Pi[n]$. Then $\bar{M}(\Phi)$ is homologically $(r+h-2)$-connected, where $h=h(\Phi)$.*

Proof. We shall give the proof by induction on h. The lemma is clear from (5. 9) if $h=1$. Assume the the validity of (5. 11) for every regular sequence of height $h-1$, and let $\Phi=(u_1, u_2, \cdots, u_l)$ be a regular sequence of height h.

Case I: $l=1$.

It follows from (5. 10) that $\bar{M}(\Phi)=\bar{M}(u_1)$ is homologically $(hr-1)$-connected. Since $hr-1 \geqq r+h-2$, $\bar{M}(\Phi)$ is homologically $(r+h-2)$-connected.

Case I: $l \geqq 2$.

By the definition, each $D_i\Phi = (u_1 \cap u_i, u_2 \cap u_i, \cdots, u_{i-1} \cap u_i)$ is a regular sequence of height $h-1$. Therefore, by the assumption of induction, we have

(A) $\bar{M}(D_i\Phi) (2 \leqq i \leqq l)$ is $(r+h-3)$-connected.

Let Φ_i $(1 \leqq i \leqq l)$ denote the subsequence $(u_1, u_2, \cdots, u_i)$ of Φ. Then we have

$$\bar{M}(\Phi_{i-1}) \cup \bar{M}(u_i) = \bar{M}(\Phi_i),$$

$$\begin{aligned} &\bar{M}(\Phi_{i-1}) \cap \bar{M}(u_i) \\ &= (\bar{M}(u_1) \cup \cdots \cup \bar{M}(u_{i-1})) \cap \bar{M}(u_i) \\ &= (\bar{M}(u_1) \cap \bar{M}(u_i)) \cup \cdots \cup (\bar{M}(u_{i-1}) \cap \bar{M}(u_i)) \\ &= \bar{M}(u_1 \cap u_i) \cup \cdots \cup \bar{M}(u_{i-1} \cap u_i) \qquad \text{(by (5. 11))} \\ &= \bar{M}(D_i \Phi). \end{aligned}$$

Therefore the (integral coefficient) homology Mayer-Vietoris exact sequence for $(\bar{M}(\Phi_i); \bar{M}(\Phi_{i-1}), \bar{M}(u_i))$ becomes as follows:[12]

$$\cdots \to H_q(\bar{M}(\Phi_{i-1})) + H_q(\bar{M}(u_i)) \xrightarrow{\phi_{i-1}} H_q(\bar{M}(\Phi_i)) \to H_{q-1}(\bar{M}(D_i \Phi)) \to \cdots.$$

It follows from (A) that $H_{q-1}(\bar{M}(D_i\Phi)) = 0$ for $q \leqq r+h-2$, and from (5. 9) that $H_q(\bar{M}(u_i)=0$ for $q \leqq hr-1$. Therefore if $q \leqq r+h-2$, then the homomorphism ϕ_{i-1}: $H_q(\bar{M}(\Phi_{i-1})) \to H_q(\bar{M}(\Phi_i))$ is onto. This holds for $i = 1, 2, \cdots, l-1$, and $\Phi_l = \Phi$, $\Phi_1=(u_1)$. Therefore we have that $\phi_{r-1}\phi_{r-2} \cdots \phi_1 : H_q(\bar{M}(u_1)) \to H_q(\bar{M}(\Phi))$ is onto. As was shown in Case I, $H_q(\bar{M}(u_1))=0$ for $q \leqq r+h-2$. This implies that $H_q(\bar{M}(\Phi))=0$ for $q \leqq r+h-2$. Namely we conclude the proof.

We are now in a position to prove the proposition (4. 1).

