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Plancherel's Theorem on General Locally Compact Groups

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Introduction: The Plancherel's theorem which was established on an abelian LC (locally compact) group by M. Krein, D. Raikov, H. Cartan and R. Godement, etc., has been developed in recent years to the case that groups are unimodular LC; F. I. Mautner[1] and I. E. Segal[2] have constructed their theories under a little restriction, that is, the separability condition on account of their method of using von Neumann's reduction theory.[3] More recently, R. Godement[4] and otherwise H. Sunouchi[5] have constructed their theories excepting such restriction and availing themselves of Radon's measure to the full.

Our object of this paper is, however, to pursue the same theme in a general situation without both separability and unimodularity in refering to the theory of functionals on B-algebras, established in the author's pervious paper.[6]

§1. Let G be a general LC group with left-invariant Haar's measure dx, and $L(G)$ the group algebra over G with respect to the measure dx in the sense of I. E. Segal; $L(G)$ is considered as an involutive Banach algebra (B-algebra) over the complex field $\mathfrak{K}$ with the norm $\| f \| = \int_G |f(x)| dx$, multiplication

$$(1.1) \qquad fg(x) = \int_G f(y) g(y^{-1}x) dy \text{ (convolution),}$$

and conjugation $f^*(x) = \overline{f(x^{-1})} \cdot \rho(x)$, where $\rho(x)$ is the Weil's density of right-invariant Haar's measure, $dx^{-1} = \rho(x) dx$.[7]

We see easily that such $*$-operation is conjgate linear, involutely, anti-automorphism on $L(G)$; $(\alpha f+g)^* = \bar{\alpha} f^* + g^*$ for $\alpha \in \mathfrak{K}$, $f^{**} = f$, $(fg)^* = g^* f^*$ and, in addition, $\| f^* \| = \| f \|$. If $f = f^*$, we call f *self-adjoint* or *hermitian* and the set of all hermitian elements the *hermitian kernel* of $L(G)$, denoting by $H(G)$.

$H(G)$ is not necessarily a sub-algebra of $L(G)$ but is a vector subspace of it over the reals $\mathfrak{R}$.[8]

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1)～5) See the bibligraphy at the end of the paper.

6) The author's work; 2).

7) *A. Weil*, L'intègration dans les groupes topologiques et ses applications (Actual. Scient. et Ind.), n° 869, Paris, 1940, p. 40 and *I. E. Segal* 5) pp. 76-77.

8) $H(G)$ forms a commutative, distributive, but non-associative normed algebra by the product $f \circ g = \frac{1}{2}(fg + fg)$; evidently $f \circ f = f^2$ for $f \in H(G)$.

In other words, $H(G)$ is a *special Jordan algebra* with the operation $\circ$,

Lemma 1. *Denoting the collection of all such linear functionals* φ *as satisfy the condition i) or ii)*

i) $\varphi(f^*f)\geqq 0$ *and* $\varphi(f^*)=\overline{\varphi(f)}$ *for* $f\in L(G)$,

ii) $2|\varphi([f,g])|\leqq\varphi(f^2)+\varphi(g^2)$ *for* $f,g\in H(G)$,

by $\hat{\Pi}(L(G))$ *or* $\tilde{\Pi}(H(G))$ *respectively, it holds that*

a) $\hat{\Pi}(L(G))\equiv\tilde{\Pi}(H(G))$,

b) $\hat{\Pi}(L(G))$ *is closed and convex in the conjugate space of* $L(G)$ *as a vector space with respect to the weak topology as functionals, and so is* $\tilde{\Pi}(H(G))$ *in that of* $H(G)$.

Here $[f,g]=\frac{1}{2i}(fg-gf)$ (special Poisson's product) for $f,g\in H(G)$ and the sign $\equiv$ means the biunique topological correspondence between the both terms in such a manner that each functional φ in the right has a unique extension $\hat{\varphi}$ over the whole $L(G)$ which coincides with the corresponding one in the left.

