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THE UNKNOTTING NUMBER AND BAND-UNKNOTTING NUMBER OF A KNOT

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Abstract

We show some results on the unknotting number and the band-unknotting number. Taniyama characterized knots whose unknotting number is half the crossing number minus one. We show that if the unknotting number of a knot is half the crossing number minus two, then the knot is the figure-eight knot, a positive 3-braid knot, a negative 3-braid knot or the connected sum of a $(2, r)$-torus knot and a $(2, s)$-torus knot for some odd integers $r, s \neq \pm 1$. In particular, we show that it is a 3-braid knot. Taniyama and Yasuhara showed that the band-unknotting number of a knot is less than or equal to half the crossing number of the knot under our notation. We show that the equality holds if and only if the knot is the trivial knot or the figure-eight knot.

1. Introduction

Throughout this paper, we assume that all links and link diagrams are oriented unless otherwise stated. A *crossing change* is a local move on a diagram of a link as in Fig. 1 (a). The *unlinking number* of a link diagram D, denoted by $u(D)$, is the minimal number of crossing changes of D which convert D into a diagram of a trivial link. The *unlinking number* of a link L is the minimal number of $u(D)$, where D is a diagram of L and it is taken over all diagrams of L. If D is a knot diagram, we call $u(D)$ the *unknotting number* of D and if K is a knot, we call $u(K)$ the *unknotting number* of K.

In general, it is very difficult to determine the unknotting number. However, the following estimations are well known. Let $c(D)$ and $c(K)$ be the crossing number of a diagram D and a knot K, respectively. Then

$$u(D) \leq \frac{c(D) - 1}{2}, \tag{1.1}$$

$$u(K) \leq \frac{c(K) - 1}{2}, \tag{1.2}$$

where D is a non-trivial diagram (i.e. a diagram with at least one crossing) and K is a non-trivial knot. It is also known that the equalities hold for diagrams illustrated in Fig. 2 and $(2, r)$-torus knots, respectively. Taniyama proved the converse.

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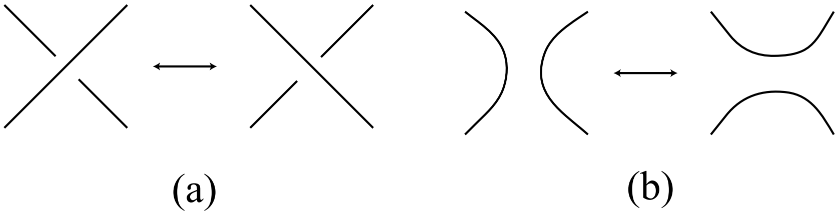


Fig. 1. A crossing change and a band-move.

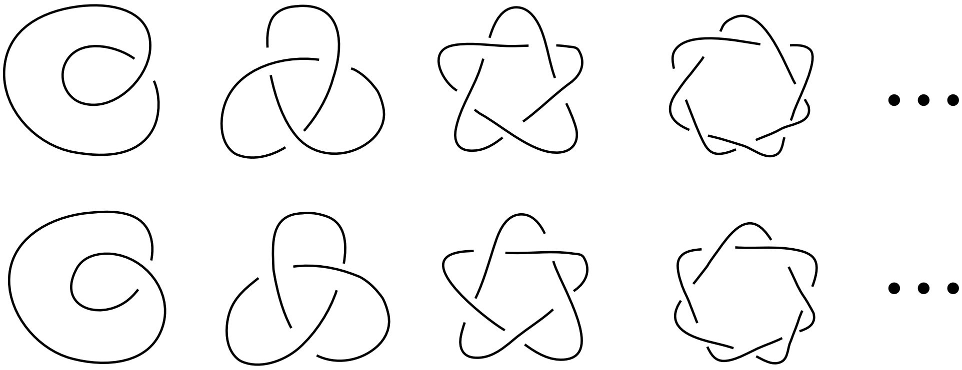


Fig. 2.

Theorem 1.1 ([24]). (1) *If D is a diagram of a knot with $u(D) = (c(D) - 1)/2$, then D is one of the diagrams illustrated in* Fig. 2.
(2) *If K is a knot with $u(K) = (c(K) - 1)/2$, then K is a $(2, r)$-torus knot for some odd integer $r \neq \pm 1$.*

Recall that the braid index of a knot is equal to two if and only if the knot is a $(2, r)$-torus knot for some odd integer $r \neq \pm 1$. The second author and Kanadome [5] (see also [24]) characterized a link diagram D with $u(D) = (c(D) - 1)/2$ and asked the following.

PROBLEM. Characterize the knot diagrams D with $u(D) = (c(D) - 2)/2$.

In this paper, we solve the above problem.

Theorem 2.12. *Let D be a reduced knot diagram. Then*

$$u(D) = \frac{c(D) - 2}{2}$$

if and only if D is the figure-eight knot diagram as in Fig. 3 (a), *the positive* 3*-braid*

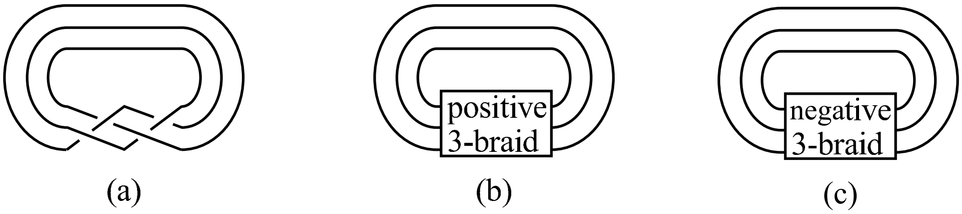


Fig. 3.

knot diagrams as in Fig. 3 (b), *the negative* 3-*braid knot diagrams as in* Fig. 3 (c) *or the connected sum of a* $(2, r)$-*torus knot diagram and a* $(2, s)$-*torus knot diagram for some odd integers* $r, s \neq \pm 1$.

Note that the braid index of a knot with a positive 3-braid diagram may be two. Let $b(K)$ be the braid index of a knot K. Then the following is a corollary of Theorem 2.12.

Corollary 2.14. *Let* K *be a knot. Then we obtain the following.*
(1) *If* $u(K) = (c(K) - 2)/2$, *then* $b(K) = 3$. *Precisely,* K *is the figure-eight knot, a positive* 3-*braid knot, a negative* 3-*braid knot or the connected sum of a* $(2, r)$-*torus knot and a* $(2, s)$-*torus knot for some odd integers* $r, s \neq \pm 1$.
(2) *If* $b(K) \geq 4$, *then* $u(K) \leq (c(K) - 3)/2$.
(3) *If* K *is prime, then* $u(K) = (c(K) - 2)/2$ *if and only if* K *is the figure-eight knot, a positive* 3-*braid knot or a negative* 3-*braid knot.*

As the authors know, the following is open.

QUESTION 1. Let K be the connected sum of a $(2,r)$-torus knot and a $(2,s)$-torus knot for some odd integers $r, s \neq \pm 1$. Is it true that

$$u(K) = \frac{c(K) - 2}{2}?$$

Note that an affirmative answer to Question 1 solves the following question since

$$u(T_{2,r} \# T_{2,s}) = \frac{c(T_{2,r} \# T_{2,s}) - 2}{2} = \frac{c(T_{2,r}) - 1}{2} + \frac{c(T_{2,s}) - 1}{2} = u(T_{2,r}) + u(T_{2,s}),$$

where we denote by $T_{2,t}$ a $(2, t)$-torus knot for some odd integer t and used the additivity of the crossing number of alternating knots under the connected sum operation (see [12], [15] and [26]).

QUESTION 2. Let r and s be some odd integers with $r, s \neq \pm 1$. Is it true that

$$u(T_{2,r} \# T_{2,s}) = u(T_{2,r}) + u(T_{2,s})?$$

Conversely, an affirmative answer to Question 2 implies that

$$u(T_{2,r} \# T_{2,s}) = u(T_{2,r}) + u(T_{2,s}) = \frac{c(T_{2,r}) - 1}{2} + \frac{c(T_{2,s}) - 1}{2} = \frac{c(T_{2,r} \# T_{2,s}) - 2}{2}.$$

Therefore Questions 1 and 2 are equivalent. If both r and s are positive or negative, we see that the equality holds. In general, the above question seem to be very difficult to answer since the connected sum of a $(2, r)$-torus knot and a $(2, -r)$-torus knot for some odd integers $r \neq \pm 1$ is ribbon (therefore slice). For example, the unknotting number of the (2, 3)-torus knot and the (2, -3) torus knot is equal to two (since unknotting number one knots are prime [21]), however the authors do not know whether or not the unknotting number of the (2, 5)-torus knot and the (2, -5)-torus knot is equal to four.

