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A characterization of a general uniform space by a system of uniformly continuous functions

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The author once concerned himself to generalize the theories of bicompact spaces by I. Gelfand, A. N. Kolmogoroff[1] and by I. Kaplansky,[2] and he characterized complete metric spaces and totally bounded uniform spaces by some systems of real-valued functions.[3] About the same problem, the recent paper by T. Shirota has got more refined results.[4] It seems, however, that their attentions have not yet been devoted to general complete uniform spaces. In this paper we shall characterize a general complete uniform space by a directed system of uniformly continuous functions from the space in a parallelotope and shall consider a more general case.

Let R be a complete uniform space and let $\{\mathfrak{U}_{\alpha n} | \alpha \in A ; n = 1, 2, \cdots\}$ be a uniform basis of this space satisfying the condition that $\mathfrak{U}^*_{\alpha n+1} < \mathfrak{U}_{\alpha n}$.[5] We denote by $D(R)$ the totality of uniformly continuous functions $u(x)$ from R in to the parallelotope $P\{I_\alpha | \alpha \in A\}$ $(I_\alpha = \{x | x \geqq 0\})$[6] satisfying the following two conditions,

A) $x \in S(y, \mathfrak{U}_{\alpha n})$ *implies* $|u_\alpha(x) - u_\alpha(y)| \leqq \frac{1}{2^n}$ *(a sort of Lipshitz's condition)*,

B) there exist a finite number of $\alpha_i \in A$ $(i = 1, \cdots, h)$ *and a definite positive number* l *such that* $\bigcup_{i=1}^{h} u_{\alpha_i}(x) \geqq l$,

where we denote by $u_\alpha(x)$ the α-coordinate of the function $u(x)$; hence $u_\alpha(x)$ is a real-valued uniformly continuous function.

Remarks. We can replace condition A) with the condition, $|u_\alpha(x) - u_\alpha(y)| \leqq \frac{k_\alpha}{2^n}$, where $k_\alpha (\alpha \in A)$ are positive definite numbers. Condition B) is a stronger condition than the condition that $u(x) \neq \{0\}$ for every $x \in R$, *i.e.* $\bigcup_{\alpha \in A} u_\alpha(x) > 0$.

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1) I. Gelfand and A. N. Kolmogoroff, On rings of continuous functions on topological spaces, C. R. URSS, 22 (1939).
2) I. Kaplansky, Lattices of continuous functions, Bull. Amer. Math. Soc. 55 (1947). This paper was unknown by the author at the time.
3) J. Nagata, On lattices of functions on topological spaces and of functions on uniform spaces, Osaka Math. J. 1 (1949).
4) T. Shirota, A generalization of a theorem of I. Kaplansky, Osaka Math. J. 4 (1952).
5) Notations and notions in this paper are chiefly due to J. W. Tukey, Convergences and uniformity in topology, 1940.
6) The notation $P\{I_\alpha | \alpha \in A\}$ means the product space of $I_\alpha (\alpha \in A)$.

Now we consider an ordering relation on $D(R)$ by defining $u(x) \leqq u'(x)$ when and only when $u_\alpha(x) \leqq u_\alpha'(x)$ for all $\alpha \in A$. Then the join $u \vee v$ of two elements u, v of $D(R)$ exists always in $D(R)$, but the meet does not always exist in $D(R)$.[7] Hence $D(R)$ is a partially ordered system and a directed system, but it is not a lattice.

Definition. We call a non-vacuous subset μ of $D(R)$ *a characteristic ideal or c-ideal*, if

I) $m' \geqq m \in \mu$ implies $m' \in \mu$,

II) $m \in \mu$, $m' \in \mu$ imply $m \wedge m' \in D(R)$ and $m \wedge m' \in \mu$,

III) for every $u_0 \in D(R)$ there exists $m \in \mu$ such that $m \not\geqq u_0$.

Definition. If for a family $\{\mu\}$ of c-ideals and for two elements u, u' of $D(R)$, there exists $\mu \in \{\mu\}$ such that $m \in \mu$ implies $u' \leqq u \vee m$, then we denote this relation by $u' < u$ $(\{\mu\})$. This relation coincides with $u' \leqq u$ if every element of $\{\mu\}$ contains an element m such that $m \leqq u$.