The unit sphere E_0 in the conjugate space of $H(G)$ is weakly compact owing to S. Kakutani-Dieudonné and convex,[1] so that the intersection

$$\tilde{E}_0\equiv E_0\cap\tilde{\Pi}(H(G)) \tag{1.1}$$

is also weakly compact and convex, and is moreover metrically bounded, $\leqq 1$, that is, $\tilde{E}_0$ is regularly convex and hence has sufficiently many extreme points, whose convex hull being weakly dense in $\tilde{E}_0$, due to the noted theorem of Krein and Milman. The totality of non-zero extreme points of $\tilde{E}_0$ is denoted by $S(\tilde{E}_0)$ and the origine (zero functional) by θ; the closure V_0^a of $V_0=S(\tilde{E}_0)\cup(\theta)$ is compact, so that $\tilde{V}_0=V_0^a-(\theta)$ is locally compact. We call $\tilde{V}_0$ the *character space* of G (if G is abelian, $\tilde{V}_0$ coincides with the dual group $\hat{G}$).

Lemma 2. *For every* φ *in* $\hat{\Pi}(L(G))$, *the set of all such elements* f *that* $\varphi(g^*f)=0$ *for all* $g\in L(G)$ *forms a closed left ideal* $'\mathfrak{J}_\varphi$ *of* $L(G)$, *for which the quotient space* $L(G)/'\mathfrak{J}_\varphi$ *forms a pre-Hilbert space with the inner product defined by*

$$(X_f,\ X_g)_\varphi=\varphi(g^*f)\,. \tag{1.2}$$

where X_f *is the residue class of* $L(G)/'\mathfrak{J}_\varphi$ *containing* $f\in L(G)$.

The proofs of Lemma 1 and 2 are not so much difficult and are omitted; about the details, see 2).

Lemma 3. *The set of all continuous positive definite (c. p. d.) functions on* G, $\mathfrak{P}(G)$, *is one-to-one corresponding to* $\hat{\Pi}(L(G))$ *and so to* $\tilde{\Pi}(H(G))$; *if* $\varphi\to\xi_\varphi$ *and* $\xi\to\varphi_\xi$, *where* $\varphi,\varphi_\xi\in\hat{\Pi}(L(G))$ *and* $\xi,\xi_\varphi\in\mathfrak{P}(G)$), *it holds*

i) $\varphi_{\xi_\varphi}=\varphi.\qquad \xi_{\varphi_\xi}=\xi$,

1) S. Kakutani, Proc. Imp. Acad., 16 (1940) and J. Dieudonné, Ann. Ec. norm. Sup. ,59 (1942), Thr. 22.

ii) $\varphi_{\alpha\xi_1+\beta\xi_2}=\bar{\alpha}\varphi_{\xi_1}+\bar{\beta}\varphi_{\xi_2}$, $\quad \xi_{\alpha\varphi_1+\beta\varphi_2}=\bar{\alpha}\xi_{\varphi_1}+\bar{\beta}\xi_{\varphi_2}$,

iii) $\varphi_\xi(f)=\int_G \overline{\xi(x)}f(x)\,dx$, $\quad f\in L(G)$,

vi) $\varphi(f)=\int_G \overline{\xi_\varphi(x)}f(x)\,dx$, $\quad f\in L(G)$,

and $\xi_\varphi(x)$ is represented as follows;

v) $\xi_\varphi(x)=(X_e, U_x X_e)_\varphi$,

where U_x is the unitary operator on the Hilbert space $\mathfrak{H}_\varphi$ of the completion of $L(G)/'\mathfrak{J}_\varphi$, defined by $U_x X_f = X_{f_x}$, $f_x(\cdot)=f(x^{-1}\cdot)$ and X_e is the strong limit in $\mathfrak{H}_\varphi$ of $\{X_{e^\lambda}\}$, in which $\{e^\lambda\}$ being the approximate identity of $L(G)$.

