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*On the Non-existence of Solution of Field Equations in Quantum Mechanics**

Osamu MIYATAKE

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Summary

The Hamiltonian operator of a system consisting of a nucleon and a scalar meson field, has no domain in the space whose vector is a superposition of states consisting of a finite number of mesons and a nucleon, in which the states of each meson and the nucleon being arbitrary.

§1. Introduction. As physicists understand, the field equations in quantum dynamics have no solution when the interaction terms are taken into account. But the rigorous proof of the non-existence of the solution seems not to have been given so tar.

In the present paper, first we determine the space in which the Hamiltonian of a system is defined, and then prove that the Hamiltonian operator has no domain in its subspace whose state vector is a superposition of states consisting of a finite number of particles, in which the state of each particle being arbitrary. For the sake of the simplicity of the treatment, we take, as an example, a system consisting of a nucleon and a scalar meson field.

In the course of the treatment, we assume that the wave functions of the nucleon and the scalar meson field are periodical with respect to a unit cube in the coordinate space. This assumption is not satisfactory in the scope of the relativity. The relativistically complete treatment will be made in another place.

§2. Determination of the space. First we state the results obtained by v. Neumann[1], in a form suitable for our purpose.

Let I be a set of indices α, whose number is enumerably infinite. $\mathfrak{H}_\alpha, \alpha \in I$ is a sequence of Hilbert spaces.

Definition 1. $z_\alpha, \alpha \in I$ and a are arbitrary complex numbers. Then the product $\Pi_{\alpha \in I} z_\alpha$ is convergent and its value is a when the following condition is satisfied. Let δ be an arbitrary positive number, then, corresponding to this δ, a finite set $I_0 = I_0(\delta) \subset I$ of α's can be determined in such a way that the difference

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$|z_{\lambda_1}z_{\lambda_2}\dots z_{\lambda_n}-a|$ is always made smaller than δ for any finite set of α $J=\mathfrak{S}(\alpha_1, \alpha_2, \dots, \alpha_n)$ which satisfies the conditions $I_0\subset J\subset I$.
According to this definition, an infinite product may have 0 as its covergent value.

Definition 2. When $\Pi_{\alpha\in I}\|\varphi_\alpha\|$, $\varphi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$ is convergent, the sequence of vectors φ_α, $\alpha\in I$ is called C-sequence.

Definition 3. When $\sum_{\alpha\in I}|\|\varphi_\alpha\|-1|$ is covergent, the sequence φ_α, $\alpha\in I$ is called C_0-sequence.

A C_0-sequence is a C-sequence. But the converse does not always hold good.

Definition 4. When $\varphi_\alpha, \psi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$ are both C_0-sequences and $\sum_{\alpha\in I}|(\varphi_\alpha, \psi_\alpha)-1|<\infty$, the two sequences $\{\varphi_\alpha\}$ and $\{\psi_\alpha\}$ are called equivalent.

v. Neumann has proved that the equivalency defined here is reflexive, symmetric and transitive. Thus the whole of the C_0-sequences is classified by the equivalency. The whole of these classes is called Γ.

In the next place, we define a direct product space of $\mathfrak{H}_\alpha$, $\alpha\in I$. Let Φ be a functional of $\varphi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$ and its value is a complex number. Φ is defined only when φ_α, $\alpha\in I$ is a C-sequence and its value is denoted as $\Phi(\varphi_\alpha\,;\alpha\in I)$. Instead of the notation $\Phi(\varphi_\alpha\,;\alpha\in I)$, the notation $\Phi(\varphi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)$ is also used when a special vector φ_{α_0} is taken into consideration.

Definition 5. $\Phi(\varphi_\alpha\,;\alpha\in I)$ is a functional of C-sequences $\varphi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$, and satisfies the following conditions.

(I) $\Phi(z\varphi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)=\bar{z}\,\Phi(\varphi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)$

(II) $\Phi(\varphi_{\alpha_0}+\psi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)=\Phi(\varphi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)$
$+\Phi(\psi_{\alpha_0}|\varphi_\alpha\,;\alpha\in I,\ \alpha\neq\alpha_0)$,

where z is a complex number and $\bar{z}$ is conjugate to z. The set of Φ's satisfying these conditions is denoted as $\Pi\bullet_{\alpha\in I}\mathfrak{H}_\alpha$.

Definition 6. A product $\Pi_{\alpha\in I}z_\alpha$ is called quasi-convergent when the product $\Pi_{\alpha\in I}|z_\alpha|$ is convergent. And

(I) when $\Pi_{\alpha\in I}z_\alpha$ is convergent, its limit is the value of the quasi-convergent product $\Pi_{\alpha\in I}z_\alpha$,

(II) when $\Pi_{\alpha\in I}z_\alpha$ is not convergent, 0 is adopted as the limit of the quasi-convergent product $\Pi_{\alpha\in I}z_\alpha$.

It can be proved that when $\varphi_\alpha, \psi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$ are C-sequences, the product $\Pi_{\alpha\in I}(\varphi_\alpha, \psi_\alpha)$ is quasi-convergent.