Proof of (4. 1). Consider the partition w_i^n defined in (5. 5). Then it is obvious that $\bar{M}(w_i^n)$ is a subset of $\bar{F}_n(X)$ and $\bar{F}_n=\bar{M}(w_1^n) \cup \bar{M}(w_2^n) \cup \cdots \cup \bar{M}(w_{\beta(n-1,n)}^n)$. Therefore, for the sequence $Q[n]$ defined in (5. 7), we have $\bar{M}(Q[n])=\bar{F}_n$. As was

12) We take the reduced homology groups for dimension 0.

proved in (5. 8), $Q[n]$ is a regular sequence of height $n-1$. Thus (4. 1) follows directly from (5. 11).

6. Symmetric products of spheres

Take an r-sphere S^r as X and let $n=2$ or 3. By applying the method stated in § 3 to such a case, we shall in this section give another proof of some of the results which we have had in [7].[13] Throughout this section, we assume that $r \geqq 2$.

THEOREM (6. 1). *Let $p<2r$, then*

$$H^p(\overline{SP}_2(S^r)) \approx \begin{cases} Z_2 \text{ for } p=r+2k+1 \quad (k=1, 2, \cdots),^{14)} \\ 0 \quad \text{for other } p. \end{cases}$$

Proof. It is obvious that $\overline{F}_2$ is an r-sphere and $\overline{P}_2$ is a $2r$-sphere. Therefore the cohomology exact sequence for $(\overline{P}_2, \overline{F}_2)$ yields the following:

$$H^q(\overline{P}_2, \overline{F}_2) \approx \begin{cases} Z & \text{for } q=r+1, \\ 0 & \text{for } q<r+1 \text{ and } r+1<q<2r. \end{cases}$$

As was seen in the proof of (4. 4), $\mathfrak{S}_2$ operates on $H^{r+1}(\overline{P}_2, \overline{F}_2)$ trivially. Therefore, in the Cartan-Leray spectral sequence (3. 3), we have that

$$E_2^{p,q} = 0 \qquad \text{for } q<r+1 \text{ and } r+1<q<2r,$$

$$E_2^{p,r+1} = H^p(\mathfrak{S}_2; Z),$$

where $\mathfrak{S}_2$ operates on Z trivially. It is known [3] that

$$H^p(\mathfrak{S}_2; Z) \approx \begin{cases} Z & \text{if } p=0, \\ Z_2 & \text{if } p \text{ is even}>0, \\ 0 & \text{otherwise.} \end{cases}$$

We can thus prove by some usual arguments in spectral sequence that

$$H^p(\overline{SP}_2, \overline{SF}_2) = E_\infty^{p-r-1, r+1} = E_2^{p-r-1, r+1} \qquad (p<2r),$$

so that

$$H^p(\overline{SP}_2, \overline{SF}_2) \approx \begin{cases} Z & \text{if } p=r+1, \\ Z_2 & \text{if } p=r+2k+1 \qquad (k=0, 1, 2, \cdots), \\ 0 & \text{otherwise} \end{cases}$$

for $p<2r$. Since $\overline{SF}_2$ is an r-sphere, the cohomology exact sequence for $(\overline{SP}_2, \overline{SF}_2)$ gives that

$$H^p(\overline{SP}_2, \overline{SF}_2) \approx H^p(\overline{SP}_2)$$

for $0<p<r$ and $r+1<p<2r$. This proves the results for $p \neq r, r+1$. Since it follows from (4. 4) that $H^p(\overline{SP}_2) = 0$ for $p \leqq r+1$, the proof of the theorem completes.

Since $\overline{P}_1(S^n) = S^n$, we have by (2. 9), (3. 1) and (6. 1) the following:

COROLLARY (6. 2). *Let $p<2r$, then*

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13) More results than in the present paper are obtained in [7], but it is impossible for us to prove all of them by the present method.

14) We shall write Z and Z_p respectively for the group of integers and the group of integers mod p.

$$H^p(SP_2(S^r)) \approx \begin{cases} Z & \text{if } p = r, \\ Z_2 & \text{if } p = r+2k+1 \quad (k = 1, 2, \cdots), \\ 0 & \text{otherwise.} \end{cases}$$

We shall next consider the 3-fold symmetric product of S^r. For this purpose, we shall first study the homology group of $\bar{F}_3(S^r)$.