The same or analogical result as above has been obtained by the author and and others.[1)]

Combining Lemma 1 and 3, we have a modified formular of ii) as follows,

ii)' $\check{\varphi}_{\alpha\xi_1+\beta\xi_2}=\alpha\check{\varphi}_{\xi_1}+\beta\check{\varphi}_{\xi_2}$, $\quad \xi_{\alpha\check{\varphi}_1+\beta\check{\varphi}_2}=\alpha\xi_{\check{\varphi}_2}+\beta\xi_{\check{\varphi}_2}$,

where $\check{\varphi}$ is an element of $\tilde{\Pi}(H(G))$ with the extended φ in $\hat{\Pi}(L(G))$ and $\alpha,\beta\in\mathfrak{K}$. We see easily that $\|\xi\|_\infty=\sup_{x\in G}|\xi(x)|=\xi(e)\leqq 1$ if and only if $\varphi_\xi\in\tilde{E}_0$: more precisely, we hold

Lemma 4. *The following three conditions are mutually equivalent: i) $\xi(x)$ is an elementary c. p. d. function with $\|\xi\|_\infty=1$ on G, ii) φ_ξ is an extreme point of $\tilde{E}_0$, iii) $\mathfrak{H}_{\varphi_\xi}$ is simple for the family of unitary operators U_x, $x\in G$.*

Here the term "simple" means that there exists no proper closed submanifold in $\mathfrak{H}_\varphi$ which is invariant under $\{U_x\}$, $x\in G$.

§**2.** We shall begin with a preliminary lemma;

Lemma 5. *Every product fg, for $f,g\in L(G)$, may be written in the form*

$$fg=h_1^*h_1+ih_2^*h_2-(h_3^*h_3+ih_4^*h_4), \tag{2.1}$$

so that, by means of the approximate identity, $L(G)$ is strongly approximated by the complex linear combinations of elements in $\mathfrak{B}(G)$, the set of all such elements as in the forms f^*f, $f\in L(G)$.

In fact, we have only to put $h_k=\frac{1}{2}(f^*+(-i)^{k-1}\cdot g)$ for $k=1,2,3,4$. In other words, $\mathfrak{B}(G)$ forms a basis of $L(G)$ as a vector space.

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1) 2) Cf. the author's 2); *H. Cartan and R. Godement,* Théorie de la dualité et analyse harmonique sur les groups abeliens localement compacts, Ann. Ec. Norm., 3, LXIV (1946). *R. Godement,* Les fonctions de type positif et la theorie des groupes, Trans. Amer. Math. Soc. 6, 3 (1948). *H. Yoshizawa,* Unitary representations of locally compact groups, Osaka Math. Jour. 1, 1 (1949).

Analogously, the set of all such elements as in the forms f^*f for $f\in L^{1,2}(G) = L(G)\cap L^2(G)$ is denoted by $\mathfrak{V}^0(G)$, then each product f^*g; $f, g\in L^{1,2}(G)$, is decomposed as in (2.1) and moreover is continuous on G, since so is every element of $\mathfrak{V}^0(G)$. (f^*g is not necessarily $\in L^{1,2}(G)$). $\tilde{V}_0$ being the character space of G, $\tilde{f}(\varphi)$ defined by

$$(2.2)\qquad \tilde{f}(\varphi)=\int_G \overline{\xi_\varphi(x)}\, f(x)\, dx\,;\qquad \varphi\in\tilde{V}_0,$$

is called the *generalized Fourier transform* (*g.F.t.*) of $f\in L(G)$, which is clearly a continuous function on $\tilde{V}_0$ (and equal to $\varphi(f)$ owing to V) in Lemma 3).

