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On the Foundation of Orders in Groups

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0. Orderd groups, or o-groups, have been studied by G. Birkhoff, A. H. Clifford, H. Cartan, T. Nakayama, C. J. Everett and S. Ulam, J. von Neuman and others, while lattice ordered groups, or l-groups, also discussed by many mathematicians, G. Birkhoff and others.[1]

The present work is to establish the structure or order-buds (cf. below) in groups. This notion is closely conjugated with that of algebraic systems[2]; in other words, an order-bud is nothing but the modification of an algebraic order in groups.

1. We shall begin with some definitions. Let **G** be a group, e being its group identity, and P a subset of **G** with the following properties;

i) $e \in P$,

ii) $PP \subset P$.

We call such P *an order-bud in* **G**; in facts, we can define an order in **G**, $x \leqq y\ (P,l)$, $x, y \in G$, when $x^{-1}y \in P$, and another order, $x \leqq y(P,r)$, when $yx^{-1} \in P$.

The former, $\leqq(P.l)$, is called a, *left order* in G, while the latter, $\leqq(P,r)$, a *right order*.

(1. 1) If $x \leqq y(P,l)$ or $x \leqq y(P,r)$, then for all $t \in G$, $tx \leqq ty(P,l)$ or $xt \leqq yt(P,r)$ respectively.

It comes from the equalities $(tx)^{-1}\cdot ty = x^{-1}t^{-1}ty = x^{-1}y$ and $yt \cdot (xt)^{-1} = ytt^{-1}x^{-1} = yx^{-1}$.

(1. 2) The set of all t such that $x \leqq t(P,l)$ or $x \leqq t(P,r)$ coincides with $x \cdot P$ or $P \cdot x$ respectively.

We denote that $x \cdot P = P^l_x$, $P \cdot x = P^r_x$, then we have

(1. 3) $P^l_e = P^r_e = P$.

(1. 4) $y \in P^l_x$ implies $P^l_y \subset P^l_x$ and $y \in P^r_x$ implies $P^r_y \subset P^r_x$.

(1. 5) $a \cdot P^x_l = P^l_{ax}$, $P^r_x \cdot a = P^r_{xa}$,

If an order-bud P in **G** fulfils the further condition; for every $t \in G$,

iii) $tPt^{-1} \subset P$,

then we call P *normal*.

(1. 6) $P^l_a = P^r_a = a \cdot P = P \cdot a$ for normal P.

We put $P^l_a (= P^r_a) = P_a$ for normal P.

Let $\{P^\lambda\}_{\lambda \in \Lambda}$ be a family of order-buds in **G**, then the set-intersection of them

$$\cap_{\lambda \in \Lambda} P^\lambda$$

is also an order-bud in G, and if all P^λ; $\lambda \in \Lambda$, are normal, then $\bigcap_{\lambda \in \Lambda} P^\lambda$ is also normal. We next define $\bigcup_{\lambda \in \Lambda} P^\lambda$ for (normal) order-buds P^λ; $\lambda \in \Lambda$, as the intersection $\bigcap_{\lambda,\mu} P^{\lambda,\mu}$ for all (normal) $P^{\lambda,\mu}$ such that $P^\lambda \subset P^{\lambda,\mu}$.

(1. 7) $P \cup P' = [P, P']$, where $[X, Y]$ means the subsemi-group generated from the set-join $X+Y$ which is closed under multiplication.

The above assertions, taking together the E. H. Moore's theorem, suggests the following

Theorem 1. *The totality of (normal) order-buds in* **G**, *denoting by* **P** ($\mathbf{P}^N$), *forms a complete lattice with respect to the above defined operations* $\cup$ *and* $\cap$; $\mathbf{P}^N \subset \mathbf{P}$.

We remark that **G** is the greatest element of **P** and $\mathbf{P}^N$, while e the smallest of them.

2. We now denote the set of all elements x^{-1} such as belong to P by P^*, that is $P^* = P^{-1}$, and put $a \cdot P^* = P_a^{l*}$, $P^* \cdot a = P_a^{r*}$ for $P \in \mathbf{P}$.

(2. 1) If $P \in \mathbf{P}$(or $\mathbf{P}^N$), $P^* \in \mathbf{P}$ (or resp. $\mathbf{P}^N$).

(2. 2) P_a^{l*} (or P_a^{r*}) concides with the set of all t such that $t \leqq a(P, l)$ (or resp. $t \leqq a(P, r)$).

(2. 3) $P_a^{**} = P_a$.

(2. 4) $P_a \subset P'_a$ implies $P_a^* \subset P_a'^*$.

(2. 5) $P_a \subset P_b$ impiles $P_a^* \subset P_b^*$.

Theorem 2. $(P \cap P')^* = P^* \cap P'^*$ and $(P \cup P')^* = P^* \cup P'^*$.