A *band-move* (or *$H(2)$-move*) is a local move on a diagram of a link as in Fig. 1 (b). Here we note that a band-move on a link diagram may not preserve the number of components of the diagram. We introduce a numerical invariant, the *band-unknotting number* of a knot K, denoted by $u_b(K)$, to be the minimal number of band-moves to deform a diagram of K into that of the unknot by Reidemeister moves and band-moves.

The band-unknotting number of a knot behaves rather differently from the unknotting number of a knot. Scharlemann proved that unknotting number one knots are prime [21]. On the other hand, band-unknotting number one knots may not be prime. Indeed, Scharlemann also showed that the connected sum of the trefoil knot and the figure eight knot has band-unknotting number one. More examples are given by Hoste, Nakanishi and Taniyama in [9] and Kanenobu and Miyazawa in [10].

Of course, some restrictions are known. Lickorish [13] gave a restriction on the linking form on the first homology group of the double cover of the 3-sphere S^3 branched along a knot with band-unknotting number one. As a corollary, he showed that 4_1 has band-unknotting number two, whereas the unknotting number of 4_1 is one. Kanenobu and Miyazawa [10] also gave a restriction on the q-polynomial of a knot with band-unknotting number one. Another restriction was given by Bao [1]. One of the natural questions on the band-unknotting number is which knots have band-unknotting number one. We answer this question for the class of twist knots.

Theorem 3.3. *Let K be a twist knot. If $u_b(K) = 1$, then $K = 3_1$, 5_2, 6_1 or 7_2 up to mirror images.*

The idea of the proof of Theorem 3.3 is same as that of Kanenobu and Murakami [11], where they determined two-bridge knots with unknotting number one. The key tool to prove this theorem is results from the Heegaard Floer homology theory which strongly restricts possible integral surgeries of a knot in S^3 which produce lens spaces, whereas Kanenobu and Murakami [11] used the cyclic surgery theorem.

We can understand the band-unknotting number of a knot in terms of surfaces in the 3-space and a 4-dimensional space. Two knots K_1 and K_2 are *g-bordant* if there is a compact connected (possibly non-orientable) surface F in S^3 with the first Betti number $\beta_1(F) = g + 1$ whose boundary has two components, K_1 and K_2. Let

$$\tilde{g}_C(K) = \min\{g \mid K \text{ is g-bordant to the unknot}\}.$$

Let $\tilde{c}(K)$ be the minimal number of elementary critical points of locally flat surface F embedded in $S^3 \times [0, 1]$ such that $F \cap S^3 \times \{0\} = K$ and $F \cap S^3 \times \{1\} =$ the unknot. Taniyama and Yasuhara [25] gave a fundamental property of the band-unknotting number of a knot, that is,

$$u_b(K) = \tilde{g}_C(K) = \tilde{c}(K)$$

for any knot K. The band-unknotting number of a knot is closely related to the *crosscap number* of a knot. The crosscap number of the trivial knot is defined to be zero and the crosscap number of a non-trivial knot is defined to be the minimal number of $\beta_1(F)$, where F is a compact connected non-orientable surface with $\partial F = K$ and it is taken over all compact, connected and non-orientable surfaces bounding K. We denote the crosscap number of a knot K by $\tilde{g}(K)$. For a knot K, Taniyama and Yasuhara [25] also showed

$$u_b(K) \ (= \tilde{g}_C(K)) \leq \tilde{g}(K) \leq \frac{c(K)}{2}.$$

This estimation is best possible since the equality holds for the trivial knot and the figure-eight knot. In this paper, we prove the converse, which is an analog of Theorem 1.1 for the band-unknotting number of a knot.

Theorem 5.1. *Let K be a knot. Then*

$$u_b(K) \leq \frac{c(K)}{2}.$$

The equality holds if and only if K is the trivial knot or the figure-eight knot.

The following lemma gives a relation between the band-unknotting number and the unknotting number of a knot, which is the key in the proof of Theorem 5.1. Note that it is immediately obtained from the result in [10]. For completeness, we give a proof.

Lemma 5.2. *Let K be a knot. Then*

$$u_b(K) \leq u(K) + 1.$$

Here we give the outline of the proof of Theorem 5.1. By combining Theorem 1.1, Corollary 2.14 and Lemma 5.2, it is easy to prove that Theorem 5.1 holds for knots K with $b(K) \neq 3$. It is essential to prove that Theorem 5.1 holds for the knots K with $b(K) = 3$. When K is the figure-eight knot, the equality holds. Otherwise, we can prove $u_b(K) < c(K)/2$ by using a property of a 3-braid knot diagram of K (see Lemma 4.2).

2. The knots whose unknotting number is half the crossing number minus two

In this section, we prove Theorem 2.12 which is one of the main results in this paper.

The second author [4] introduced the notion of a pseudo diagram and the trivializing number of a projection. We recall these definitions to prove Theorem 2.12. First, recall that a diagram consists of the underlying curves and over/under information of crossings of the underlying curves. A *pseudo diagram* Q is a diagram D in which we forget over/under information of some (possibly, all) crossings. Here, we allow the possibility that a pseudo diagram is indeed a diagram. Then we say that D *is obtained from* Q and a crossing without over/under information is called a *pre-crossing*. In particular, we define that a *projection* P is a diagram D in which all crossings do not have over/under information. Then we say that P is the *projection of* D.

A pseudo diagram Q is *trivial* if every diagram obtained from Q represents a trivial link. For example, the pseudo diagram (a) in Fig. 4 is trivial and both pseudo diagrams (b) and (c) in Fig. 4 are not trivial. Let Q and Q' be pseudo diagrams of a diagram, respectively. Then we say that *a pseudo diagram* Q' *is obtained from a pseudo diagram* Q if each crossing of Q has the same over/under information with Q'. The *trivializing number* of a projection P, denoted by $tr(P)$, is the minimal number of the crossings of Q, where Q varies over all trivial pseudo diagrams obtained from P.

A relation between the unlinking number and trivializing number is given in the following proposition. It follows from the definition of the trivializing number and the fact that the mirror diagram of a trivial link is also trivial. For a pseudo diagram Q, the mirror pseudo diagram, denoted by $\bar{Q}$, is the pseudo diagram with opposite over/under information at all crossings in Q.

Proposition 2.1 ([7]). *Let P be a projection and D a diagram obtained from P. Then $u(D) \leq tr(P)/2$.*

Proof. Let Q be a trivial pseudo diagram obtained from P which realizes $tr(P)$. Let $p_1, \dots, p_{tr(P)}$ be the pre-crossings of P which have given over/under information in Q. By applying n ($\leq tr(P)$) crossing changes, we deform D into the diagram D' so that over/under information of $p_1, \dots, p_{tr(P)}$ in Q and that of $p_1, \dots, p_{tr(P)}$ in D' agree. Then D' represents a trivial link. Let $\bar{Q}$ be the mirror pseudo diagram of Q. Then $\bar{Q}$ is also trivial. By applying $tr(P) - n$ crossing changes, we deform D into the diagram D'' such that over/under information of $p_1, \dots, p_{tr(P)}$ in $\bar{Q}$ and that of $p_1, \dots, p_{tr(P)}$

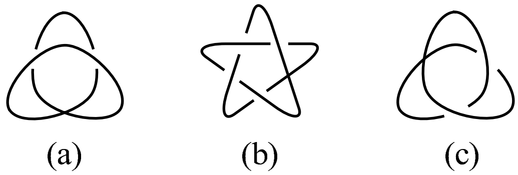


Fig. 4. Pseudo diagrams.

in D'' agree. Then D'' also represents a trivial link. Therefore

$$u(D) \leq \min\{n, tr(D) - n\} \leq \frac{tr(P)}{2}. \qquad \square$$

Let P be a knot projection. A simple closed curve l in the 2-sphere S^2 is a *decomposing circle* of P if the intersection of P and l is the set of just two transversal double points. Then the following proposition holds.

Proposition 2.2 ([4]). *Let P be a knot projection and l a decomposing circle of P. Let $\{q_1, q_2\} = P \cap l$. Let B_1 and B_2 be the disks such that $B_1 \cup B_2 = S^2$ and $B_1 \cap B_2 = l$. Let α be one of the two arcs on l joining q_1 and q_2. Let $P_1 = (P \cap B_1) \cup \alpha$, $P_2 = (P \cap B_2) \cup \alpha$ be the knot projections. Then $tr(P) = tr(P_1) + tr(P_2)$.*

Here, a knot projection P is *prime* if, for any decomposing circle, one of P_1 and P_2 has no pre-crossings. Also, a knot diagram D is *prime* if the projection of D is prime. We give some definitions. A pre-crossing p of a projection P is said to be *nugatory* if the number of connected components of $P - p$ is greater than that of P. A crossing c of a diagram D obtained from a projection P is also said to be *nugatory* if the pre-crossing corresponding to c is nugatory in P. A projection P (resp. a diagram D) is said to be *reduced* if P (resp. D) has no nugatory pre-crossings (resp. no nugatory crossings). We have the following from each of results of [3], [18] and [23].