For every $f\in\mathfrak{V}_c$, the intersection $\mathfrak{V}(G)\cap C(G)$,[1] putting

$$(2.3)\qquad \pi(\tilde{f})=f(e),$$

π is evidently a positive and additive functional on $\tilde{\mathfrak{V}}_c$ and is extensible to the whole $(\mathfrak{V}_c)_{\mathfrak{K}}$, the positiveness.[2] [3] π defines a Radon's measure $d\varphi$ on $\tilde{V}_0$ such as

$$(2.3)'\qquad \pi(\tilde{f})=\int_{\tilde{V}_0}\tilde{f}(\varphi)\,d\varphi,$$

which will be precisely proved in the later paragraph, §4.

Now, we consider the series of sub-pre-Hilbert spaces

$$\{\mathfrak{H}^0_\varphi\},\ \mathfrak{H}^0_\varphi=L^{1,2}(G)/'\mathfrak{J}_\varphi \text{ of } L(G)/'\mathfrak{J}_\varphi \text{ for } \varphi\in\tilde{V}_0.$$

Then, the direct sum $\mathfrak{H}^0_{\tilde{V}}=\sum\oplus\mathfrak{H}^0_\varphi$, $\varphi\in\tilde{V}_0$, is also a pre-Hilbert space with the inner product

$$(2.4)\qquad (\hat{f}_1,\hat{f}_2)_{\tilde{V}}=\int_{\tilde{V}_0}(X_{f_1},X_{f_2})_\varphi\,d\varphi=\int_{\tilde{V}_0}\varphi(f_2^*f_1)\,d\varphi^{4)}$$

for $\hat{f}_i=\sum\oplus X^{(\varphi)}_{f_i}$, $f_i\in L^{1,2}(G)$. Consequently, we have

$$(2.5)\qquad \begin{aligned}\|\hat{f}\|^2_{\tilde{V}_0}&=(\hat{f},\hat{f})_{\tilde{V}_0}\\ &=\int_{\tilde{V}_0}\|X_f\|^2_\varphi\,d\varphi=\int_{\tilde{V}_0}\varphi(f^*f)\,d\varphi=f^*f(e),\end{aligned}$$

and, on the other hand,

$$(2.6)\qquad f^*f(e)=\int_G f^*(x)\,f(x^{-1})\,dx=\int_G|f(x)|^2\,dx,$$

that is,

$$(2.7)\qquad \|\hat{f}\|_{\tilde{V}_0}=\|f\|_{L^2(G)}.$$

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1) $C(G)$ means the B-space of all complex-valued bounded continuous functions on G.

2) $(\cdot)_{\mathfrak{P}}$, $(\cdot)_{\mathfrak{R}}$, $(\cdot)_{\mathfrak{K}}$ means the linear envelope of the space $(\cdot)$ over the positive nembers, the reals $\mathfrak{R}$, or the complex field $\mathfrak{K}$ respectively.

3) $\tilde{X}$ means the collection of all *g. F. t.* of $f\in X$, $X\subset L(G)$.

4) $\pi(g^*f)$ is reasonable for $f, g\in L^{1,2}(G)$ owing to Lemma 5, since $(\mathfrak{V}_0)_{\mathfrak{K}}\subset(\mathfrak{V}_c)_{\mathfrak{K}}$.

Thus, the correspondence $f \to \hat{f}$ give rise to a norm-preserving linear operator P from $L^{1,2}(G)$ onto $\mathfrak{H}^0_{\tilde{V}_0}$, and $\mathfrak{H}_{\tilde{V}_0}$ being the completion of $\mathfrak{H}^0_{\tilde{V}_0}$ with respect to the norm (2.5), P comes up to the isometrical isomorphism from $L^2(G)$ onto $\mathfrak{H}_{\tilde{V}_0}$, since $L^{1,2}(G)$ is dense in $L^2(G)$. Thus, we hold