Dfinition 7. $\varphi_\alpha^0\in\mathfrak{H}_\alpha$, $\alpha\in I$ is a fixed C-sequence, and $\varphi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$ is an arbitrary C-sequence. Then, $\Phi(\varphi_\alpha\,;\alpha\in I)=\Pi_{\alpha\in I}(\varphi_\alpha^0, \varphi_\alpha)$ is quasi-convergent and an element of $\Pi\bullet_{\alpha\in I}\mathfrak{H}_\alpha$. Let this Φ be denoted as $\Phi=\Pi\otimes_{\alpha\in I}\varphi_\alpha^0$.

Definition 8. In $\Phi=\sum_{\nu=1}^{p}\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\nu}^0$, $p=0, 1, 2, \dots$, and $\varphi_{\alpha,\nu}^0\in\mathfrak{H}_\alpha$, $\alpha\in I$ are C-sequences for every ν which vary from 1 to p. The set of Φ's is denoted as $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Of course, $\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is a subset of $\Pi\bullet_{\alpha\in I}\mathfrak{H}_\alpha$. The inner product of $\Phi=\sum_{\nu=1}^{p}\Pi\otimes_{\alpha\in I}\varphi_{\alpha,\nu}^0$ and $\Psi=\sum_{\mu=1}^{q}\Pi\otimes_{\alpha\in I}\psi_{\alpha,\mu}^0$ is given by $(\Phi,\Psi)=\sum_{\nu=1}^{p}\sum_{\mu=1}^{q}\Pi_{\alpha\in I}(\varphi_{\alpha,\nu}^0,\psi_{\alpha,\mu}^0)$. The norm of Φ is defined by $(\Phi,\Phi)^{\frac{1}{2}}$ and is denoted as $\|\Phi\|$. The distance between Φ and Ψ is defined as $\|\Phi-\Psi\|$.

Definition 9. Let Φ be a functional for which exists a sequence $\Phi_1,\Phi_2,\ldots \in\Pi'\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ satisfying the following conditions:

(I) $\Phi(\varphi_\alpha;\alpha\in I)=\lim_{r\to\infty}\Phi_r(\varphi_\alpha;\alpha\in I)$ for all C-sequences $\varphi_\alpha\in\mathfrak{H}_\alpha$, $\alpha\in I$,

(II) $\lim_{r,s\to\infty}\|\Phi_r-\Phi_s\|=0$,

The set of such Φ's is denoted as $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

When $\lim_{n\to\infty}\Phi_n=\Phi$ and $\lim_{n\to\infty}\Psi_n=\Psi$, the inner product (Φ,Ψ) of Φ and Ψ is defined as $\lim_{n\to\infty}(\Phi_n,\Psi_n)$. By using the inner product, the topology is introduced in $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, and the following Theorem is obtained.

Theorem 1. The space $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is linear, metric and topologically complete.

Definition 10. $C\in\Gamma$ is an equivalent class. φ_α, $\alpha\in I$ is an arbitrary C_0-sequence contained in C. A closed linear set determined by the whole of the products such as $\Pi\otimes_{\alpha\in I}\varphi_\alpha$, is called the incomplete direct product space and denoted as $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$. Of course, this is a subspace of $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

When C and C' are different equivalent classes, $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and $\Pi^{C'}\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ are mutually orthogonal. The space determined by the whole of the incomplete direct product spaces $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $C\in\Gamma$ is $\Pi\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Lemma 1. Let C be an equivalent class. Then we can choose a C_0-sequence φ_α^0, $\alpha\in I$ in C such as $\|\varphi_\alpha^0\|=1$, $\alpha\in I$. And the closed linear set determined by the whole of the products $\Pi\otimes_{\alpha\in I}\varphi_\alpha$ whose factors φ_α are respectively equal to φ_α^0 for all but a finite number of the indices α, is the space $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$, $C\in\Gamma$.

Lemma 2. Let C be an equivalent class, and φ_α^0, $\alpha\in I$ be a C_0-sequence in C such that $\|\varphi_\alpha^0\|=1$. A sequence $\varphi_{\alpha,\beta(\alpha)}$, $\beta(\alpha)=0,1,2,\ldots$ is a c.n.o.s. (complete normalized orthogonal set) of $\mathfrak{H}_\alpha$, and $\varphi_{\alpha,0}=\varphi_\alpha^0$. When a finite number of $\beta(\alpha)$, $\alpha\in I$ are different from zero, we say that the set of $\beta(\alpha)$ belongs to the class $\boldsymbol{F}$, and denote it as $\beta(\alpha)\in\boldsymbol{F}$, or simply as $\beta\in\boldsymbol{F}$. When the set $\beta(\alpha)$, $\alpha\in I$ varies on the whole of $\boldsymbol{F}$, the whole of $\Pi_{\beta\in F}\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)}$ is a c.n.o.s. of $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$.

Theorem 2. When we use the notations in *Lemma* 2, there is one-to-one correspondence between elements Φ in $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ and coefficients $a[\beta(\alpha);\alpha\in I]$ in the following way:

(I) $a[\beta(\alpha);\alpha\in I]$'s are complex numbers and defined for only the functions $\beta(\alpha)\in\boldsymbol{F}$,

(II) $\sum_{\beta(\alpha)\in F}|a[\beta(\alpha);\alpha\in I]|^2$ is convergent,

(III) $a[\beta(\alpha);\alpha\in I]=(\Phi,\Pi_{\beta\in F}\otimes_{\alpha\in I}\varphi_{\alpha,\beta(\alpha)})=\Phi(\varphi_{\alpha,\beta(\alpha)};\alpha\in I)$,

(IV) when Φ and Ψ correspond to $a[\beta(\alpha);\alpha\in I]$ and $b[\beta(\alpha);\alpha\in I]$, respectively,

$$(\Phi,\Psi)=\sum_{\beta(\alpha)\in F}a[\beta(\alpha);\alpha\in I]\,b[\beta(\alpha);\alpha\in I].$$

Theorem 3. The space $\Pi^C\otimes_{\alpha\in I}\mathfrak{H}_\alpha$ is linear, metric and topologically complete.