As was seen in the proof (4. 1), we have

$$\bar{F}_3(S^r) = \bar{M}(w_1^3) \cup \bar{M}(w_2^3) \cup \bar{M}(w_3^3).$$

(See § 5.) For the sake of the brevity, we write

$$\bar{F} = \bar{F}_3(S^r), \qquad \bar{N}_i = \bar{M}(w_i^3) \qquad (i = 1, 2, 3).$$

Let 0 denote the minimal element in the lattice $\Pi[3]$, then we write also

$$\bar{D} = \bar{M}(0).$$

Then $\bar{N}_i$ is a $2r$-sphere, and $\bar{D}$ is an r-sphere.

We shall prove

LEMMA (6. 3).

$$H_q(\bar{F}_3(S^r)) \approx \begin{cases} Z+Z & \text{for } q = r+1, \\ 0 & \text{for } 0<q<r+1 \text{ and } r+1<q<2r. \end{cases}$$

Proof. Consider the homology Mayer-Vietoris sequence for $(\bar{N}_2 \cup \bar{N}_3; \bar{N}_2, \bar{N}_3)$. Since $\bar{N}_2 \cap \bar{N}_3 = \bar{D}$, we have then the following exact sequence :[15]

$$\cdots\cdot \to H_q(\bar{D}) \xrightarrow{\psi} H_q(\bar{N}_2)+H_q(\bar{N}_3) \xrightarrow{\phi} H_q(\bar{N}_2 \cup \bar{N}_3) \xrightarrow{\Delta} H_{q-1}(D) \to \cdots\cdot.$$

This proves that

(A)
$$H_q(\bar{N}_2 \cup \bar{N}_3) = 0 \text{ for } 0<q<r+1 \text{ and } r+1<q<2r,$$
$$\Delta : H_{r+1}(\bar{N}_2 \cup \bar{N}_3) \approx H_r(\bar{D}) \ (\approx Z).$$

Consider next the homology Mayer-Vietoris sequence for $(\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3; \bar{N}_1, \bar{N}_2 \cup \bar{N}_3)$. Since $\bar{N}_1 \cup (\bar{N}_2 \cup \bar{N}_3) = \bar{F}$ and $\bar{N}_1 \cap (\bar{N}_2 \cup \bar{N}_3) = \bar{D}$, we have then the following exact sequence :[15]

$$\cdots\cdot \to H_q(\bar{D}) \xrightarrow{\psi} H_q(\bar{N}_1)+H_q(\bar{N}_2 \cup \bar{N}_3) \xrightarrow{\phi} H_q(\bar{F}) \xrightarrow{\Delta} H_{q-1}(\bar{D}) \to \cdots\cdot.$$

This, together with (A), proves that

$$H_q(\bar{F}) = 0 \text{ for } 0<q<r+1 \text{ and } r+1<q<2r,$$

and that the sequence

(B)
$$0 \to H_{r+1}(\bar{N}_2 \cup \bar{N}_3) \xrightarrow{\phi} H_{r+1}(\bar{F}) \xrightarrow{\Delta} H_r(\bar{D}) \to 0$$

is exact, and hence $H_{r+1}(\bar{F}) \approx Z+Z$. Thus we complete the proof of (6. 3).

REMARK. By using of the method similar as in the proof of (4. 1) stated in § 5, we can prove the follwing :

$H_q(\bar{F}_n(S^r))=0$ *for* $0<q<r+n-2$ *and* $r+n-2<q<2r+n-3$; $H_{r+n-2}(\bar{F}(S^r))$ *is a finitely generated free abelian group.*

Let (i, j, k) be any permutation of $(1, 2, 3)$, and consider the diagram :

15) We use the same notations as in p. 39 of [4].