Definition. We call a family $\{\mu\}$ of c-ideals a *max family* if $\{\mu\}$ satisfies the following conditions,

1) for every $u \in D(R)$ there exists $u' \in D(R)$ such that $u' < u$ $(\{\mu\})$; $u' \not< p$ $(\{\mu\})$ and $u' \not< q$ $(\{\mu\})$[8] imply $u \not\leqq p \vee q$ for every elements p, q of $D(R)$,

2) $\nu \supset \mu \in \{\mu\}$ implies $\nu \in \{\mu\}$ for every c-ideal ν,

3) if $\mu_\gamma \in \{\mu\}$ and $\bigcap_\gamma \mu_\gamma \neq \phi$, then $\bigcap_\gamma \mu_\gamma$ is a c-ideal and it is contained in $\{\mu\}$,

4) $\{\mu\}$ is a maximum family satisfying 1), 2), 3).

Definition. We put $\{\mu\}(x_0) = \{\mu \mid$ for all α_0 and $\varepsilon > 0$, there exists $u \in \mu$ such that $u_{\alpha_0}(x_0) < \varepsilon$; μ is a c-ideal$\}$.

Lemma 1. *For every α_0 and x_0, there exists a c-ideal μ such that $\mu \in \{\mu\}(x_0)$; $m \in \mu$ implies $m_{\alpha_0}(x) \geqq \frac{1}{2^n}$ for $x \notin S(x_0, \mathfrak{U}_{\alpha_0 n})$.*

Proof. For every positive integers i, n and for each $\alpha_0 \in A$, we can define a real-valued uniformly continuous function $f(x)$ such that $0 < f(x_0) < \frac{1}{i}$, $f(x) \geqq \frac{1}{2^n}$ $(x \notin S(x_0, \mathfrak{U}_{\alpha_0 n}))$, $f(x) \geqq f(x_0)$; $x \in S(y, \mathfrak{U}_{\alpha_0 n'})$ implies $|f(x) - f(y)| < \frac{1}{2^{n'}}$. The method of defining such a function is the same as in the proof of Urysohn's lemma.

Using such a function, we define $m^{(i)}(x) \in D(R)$ so that $0 < m^{(i)}_{\alpha_0}(x_0) < \frac{1}{i}$, $m^{(i)}_{\alpha_0}(x) \geqq \frac{1}{2^{n+1}}$ $(x \notin S(x_0, \mathfrak{U}_{\alpha_0 n+1}))$, $m^{(i)}_\alpha(x) = 0$ $(\alpha \neq \alpha_0)$, and so that $m^{(i+1)}(x) \leqq m^{(i)}(x)$ $(i = 1, 2, \cdots)$. Then the family. $\mu = \{m \mid D(x) \ni m \geqq m^{(i)}$ for some $i\}$

7) We use the symbols $\vee$, $\wedge$ for joins and meets of elements of $D(R)$ and for uniform coverings, and we use $\cup$, $\cap$ for joins and meets of real numbers and for sets.

8) $u' \not< p$ means the negation of $u' < p$.

is a c-ideal satisfying the condition of this lemma.

Lemma 2. *$\{\mu\}(x_0)$ is a max family for each $x_0 \in R$.*

Proof. We shall prove firstly that $\{\mu\}(x_0)$ satisfies condition 1). Let u be an arbitrary element of $D(R)$, then from condition B) of $D(R)$, there exist $\alpha_0 \in A$ and a positive integer n such that $u_{\alpha_0}(x_0) > \frac{1}{2^n} > 0$. For this n we define $u' \in D(R)$ so that $u'_{\alpha_0}(x) = \frac{1}{2^{n+1}}$, $u_\alpha' = 0$ $(\alpha \neq \alpha_0)$. By Lemma 1 we denote by μ a c-ideal of $\{\mu\}(x_0)$ such that $x \notin S(x_0\ \mathfrak{U}_{\alpha_0 n+1})$ implies $m_{\alpha_0}(x) \geqq \frac{1}{2^{n+1}}$ for every $m \in \mu$. Then since $u_{\alpha_0}(x) > \frac{1}{2^n} - \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$ for $x \in S(x_0, \mathfrak{U}_{\alpha_0 n+1})$ from condition A), we get $u' \leqq m \vee u$ for every $m \in \mu$. Hence $u' < u$ $(\{\mu\})$.