The introduction of v. Neumann's results has finished. We shall apply these results to our problem.

As already has been known[2], a scalar meson field is described by a scalar function $U(x, y, z, t)$ and its conjugate function $\bar{U}(x, y, z, t)$.

In the presence of a nucleon, the Hamiltonian of the total system is given by

$$H=H_0+H', \tag{2}$$

where

$$H_0=H_M+H_U. \tag{2}$$

H_M is the Hamiltonian of the nucleon:

$$H_M=\alpha\boldsymbol{p}+\rho_3\left(\frac{1}{2}(1+\tau_3)M_N+\frac{1}{2}(1-\tau_3)M_P\right). \tag{3}$$

H_U is the Hamiltonian of the meson field:

$$H_U=\sum_k(N^+(k)+N^-(k))E_k. \tag{4}$$

H' is the interaction term:

$$H'=-ig\sum_k\sqrt{\frac{2\pi}{E_k}}\left\{(a_k^*-b_k)\exp(-i\boldsymbol{kr})Q^*-(a_k-b_k^*)\exp(i\boldsymbol{kr})Q\right\}\rho_3. \tag{5}$$

In the present paper, we use the natural unit, i.e., the light velocity c and Planck constant $\hbar$ are both equal to 1. The neutron mass M_N and the proton mass M_P are both assumed to be equal to one and the same value M. (α, ρ_3) are Dirac's spin matrices. We use ρ_3 instead of the usual notation β. (τ_1, τ_2, τ_3) are the isotopic spin matrices. We use the usual matrix representations

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{6}$$

E_k in (4) is equal to $\sqrt{k^2+m^2}$, where $\boldsymbol{k}$ and m is the momentum and the mass of a meson, respectively. The k in the formulas (4) and (5) stands for the momentum $\boldsymbol{k}$. The operator $N^+(k)=a_k^*a_k$ is an infinite matrix having the eigenvalues 0, 1, 2, ..., and is an operator representing the number of mesons whose charge, momentum and energy are $+e$, $\boldsymbol{k}$ and E_k, respectively. The operator $N^-(k)=b_k^*b_k$ represents the number of mesons whose charge, momentum and energy are $-e$, $-\boldsymbol{k}$ and E_k, respectively. We shall call $N^+(k)$-meson a meson belonging to the operator $N^+(k)$, and $N^-(k)$-meson a meson belonging to the operator $N^-(k)$. In the above formulas

$$\boldsymbol{p} = -i \text{ grad}. \quad Q = \frac{1}{2}(\tau_1 + i\tau_2), \quad Q^* = \frac{1}{2}(\tau_1 - i\tau_2).$$

The components of $\boldsymbol{k}$ are either zero or integers multiplied by 2π. The operators a_k^* and a_k are the creation and the annihilation operators of $N_+(k)$-meson, respectively. b_k^* and b_k are those of $N^-(k)$-meson.

A state in which $\beta^+(k)$ $N^+(k)$-mesons are there, is represented by a vector whose $(\beta^+(\boldsymbol{k})+1)$-th component is 1, the other components being all zero, where $\beta^+(k)=0, 1, 2, \ldots$. We shall denote this vector as $\varphi(\beta^+(k))$. Whole of $\varphi(\beta^+(k))$, $\beta^+(k)=0, 1, 2, \ldots$ determines a Hilbert space, $\mathfrak{H}^+(k)$. In the same way, we define $\varphi(\beta^-(k))$, $\beta^-(k)=0, 1, 2, \ldots$ for each value of $\boldsymbol{k}$. Whole of $\varphi(\beta^-(k))$, $\beta^-(k)=0, 1, 2, \ldots$ determines a Hilbert space. $\mathfrak{H}^-(k)$.

It can easily be proved that the operators $N^+(k)$ and $N^-(k)$ are self-adjoint in the respective spaces $\mathfrak{H}^+(k)$ and $\mathfrak{H}^-(k)$.

Let the set of the momenta $\boldsymbol{k}$'s be K. The sequence $\varphi(\beta^+(k)=0)$, $\boldsymbol{k}\in K$ is a C_0-sequence. Let this sequence correspond to the C_0-sequence $\varphi_\alpha{}^0$, $\alpha\in I$ in *Lemma* 2, and let $\varphi(\beta^+(k))$, $k\in K$, $\beta^+(k)=0, 1, 2, \ldots$ correspond to $\varphi_{\alpha,\beta(\alpha)}$, $\alpha\in I$. The fact that only a finite number of $\beta^+(k)$, $k\in K$ are non-zero, is expressed by a symbol $\beta^+\in\boldsymbol{F}$. Then, it is concluded that the whole of the vectors