Proof. The former is obvious. The latter is obtained from the following relations: As $(P \cup P')^* \supset P^*$ and P'^*, we have

$$P \cup P' = P^{**} \cup P'^{**} \subset (P^* \cup P'^*)^*$$
$$\subset (P \cup P')^{**} = P \cup P'.$$

We say that P^* is the reciprocal order-bud of P and P is self-reciprocal, if $P = P^*$, The totality of self-reciprocal elements of **P** or $\mathbf{P}^N$ is denoted by **K** or $\mathbf{K}^N$ respectively.

(2. 6) $G \in \mathbf{K}$, $\mathbf{K}^N$ and $e \in \mathbf{K}$, $\mathbf{K}^N$.

(2. 7) For every $P \in \mathbf{P}(\mathbf{P}^N)$, $P \cap P^*$ and $P \cup P^* \in \mathbf{K}(\mathbf{K}^N)$.

(2. 8) If $P \in \mathbf{K}(\mathbf{K}^N)$, P is a (normal) subgroup of **G** belongs to **K** ($\mathbf{K}^N$).

Theorem 3. **K** *is a complete lattice which is equal to the lattice of all subgroups of* **G**, *and* $\mathbf{K}^N$ *is a complete modular sublattice which is equal to the lattice of all normal subgroups of* **G**[3]).

3. We call the order which is generated from an order-bud P the order of P and denote the group **G** in which the order of P is defined by **G** (P).

We next introduce an order-bud P^S in a subgroup **S** of **G**(P) by putting $P^S = P \cap \mathbf{S}$, and define an order-bud P_H in the factor-group **G**/**H**, where **H** is a normal subgroup of **G**, by taking the set of all $t \cdot \mathbf{H}$, $t \in P$, in **G**/**H**. We call P^S or P_H the derived order-bud of P in $\mathbf{S}$ or in $\mathbf{G/H}$ respectively, and abbreviate the order of the derived order-bud to the derived order.

If P is normal, then P^S and P_H are both normal.

We say that, if $P \cup P^* = \mathbf{G}$, $= e$, $P \cap P^* = \mathbf{G}$, or $= e$, the order of P *connected*, *discrete*, *trivial*, or *proper* respectively.

(3. 1) The derived order of P in $\mathbf{G}/P\cup P^*$, $\mathbf{G}/P\cap P^*$, $P\cup P^*$, or $P\cap P^*$ is *descrete*, *proper*, *connected*, or *trivial* respectively.

(3. 2) If $\mathbf{S}\in\mathbf{K}$, the derived order $\mathbf{S}^S$ is trivial, and if $\mathbf{H}\in\mathbf{K}^N$, the derived order $\mathbf{H}_H$ is discrete.

Theorem 4. *That normal P is connected in $\mathbf{G}(P)$ implies that $\mathbf{G}(P)$ is a directed set with respect to the order of P, and vice versa.*[4]

Proof. We denote the set of all x such that $x \in P^*_t, t \in P$, by R, then we have

i) $e \in R$,

ii) $R\cdot R \subset R$,

iii) $aRa^{-1} \subset R$ for every $a \in \mathbf{G}$.

Hence R is a normal order-bud in $\mathbf{G}$. Since it is clear that $P\subset R$ and $P^*\subset R$, we have $\mathbf{G}=P\cup P^*\subset R$, that is $R=\mathbf{G}$. Consequently, for every pair a, b in $\mathbf{G}$, we can find two elements t and s such that $e, a \leqq t(P)$, $e, b \leqq s(P)$ and so we have $a \leqq ts(P)$ and $b \leqq ts(P)$.

Coversely, if $\mathbf{G}$ is directed set for the order of P, for every $a \in \mathbf{G}$, there exists an element t of P such that $a \leqq t$, and so we have $at^{-1} \in P^*$ and $t \in P$, that is

$$a = at^{-1}t \in P \cup P^*,$$

and consequently, a being arbitrary, it concludes that $\mathbf{G}\subset P\cup P^*$ and so $P\cup P^*=\mathbf{G}$.

Each lefft (right) coset $\bar{C}^l_x(P)$ ($\bar{C}^r_x(P)$) in $\mathbf{G}/P\cup P^*$, which contains x, is said to be the *left* (*right*) *component* of x by P, while each left (right) coset $\bar{C}^l_x(P)$ ($\underline{C}^r_x(P)$) in $\mathbf{G}/P\cap P^*$, which contains x, the left (right) trivialkernel of x by P.

(3. 3) $a \in \bar{C}^l_x(P)$ implies $\bar{C}^l_a(P)=\bar{C}^l_x(P)$, and so for $\bar{C}^r_x(P)$.

(3. 4) $a\in\bar{C}^l_x(P)$ implies $\underline{C}^l_a(P)=\underline{C}^l_x(P)$, and so for $\underline{C}^r_x(P)$.