$$H_r(\bar{D}) \xleftarrow{\partial_i} H_{r+1}(\bar{N}_i, \bar{D}) \xrightarrow{l_{ij*}} H_{r+1}(\bar{N}_i \cup \bar{N}_j, \bar{N}_j)$$
$$\xleftarrow{m_{ij*}} H_{r+1}(\bar{N}_i \cup \bar{N}_j) \xrightarrow{n_{k*}} H_{r+1}(\bar{F}),$$

where ∂_i is the boundary homomorphism, and $l_{ij}*$, m_{ij*}, n_{k*} are the homomorphisms induced by the inclusion maps. Then ∂_i, l_{ij*} and m_{ij*} are onto-isomorphisms. Write

$$\sigma_{ij} = n_{k*} m_{ij*}^{-1} l_{ij*} \partial_i^{-1}.$$

Then we can prove the following by some usual arguments in homology theory [4].

LEMMA (6. 4). $\qquad \sigma_{ij} = -\sigma_{ji}.$

Let $s^r \in H_r(\bar{D})$ denote a generator, and write

$$e_k = \sigma_{ij}(s^r) \in H_{r+1}(\bar{F}_3(S^r)) \qquad (i<j).$$

Then we have

LEMMA (6. 5). *$H_{r+1}(\bar{F}_3(S^r))$ is generated by e_1 and e_2.*

Proof. Recall the definitions of Δ and ϕ in the Mayer-Vietoris sequence [4]. Then it follows that the isomorphism Δ in (A) can be written explicitely as

$$\partial_2 l_{23*}^{-1} m_{23*},$$

so that the image of ϕ in (B) is generated by

$$n_{1*} (\partial_2 l_{23*}^{-1} m_{23*})^{-1} (s^r) = \sigma_{23}(s^r) = e_1.$$

Therefore if we can show that

$$\Delta\sigma_{13} = \text{the identity isomorphism of } H_r(\bar{D})$$

for the homomorphism Δ in (B), the proof is complete. This is proved as follows. Consider the commutative diagram

$$\begin{array}{ccc}
H_{r+1}(\bar{N}_1 \cup \bar{N}_3) & \xrightarrow{n_{2*}} & H_{r+1}(\bar{F}) \\
\downarrow{\scriptstyle m_{13*}} & & \downarrow{\scriptstyle \beta_*} \\
H_{r+1}(\bar{N}_1 \cup \bar{N}_2 \cup \bar{N}_3) & \xrightarrow{\gamma_*} & H_{r+1}(\bar{F}, \bar{N}_1 \cup \bar{N}_2) \\
\nwarrow{\scriptstyle l_{13*}} & & \nearrow{\scriptstyle \alpha_*} \\
& H_{r+1}(\bar{N}_1, \bar{D}) & \\
& \downarrow{\scriptstyle \partial_1} & \\
& H_r(\bar{D}) &
\end{array}$$

where α_*, β_* and γ_* are the homomorphisms induced by the inclusion maps. Then α_* is an onto-isomorphism, and the homomorphism Δ in (B) can be written explicitely as

$$\partial_1 \alpha_*^{-1} \beta_*.$$

Therefore we have

$$\Delta\sigma_{13} = (\partial_1 \alpha_*^{-1} \beta_*)(n_{2*} m_{13*}^{-1} l_{13*} \partial_1^{-1})$$
$$= \partial_1 \alpha_*^{-1} \gamma_{13*} l_{13*} \partial_1^{-1} \qquad (\text{by } \beta_* n_{2*} = \gamma_* m_{13*})$$
$$= \text{the identity} \qquad (\text{by } \gamma_* l_{13*} = \alpha_*).$$

Thus we complete the proof.