$$\varphi_{\beta^+}(U) = \Pi_{\beta^+\in\boldsymbol{F}} \otimes_{\boldsymbol{k}\in K} \varphi(\beta^+(k)) \tag{7}$$

is a c.n.o.s. of the incomplete direct product space

$$\mathfrak{H}^+(U) = \Pi^C \otimes_{k\in K} \mathfrak{H}^+(k), \tag{8}$$

where C is the equivalent class to which the C_0-sequence $\varphi(\beta^+(k)=0)$, $k\in K$ belongs. In like manner, we define

$$\varphi_{\beta^-}(U) = \Pi_{\beta^-\in\boldsymbol{F}} \otimes_{k\in K} \varphi(\beta^-(k)) \tag{9}$$

and the whole of $\varphi_{\beta^-}(U)$'s is a c.n.o.s. of the incomplete direct product space

$$\mathfrak{H}^-(U) = \Pi^C \otimes_{k\in K} \mathfrak{H}^-(k), \tag{10}$$

where C is the equivalent class to which the C_0-sequence $\varphi(\beta^-(k)=0)$, $k\in K$ belongs. From (7) and (9), we define a vector

$$\varphi_\beta(U) = \varphi_{\beta^+}(U)\otimes\varphi_{\beta^-}(U) = \Pi_{\beta^+\in\boldsymbol{F}} \otimes_{k\in K} \varphi(\beta^+(k)) \otimes \Pi_{\beta^-\in\boldsymbol{F}} \otimes_{k\in K} \varphi(\beta^-(k)), \tag{11}$$

which is simply written as

$$\varphi_\beta(U) = \Pi_k \otimes \varphi(\beta^+(k)) \otimes \Pi_k \otimes \varphi(\beta^-(k)). \tag{12}$$

The whole of $\varphi_\beta(U)$'s is the c.n.o.s. of the incomplete direct product space

$$\mathfrak{H}(U) = \mathfrak{H}^+(U)\otimes\mathfrak{H}^-(U) = \Pi_k^C \otimes \mathfrak{H}^+(k) \otimes \Pi_k^C \otimes \mathfrak{H}^-(k). \tag{13}$$

The vector $\varphi_\beta(U)$ represents a state of a system consisting of $\beta^+(k)$, $k\in K$,

$N^+(k)$-mesons and $\beta^-(k)$, $k\in K$, $N^-(k)$-mesons. The condition $\beta^\pm\in \boldsymbol{F}$ means that the total number of mesons contained in the system, is finite. This condition is necessary in order that the total energy of the meson field is finite.

It is easy to verify that $\varphi_\beta(U)$ is an eigenfunction of H_U and that the corresponding eigenvalue is $\sum(\beta^+(k))+\beta^-(k)E_k$, which is finite by the condition $\beta^\pm\in F$.

In the next place, we shall consider the space of vectors which represent the states of the nucleon.

When we put as

$$\psi=\psi^{(N)}+\psi^{(P)},\ \psi^{(N)}=\begin{pmatrix}u\exp(i\boldsymbol{pr})\\0\end{pmatrix},\quad \psi^{(P)}=\begin{pmatrix}0\\u\exp(i\boldsymbol{pr})\end{pmatrix},$$
$$\widetilde{H}=\alpha\boldsymbol{p}+\rho_3 M,$$

the eigenvalue problem

$$H_M\psi=E\psi \tag{14}$$

is reduced to that of the equation

$$\widetilde{H}u=Eu, \tag{15}$$

which has four solutions orthogonal with one another:

$$\widetilde{H}u^{(1)}=E(p)u^{(1)},\ \ e\sigma u^{(1)}=u^{(1)}\ ;\ \widetilde{H}u^{(2)}=E(p)u^{(2)},\ \ e\sigma u^{(2)}=-u^{(2)},$$
$$\widetilde{H}u^{(3)}=-E(p)u^{(3)},\ \ e\sigma u^{(3)}=u^{(3)}\ ;\ \widetilde{H}u^{(4)}=-E(p)u^{(4)},\ \ e\sigma u^{(4)}=-u^{(4)}.$$

Here, $\boldsymbol{e}$ is a unit vector in the direction of $\boldsymbol{p}$, and $E(p)^2=\boldsymbol{p}^2+M^2$.

As the wave function ψ is periodic with respect to a unit cube, the number of $\boldsymbol{p}$'s is enumerably infinite, and the four vectors $u^{(i)}$, $i=1, 2, 3, 4$ are determined for each $\boldsymbol{p}$. Thus we obtain enumerably infinite number of $\psi^{(N)}$'s and $\psi^{(P)}$'s. And when we put as

$$\psi^{(N)}_{m,n}=\begin{pmatrix}u_n^{(m)}\exp(i\boldsymbol{p}_n\boldsymbol{r})\\0\end{pmatrix},\quad \psi^{(P)}_{m,n}=\begin{pmatrix}0\\u_n^{(n)}\exp(i\boldsymbol{p}_n\boldsymbol{r})\end{pmatrix},$$
$$m=1,2,3,4\ ;\ n=1,2,3,4,\dots,$$

we obtain the orthogonality relations

$$(\psi^{(N)}_{m,n},\psi^{(N)}_{m',n'})=\delta_{mm'}\delta_{nn'},\quad (\psi^{(P)}_{m,n},\psi^{(P)}_{m',n'})=\delta_{mm'}\delta_{nn'},$$
$$(\psi^{(N)}_{m,n},\psi^{(P)}_{m',n'})=0,$$
$$(\psi^{(N)}_{m,n},Q\psi^{(P)}_{m',n'})=\delta_{mm'}\delta_{nn'},\quad (Q^*\psi^{(N)}_{m,n},\psi^{(P)}_{m',n'})=\delta_{mm'}\delta_{nn'}.$$

The orthonormal set $\{\psi^{(N)},\psi^{(P)}\}$ thus obtained is denoted simply as $\{\varphi(M)\}$. The c.n.o.s. $\{\varphi(M)\}$ determines a Hilbert space, $\mathfrak{H}(M)$.