Theorem 1. (An extension formula of Plancherel's theorem) *Let G be a general LC group with left-invariant Haar's measure dx; with the same definitions as above, two Hilbert spaces $L^2(G)$ and $\mathfrak{H}_{\tilde{V}_0}$ are mutually isometrically isomorphic.*

§3. Since $(\mathfrak{B}_c)_{\mathfrak{R}}$ is dense in $L_c = L(G) \cap C(G)$, if $f \to f_0$ strongly in $L(G)$, $f_0 \in L_c$ and $f \in (\mathfrak{B}_c)_{\mathfrak{R}}$, then $f(e) \to f_0(e)$ (both f and f_0 being continuous) and $\tilde{f} \to \tilde{f}_0$ uniformly on $\tilde{V}_0$ by means of $\|\tilde{f}\|_\infty \leqq \|f\|$, where $\|\tilde{f}\|_\infty$ is the uniform norm on $\tilde{V}_0$, *i. e.* $\|\tilde{f}\|_\infty = \sup_{\tilde{V}_0} |\tilde{f}(\varphi)|$.

If $f \in L_c$, then f_x is also in L_c and

(3.1) $f(x) = f_{x^{-1}}(e) = \int_{\tilde{V}_0} \tilde{f}_{x^{-1}}(\varphi)\, d\varphi$; putting $\varphi_f(x) = \tilde{f}_{x^{-1}}(\varphi)/\tilde{f}(\varphi)$ for $\tilde{f}(\varphi) \neq 0$, otherwise $\varphi_f(x) = 1$, we have always

(3.2) $\tilde{f}_{x^{-1}}(\varphi) = \varphi_f(x)\, \tilde{f}(\varphi)$ for all x in G,
since $\tilde{f}(\varphi) = 0$ implies $\tilde{f}_{x^{-1}}(\varphi) = 0$ for all $x \in G$.[1]

Summerizing these, we can formulate

Theorem 2. (Inversion formula of Fourier transform) *Let $\tilde{f}(\varphi)$ be the g.F.t. of $f \in L_c(G)$, i.e.*

$$\tilde{f}(\varphi) = \int_G \xi_\varphi(x)\, f(x)\, dx\,,$$

then $f(x)$ is representable in the form;

$$f(x) = \int_{\tilde{V}_0} \varphi_f(x)\, \hat{f}(\varphi)\, d\varphi\,. \tag{3.3}$$

Theorem 3. *Let Q be the inverse operator of P as in Theorem 1. Then, $Pf = \tilde{f}$ and $Q\tilde{f} = f$ for $f \in L^2(G)$ and, for $f^\lambda \to f$ in $L^2(G)$,*

$$\tilde{f}(\varphi) = l.i.m. \int_G \overline{\xi_\varphi(x)}\, f^\lambda(x)\, dx\,,$$

$$f(x) = l.i.m. \int_{\tilde{V}_0} \varphi_f(x)\, \tilde{f}^\lambda(\varphi)\, d\varphi\,.$$

where "l. i. m." (limit in the mean) should be taken as the sense of strong convergence in the respective Hilbert spaces. (See Appendix II).

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1) In $f \ni {}'\mathfrak{J}_\varphi$, then $(e^\lambda)_x \cdot f \to f_x \in {}'\mathfrak{J}_\varphi$ since ${}'\mathfrak{J}_\varphi$ is closed.

§4. Now we shall investigate a Radon's measure defined on $\tilde{V}_0$: Denoting the set of all positive continuous functions on $\tilde{V}_0$ with compact supports by $L^+(\tilde{V}_0)$, every $\tilde{g} \in L^+(\tilde{V}_0)$ is confined by some element of $(\tilde{\mathfrak{B}}_c)_{\mathfrak{P}}$; in fact, suppose K being the compact support of $\tilde{g}$ in $\tilde{V}_0$ and let ε be an arbitarily given positive number, then there exists a function f_φ of $\mathfrak{B}_c$ such that $\varphi(f_\varphi) > 2\varepsilon$ for each φ in $\tilde{V}_0$,[1] so that K is covered by finite numbers of $U_{\varphi_i}(f_i\,;\,\varepsilon)$, $f_i = f_{\varphi_i}$ and $i = 1.2, \cdots, n$; consequently, we see that $\tilde{g} \leq \sum_{i=1}^{n} k_i \hat{f}_i$, where k_i is a suitable positive number. For every $\tilde{g} \in \mathfrak{M}$, the real vector lattice generated from $(\tilde{\mathfrak{B}}_c \cup L^+(V_0))_{\mathfrak{R}}$, putting