Let S and T denote the elements of $\mathfrak{S}_3$ defined as follows:

$$S=\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix},\quad T=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}.$$

Then both S and T are the identity on $\bar{D}$, and

$$S(\bar{N}_1)=\bar{N}_1,\quad S(\bar{N}_2)=\bar{N}_3,\quad S(\bar{N}_3)=\bar{N}_2,$$
$$T(\bar{N}_1)=\bar{N}_2,\quad T(\bar{N}_2)=\bar{N}_3,\quad T(\bar{N}_3)=\bar{N}_1.$$

Therefore, by using of some commutative diagrams, we can easily assert that

$$S_*\sigma_{12}=\sigma_{13},\quad S_*\sigma_{13}=\sigma_{12},\quad S_*\sigma_{23}=\sigma_{32},$$
$$T_*\sigma_{12}=\sigma_{23},\quad T_*\sigma_{13}=\sigma_{21},\quad T_*\sigma_{23}=\sigma_{31}.$$

This, together with (6. 4), implies that

(6. 6) $$S_*(e\ )=e_2,\quad S_*(e_2)=e_3,\quad S_*(e_1)=-e_1,$$
$$T_*(e_3)=e_1,\quad T_*(e_2)=-e_3,\quad T_*(e_1)=-e_2.$$

We shall prove

LEMMA (6. 7). $e_3=-e_1+e_2$.

Proof. It follows from (6. 3) and (6. 5) that e_3 can be represented as

$$e_3=p\,e_1+q\,e_2\qquad (p,q:\text{integers}).$$

Then we have by (6. 6) that

$$e_1=T_*(e_3)=T_*(p\,e_1+q\,e_2)=-p\,e_2-q\,e_3$$
$$=-p\,e_2-q\,(p\,e_1+q\,e_2)=-p\,q\,e_1-(p+q^2)\,e_2.$$

This impies that

$$p\,q=-1 \text{ and } p+q^2=0,$$

so that $p=-1$ and $q=1$. Thus we have (6. 7).

From (6.6) and (6.7), we obtain

LEMMA (6. 8). $S_*(e_1)=-e_1,\quad S_*(e_2)=-e_1+e_2,$
$T_*(e_1)=-e_2,\quad T_*(e_2)=e_1-e_2.$

We shall here pass to the cohomology. It follows from (6. 3) that

$$H^q(\bar{F}_3(S^r))\approx\begin{cases}Z+Z & \text{for } q=r+1,\\ 0 & \text{for } 0<q<r+1 \text{ and } r+1<q<2r.\end{cases}$$

Since $H^q(\bar{P}_3(S^r))=0$ for $0<q<3r$ it follows that

$$\delta: H^q(\bar{F}_3(S^r))\approx H^{q+1}(\bar{P}_3(S^r),\ \bar{F}_3(S^r))$$

for $0<q<3r-1$. Denote by $e_1^*, e_2^*\in H^{r+1}(\bar{F}_3(S^r))=\mathrm{Hom}\,(H_{r+1}(\bar{F}_3(S^r)),Z)$ the dual of e_1 and e_2 respectively, and write

$$c_i=\delta\,e_i^*\in H^{r+2}(\bar{P}_3(S^r),\bar{F}_3(S^r))\qquad (i=1,2).$$

Then the following is obvious from the above consideration.

PROPOSITION (6. 9).

$$H^q(\bar{P}_3(S^r),\bar{F}_3(S^r))\approx\begin{cases}Z+Z & \textit{for } q=r+2,\\ 0 & \textit{for } q<r+2 \textit{ and } r+2<q<2r.\end{cases}$$

Further $H^{r+2}(\bar{P}_3(S^r),\bar{F}_3(S^r))$ *is generated by* c_1 *and* c_2.