Next assume that $u' \nless p$ $(\{\mu\})$, then from the above arguement we see that there exists $x \in S(x_0, \mathfrak{U}_{\alpha_0 n+1})$ such that $p_{\alpha_0}(x) < u'_{\alpha_0}(x) = \frac{1}{2^{n+1}}$. Hence $p_{\alpha_0}(x_0) < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$ from condition A). In the same way we see that $u' \nless q$ implies $q_{\ 0}(x_0) < \frac{1}{2^n}$. Hence $p_{\alpha_0}(x_0) \cup q_{\alpha_0}(x_0) < \frac{1}{2^n} < u_{\alpha_0}(x_0)$, and hence $u \nleqq p \vee q$.

It is obvious that $\{\mu\}(x_0)$ satisfies condition 2). Next we prove that $\{\mu\}(x_0)$ satisfies also 3). Let $\mu_\gamma \in \{\mu\}(x_0)$ and $\bigcap_\gamma \mu_\gamma \neq \phi$. then for $n' \in \bigcap_\gamma \mu_\gamma$ and for every $m^{(\gamma)} \in \mu_\gamma$ we get $m^{(\gamma)} \wedge n' \in \mu_\gamma$ from condition II) of c-ideal. Take $m^{(\gamma)} \in \mu_\gamma$ such that $m^{(\gamma)}_{\alpha_0}(x_0) < \varepsilon$ for arbitrary α_0 and $\varepsilon > 0$, then for $\sup_\gamma (m^{(\gamma)} \wedge n') = m$, $m^{(\gamma)}_{\alpha_0}(x_0) = \sup_\gamma m^{(\gamma)}_{\alpha_0}(x_0) \leqq \varepsilon$. Since $m \in D(R)$, we get $m \in \bigcap_\gamma \mu_\gamma$ from condition I). Hence $\bigcap_\gamma \mu_\gamma$ satisfies condition III) of c-ideal. Since $\bigcap_\gamma \mu_\gamma$ satisfies obviously I), II), $\bigcap_\gamma \mu_\gamma$ is a c-ideal and is an element of $\{\mu\}(x_0)$ from the above arguement.

Lemma 3. *If every μ_γ are elements of a max famlly $\{\mu\}$, then for a definite positive number k, $\mu_\gamma^k = \{m' \mid D(R) \ni m' \geqq m \wedge k = \{m_\alpha(x) \cap k\},\ \{m_\alpha(x)\} = m \in \mu_\gamma\}$ and $\bigcap_\gamma \mu_\gamma^k$ are c-ideals and are elements of $\{\mu\}$.*

Proof. It is obvious that μ_γ^k satisfies conditions I), III) of c-ideal. Let $m', m'' \in \mu_\gamma$ and let $m' \geqq m \wedge k$, $m'' \geqq m \wedge k$; $m, n \in \mu_\gamma$, then $m' = m' \vee (m \wedge k)$ $= (m' \vee m) \wedge (m' \vee k) \geqq (m' \vee m) \wedge k$, $m'' \geqq (m'' \vee n) \wedge k$. Hence $(m' \wedge m'')$ $\geqq ((m' \vee m) \wedge (m'' \vee n)) \wedge k$. Since $m, n \in \mu_\gamma$, from conditions I), II), $(m' \vee m)$ $\wedge (m'' \vee n) \in \mu_\gamma$. Therefore $m' \wedge m'' \in D(R)$ and $m' \wedge m'' \in \mu_\gamma^k$. Hence μ_γ^k satisfies also condition II) and is a c-ideal. Since $\mu_\gamma^k \supset \mu_\gamma$, from condition 2) we get $\mu_\gamma^k \in \{\mu\}$. If we take $m_0^\gamma = m^{(\gamma)} \wedge k$, $m^{(\gamma)} \in \mu_\gamma$, then $\sup_\gamma m_0^{(\gamma)} \in \bigcap_\gamma \mu_\gamma^k$; hence $\bigcap_\gamma \mu_\gamma^k \neq \phi$. Therefore from condition 3) $\bigcap_\gamma \mu_\gamma^k$ is a c-ideal and it is an element of $\{\mu\}$.