Proposition 2.3. *Let P be a reduced knot projection. Then $tr(P) = 0$ if and only if P is the projection without pre-crossings.*

We associate a chord diagram to a knot pseudo diagram as follows. Let Q be a pseudo diagram with n pre-crossings. A *chord diagram* of Q is a circle with n chords marked on it by dashed line segment where the preimage of each pre-crossing is connected by a chord. Then we denote it by CD_Q. For example, let Q be the pseudo diagram (a) in Fig. 5. Then a chord diagram (b) in Fig. 5 is CD_Q. Many results in [4] are restated in terms of the chord diagram associated to a pseudo diagram as follows.

Let Q be a knot pseudo diagram. If CD_Q contains a sub-chord diagram as (c) in Fig. 5, we can construct a diagram obtained from Q such that the arf invariant

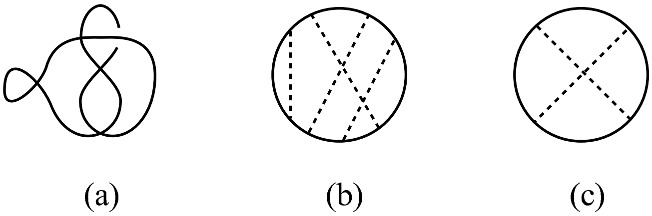


Fig. 5.

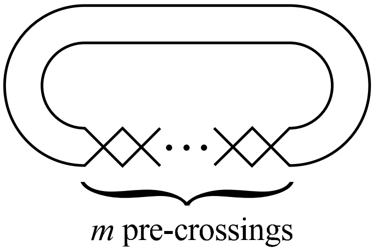


Fig. 6.

of the knot represented by the diagram is non-trivial (cf. [4]). Therefore we obtain the following.

Proposition 2.4 ([4]). *Let Q be a knot pseudo diagram. If CD_Q contains a sub-chord diagram as* (c) *in* Fig. 5, *then Q is not trivial.*

Theorem 2.5 ([4]). *Let P be a knot projection. Then, $tr(P) = \min\{n \mid$ there is a chord diagram obtained from CD_P by deleting n chords does not contain a sub-chord diagram as* (c) *in* Fig. 5$\}$ *and $tr(P)$ is even.*

Theorem 2.6 ([4]). *Let P be a knot projection with at least one pre-crossing. Then it holds that $tr(P) \leq p(P) - 1$, where $p(P)$ is the number of the pre-crossings of P. The equality holds if and only if P is one of the projections as illustrated in* Fig. 6 *where m is some positive odd integer.*

Note that we recover Theorem 1.1 using Proposition 2.1 and Theorem 2.6 (cf. [6]). Smoothing a pre-crossing is the deformation as (a) in Fig. 7. Smoothing a crossing is the deformation as (b) or (c) in Fig. 7. We prove the following.

Lemma 2.7. *Let P be a reduced knot projection. Then, $tr(P) = p(P) - 2$ if and only if P is one of the projections of positive or negative* 3*-braid knot diagrams as illustrated in* Fig. 3 *and the projections of the connected sum of a* $(2, r)$*-torus knot diagram and a* $(2, s)$*-torus knot diagram for some odd integers $r, s \neq \pm 1$.*

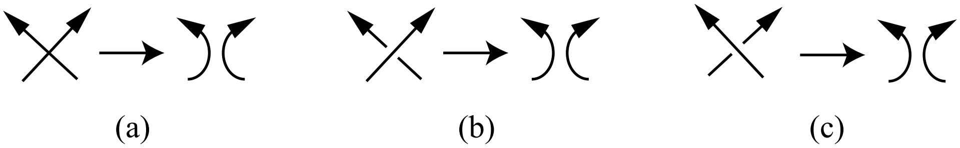


Fig. 7.

Proof. First, we show the ‘if’ part. If P is one of the projections of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram, it follows from Theorem 2.6 and Proposition 2.2 that $tr(P) = p(P) - 2$. Suppose that P is one of the projections of positive 3-braid diagrams. By Theorem 2.6, $tr(P) \leq p(P) - 2$. Assume that $tr(P) < p(P) - 2$. Let Q be a trivial pseudo diagram which realizes the trivializing number of P. Let $p_1, p_2, \ldots, p_n$ be the pre-crossings of Q. Then $n \geq 3$ since $tr(P) < p(P) - 2$. Let P' be the projection obtained from P by smoothing $p_1, p_2, \ldots, p_n$. Then P' is a projection of $(n+1)$-component link diagram from Proposition 2.4. This contradicts that P is one of the projections of positive 3-braid knot diagrams.

Next, we show the ‘only if’ part. If P is not prime, P is the projection of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$ from Proposition 2.3, Theorem 2.6 and Proposition 2.2.

Suppose that P is prime. We show that one of the components of P_p is a projection of a $(2, t)$-torus knot diagram for some odd integer t and the other component of P_p has no self pre-crossings for any pre-crossing p where P_p is the projection obtained from P by smoothing p. Namely, for any chord d there exists a chord which does not cross d in CD_P. Let P_1 and P_2 be the knot projections of P_p. If each of P_1 and P_2 has no pre-crossings, this implies that $p(P)$ is odd. This contradicts that $tr(P)$ is even by Theorem 2.5. If each of P_1 and P_2 has a pre-crossing, this implies that $tr(P) < p(P) - 2$. Without loss of generality, we may assume that P_1 has a pre-crossing. If P_1 is not one of the projections of $(2, t)$-torus knot diagrams, $tr(P_1) < p(P_1) - 1$ by Theorem 2.6. This implies that $tr(P) < p(P) - 2$ and it contradicts our assumption. Therefore, one of the components of P_p is the projection of a $(2, t)$-torus knot diagram for some odd integer t and the other component of P_p has no self pre-crossings for any pre-crossing p.

We can suppose that P_1 is the projection of a $(2, t)$-torus knot diagram. Let p' be a self pre-crossing of P_1 and P_1' and P_1'' the knot projections obtained from P_1 by smoothing p'. Note that each of P_1' and P_1'' does not have pre-crossings. Let $a_1, a_2, \ldots, a_n$ (resp. $b_1, b_2, \ldots, b_m$) be the pre-crossings of P_1' (resp. P_1'') and P_2 which appear on P_2 from p in this order along the orientation. Here, $a_1, a_2, \ldots, a_n$ appear on P_1' from a certain point in this order along the orientation and also $b_1, b_2, \ldots, b_m$ appear on P_1'' from a certain point in this order along the orientation. If this is not the case, there exists a part of a chord diagram as illustrated in Fig. 8. This contradicts $tr(P) = p(P) - 2$ by Theorem 2.5. Therefore, P is one of the projections of positive or negative 3-braid knot diagrams. $\square$

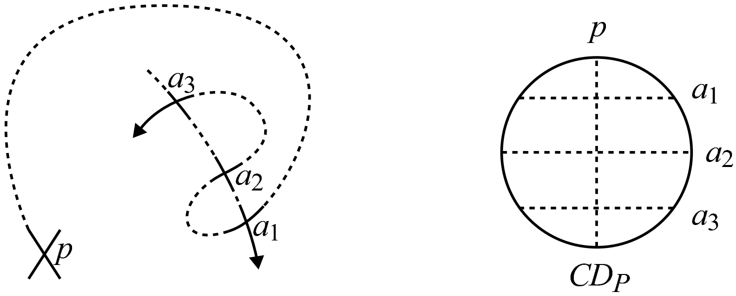


Fig. 8.