$$\hat{\pi}(\tilde{g}) = \inf_{\tilde{g}^+ \leq \tilde{f}} \pi(\hat{f}) \quad \text{for} \quad \hat{f} \in (\tilde{\mathfrak{B}}_c)_{\mathfrak{P}}, \qquad \text{(See Appendix I)} \tag{4.1}$$

and $\hat{\pi}(-\tilde{g}) = \hat{\pi}(\tilde{g})$, $\hat{\pi}$ is definable on $\mathfrak{M}$ on which it is surely a subadditive positive-domogeneous function, *i.e.*

$$\begin{cases} \hat{\pi}(\tilde{g}_1 + \tilde{g}_2) \leq \hat{\pi}(g_1) + \hat{\pi}(g_2). \\ \hat{\pi}(\alpha\tilde{g}) = \alpha\hat{\pi}(\tilde{g}) \quad \text{for} \quad \alpha \geqq 0, \end{cases} \tag{4.2}$$

where $\tilde{g}$, $\tilde{g}_1$, $\tilde{g}_2 \in \mathfrak{M}$, and $\pi \leq \hat{\pi}$ on $(\tilde{\mathfrak{B}}_c)_{\mathfrak{R}}$. Owing to Hahn-Banach's extension theorem, there exists an additive functional π^0 on $\mathfrak{M}$ such that $\hat{\pi} = \pi^0$ on $(\tilde{\mathfrak{B}}_c)_{\mathfrak{R}}$: we can always assume tnat π^0 is positive on $L^+(\tilde{V}_0)$; if this were not so, we may put again

$$\begin{cases} \hat{\hat{\pi}}(\hat{f}) = \pi_0^+(\hat{f}^+) - \pi_0^+(-\hat{f}^-) \\ \pi_0^+(\hat{f}^+) = \sup\limits_{0 \leqq \tilde{g} \leqq \tilde{f}} \pi^0(\tilde{g}), \quad \text{for} \quad \hat{f}, \tilde{g} \in \mathfrak{M}. \end{cases} \tag{4.3}$$

and see immediately that such $\hat{\hat{\pi}}$ satisties the all conditions mentioned above. Then, $\hat{\hat{\pi}}$ defines a Radon's measure on $\tilde{V}_0$ and every element of $(\mathfrak{B}_c)_{\mathfrak{R}}$ is integrable with respect to such $\hat{\hat{\pi}}$; $\pi = \pi^0 = \hat{\hat{\pi}}$ on it.

Thus the integrations in (2.3), (2.4), (3.3), etc. should be considered in the sense of Radon.

§5. Finally, we shall make a slight survey of the case that G is abelian, from which we might recognize that our extension is a natural one of abelian cases.

Let G be LC abelian; then, $\tilde{V}_0 = S(\tilde{E}_0)$ and the character group $\hat{G}$ of G coincides with $\tilde{V}_0$[2], so that (3.2) is representable as follows

$$\hat{f}_x(\chi) = \overline{\chi(x)}\,\hat{f}(\chi). \tag{5.1}$$

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1) This comes from the fact: $\mathfrak{B}_c$ is dense in $L(G)$ and forms a basis of it, since $\mathfrak{B}^+$ the collection of such elements in the forms f^*f, $f \in L^+(G)$, is a basis of $L(G)$, *i.e.* $(\mathfrak{B}^+)_{\mathfrak{R}}$ is dense in $L(G)$ and $\mathfrak{B}^+ \subset \mathfrak{B}_c$.