Let us assume that $\{\mu\}$ is a max family, then for each $\alpha_0 \in A$ and each positive integer n putting $u^{(\alpha_0 n)} = \{u_\alpha(x)\}$, $u_{\alpha_0}(x) = \frac{1}{2^n}$, $u_\alpha(x) = 0$ $(\alpha \neq \alpha_0)$, we get $u'^{(\alpha_0 n)}$ satisfying condition 1) of max family for this $u^{(\alpha_0 n)}$ and $\{\mu\}$. For these $\{\mu\}$, $u^{(\alpha_0 n)}$, $u'^{(\alpha_0 n)}$, we shall prove the following lemmas.

Lemma 4. *There exists* $x_0 = x_0(\alpha_0\ n) \in R$ *such that* $u'^{(\alpha_0 n)}_{\alpha_0}(x_0) \leqq u^{(\alpha_0 n)}_{\alpha_0}(x) = \frac{1}{2^n}$, *and for every* $\mu \in \{\mu\}$ *there exists* $m \in \mu$: $m_{\alpha_0}(x_0) < u'^{(\alpha_0 n)}_{\alpha_0}(x_0)$,

Proof. Firstly we show the existence of $x_0 \in R$ and $\alpha' \in A$ such that for every $\mu \in \{\mu\}$ there exists $m \in \mu$: $m_{\alpha'}(x_0) < u'_{\alpha'}(x_0)$, where we denote $u'^{(\alpha_0 n)}$ by u' and $u^{(\alpha_0 n)}$ by u for simplicity. For if we assume the contrary, then for every x, α we get $\mu_{x,\alpha} \in \{\mu\}$ such that $m \in \mu_{x,\alpha}$ implies $m_\alpha(x) \geqq u_{\alpha'}(x)$. For these $\mu_{x,\alpha}$, $\bigcap_{x \in R,\ \alpha \in A} \mu^k_{x,\alpha} = \mu_0$ is to be a c-ideal from Lemma 3. If $m \in \mu_0$, then for each x and each α $m \in \mu^k_{x,\alpha}$, and hence $m_\alpha(x) \geqq u'_\alpha(x) \frown k$. Therefore $m \geqq u' \wedge k$, but since $u' \wedge k \in D(R)$ is obvious, this contradicts condition III) of c-ideal. Hence we see that for every $\mu \in \{\mu\}$ there exists $m \in \mu$ such that $m_\alpha'(x_0) < u'_{\alpha'}(x_0)$.

Next we show that $u'_{\alpha'}(x_0) \leqq u_{\alpha'}(x_0)$ for such α', x_0. For if we assume the contrary: $u'_{\alpha'}(x_0) > u_{\alpha'}(x_0)$, then this formula combining with the above conclusion implies that for every $\mu \in \{\mu\}$ there exists $m \in \mu$ such that $u_{\alpha'}(x_0) \smile m_{\alpha'}(x_0) < u'_{\alpha'}(x_0)$. Hence $u' \not< u(\{\mu\})$. This conclusion contradicts condition 1) of max family, and hence it must be $u'_{\alpha'}(x_0) \leqq u_{\alpha'}(x_0)$.

If we assume that $\alpha' \neq \alpha_0$, then from $u_{\alpha'}(x_0) = 0$, $u'_{\alpha'}(x_0) = 0$ holds, but this contradicts the existence of $m \in \mu$ such that $m_{\alpha'}(x_0) < u'_{\alpha'}(x_0)$. Hence for every α_0, n there exists $x_0 = x_0(\alpha_0\ n) \in R$ such that for every $\mu \in \{\mu\}$ there exists $m \in \mu$ such that $m_{\alpha_0}(x_0) < u'_{\alpha_0}(x_0) \leqq u_{\alpha_0}(x_0) = \frac{1}{2^n}$, and the proof of this lemma is complete.

Definition. Put $V_{\alpha_0 n} = \{x \mid$ for every $\mu \in \{\mu\}$ there exists $m \in \mu$ such that $m_{\alpha_0}(x) < u'^{(\alpha_0 n)}(x)\}$, then by Lemma 4 $x_0(\alpha_0\ n) \in V_{\alpha_0 n}$.

Lemma 5. $V_{\alpha_0 n} \subset S^2(x_0(\alpha_0\ n), \mathfrak{U}_{\alpha_0 n})$.