Lemma 2.8. *Let P be a non-prime projection with $tr(P) = p(P) - 2$ and D a diagram obtained from P. Suppose that D is not the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for any odd integers $r, s \neq \pm 1$. Then,*

$$u(D) < \frac{c(D) - 2}{2}.$$

Proof. By Lemma 2.7, P is one of the projections of the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$. Immediately, we see that

$$u(D) < \frac{c(D) - 2}{2}. \qquad \square$$

Lemma 2.9. *Let P be a prime projection with $tr(P) = p(P) - 2$ and D a diagram obtained from P which is neither positive nor negative and does not represent the figure-eight knot. Then,*

$$u(D) < \frac{c(D) - 2}{2}.$$

Proof. We show that there exists a crossing c in D such that the mutual crossings of D_c contain both a positive crossing and a negative crossing where a *mutual* crossing lies on between two component and D_c is the diagram obtained from D by smoothing c. There exists a chord corresponding to a positive crossing c_+ which crosses a chord corresponding to the negative crossing in CD_P as (c) in Fig. 5 since P is prime where a chord corresponding to a crossing means that the pre-crossing of the crossing represents the chord in CD_P. We concentrate on c_+. If the chord corresponding to c_+ crosses a chord corresponding to a positive crossing, we set $c = c_+$. If the chord corresponding to c_+ crosses two chords corresponding to negative crossings which cross each other, we set c to be the crossing corresponding to one of the two chords. Assume that the chord corresponding to c_+ crosses more than two chords corresponding

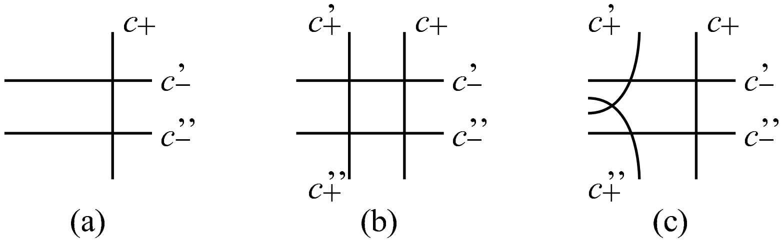


Fig. 9.

to the negative crossings. Two of the chords cross each other since $tr(P) = p(P) - 2$. Suppose that the chord corresponding to c_+ crosses just two chords corresponding to the negative crossings, say c'_- and c''_-, in CD_P as illustrated in Fig. 9 (a). If the chord corresponding to c'_- (resp. c''_-) crosses a chord corresponding to the negative crossing, we set $c = c'_-$ (resp. $c = c''_-$). Assume that the chord corresponding to c'_- or c''_- crosses more than two chords corresponding to the positive crossings. Similarly, we see that two of the chords cross each other since $tr(P) = p(P) - 2$. We set c to be the crossing corresponding to one of the two chords. Assume that each of the chord corresponding to c'_- and the chord corresponding to c''_- crosses just two chords corresponding to the positive crossings. Let c'_+ (resp. c''_+) be the crossing corresponding to the chord which does not represent c_+ and crosses the chord corresponding to c'_- (resp. c''_-). If $c'_+ = c''_+$ as illustrated in Fig. 9 (b), it implies that D represents the figure-eight knot diagram. Assume that this is not the case. Since $tr(P) = p(P) - 2$, c'_+ and c''_+ cross each other as illustrated in Fig. 9 (c). We set $c = c'_+$.

We consider D_c and note that one component of D_c, say D'_c, is obtained from a $(2, r)$-torus knot diagram by some crossing changes where r is some odd integer and another, say D''_c, does not have a crossing. If D'_c is not a $(2,r)$-torus knot diagram then we see that

$$u(D) < \frac{c(D) - 2}{2}.$$

Suppose that D'_c is a $(2, r)$-torus knot diagram. There exist at least two arcs on D'_c which have the end points as a positive mutual crossing and a negative mutual crossing. From a property of D'_c, there exists a simple arc of D'_c, say l_1, in such arcs. See Fig. 10. Let c_1 (resp. c_2) be the negative (resp. positive) mutual crossing as end points of l_1. We can suppose that l_1 has exactly two mutual crossings c_1 and c_2 (possibly, has other crossings which are not mutual). Let l_2 be the arc such that $l_1 \cup l_2 = D'_c$. Note that all crossings of D_c except c_1 and c_2 lie on l_2. By abuse of notation, the part of D corresponding to l_1 (resp. l_2) is also denoted by l_1 (resp. l_2). We consider the following two ways to change the crossings on l_2 at D.

(i) The crossings on l_2 are over than the other, and for the self-crossings on l_2 we change crossings by descending from c_1 to c_2 on l_2, that is, we change crossings so

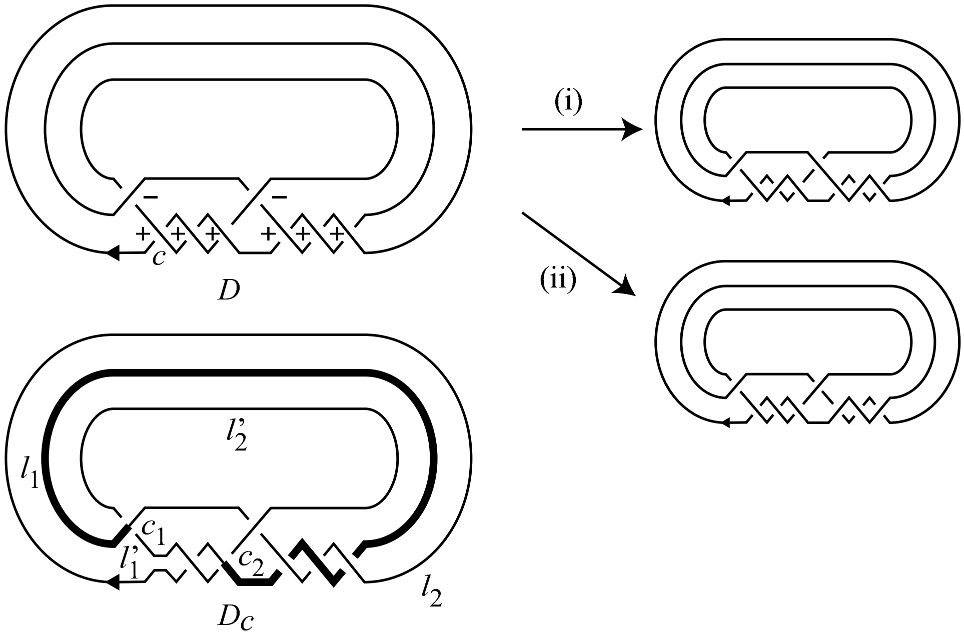


Fig. 10.

that every crossing may be first traced as an over-crossing.
(ii) The crossings on l_2 are under than the other, and for the self-crossings on l_2 we change crossings by descending from c_2 to c_1.

Here, each crossing except c, c_1 and c_2 is changed exactly once in (i) or (ii). Therefore, the number of crossing changes in (i) or (ii) is less than $(c(D)-2)/2$. We show that each diagram obtained in (i) and (ii) represents the trivial knot.

Let l'_1 be the arc on D''_c such that the end points of l'_1 are c_1 and c_2 and c exists on l'_1 at D. Let l'_2 be the arc such that $l'_1 \cup l'_2 = D''_c$. By abuse of notation, the part of D corresponding to l'_1 (resp. l'_2) is also denoted by l'_1 (resp. l'_2). Since l_1 does not contain the mutual crossings except c_1 and c_2, D'_c is over than D''_c or D''_c is over than D'_c at both c_1 and c_2.

Suppose that D''_c is over than D'_c at both c_1 and c_2 and (ii). Assume that c sits between l_1 and l'_1. See Fig. 11 (a). We can remove c_1 and c_2 and see that there exists a disk whose boundary contains both l'_1 and l'_2. Therefore, we see that D represents the trivial knot. Assume that c sits between l_2 and l'_1. Similarly, we can remove c_1 and c_2 and see that there exists a disk whose boundary contains both l'_1 and l'_2. Therefore, we see that D represents the trivial knot. Similarly, we can show that D represents the trivial knot in other cases. $\square$

We recall the theorem and the proposition to estimate the unknotting number.

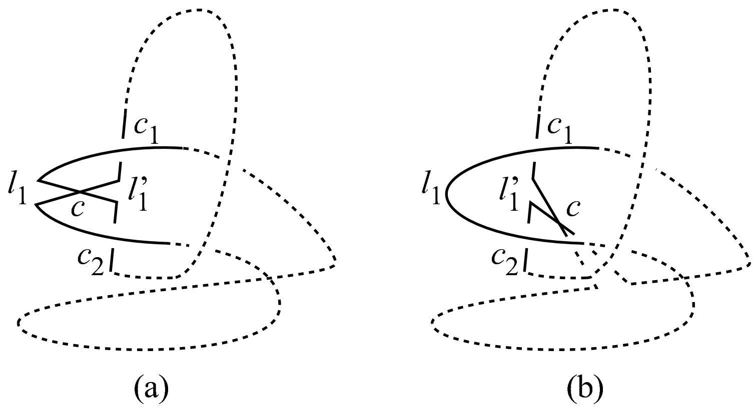


Fig. 11.

Theorem 2.10 ([16, 20]). *Let D be a positive diagram or a negative diagram and K the knot represented by D. Then $2g_4(K) = 2g(K) = c(D) - O(D) + 1$ holds where $O(D)$ is the number of the Seifert circles and $g_4(K)$ is the 4-ball genus of K.*

We note that $s(K) = c(D) - O(D) + 1$ for a positive knot K and a positive diagram D of K where $s(K)$ denotes the Rasmussen invariant. The following is well-known.