By (6. 8) and the naturality of δ, we can assert easily that

$$S^*(c_1)=-c_1-c_2,\quad S^*(c_2)=c_2,$$
$$T^*(c_1)=c_2,\qquad T^*(c_2)=-c_1-c_2.$$

Therefore the definition (3. 2) gives

PROPOSITION (6. 10). *The group $\mathfrak{S}_3$ operates on $H^{r+2}(\bar{P}_3(S^r), \bar{F}_3(S^r))$ as follows*:

$$S(c_1) = -c_1 - c_2, \quad S(c_2) = c_2,$$
$$T(c_1) = -c_1 - c_2, \quad T(c_2) = c_1.$$

We shall prove

THEOREM (6. 11). *Let $p \leqq 2r-1$, then*

$$H^p(\overline{SP}_3(S^r)) = \begin{cases} Z_3 & \text{if } p = r+4k+1 \quad (k = 1, 2, \cdots), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By making use of the theorem 1 of Chap III in [5], or of the projective resolution for $\mathfrak{S}_3$ given in § 7, we can compute the cohomology groups of $\mathfrak{S}_3$ with coefficients in the $\mathfrak{S}_3$-group $H^{r+2}(\bar{P}_3(S^r), \bar{F}_3(S^r))$ described in (6. 10). This computation is straightforward, and is left to the reader. The result is as follows:

$$H^p(\mathfrak{S}_3; H^{r+2}(\bar{P}_3(S^r), \bar{F}_3(S^r))) \approx \begin{cases} Z_3 & \text{if } p = 4k+3 \quad (k = 1, 2, \cdots), \\ 0 & \text{otherwise.} \end{cases}$$

Consider the Cartan-Leray spectral sequence (3. 3), then it follows from (6. 9) and the fact just stated that

$$E_2^{p,q} = 0 \quad \text{for } q < r+2 \text{ and } r+2 < q \leqq 2r,$$

$$E_2^{p,r+2} \approx \begin{cases} Z_3 & \text{for } p = 4k+3 \quad (k = 0, 1, 2, \cdots), \\ 0 & \text{for other } p. \end{cases}$$

Thus, by some usual arguments in spectral sequence, we can prove that if $p \leqq 2r$ then

$$H^p(\overline{SP}_3(S^r), \overline{SF}_3(S^r)) = E_\infty^{p-r-2,r+2} = E_2^{p-r-2,r+2}$$
$$\approx \begin{cases} Z_3 & \text{for } p = r+4k+1 \quad (k = 1, 2, \cdots), \\ 0 & \text{for other } p. \end{cases}$$

Since it follows from (3. 5) that $H^p(\overline{SF}_3(S^r)) = 0$ for $0 < p < 2r$, the exact sequence for $(\overline{SP}_3(S^r), \overline{SF}_3(S^r))$ yields that

$$H^p(\overline{SP}_3(S^r), \overline{SF}_3(S^r)) \approx H^p(\overline{SP}_3(S^r)) \text{ for } p < 2r.$$

This, together with the above fact, proves the theorem.

As a direct consequence of (2. 9), (3. 1), (6. 1) and (6. 11), we have

COROLLARY (6. 12). *Let $p < 2r$, then*

$$H^p(SP_3(S^r)) \approx \begin{cases} Z & \text{for } p = r, \\ Z_6 & \text{for } p = r+4k+1 \quad (k = 1, 2, \cdots), \\ Z_2 & \text{for } p = r+4k+3 \quad (k = 0, 1, \cdots), \\ 0 & \text{for other } p. \end{cases}$$

we shall finally prove

THEOREM (6. 13). *It holds that*

$$H^{r+2}(SP_n(S^r)) = 0 \text{ for } r \geqq 3 \text{ and } n \geqq 1.$$

Proof. Note first that

$$H^{r+2}(\overline{SP}_n(S^r)) = 0 \text{ for } r \geqq 3 \text{ and } n = 2 \text{ or odd.}$$

In fact, this is trivial for $n = 1$, and follows from (6. 1) for $n = 2$, from (6. 11) for $n=3$ and from (i) of (4. 5) for odd $n \geqq 5$. Recall next (ii) of (4. 5). Then it follows from the above fact inductively that

$$H^{r+2}\left(\overline{SP}_n(S^r)\right)=0 \text{ for } r\geqq 3 \text{ and } n\geqq 1.$$

This, together with (2. 9), yields the theorem.