Proof. Let us assume the contrary and let us assume that $S(x_0, \mathfrak{U}_{\alpha_0 n}) \frown S(x', \mathfrak{U}_{\alpha_0 n}) = \phi$, $x' \in V_{\alpha_0 n}$. For these x_0, x' we define the following two functions p, q in $D(R)$: $p = \{p_\alpha(x)\}$, $p_{\alpha_0}(x_0) < u'_{\alpha_0}(x_0)$, $p_{\alpha_0}(x) \geqq \frac{1}{2^n}$ for $x \notin S(x_0, \mathfrak{U}_{\alpha_0 n})$; $q = \{q_\alpha(x)\}$, $q_{\alpha_0}(x') < u'_{\alpha_0}(x')$, $q_{\alpha_0}(x) \geqq \frac{1}{2^n}$ for $x \notin S(x', \mathfrak{U}_{\alpha_0 n})$. Then since $x_0, x' \in V_{\alpha_0 n}$, obviously $u' \not< p(\{\mu\})$, $u' \not< q(\{\mu\})$. From $S(x_0, \mathfrak{U}_{\alpha_0 n}) \frown S(x', \mathfrak{U}_{\alpha_0 n}) = \phi$, $u \leqq p \vee q$ is obvious, but this contradicts condition 1) of max family. Hence it must be $V_{\alpha_0 n} \subset S^2(x_0, \mathfrak{U}_{\alpha_0 n})$.

Lemma 6. *For every* α_i, n_i $(i = 1, \cdots, h)$, $\bigcap_{i=1}^{h} S^2(x_0(\alpha_i\ n_i), \mathfrak{U}_{\alpha_i n_i}) \neq \phi$ *holds.*

Proof. 1. Let us assume the contrary, *i.e.* $\bigcap_{i=1}^{h} S^2(x_0(\alpha_i\ n_i), \mathfrak{U}_{\alpha_i n_i}) = \phi$, then for every $x \in R$ there exists i such that $x \notin S^2(x_0(\alpha_i\ n_i), \mathfrak{U}_{\alpha_i n_i})$. For each $x \in R$, we fix one of such i. Since from Lemma 5, for such i, $x \notin V_{\alpha_i n_i}$ holds, noting the definition of $V_{\alpha_i n_i}$, we see that there exists $\mu(x, i) \in \{\mu\}$ such that $m \in \mu(x, i)$ implies $m_{\alpha_i}(x) \geqq u'^{(\alpha_i n_i)}_{\alpha_i}(x)$. For $\alpha \neq \alpha_i$, from the proof of Lemma 4, there exists $\mu(x, \alpha) \in \{\mu\}$ such that $m \in \mu(x, \alpha)$ implies $m_\alpha(x) \geqq u'^{(\alpha_i n_i)}_\alpha(x)$. Putting $\mu(x, \alpha_i) = \mu(x, i)$ for $\alpha = \alpha_i$, we get $\mu(x, \alpha) \in \{\mu\}$ for every $\alpha \in A$ having the property that $m_\alpha(x) \geqq u'^{(\alpha_i n_i)}_{\alpha_i}(x)$ holds for every $m \in \mu(x, \alpha)$. From Lemma 3 we get $\bigcap_{\alpha \in A} \mu^k(x, \alpha) = \mu'(x, i) \in \{\mu\}$. For this $\mu'(x, i)$, $m \in \mu'(x, i)$ implies $m_\alpha(x) \geqq u'^{(\alpha_i n_i)}_\alpha \frown k$ for every α.

2. Next putting $v = sup\ \{inf\ \{m | m \in \mu'(x, i)\} | x \in R\}$, we show that $v \in D(R)$. It is obvious that v has a definite value for each point of R, and that v satisfies condition A) of $D(R)$. We shall show that $D(R)$ satisfies also B). Since $u'^{(\alpha_i n_i)} \in D(R)$ $(i = 1, \cdots, h)$, there exist finite subsets F_i $(i = 1, \cdots, h)$ of A and positive numbers l_i $(i = 1, \cdots, h)$ such that $\bigcup_{\alpha \in F_i} u'^{(\alpha_i n_i)}_\alpha(x) \geqq l_i > 0$ $(i = 1, \cdots, h)$. We put $\bigcup_{i=1}^{h} F_i = F$; F is a finite subset of A. Let x be an arbitrary point of R and let i be the fixed number for x such that $x \notin S^2(x_0(\alpha_i\ n_i), \mathfrak{U}_{\alpha_i n_i})$, then $\bigcup_{\alpha \in F_i} inf\ \{m_\alpha(x) | m \in \mu'(x, i)\} \geqq \bigcup_{\alpha \in F_i} (u'^{(\alpha_i n_i)}_\alpha(x) \frown k) \geqq l_i \frown k$ from the property of $\mu'(x, i)$ decided in 1. Hence for each $x \in R$, $\bigcup_{\alpha \in F} v_\alpha(x) \geqq \bigcup_{\alpha \in F_i} inf\ \{m_\alpha(x) | m \in \mu'(x, i)\} \geqq l_i \frown k$ for some i. Therefore for every x we get $\bigcup_{\alpha \in F} v_\alpha(x) \geqq (\bigcap_{i=1}^{h} l_i) \frown k > 0$, *i.e.* v satisfies condition B), and hence $v \in D(R)$.