Using $\psi^{(N)}$, $\psi^{(P)}$ and $\varphi_\beta(U)$ in (12), we obtain two vectors

$$\phi^{(N)}_\beta=\psi^{(N)}\otimes\varphi_\beta(U),\quad \phi^{(P)}_\beta=\psi^{(P)}\otimes\varphi_\beta(U).$$

When we need not to distinguish between $\psi^{(N)}$ and $\psi^{(P)}$, instead of these notations, we use the notation

$$\phi_\beta = \varphi(M) \otimes \varphi_\beta(U).$$

The orthonormal set $\{\phi_\beta\}$ determines an incomplete direct product space

$$\mathfrak{H} = \mathfrak{H}(M) \otimes \mathfrak{H}(U)$$

§3. Singularity of the Hamiltonian operator.

Theorem 1. *The vectors $\phi_\beta^{(N)}$ ond $\phi_\beta^{(P)}$ do not belong to the domain of the interaction term H'.*

Proof. For brevity, we put as

$$\sqrt{\frac{2\pi}{E_k}}\left\{(a_k^* - b_k)\exp(-i\boldsymbol{kr})Q^* - (a_k - b_k^*)\exp(i\boldsymbol{kr})Q\right\}\rho_3 \equiv H_k. \tag{16}$$

Then

$$H_k\phi_\beta^{(N)} = \sqrt{\frac{2\pi}{E_k}}\left\{\rho_3\sqrt{\beta^+(k)+1}\,\phi_\beta^{(N)}\left[Q^*\psi^{(N)};\ \cdots +1(k)\cdots;\ \cdots\cdots\cdots\cdots\right]\exp(-iE_kt)\right.$$
$$\left. -\rho_3\sqrt{\beta^-(k)}\,\phi_\beta^{(N)}\left[Q^*\psi^{(N)};\ \cdots\cdots\cdots;\ \cdots -1(k)\cdots\right]\exp(iE_kt)\right\}\exp(-i\boldsymbol{kr}), \tag{17}$$

where

$$\phi_\beta^{(N)}\left[Q^*\psi^{(N)};\ \cdots +1(k)\cdots;\ \cdots\cdots\right]$$
$$= Q^*\psi^{(N)}\otimes\varphi(\beta^+(k)+1)\otimes\prod_{k'\neq k}\otimes\varphi(\beta^+(k'))\otimes\prod_{k'}\otimes\varphi(\beta^-(k')), \tag{18}$$

$$\phi_\beta^{(N)}\left[Q^*\psi^{(N)};\ \cdots\cdots;\ \cdots -1(k)\cdots\right]$$
$$= Q^*\psi^{(N)}\otimes\varphi(\beta^-(k)-1)\otimes\prod_{k'}\otimes\varphi(\beta^+(k'))\otimes\prod_{k'\neq k}\otimes\varphi(\beta^-(k')) \tag{19}$$

when

$$\phi_\beta^{(N)} = \psi^{(N)}\otimes\Pi\,\varphi(\beta^+(k))\otimes\Pi\otimes\varphi(\beta^-(k)).$$

The two vectors (18) and (19) are mutually orthogonal, and $H_k\phi_\beta^{(N)}$ and $H_{k'}\phi_\beta^{(N)}$ are mutually orthogonal when $k\neq k'$. So that we obtain

$$\|H'\phi_\beta^{(N)}\|^2 = g^2\left(\textstyle\sum_{k\in K}H_k\phi_\beta^{(N)},\ \sum_{k\in K}H_k\phi_\beta^{(N)}\right) = 2\pi g\textstyle\sum_{k\in K}\frac{1}{E_k}(\beta^+(k)+\beta^-(k)+1) = \infty.$$

i.e., $\|H'\phi_\beta^{(N)}\| = \infty$. In like manner, we obtain $\|H'\phi_\beta^{(P)}\| = \infty$.