7. Miscellany

I. *Projective resolution for the group* $\mathfrak{S}_3$.

For each integer $q\geqq 0$, construct an $\mathfrak{S}_3$-free abelian group K_q having as an $\mathfrak{S}_3$-basis a set of $q+1$ abstract elements $e_{0,q}, e_{1,q-1}, \cdots, e_{q,0}$. Define a homomorphism $\partial : K_q\to K_{q-1}$ by

$$\partial(e_{2i,j})=(1+T+T^2)\,e_{2i-1,j}+\left(1+(-1)^{i+j}S\right)e_{2i,j-1},$$

$$\partial(e_{2i+1,j})=(1-T)\,e_{2i,j}-\left(1-(-1)^{i+j}TS\right)e_{2i+1,j-1},$$

where $S,T\in\mathfrak{S}_3$ are the elements defined in § 6. Then the verification that $\partial\partial=0$ is straightforward. Therefore we have an $\mathfrak{S}_3$-complex $K=\{K_q,\partial\}$ which is $\mathfrak{S}_3$-free. Further we can easily prove that K is acyclic, by using the following contracting homotopy h : Let $k=0,1,2,$ and $l=0,1,2,3,$ then

$$h(T^k e_{i,j})=0 \quad \text{if } j>0,$$

$$h(T^k e_{2i,0})=\begin{cases}0 & \text{if } k=0,\\ -e_{2i+1,0} & \text{if } k=1,\\ -(1+T)\,e_{2i+1,0} & \text{if } k=2,\end{cases}$$

$$h(T^k e_{2i+1,0})=\begin{cases}0 & \text{if } k=0,1,\\ e_{2i+2,0} & \text{if } k=2,\end{cases}$$

$$h(T^kS\,e_{4i+l,j})=(-1)^{[l/2]+j+1}T^{k+1-(-1)^l}e_{4i+l,j+1} \quad \text{if } j>0,$$

$$h(T^kS\,e_{4i+l,0})=(-1)^{[l/2]+1}T^{k+1-(-1)^l}e_{4i+l,1}+(-1)^{[(l+1)/2]}h\left(T^{k+1-(-1)^l}e_{4i+l,0}\right),$$

where $[m]$ stands for the greatest integer $\leqq m$. Thus K is a projective resolution for $\mathfrak{S}_3$.

II. We have seen in (3. 3) that there is the spectral sequence relating $\left(\bar{P}_n(X), \bar{F}_n(X)\right)$ to $\left(\overline{SP}_n(X), \overline{SF}_n(X)\right)$.

The same holds between $(P_n(X),F_n(X))$ and $(SP_n(X),SF_n(X))$. Namely we can assert the following : There exists a cohomology spectral sequence (E_r) in which the term $E_2^{p,q}$ is isomorphic with $H^p\left(\mathfrak{S}_n;\ H^q(P_n(X), F_n(X);\ G)\right)$, and $E_\infty^{p,q}$ is isomorphic with the graduated group associated with $H^{p+q}(SP_n(X), SF_n(X);\ G)$, appropriately filtered. By using this spectral sequence, we can prove that if $p<2r$ then

$$H^p\left(SP_3(S^r), SF_3(S^r);\ Z\right)\approx H^{p-r-1}(\mathfrak{S}_3;\ Z),$$

where $\mathfrak{S}_3$ operates on Z trivially. From this, we can also obtain (6. 12).

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—— 125 頁 下段 9 行目を下記の如く訂正下さい. ——

$$\beta \psi_{i_1, i_2, \cdots, i_m} = \psi_{i_{\beta(1)}, i_{\beta(2)}, \cdots, i_{\beta(m)}} \; (\beta \in \mathfrak{S}_m),$$

we have

$$\psi^{\divideontimes}_{i_{\beta(1)}, i_{\beta(2)}, \cdots, i_{\beta(m)}}(c) = \psi^{\divideontimes}_{i_1, i_2, \cdots, i_m} \beta^{\divideontimes}(c) = \psi^{\divideontimes}_{i_1, i_2, \cdots, i_m}(c)$$