3. From Lemma 3 and from 1 of this proof, $\bigcap_{x \in R} \mu'(x, i)$ is to be a c-ideal. $m \in \bigcap_{x \in R} \mu'(x, i)$, however, implies $m \in \mu'(x, i)$ for every x, and hence $m \geqq inf\ \{m | m \in \mu'(x, i)\}$. Therefore $m \geqq v$ for every $m \in \bigcap_{x \in R} \mu'(x, i)$, but this contracts condition III) of c-ideal. Thus the first assumption: $\bigcap_{i=1}^{h} S^2(x_0(\alpha_i\ n_i), \mathfrak{U}_{\alpha_i n_i}) = \phi$ is impossible. By this lemma $\{S^2(x_0(\alpha\ n), \mathfrak{U}_{\alpha n}) | \alpha \in A, n = 1, 2, \cdots\}$ is a chauchy filter, and hence by the completeness of R, this filter converges to a point of R.

Lemma 7. *If the chauchy filter* $\{S^2(x_0(\alpha\ n), \mathfrak{U}_{\alpha n}) | \alpha \in A, n = 1, 2, \cdots\}$ *converges to* $a \in R$, *then* $\{\mu\} = \{\mu\}(a)$.

Proof. Take arbitrary $\alpha_0 \in A$ and a positive integer n, then for $x_0 = x_0(\alpha_0\ n)$, there exists $m \in \mu$ such that $m_{\alpha_0}(x_0) < \frac{1}{2^n}$ for every $\mu \in \{\mu\}$. Since $a \in \overline{S^2(x_0(\alpha_0\ n), \mathfrak{U}_{\alpha_0 n})}$, from condition A) we get $m_{\alpha_0}(a) < \frac{1}{2^n} + \frac{1}{2^n} + m_{\alpha_0}(x_0) < \frac{1}{2^{n-2}}$. Hence $\{\mu\} \subset \{\mu\}(a)$. Therefore from condition 4) of max family. we get $\{\mu\} = \{\mu\}(a)$.

If $\{\mu\}(x_0) \subset \{\mu\}$ and if $\{\mu\}$ is a family of c-ideal satisfying conditions 1), 2), 3), then $\{\mu\} \subset \{\mu\}(a)$ for some $a \in R$. Hence $\{\mu\}(x_0) \subset \{\mu\}(a)$, but this is possible obviously only when $x_0 = a$. Hence $\{\mu\}(x_0) = \{\mu\}$; hence $\{\mu\}(x_0)$ satisfies 4), too.

Denoting by $\mathfrak{D}(R)$ the totality of max families of R, we get a one-to-one correspondence between R and $\mathfrak{D}(R)$ from the above conclusions. We shall denote by $\mathfrak{D}(A)$ the image of the subset A of R in $\mathfrak{D}(R)$ by this correspondence,

Definition. We call a covering $\{\mathfrak{D}(U_\gamma) \mid \gamma \in C\}$ of $\mathfrak{D}(R)$ a *uniform covering of* $\mathfrak{D}(R)$, if and only if there exists $u \in D(R)$ such that if $\{\mu\}(a_\gamma) \notin D(U_\gamma)$ $(\gamma \in C)$, then for some two points a_1, a_2 of a_γ $(\gamma \in C)$, there exist $\mu_1 \in \{\mu\}(a_1)$, $\mu_2 \in \{\mu\}(a_2)$ such that $m_1 \in \mu_1$, $m_2 \in \mu_2$ imply $u \leqq m_1 \vee m_2$.

Lemma 8. *In order that* $\{\mathfrak{D}(U_\gamma)\}$ *is a uniform covering of* $\mathfrak{D}(R)$ *it is necessary and sufficient that* $\{U_\gamma\}$ *is a uniform covering of* R.