Proposition 2.11. *Let K be a knot. Then $u(K) \geq g_4(K)$.*

Now we prove the following.

Theorem 2.12. *Let D be a reduced knot diagram. Then*

$$u(D) = \frac{c(D) - 2}{2}$$

if and only if D is the figure-eight knot diagram as in Fig. 3 (a), *the positive* 3*-braid knot diagrams as in* Fig. 3 (b), *the negative* 3*-braid knot diagrams as in* Fig. 3 (c) *or the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$.*

Proof. First, we show the 'if' part. If D is one of the figure-eight knot diagram and the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram, it is obvious. Suppose that D is one of the positive 3-braid knot diagrams and the negative 3-braid knot diagrams. Let P be the projection of D. By Lemma 2.7, $tr(P) = p(P) - 2$ and so $u(D) \leq (c(D) - 2)/2$ by Proposition 2.1. Let K be the knot represented by D. Then $u(D) \geq u(K) \geq (c(D) - O(D) + 1)/2 = (c(D) - 2)/2$ by Theorem 2.10 and Proposition 2.11. Therefore, $u(D) = (c(D) - 2)/2$.

Next, we show the 'only if' part. It is obvious that $c(D)$ is even. Hence, it is sufficient to consider the diagrams obtained from the projections P with $tr(P) = p(P) - 2$ by Proposition 2.1 and Theorem 2.6. Then we see from Lemmas 2.8 and 2.9. □

There exists a knot K which does not have a minimal crossing diagram D of K with $u(D) = u(K)$. Let K be the pretzel knot of type (5,1,4). Bleiler [2] and Nakanishi [17] independently discovered that K does not have a minimal crossing diagram D of K with $u(D) = u(K)$. Here we note that $2 = u(K) = (c(K) - 6)/2$. The second author and Kanadome [5] asked the following.

PROBLEM. Find the number $n_{\min}$ which is defined to be the minimal number of n such that there exists a prime knot K with $u(K) = (c(K) - n)/2$ which has no minimal diagrams D of K with $u(D) = u(K)$.

Nakanishi and Bleiler's example implies that $n_{\min} \leq 6$. The second author and Kanadome [5] partially solve this problem as follows.

Lemma 2.13 ([5]). *Let K be a knot with $u(K) \geq (c(K) - 2)/2$ and D a minimal crossing diagram of K. Then $u(K) = u(D)$.*

Therefore, we have $3 \leq n_{\min} \leq 6$. By Theorem 2.12 and Lemma 2.13, we obtain the following.

Corollary 2.14. *Let K be a knot. Then we obtain the following.*
(1) *If $u(K) = (c(K) - 2)/2$, then $b(K) = 3$. Precisely, K is the figure-eight knot, a positive 3-braid knot, a negative 3-braid knot or the connected sum of a $(2, p)$-torus knot and $(2, q)$-torus knot for some odd integers $p, q \neq \pm 1$.*
(2) *If $b(K) \geq 4$, then $u(K) \leq (c(K) - 3)/2$.*
(3) *If K is prime, then $u(K) = (c(K) - 2)/2$ if and only if K is the figure-eight knot, a positive 3-braid knot or a negative 3-braid knot.*

Proof. (1) Let D be a minimal crossing diagram of K. By Lemma 2.13,

$$u(D) = u(K) = \frac{c(K) - 2}{2} = \frac{c(D) - 2}{2}.$$

By Theorem 2.12, D represents one of the figure-eight knot, the positive 3-braid knots, the negative 3-braid knots or the connected sum of a $(2, r)$-torus knot and $(2, s)$-torus knot. Therefore the braid index of K is three.

(2) If $u(K) \geq (c(K) - 2)/2$, then $b(K) = 1$, $b(K) = 2$ or $b(K) = 3$ by Theorem 1.1 and Corollary 2.14 (1).

(3) First we show the 'only if' part. By Corollary 2.14 (1), K is the figure-eight knot, a positive 3-braid knot, a negative 3-braid knot.

Next, we show the 'if' part. If K is the figure-eight knot, then $u(K) = (c(K) - 2)/2$. Suppose that K is one of the positive 3-braid knots and the negative 3-braid knots. Then we obtain $u(K) = (c(K)-2)/2$ by Theorem 2.10 and Proposition 2.11. □

REMARK 2.15. Let K be a prime knot up to 10 crossings with $u(K) = (c(K) - 2)/2$. Then K is $4_1, 8_{19}, 10_{124}, 10_{139}$ or 10_{152}. Note that 4_1 is the figure-eight knot and 8_{19} is the torus knot of type (3, 4).

We study the unknotting number of a minimal crossing diagram of a knot. First, we observe the diagrams D with $u(D) = (c(D) - 2)/2$. Then we have make an improvement to Lemma 2.13.

Corollary 2.16. *Let D be a prime knot diagram with $u(D) \geq (c(D) - 2)/2$ and K the knot represented by D. Then $u(K) = u(D)$ holds.*

Proof. If $u(D) = (c(D) - 1)/2$, it follows from Theorem 1.1. If D is the figure-eight knot diagram, $u(K) = u(D)$ holds. Otherwise, by Theorem 2.12, D is one of the positive 3-braid knot diagrams and the negative 3-braid knot diagrams. Then we have $(c(D)-2)/2 = u(D) \geq u(K) \geq (c(D)-2)/2$ by Theorem 2.10 and Proposition 2.11. □

Here, there is a possibility that a prime knot diagram with $u(D) \geq (c(D) - 2)/2$ represents a $(2, r)$-torus knot for some odd integer r.

Corollary 2.17. *Let D be a prime knot diagram with $u(D) \geq (c(D) - 2)/2$ and K be the knot represented by D. Then the following holds.*
(1) $c(D) - 1 \leq c(K) \leq c(D)$.
(2) $u(K) = (c(K) - 1)/2$ *or* $u(K) = (c(K) - 2)/2$.

Proof. (1) Suppose that $c(K) \leq c(D) - 2$. From the inequality (1.2) and Corollary 2.16, $u(D) = u(K) \leq (c(K) - 1)/2 \leq (c(D) - 3)/2$. This contradicts that $u(D) \geq (c(D) - 2)/2$.

(2) There are two cases where $c(K) = c(D)$ and $c(K) = c(D) - 1$ by (1). Suppose that $c(K) = c(D)$. By Corollary 2.16, we have $u(K) = u(D)$. Therefore one of the equalities above holds. Suppose that $c(K) = c(D) - 1$. By the inequality (1.2), Corollary 2.16 and the assumption,

$$\frac{c(K) - 1}{2} \geq u(K) = u(D) \geq \frac{c(D) - 2}{2} = \frac{c(K) - 1}{2}.$$

Therefore, $u(K) = (c(K) - 1)/2$. □

Corollary 2.18. *Let K be a knot and D a minimal crossing diagram of K.*

(1) *If* $u(K) = (c(K) - 3)/2$, *then* $u(K) = u(D)$.
(2) *If* K *is prime and* $u(K) = (c(K) - 4)/2$, *then* $u(K) = u(D)$.

Proof. (1) We have the following chain of inequalities.

$$\frac{c(D)-3}{2} = \frac{c(K)-3}{2} = u(K) \leq u(D) \leq \frac{c(D)-1}{2}. \tag{2.1}$$

Since $c(K)$ is odd, $u(D) = (c(D) - 1)/2$ or $u(D) = (c(D) - 3)/2$. If $u(D) = (c(D) - 1)/2$, then D is one of the diagrams illustrated in Fig. 2 by Theorem 1.1. Then K is trivial or $u(K) = (c(K) - 1)/2$ (for example, by using the signature of a knot). This contradicts our assumption. Therefore $u(D) = (c(D) - 3)/2$. We conclude that $u(D) = u(K)$ by the inequality (2.1).

(2) We have the following chain of inequalities.

$$\frac{c(D)-4}{2} = \frac{c(K)-4}{2} = u(K) \leq u(D) \leq \frac{c(D)-1}{2}. \tag{2.2}$$

Since $c(K)$ is even, $u(D) = (c(D) - 2)/2$ or $u(D) = (c(D) - 4)/2$. If $u(D) = (c(D) - 2)/2$, then, by Theorem 2.12, D is one of the figure-eight knot diagram as (a), the positive 3-braid knot diagrams as (b) illustrated in Fig. 3, the mirror diagrams of them and the connected sum of a $(2, r)$-torus knot diagram and a $(2, s)$-torus knot diagram for some odd integers $r, s \neq \pm 1$.