Theorem 2. *Let ϕ be a vector belonging to the domain of H_0, and be expanded as*

$$\phi = \sum_{j=1}^{\infty} c_{\beta_j}^{(N)}\phi_{\beta_j}^{(N)} + \sum_{j=1}^{\infty} c_j^{(P)}\phi_{\beta_j}^{(P)}. \tag{20}$$

Here $c_{\beta_j}^{(N)}$ and $c_{\beta_j}^{(P)}$ are numerical coefficients, and $\phi_{\beta_j}^{(N)}$ is the j-th vector of the set obtained by arranging the members of the set $\{\phi_\beta^{(N)}\}$ in a suitable manner. $\phi_{\beta_j}^{(P)}$ is defined in a similar manner. Then $\sum_j|c_j^{(N)}E^{(N)}(\beta_j)|^2$ and $\sum_j|c_j^{(P)}E^{(P)}(\beta_j)|^2$ are both convergent, where

$$E^{(N)}(\beta_j) = \pm E(p) + \sum_{k\in K}(\beta^+(k)+\beta^-(k))E_k, \tag{21}$$

in which $+E(p)$ *or* $-E(p)$ *is the eigenvalue corresponding to the nucleon state vector* $\psi^{(N)}$ *which is the factor of* $\phi_{\beta_j}^{(N)}$*, and* $\beta^+(k)$ *and* $\beta^-(k)$ *in the summation on the right hand side are the particle numbers in the state* $\phi_{\beta_j}^{(N)}$*.* $E^{(P)}(\beta_j)$ *is defined in a similar manner.*

Proof. In the following, we use the notation $E_I^{(N)}$ such as

$$E_I^{(N)}(\beta_j) = \pm E(p) + \sum_{|k|<I} \beta^+(k)+\beta^-(k)),$$

so that

$$\lim_{I\to\infty} E_I^{(N)}(\beta_j) = E^{(N)}(\beta_j).$$

In the same way, $E_I^{(P)}(\beta_j)$ is defined as

$$\lim_{I\to\infty} E_I^{(N)}(\beta_j) = E^{(P)}(\beta_j).$$

Then

$$\begin{aligned}
\infty > \|H_0\phi\|^2 &= \sum_{\phi,\ \beta\in F}|(H_0\phi, \phi_\beta)|^2 \\
&= \sum\Big|\lim_{I\to\infty}([H_M + \sum_{|k|<I}(N^+(k)+N^-(k))E_k]\phi, \phi_\beta)\Big|^2 \\
&= \sum\Big|\lim_{I\to\infty}\Big\{\sum_{j=1}^{\infty} c_j^{(N)}(\phi_{\beta_j}^{(N)}, [H_M+\sum_{|k|<I}(N^+(k)+N^-(k))E_k]\phi_\beta) \\
&\qquad + \sum_{j=1}^{\infty} c_j^{(P)}(\phi_{\beta_j}^{(P)}, [H_M+\sum_{|k|<I}(N^+(k)+N^-(k))E_k]\phi_\beta)\Big\}\Big|^2 \\
&\geqq \sum_{\phi_\beta^{(N)},\ \beta\in F}\Big|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j^{(N)}([H_M+\sum_{|k|<I}(N^+(k)+N^-(k))E_k)]\phi_{\beta_j}^{(N)}, \phi_\beta^{(N)})\Big|^2 \\
&= \sum\Big|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j^{(N)}(E_I^{(N)}(\beta_j)\phi_{\beta_j}^{(N)}, \phi_\beta^{(N)})\Big|^2 \\
&= \sum_{j=1}^{\infty}|c_j^{(N)}E^{(N)}(\beta_j)|^2.
\end{aligned}$$

So that

$$\sum_{j=1}^{\infty}|c_j^{(N)}E^{(N)}(\beta_j)|^2 < \infty. \tag{22}$$

In the same way, we obtain

$$\sum_{j=1}^{\infty}|c_j^{(P)}E^{(P)}(\beta_j)|^2 < \infty. \tag{23}$$

Theorem 3. *Let a vector* ϕ *expand in a series*

$$\phi = \sum c_j\phi_{\beta_j} = \sum_{j=1}^{\infty} c_j^{(N)}\phi_{\beta_j}^{(N)} + \sum_{j=1}^{\infty} c_j^{(P)}\phi_{\beta_j}^{(P)}.$$

If the number of mesons in each of the states $\phi_{\beta_j}^{(N)}$*'s and* $\phi_{\beta_j}^{(P)}$*'s is smaller than a fixed constant,* ϕ *does not belong to the domain of the interaction term* H'*.*

Proof. If ϕ belongs to the domain of H', $\|H'\phi\|$ must be finite. But, as being shown below, this assumption leads to a contradiction. We assume that ϕ belongs to the domain of H', then

$$\infty > ||H'\phi||^2 = \sum_{\beta\in F}|(H'\phi, \phi_\beta)|^2 = g^2\sum_{\beta\in F}|\lim_{I\to\infty}(\sum_{|k|<I} H_k\phi, \phi_\beta)|^2$$
$$= g^2\sum_{\beta\in F}|\lim_{I\to\infty}(\phi, \sum_{|k|<I} H_k\phi_\beta)|^2 = g^2\sum_{\beta\in F}|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j(\phi_{\beta_j}, \sum_{|k|<I} H_k\phi_\beta)|^2$$
$$= g^2\sum_{\beta\in F}|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j \sum_{|k|<I}(H_k\phi_{\beta_j}, \phi_\beta)|^2$$
$$\geqq g^2\sum_{\phi_\beta^{(P)}, \beta\in F}|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j^{(N)}\sum_{|k|<I}(H_k\phi_{\beta_j}^{(N)}, \phi_\beta^{(P)})|^2. \tag{24}$$

We assume that the meson number of the vector $\phi_{\beta_1}^{(N)}$ is not smaller than those of the other $\phi_{\beta_j}^{(N)}$ $j \neq 1$, i.e., $\sum(\beta^+(k)+\beta^-(k))$ of $\phi_{\beta_1}^{(N)}$ is the least upper bound of those of all $\phi_{\beta_j}^{(N)}$, $j=1, 2, 3, \ldots\ldots$.