Proof. *Necessity.* Let $\{U_\gamma\}$ be a uniform covering of R and let $\mathfrak{U}_{\alpha_0 n}$ be a uniform covering of the uniform basis of R such that $\mathfrak{U}^{\Delta *}_{\alpha 0 n} < \{U_\gamma\}$. For these α_0 and n, we put $u = \{u_\alpha(x)\}$, $u_{\alpha_0}(x) = \frac{1}{2^n}$, $u_\alpha(x) = 0$ $(\alpha \neq \alpha_0)$. If $\{\mu\}(a_\gamma) \notin \mathfrak{D}(U_\gamma)$, then we take a_1, a_2 from a_γ so that a_1 is an arbitrary point of a_γ and $S^2(a_1, \mathfrak{U}_{\alpha_0 n}) \subset U_\gamma \notin a_\gamma = a_2$. From Lemma 1, we get $\mu_1 \in \{\mu\}(a_1)$, $\mu_2 \in \{\mu\}(a_2)$ such that $m' \in \mu_1$ implies $m'_{\alpha 0}(x) \geqq \frac{1}{2^n}$ for $x \notin S(a_1, \mathfrak{U}_{\alpha_0 n})$, $m'' \in \mu_2$ implies $m''_{\alpha 0}(x) \geqq \frac{1}{2^n}$ for $x \notin S(a_2, \mathfrak{U}_{\alpha_0 n})$. Since $S(a_1, \mathfrak{U}_{\alpha_0 n}) \frown S(a_2, \mathfrak{U}_{\alpha_0 n}) = \phi$, $u \leqq m' \vee m''$ holds for every $m' \in \mu_1$, $m'' \in \mu_2$. Hence $\{\mathfrak{D}(\mathfrak{U}_\gamma)\}$ is a uniform covering of $\mathfrak{D}(R)$ by the above definition.

Sufficiency. Let us assume that $\{U_\gamma\}$ is not a uniform covering of R. If u is an arbitrary element of $D(R)$, then from condition B) of $D(R)$ there exist a finite number of α_i $(i = 1, \cdots, h)$ and a positive integer n such that $\bigcup_{i=1}^{h} u_{\alpha_i}(x) \geqq \frac{1}{2^n} > 0$. Since for these α_i, n, $\bigwedge_{i=1}^{h} \mathfrak{U}_{\alpha_i n+1}$ is a uniform covering of R. $\bigwedge_{i=1}^{h} \mathfrak{U}_{\alpha_i n+1} \not< \{U_\gamma\}$ holds. Hence there exists $U \in \bigwedge_{i=1}^{h} \mathfrak{U}_{\alpha_i n+1}$ such that $U \frown U^c_\gamma \neq \phi$ for all γ.

We take a definite point $x_0 \in U$. Then there exists α_i such that $u_{\alpha_i}(x_0) \geqq \frac{1}{2^n}$. Next we take a_γ so that $a_\gamma \in U \frown U^c_\gamma$, *i.e.* $\{\mu\}(u_\gamma) \notin \mathfrak{D}(U_\gamma)$. If a_1, a_2 are two points of a_γ and if $\mu_1 \in \{\mu\}(a_1)$, $\mu_2 \in \{\mu\}(a_2)$ then there exists $m' \in \mu_1$, $m'' \in \mu_2$ such that $m'_{\alpha_i}(a_1) < \frac{1}{2^{n+1}}$, $m''_{\alpha_i}(a_2) < \frac{1}{2^{n+1}}$. Since a_1, a_2, $x_0 \in U \in \bigwedge_{i=1}^{h} \mathfrak{U}_{\alpha_i n+1} < \mathfrak{U}_{\alpha_i n+1}$, from condition B) we get $m'_{\alpha_i}(x_0) < m'_{\alpha_i}(a_1) + \frac{1}{2^{n+1}} < \frac{1}{2^n}$, $m''_{\alpha_i}(x_0) < m''_{\alpha_i}(a_2) + \frac{1}{2^{n+1}} < \frac{1}{2^n}$. Hence $m'_{\alpha_i}(x_0) \cup m''_{\alpha_i}(x) < \frac{1}{2^n} \leqq u_{\alpha_i}(x_0)$, and hence $u \nleqq m' \vee m''$ for every $m' \in \mu_1$, $m'' \in \mu_2$. Therefore $\{\mathfrak{D}(U_\gamma)\}$ does not satisfy the condition of the above definition, and hence $\{\mathfrak{D}(U_\gamma)\}$ is not a uniform covering of $\mathfrak{D}(R)$.