By Corollary 2.17 (2), $u(K) = (c(K) - 1)/2$ or $u(K) = (c(K) - 2)/2$. This contradicts our assumption. Therefore $u(D) = (c(D) - 4)/2$. We conclude that $u(D) = u(K)$ by the inequality (2.2). □

Corollary 2.19. *The inequality* $5 \leq n_{\min} \leq 6$ *holds.*

Proof. As mentioned before, we have $3 \leq n_{\min} \leq 6$. Corollary 2.18 implies that $n_{\min} \neq 3$ and $n_{\min} \neq 4$. Therefore we obtain $5 \leq n_{\min} \leq 6$. □

3. The band-unknotting number of a twist knot

In this section, we determine a twist knot whose band-unknotting number is one (Corollary 3.4).

We recall some notations. Let K be a knot in S^3 and n an integer. We denote by $\lambda(K, n)$ the manifold obtained from S^3 by a Dehn-surgery along K with slope n, by $\Sigma(K)$ the double cover of S^3 branched along K and by $L(r, s)$ a lens space of type (r, s) for some coprime integers r and s. Montesinos showed the following.

Lemma 3.1 ([14]). *Let* K *be a knot. If* $u_b(K) = 1$, *then there exist a knot* K' *and an integer* n *such that* $\Sigma(K) \simeq \lambda(K', n)$, *where* $\simeq$ *means that* $\Sigma(K)$ *and* $\lambda(K', n)$ *are homeomorphic.*

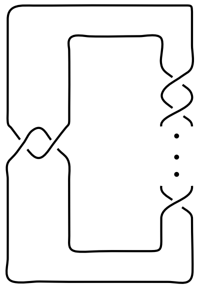


Fig. 12.

We consider all knots in this section up to mirror images. A twist knot is a knot as in Fig. 12. Note that a twist knot is a two bridge link of type $(r, 2)$ in the sense of Schubert for some positive integer r and denoted it $S(r, 2)$. In general, it is an interesting and difficult question that which lens spaces are produced by an integral surgery along a knot in S^3. Rasmussen [19] and Tange [22] showed the following.

Lemma 3.2 ([19], [22]). *Let r be a positive integer. If there exist a knot K and an integer n such that $L(r, 2) \simeq \lambda(K, n)$, then r is 3, 7, 9 or 11.*

Theorem 3.3. *Let K be a twist knot. If $u_b(K) = 1$, then $K = 3_1$, 5_2, 6_1 or 7_2 up to mirror images.*

Proof. Let r be a positive integer such that $K = S(r, 2)$. Then it is well known that $\Sigma(K) \simeq L(r, 2)$. Since $u_b(K) = 1$, by Lemma 3.1, there exist a knot K' and an integer n such that $\Sigma(K) \simeq \lambda(K', n)$. Therefore $L(r, 2) \simeq \lambda(K', n)$. By Lemma 3.2, r must be 3, 7, 9 or 11. Hence K is $S(3, 2) = 3_1$, $S(7, 2) = 5_2$, $S(9, 2) = 6_1$ or $S(11, 2) = 7_2$. $\square$

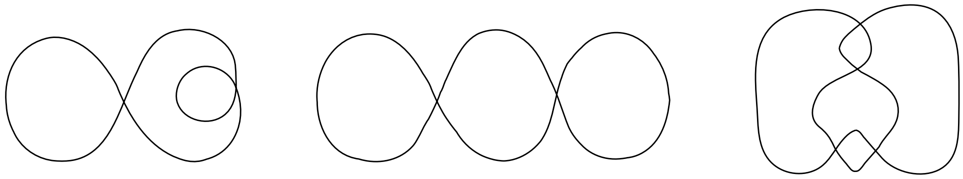
Corollary 3.4. *Let K be a non-trivial twist knot. Then $u_b(K) = 1$ if and only if $K = 3_1$, 5_2, 6_1 or 7_2 (up to mirror images). Other twist knots are knots with $u_b(K) = 2$.*

Proof. It is easy to show that $u_b(K) \leq 2$. If $u_b(K) = 1$, by Theorem 3.3, $K = 3_1$, 5_2, 6_1 or 7_2 (up to mirror images). Indeed, these knots have the band-unknotting number one [10]. $\square$

4. A property of the projection of a 3-braid knot diagram

In this section, we show Lemma 4.2 on the projection of a 3-braid knot diagram.

Let $P = P_1 \cup P_2 \cup \cdots \cup P_n$ be a link projection. We denote by $p(P_i)$ the number of self pre-crossings of P_i and by $p(P_i, P_j)$ the number of mutual pre-crossings between



(a) (b) (c)

Fig. 13.

P_i and P_j. Therefore the following equality holds.

$$p(P) = \sum_{i=1}^{n} p(P_i) + \sum_{i<j} p(P_i, P_j).$$

Let P be a knot projection and p a pre-crossing of P. We say that p satisfies the condition C_1 if one of the components of P_p has exactly one self pre-crossing and the other component of P_p has no self pre-crossings every pre-crossing of the projections illustrated in Fig. 13 satisfies the condition C_1. The converse is also true.

Lemma 4.1. *Let P be a knot projection. If every pre-crossing of P satisfies the condition C_1, then it is one of projections illustrated in* Fig. 13.

Proof. Let p be a pre-crossing of P. Then we can suppose that P is a projection as shown in Fig. 14, if necessary, by reversing the orientation of the projection. Here we let P_1 be the component of P_p which has no self pre-crossings and P_2 the component of P_p which has a self pre-crossing q. We also denote by q the pre-crossing of P which is corresponding to q of P_p. The proof of this lemma is divided into two cases.

CASE 1. P is a projection as shown in Fig. 14 (a).

By smoothing at q (of P_p), we obtain a 3-component projection and denote it by $P_1 \cup P_{21} \cup P_{22}$ as in Fig. 15 (a). Since P_2 has a self pre-crossing q and the pre-crossing p of P satisfies the condition C_1, we obtain $p(P_{21}, P_{22}) = 0$. Similarly, since pre-crossing q of P satisfies the condition C_1, we obtain $p(P_1, P_{21}) = 0$. From the configuration of $P_1 \cup P_{21} \cup P_{22}$, the equality $p(P_1, P_{22}) = 0$ holds. Therefore P must be as in Fig. 13 (a).

CASE 2. P is a projection as shown in Fig. 14 (b).

By smoothing at q (of P_p), we obtain a 3-component projection and denote it by $P_1 \cup P_{21} \cup P_{22}$ as in Fig. 15 (b). As in the Case 1, we obtain that $p(P_{21}, P_{22}) = 0$ and $p(P_1, P_{21}) = 0$. In this case, $p(P_1, P_{22})$ may not be zero. By isotopy, P is deformed into a projection as shown in Fig. 16 (a), where T is the projection of a tangle diagram which consists of two arcs without self crossings. Recall that $p(P)$ is a positive even

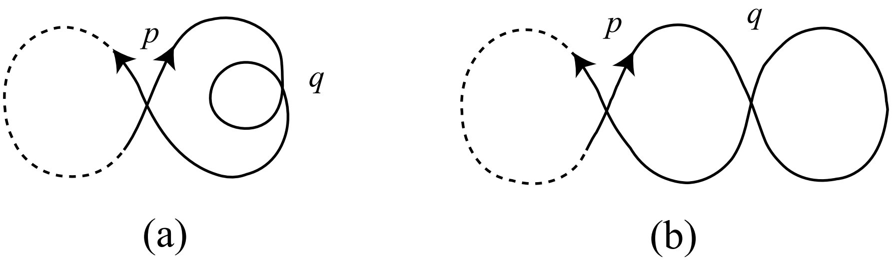


Fig. 14.

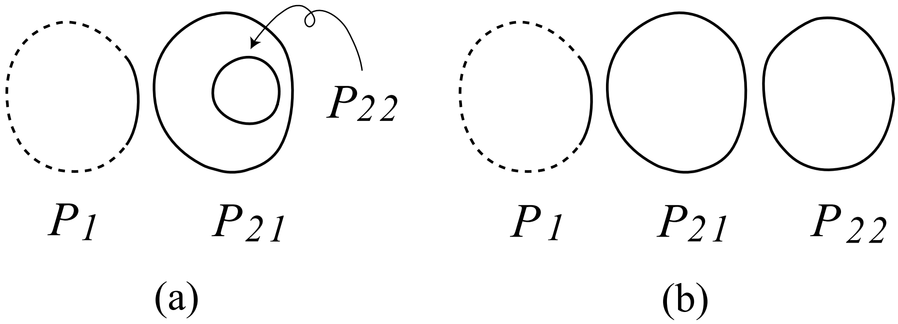


Fig. 15.

number by hypothesis. If $p(P) = 2$, P the projection as shown in Fig. 16 (b). If $p(P) = 4$, P is the projection as shown in Fig. 16 (c). To complete the proof, we show the following claim.