$$H_k\phi_{\beta_1}^{(N)} = \sqrt{\frac{2\pi}{E_k}}\Big\{\sqrt{\beta^+(k)+1}\,\rho_3\phi_{\beta_1}^{(N)}[Q^*\psi_{\beta_1}^{(N)}; \cdots +1(k)\cdots; \cdots\cdots]\exp(-iE_kt)$$
$$-\sqrt{\beta^-(k)}\,\rho_3\phi_{\beta_1}^{(N)}[Q^*\psi_{\beta_1}^{(N)}; \cdots\cdots; \cdots -1(k)\cdots]\exp(iE_kt)\Big\}\exp(-i\boldsymbol{kr}), \tag{25}$$

where $\psi_{\beta_1}^{(N)}$ is the nucleon state vector contained in $\phi_{\beta_1}^{(N)}$. When the order number of $\boldsymbol{k}$ is sufficiently large, both $\beta^+(k)$ and $\beta^-(k)$ vanish, and we obtain

$$H_k\phi_{\beta_1}^{(N)} = \sqrt{\frac{2\pi}{E_k}}\rho_3\phi_{\beta_1}^{(N)}[Q^*\psi_{\beta_j}^{(N)}; \cdots +1(k)\cdots; \cdots\cdots]\exp(-iE_kt-i\boldsymbol{kr}). \tag{26}$$

Let the domain of k for which the expression (25) is reduced to the form of (26), be denoted by $\tilde{K}$. The k's which belong to K but not to $\tilde{K}$, is of finite number.

A vector $\phi_\beta^{(P)}$ which satisfies the relation

$$(H_k\phi_{\beta_1}^{(N)}, \phi_\beta^{(P)}) = \sqrt{\frac{2\pi}{E_k}}(\rho_3 u_{\beta_1}^{(N)}, u_\beta^{(P)})\exp(-iE_kt) \tag{27}$$

is of the form

$$\phi_\beta^{(P)} = \phi_{\beta_1}^{(N)}[\psi_\beta^{(P)}; \cdots +1(k)\cdots; \cdots\cdots], \quad k\in\tilde{K} \tag{28}$$

where $\psi_\beta^{(P)}$ is a nucleon state vector whose mometum is $\boldsymbol{p}-\boldsymbol{r}$, $\boldsymbol{p}$ being the momentum of $\psi_{\beta_1}^{(N)}$. Let the assembly of such $\phi_\beta^{(P)}$'s be denoted as $\tilde{\boldsymbol{F}}$. Then, from (24),

$$\infty > g^2\sum_{\phi_\beta^{(P)}\in\tilde{\boldsymbol{F}}}|\lim_{I\to\infty}\sum_{j=1}^{\infty} c_j^{(N)}\sum_{|k|<I}(H_k\phi_{\beta_j}^{(N)}, \phi_\beta^{(P)})|^2$$
$$= g^2\sum_{\phi_\beta^{(P)}\in\tilde{\boldsymbol{F}}}|c_1^{(N)}\sum_{k\in K}(H_k\phi_{\beta_j}^{(N)}, \phi_\beta^{(P)}) + \lim_{I\to\infty}\sum_{j=2}^{\infty} c_j^{(N)}\sum_{|k|<I}(H_k\phi_{\beta_j}^{(N)}, \phi_\beta^{(P)})|^2. \tag{29}$$

And, on the one hand,

$$\infty = ||H'\phi_{\beta_1}^{(N)}||^2 = \sum_{\phi_\beta^{(P)}\bar{\in}\tilde{\boldsymbol{F}}}|(H'\phi_{\beta_1}^{(N)}, \phi_\beta^{(P)})^2 + \sum_{\phi_\beta^{(P)}\in\tilde{\boldsymbol{F}}}|(H'\phi_{\beta_1}^{(N)}, \phi_\beta^{(P)})|^2.$$

The number of $\phi_\beta^{(P)}\bar{\in}\tilde{\boldsymbol{F}}$ which satisfy the condition

$$(H'\phi_{\beta_1}^{(N)}, \phi_\beta^{(P)}) \neq 0$$

is finite, so that we obtain

$$\sum_{\phi_\beta^{(P)}\in\tilde{F}}|(H'\phi_{\beta_1}^{(N)},\ \phi_\beta^{(P)})|^2=\infty.$$

Accordingly, if we can prove that

$$\sum_{\phi_\beta^{(P)}\in\tilde{F}}|\lim_{I\to\infty}\sum_{j=2}^{\infty}c_j^{(N)}\sum_{|k|<I}(H_k\phi_{\beta_j}^{(N)},\ \phi_\beta^{(P)})|^2<\infty, \tag{30}$$

it will be concluded that the right hand side of (29) is divergent. This is contradictory to the inequality (29).