From this lemma we get the following

Theorem 1. *In order that complete uniform spaces* R_1 *and* R_2 *are uniformly homeomorphic it is necessary and sufficient that* $D(R_1)$ *and* $D(R_2)$ *are order-isomorphic, where* $D(R_1)$, $D(R_2)$ *are directed systems of all functions satisfying conditions* A), B).

Next let us consider a uniform space without completeness property. Let $\mathfrak{U}_{\alpha n}$ and $P\{I_\alpha \mid \alpha \in A\}$ have the same meaning as in the case of complete space.

We denote by $D'(R)$ the directed system of all the uniformly continuous functions from R into $P\{I_\alpha | \alpha \in A\}-\{0\}$ satisfying condition A). We regard this directed system as having real numbers β such that $0<\beta\leqq 1$ as operators, and define $\beta u = \{\beta u_\alpha(x)\}$. We call a subset μ of $D'(R)$ a c-ideal as in the previous case, if μ satisfies conditions I), II), III). In this case, max family is a family of c-ideals satisfying conditions 2), 3), 4), and condition 1) is unnecessary.

We can prove in the same way as in the previous case that $\{\mu\}(x_0) = \{\mu |$ for every $\alpha, \varepsilon > 0$ there exists $u \in \mu$ such that $u_\alpha(x_0) < \varepsilon$, μ is a c-ideal of $D'(R)\}$ satisfies condition 2), 3). Conversely, if $\{\mu\}$ is a max family of c-ideals satisfying conditions 2), 3), 4), then $\{\mu\} \subset \{\mu\}(x_0)$ for some $x_0 \in R$. For if we assume the contrary, then for every $x_0 \in R$ there exist $\alpha(x_0) \in A$, $\varepsilon(x_0) > 0$ and $\mu(x_0) \in \{\mu\}$ such that $m_{(x_0)}(x_0) \geqq \varepsilon(x_0) > 0$ for every $m \in \mu(x_0)$. Then as in the previous case $\bigcap_{x_0 \in R} \mu^k(x_0)$ is to be a c-ideal of $D'(R)$ for a positive number k. Putting $snp\ \{inf\ \{m_\wedge k | m \in \mu(x_0)\} | x_0 \in R\} = u$. we see easily that $u(x_0) \geqq \varepsilon(x_0)_\frown k > 0$. Since obviously u has a finite value at every point of R, u is an element of $D'(R)$. If $m \in \subset_{x_0 \in R} \mu^k(x_0)$, then for every x_0, $m \geqq m_0 {}_\wedge k$ for some $m_0 \in \mu(x_0)$, and hence $m \geqq inf\ \{m_\wedge k | m \in \mu(x_0)\}$ for every $x_0 \in R$. Therefore it must be $m \geqq u$, but this contradicts condition III) of c-ideal. Hence we get $\{\mu\} \subset \{\mu\}(x_0)$ for some $x_0 \in R$ and $\{\mu\} = \{\mu\}(x_0)$ from condition 4) of max family.

It is obvious that in order that for every β: $0<\beta\leqq 1$, there exists $u' \in D'(R)$ such that $\beta u' = u$ it is necessary and sufficient that $u_\alpha(x)$ is a definite number $k_\alpha \geqq 0$ for each $\alpha \in A$. We denote by $\mathfrak{D}'(R)$ the totality of max families of $D'(R)$. To define uniform coverings of $\mathfrak{D}'(R)$, we replace "there exists $u \in D(R) \cdots$" in the previous definition of uniform covering of $\mathfrak{D}(R)$ with "there exists $u \in D'(R)$ such that $\beta u' = u$ for every β and for some $u' \in D'(R)$". Then we can show easily that $\{\mathfrak{D}(U_\gamma)\}$ is a uniform covering of $\mathfrak{D}(R)$ if and only if $\{U_\gamma\}$ is a uniform covering of R. Thus for a general uniform space we get the following.

Theorem 2. *In order that two uniform spaces R_1 and R_2 are uniformly homeomorphic it is necessary and sufficient that $D'(R_1)$ and $D'(R_2)$ are operator-isomorphic.*