Claim. *If $p(P) \geq 6$, there exists a pre-crossing which does not satisfy the condition C_1.*

Since P_1 has no self crossing, arcs of P_1 in T meet P_{22} at two points r_1 and r_2 as illustrated in Fig. 17 (a). Since $p(P) \geq 6$, at least one component of $P_2 \setminus \{r_1, r_2\}$ in T contains a pre-crossing. There are two cases to consider as illustrated in Fig. 17 (b) and (c). For case (b), r_2 does not satisfy the condition C_1 and for case (c), r_1 and r_2 do not satisfy the condition C_1. □

The following lemma on the projection of a 3-braid knot diagram is used to prove Theorem 5.1.

Lemma 4.2. *Let P be the projection of a* 3*-braid knot diagram. Then we obtain the following.*

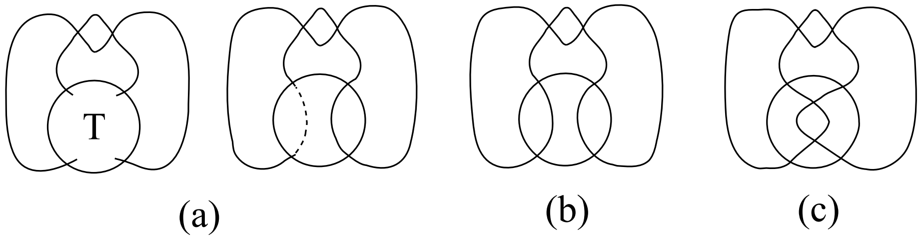


Fig. 16.

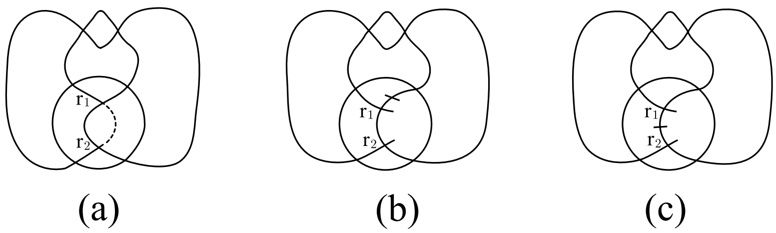


Fig. 17.

(1) *Let p be a pre-crossing of P. Then one of the components of P_p is the projection of a $(2, r)$-torus knot diagram for some odd integer r and the other component of P_p has no self pre-crossings.*
(2) *If P is not the projection as in* Fig. 13 (c), *then there exists a pre-crossing p such that one of the components of P_p is the projection of a $(2, r)$-torus knot diagram for some odd integer r with $|r| \geq 3$ and the other component of P_p has no self pre-crossings.*

Proof. It is easy to see that the statement (1) holds. We only prove the statement (2). If, for any pre-crossing p, one of the components of P_p is a projection with one pre-crossing and the other component of P_p has no self pre-crossings, then the P is one of those in Fig. 13 by Lemma 4.1. It contradicts our assumption. $\square$

5. An upper bound for the band-unknotting number of a knot

In this section, we prove the following theorem which is one of the main results in this paper.

Theorem 5.1. *Let K be a knot. Then*

$$u_b(K) \leq \frac{c(K)}{2}. \tag{5.1}$$

The equality holds if and only if K is the trivial knot or the figure-eight knot.

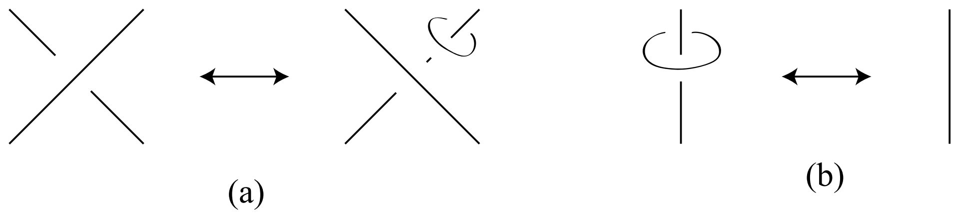


Fig. 18. A move of type 1 and a move of type 2.

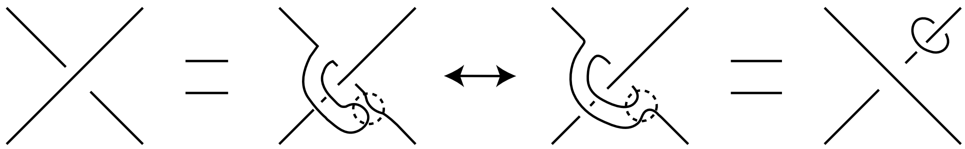


Fig. 19. A move of type 1 is achieved by a band-move and a Reidemeister move.

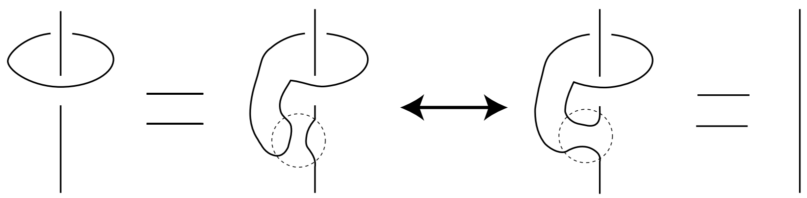


Fig. 20. A move of type 2 is achieved by a band-move and Reidemeister moves.

We define two local moves. A *move of type* 1 is a local move on a link diagram D as shown in Fig. 18 (a). This move is achieved by a band-move and a Reidemeister move (see Fig. 19). A *move of type* 2 is a local move on a link diagram as shown in Fig. 18 (b). This move is achieved by a band-move and Reidemeister moves (see Fig. 20). These moves are used in the proof of Theorem 5.1. Now we prove the following lemma. Note that it is a corollary of Theorem 3.1 in [10] and we give a direct and simple proof.

Lemma 5.2. *Let K be a knot. Then*

$$u_b(K) \leq u(K) + 1.$$

Proof. We first observe the following claim.

Claim. *A single crossing change in a link diagram is achieved by two band-moves and two crossing changes in a knot diagram are achieved by two band-moves.*

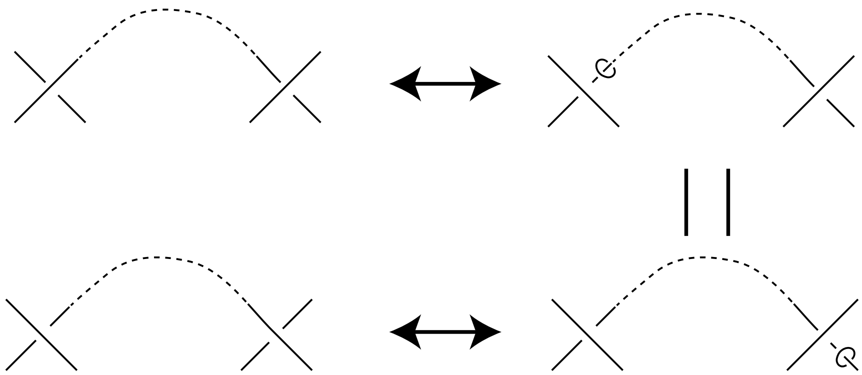


Fig. 21.

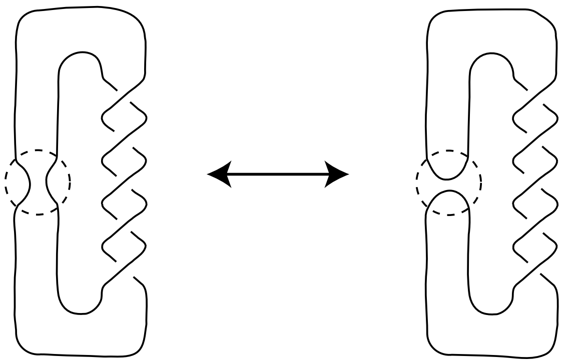


Fig. 22.

A single crossing change in a link diagram is achieved by a move of type 1 near the crossing and a move of type 2. Two crossing changes in a knot diagram are achieved by two moves of type 1, see Fig. 21.

Let D be a diagram of K with $u(K) = u(D)$. If $u(D)$ is even, then $u_b(K) \leq u(D) = u(K)$ since even number crossing changes are achieved by even number band-moves by the claim. If $u(D)$ is odd, set $u(D) = 2n + 1$ ($n \geq 0$). Since $2n$ crossing changes are achieved by $2n$ band-moves and a single crossing change is achieved by two band-moves by the claim, we have $u_b(K) \leq 2n + 2 = u(D) + 1 = u(K) + 1$. □

Recall that $u(K) \leq (c(K) - 1)/2$ for any non-trivial knot K and the equality holds if and only if K is a $(2, r)$-torus knot for some odd integer $r \neq 1$. We study the band-unknotting number of these knots.