2) In abelian cases, V_0 itself is compact in $\tilde{E}_0$; hence, $\tilde{V}_0 = V_0 - \theta$ is locally compact.

in which $\chi(x)=\chi_f(x)$ is independent of the choice of f such that $\tilde{f}(\chi)\neq 0$, since $\tilde{f}_x\cdot\tilde{g}=\tilde{f}\cdot\tilde{g}_x=\widetilde{(f\cdot g)}_x$, and hence such $\chi(x)$ as in (5.1) is always definitive.

Moreover, we hold

$$\widetilde{f^*f}(\chi)=\int\int_G \overline{f(x)}f(y)\chi(xy^{-1})\,dxdy$$
$$=\overline{\tilde{f}(\chi)}\cdot\tilde{f}(\chi)=|\tilde{f}(\chi)|^2,$$

from which it follows

$$\|\hat{f}\|^2_{\hat{G}}=\int_{\hat{G}}|\tilde{f}(\chi)|^2\,d\chi; \tag{5.2}$$

this is the modified formula of (2.5).

From this, we have $\mathfrak{H}_{\hat{G}}\subset L^2(\hat{G})$ and, by means of the Pontrjagin's duality theorem, $\mathfrak{H}_{\hat{G}}=L^2(G)$.

Appendix I. To define (4.1) in §4, we prepare a Lemma;

Lemma 6. *If* $\hat{f}(\varphi)\geqq 0$ *for all* $\varphi\in\tilde{V}_0$, $f\in L_c$, *then it holds* $f(e)\geqq 0$.

Suppase that $f(e)<0$ and let W be a neighborhood of e such that $\Re(f)(x)<0$ for $x\in W$; take further a neighborhood U of e such that $UU^{-1}\subset W$: then

$$\xi_U(x)=\int_G C_U(xy)C_U(y)dy$$

is in $\mathfrak{P}(G)$[1], and $\tilde{f}(\varphi)\geqq 0$ on $\tilde{V}_0$ implies that $\varphi(f)\geqq 0$ for all φ in $\tilde{\Pi}(H(G))$, so that

$$\varphi_{\xi_U}(f)=\int_G \xi_U(x)f(x)dx$$
$$=\int_U C_U(t)\int_{U^{-1}} C_U(s^{-1})f(ts)dtds\geqq 0.$$

These are contradictory, from which the Lemma.

Appendix II. We have another formulation of (3.4) in Thr. 3 as follows; for a suitable series $\{K_\alpha\}$ of compact sets K_α in G,

$$\tilde{f}(\varphi)=\underset{\alpha}{l.i.m.}\int_{K_\alpha}\xi_\varphi(x)f(x)dx; \tag{3.4$'$}$$

since $(L^+(G))_{\mathfrak{K}}$ is dense in $L^2(G)$ (see A. Weil, loc. cit. p. 32), for a given $\varepsilon>0$, there exists f^0 with compact carrier K such that $\|f-f^0\|_{L_2(G)}=\|\tilde{f}-\tilde{f}^0\|_{\tilde{V}_0}<\varepsilon/2$, and

$$\int_G|f-f^0|^2dx=\int_K|f_K-f^0|^2dx+\int_{G-K}|f|^2dx<\varepsilon^2/4,$$

where $f_K(x)=f(x)$, $x\in K$, and otherwise $=0$, so that $\|f_K-f^0\|_{L^2(G)}<\varepsilon/2$ and $\|f-f_K\|_{L^2(G)}=\|\tilde{f}-\tilde{f}_K\|_{\tilde{V}_0}<\varepsilon$.

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1) C_U means a characteristic function of U; see A. Weil, loc. cit.

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