(3. 5) Putting $\bar{C}^l_e(P)=\bar{C}^r_e(P)=\bar{C}(P)$, $\bar{C}(P)$ belongs to $\mathbf{K}$; $\underline{C}^l_x(P)=x\cdot\bar{C}(P)$, and $\bar{C}^r_x(P)=\bar{C}(P)\cdot x$. If $P\in\mathbf{P}^N$, $\bar{C}(P)\in\mathbf{K}^N$.

(3. 6) Putting $\underline{C}^l_e(P=\underline{C}^r_e(P)=\underline{C}(P)$, $\underline{C}(P)$ belongs to $\mathbf{K}$; $\underline{C}^l_x(P)=x\cdot\bar{C}(P)$, and $\underline{C}^r_x(P)=C(P)\cdot x$. If $P\in\mathbf{P}^N$, $\underline{C}(P)\in\mathbf{K}^N$.

(3. 7) $P\subset Q$ impiles $\bar{C}^l_x(P)\subset\bar{C}^l_x(Q)$, and so for the others, $\bar{C}^r_x(P)$, $\underline{C}^l_x(P)$, and $\underline{C}^r_x(P)$.

(3. 8) $(\bar{C}^l_x(P))^*=\bar{C}^l_x(P)$, $(\bar{C}^r_x(P))^*=\bar{C}^r_x(P)$.

(3. 9) $(\underline{C}^l_x(P))^*=\underline{C}^l_x(P)$, $(\underline{C}^r_x(P))^*=\underline{C}^r_x(P)$.

We have generally that for every $x\in G$,

(3. 10) $x \in \underline{C}^r_x(P) \subset \left\{ \begin{matrix} P^l_x \\ P^{l*}_x \end{matrix} \right\} \subset \bar{C}^l_x(P) \subset \mathbf{G}$,

and similary

(3. 11) $x \in \underline{C}^r_x(P) \subset \left\{ \begin{matrix} P^r_x \\ P^{r*}_x \end{matrix} \right\} \subset \bar{C}^r_x(P) \subset \mathbf{G}$.

As $\bar{C}^l_x(P)=\mathbf{G}(=e)$ for some x implies $\bar{C}(P)=\mathbf{G}(=e)$ and that is same for $\bar{C}^r_x(P)$, the order of P being connected or discrete is characterized by the equality $\bar{C}^l_x(P)=\mathbf{G}$ or $=x$ resp., otherwise $\bar{C}^r_x(P)=\mathbf{G}$ or $=x$ resp. for some x.

Analogously, the order of P being trivial or proper is characterized by the equality $\underline{C}^x(P)=\mathbf{G}$ or $=e$ resp., otherwise $C^r_x(P)=\mathbf{G}$ or $=e$ resp. for some x.

$\mathbf{G}(P)$ is decomposed into the direct sum of left or right components, or into that of left or right trivial-kernels; for example,

(3. 12) $\mathbf{G}(P)=\sum_{\lambda \in \Delta} \oplus \bar{C}^l_\lambda(P)$, $\mathbf{G}(P)=\sum_{\mu \in \Delta'} \oplus \underline{C}^l_\mu(P)$,

and each component $\bar{C}^l_\lambda(P)$ is also decomposed in some direct sum of trivial-kernels;

$$\bar{C}^l_\lambda(P)=\sum \oplus \underline{C}^{\lambda l}_{,\mu}(P).$$

Here every component is a directed set with respect to the order of P, while every trivial-kernel is a trivial set, *i.e.* for its arbitrary two elements x and y, it is always that $x \leqq y$ (P, l).

(3. 13) If P is self-reciprocal, then $\bar{C}^l_x(P) \doteq \underline{C}^l_x(P)$ and $\bar{C}^r_x(P)=\underline{C}^r_x(P)$.

In general, we hold some duality between components and trivial-kernels as follows;

Theorem 5. $\bar{C}_x(\underline{C}(P))=\underline{C}_x(P)$, $\underline{C}_x(\bar{C}(P)=\bar{C}_x(P)$, for $P \in \mathbf{P}^N$

References.

1) Numerous literatures on o-groups and *l*-groups are noted in the Birkhoff's work: G. Birkhoff, Lattice Theory (1948)

2) Algebraic orders in generalized algebraic systems shall be discussed later.

3) The lattice of subgroups or normal subgroups is discussed by Prof. K. Shoda, O. Ore, etc.

4) Sometimes o-groups are *a priori* defined as directed sets; *e.g.* C. Everett and S. Ulrm, On ordered groups, Trans. Amer. Math. Soc. 57(1945). But our Theorem 4 indicates the fundamental meaning of this postulation.

5) Order-buds are sometimes defined in the form as follows;

*) $e \notin P$.

This example is *e.g.* found in Iwasawa's paper: K. Iwasawa, On linearly ordered groups, Jour. of the Math. Soc. Japan, Vol. 1, No. 1 (1948) But it is not suitable for the construction of order-algebra.