For arbitrary k' and β_j, $j\geqq 2$,

$$H_{k'}\phi_{\beta_j}^{(N)}=\sqrt{\frac{2\pi}{E_{k'}}}\Big\{\sqrt{\beta^+(k')+1}\,\rho_3\phi_{\beta_j}^{(N)}[Q^*\psi_{\beta_j}^{(N)};\cdots+1(k')\cdots;\cdots\cdots]\exp(-iE_{k'}t)$$
$$-\sqrt{\beta^-(k')}\,\rho_3\phi_{\beta_j}^{(N)}[Q^*\psi_{\beta_j}^{(N)};\cdots\cdots;\cdots-1(k')\cdots]\exp(iE_kt)\Big\}\exp(-i\boldsymbol{kr}). \tag{31}$$

So that, when and only when $\phi_{\beta_j}^{(N)}$ satisfies the relation

$$\phi_{\beta_j}^{(N)}[Q^*\psi_{\beta_j}^{(N)};\cdots+1(k')\cdots;\cdots\cdots]\exp(-i\boldsymbol{k'r})=\phi_\beta^{(P)}=\phi_{\beta_1}^{(N)}[\psi_\beta^{(P)};\cdots+1(k)\cdots;\cdots\cdots] \tag{32}$$

or

$$\phi_{\beta_j}^{(N)}[Q^*\psi_{\beta_j}^{(N)};\cdots\cdots;\cdots-1(k')\cdots]\exp(-i\boldsymbol{k'r})=\phi_\beta^{(P)}=\phi_{\beta_1}^{(N)}[\psi_\beta^{(P)};\cdots+1(k)\cdots;\cdots\cdots], \tag{33}$$

the condition

$$(H_{k'}\phi_{\beta_j}^{(N)},\ \phi_\beta^{(P)})\neq 0,\quad \phi_\beta^{(P)}\in\tilde{F} \tag{34}$$

holds valid. (32) is equivalent to the condition:

$$\psi_{\beta_j}^{(N)}=\psi_\beta^{(P)}\exp(i\boldsymbol{k'r}),\quad (\beta^+(k')\text{ of }\phi_{\beta_j}^{(N)})=(\beta^+(k')\text{ of }\phi_{\beta_1}^{(N)})-1,$$
$$(\beta^+(k)\text{ of }\phi_{\beta_j}^{(N)})=(\beta^+(k)\text{ of }\phi_{\beta_1}^{(N)})+1=1,$$
$$(\beta^\pm(k'')\text{ of }\phi_{\beta_j}^{(N)})=(\beta^\pm(k'')\text{ of }\phi_{\beta_1}^{(N)}),\quad k''\neq k,k'. \tag{35}$$

(33) is equivalent to the condition

$$\psi_{\beta_j}^{(N)}=\psi_\beta^{(P)}\exp(i\boldsymbol{k'r}),\quad (\beta^-(k')\text{ of }\phi_{\beta_j}^{(N)})=(\beta^-(k')\text{ of }\phi_{\beta_1}^{(N)})+1,$$
$$(\beta^+(k)\text{ of }\phi_{\beta_j}^{(N)})=(\beta^+(k)\text{ of }\phi_{\beta_1}^{(N)})+1=1,$$
$$(\beta^\pm(k'')\text{ of }\phi_{\beta_j}^{(N)})=(\beta^\pm(k'')\text{ of }\phi_{\beta_1}^{(N)}),\quad k''\neq k,k'. \tag{36}$$

From the assumption that $\sum(\beta^+(k)+\beta^-(k))$ of $\phi_{\beta_1}^{(N)}$ is not smaller than those of the other $\phi_{\beta_j}^{(N)}$, no $\phi_{\beta_j}^{(N)}$ satisfies the condition (36). The number of $\phi_{\beta_j}^{(N)}$'s which satisfy the condition (35) is bounded for all $\phi_\beta^{(P)}\in\tilde{F}$.

When the nucleon momenta of $\phi_\beta^{(P)}\in\tilde{F}$ and $\phi_{\bar\beta}^{(P)}\in\tilde{F}$ are different from each other, one and the same vector $\phi_{\beta_j}^{(N)}$ can not satisfy simultaneously the two conditions

$$(H_{k'}\phi_{\beta_j}^{(N)},\ \phi_\beta^{(P)})\neq 0,\quad (H_{k''}\phi_{\beta_j}^{(N)},\ \phi_{\bar\beta}^{(P)})\neq 0. \tag{37}$$

for arbitrary k' and k''.

When the nucleon momenta of $\phi_{\beta}^{(P)}$ and $\phi_{,}^{(P)}$ are equal, the condition (37) can hold valid simultaneously. But the number of $\phi_{\beta}^{(P)}$'s for each $\boldsymbol{k}$, is at most four. By using these results and the condition $\sum_{j=1}^{\infty}|c_j^{(N)}|^2<\infty$, we can conclude that the inequality (30) holds valid. Q.E.D.

§4. Cnclusion. The interactinn operator H' has no domain in a space whose state vector is a superposition of states consisting of a finite number of mesons and a nucleon, the states of each meson and the nucleon being arbitrary. So that, the total Hamiltonian H has no domain in this space. But we cannot prove whether H' has a domain in the whole incomplete direct product space $\mathfrak{H}$ or not. The space $\mathfrak{H}$ contains a sequence of vectors whose meson numbers increase infinitely. In this case, the number of $\phi_{\beta_j}^{(N)}$'s which satisfy the condition (36) in the preceding paragraph, may be infinite.

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References

1) J. V. Neumann, Comp. Math. 46, 1 (1939).
2) H. Yukawa and S. Sakata, Proc. Phys. Math. Soc. Japan 19, 1048 (1937)