EXAMPLE 5.3. Let K be a $(2, r)$-torus knot for some odd integer $r \neq 1$. Then $u_b(K) = 1$ ($< c(K)/2$). Fig. 22 illustrates the case $r = 5$.

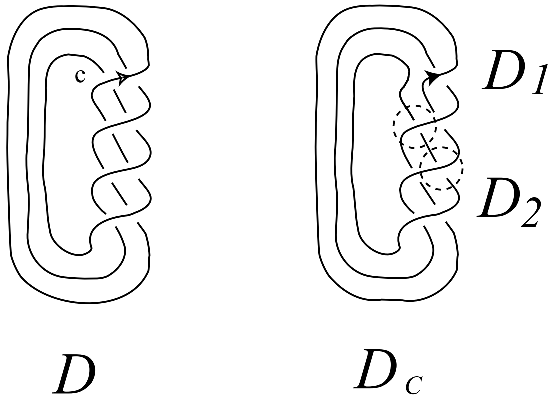


Fig. 23.

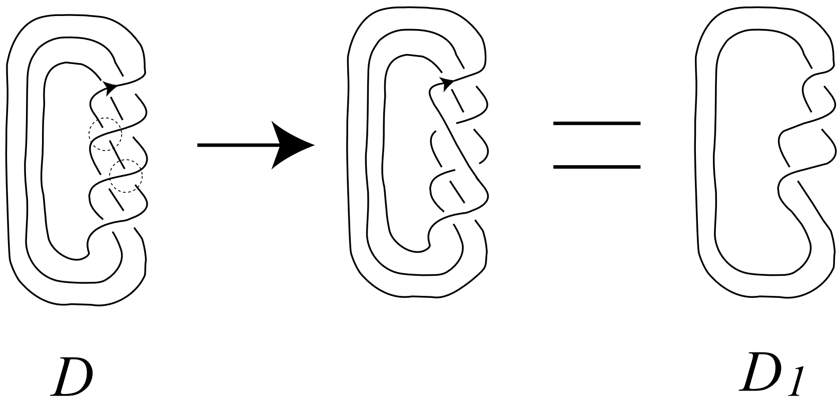


Fig. 24.

Next, we study the band-unknotting number of knots K with $u(K) = (c(K)-2)/2$.

EXAMPLE 5.4. Let K be the figure-eight knot. Then $u(K) = (c(K) - 2)/2$ and Lickorish [13] showed that $u_b(K) = 2$ $(= c(K)/2)$.

EXAMPLE 5.5. Let K be 8_{19}. Then $u(K) = (c(K)-2)/2$. We show that $u_b(K) \leq 3$ $(< c(K)/2)$. Let D be the minimal crossing diagram of K and c the crossing of D as shown in Fig. 23. One of the components of D_c is the trefoil knot diagram D_1 and the other is the trivial knot diagram D_2 (i.e. the diagram without crossings). We change the over/under information of D so that D_2 is over than D_1 at the mutual crossings between D_1 and D_2 (see Figs. 23 and 24). In this process, we need 2 $(= c(D_1, D_2)/2)$ crossing changes. By the claim in Lemma 5.2, we obtain D_1 from D by two band-moves (see Fig. 24). Therefore $u_b(K) \leq 3$ $(< c(K)/2)$.

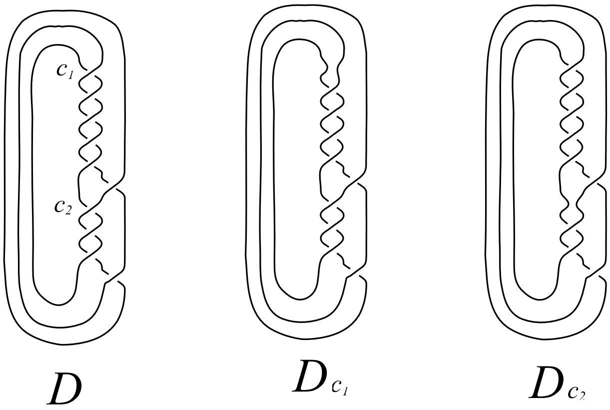


Fig. 25.

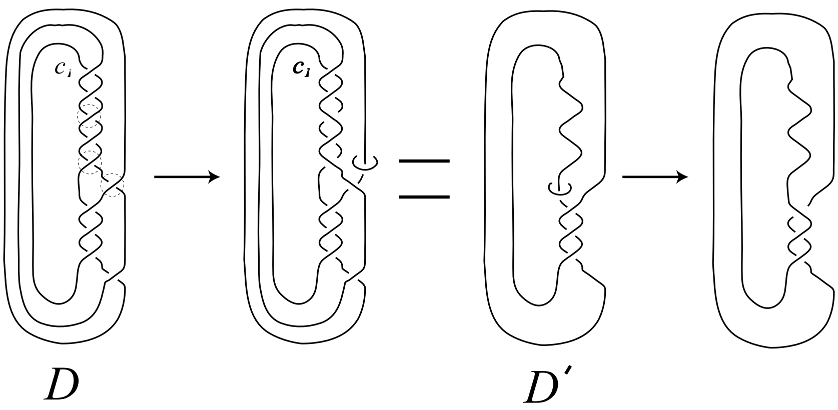


Fig. 26.

EXAMPLE 5.6. Let K be 10_{124}. Then $u(K) = (c(K) - 2)/2$. Let D be the minimal crossing diagram of K and c_1 and c_2 the crossings of D as shown in Fig. 25.

Now we consider D_{c_1} and show that $u_b(K) \leq 4$ $(< c(K)/2)$. One of the components of D_{c_1} is the trefoil knot diagram D_1 and the other is the trivial knot diagram D_2. We change the over/under information of D so that D_2 is over than D_1 at the mutual crossings between D_1 and D_2 (see Fig. 26). In this process, we need 3 $(= c(D_1, D_2)/2)$ crossing changes and we obtain the diagram D' as in Fig. 26 from D by two moves of type 1 and a move of type 2. By a move of type 1 near the crossing of D' as in Fig. 26, we obtain a diagram of the trivial knot. Therefore $u_b(K) \leq 4$ $(< c(K)/2)$.

We also consider D_{c_2} and show that $u_b(K) \leq 3$ $(< c(K)/2)$. One of components of D_{c_2} is the $(2, 5)$-torus knot diagram D_1 and the other is the trivial knot diagram D_2. We change the over/under information of D so that D_2 is over than D_1 at the mutual

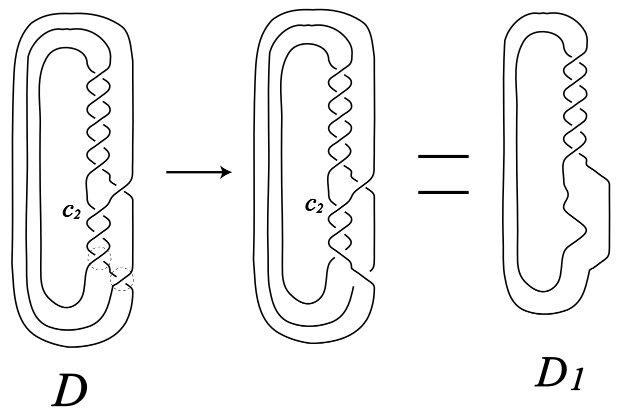


Fig. 27.

crossings between D_1 and D_2 (see Fig. 27). In this process, we need 2 $(= c(D_1, D_2)/2)$ crossing changes. By the claim in Lemma 5.2, we obtain D_1 from D by two band-moves (see Fig. 27). Therefore $u_b(K) \leq 3$ $(< c(K)/2)$.

Let $D = D_1 \cup D_2 \cup \cdots \cup D_n$ be an n-component link diagram. We denote by $c(D_i)$ the number of the self crossings of D_i and by $c(D_i,\, D_j)$ the number of mutual crossings which lie on between D_i and D_j. Therefore the following equality holds.

$$c(D) = \sum_{i=1}^{n} c(D_i) + \sum_{i<j} c(D_i,\, D_j).$$

Now we prove Theorem 5.1.

Proof of Theorem 5.1. First, we prove the inequality (5.1). The inequality holds for the trivial knot and a $(2,r)$-torus knot for some odd integer $r \neq \pm 1$ (see Example 5.3). Therefore we may assume that K is not a $(2,\, r)$-torus knot for any odd integer r. Then, by Theorem 1.1, the inequality $u(K) \leq (c(K) - 2)/2$ holds. By Lemma 5.2, we obtain

$$u_b(K) \leq u(K) + 1 \leq \frac{c(K)}{2}.$$

Next, we prove that the equality holds if and only if K is the trivial knot or the figure-eight knot. The ‘if’ part is trivial (see Example 5.4). Therefore we may assume that K is neither the trivial knot nor the figure-eight knot. If $u(K) \neq (c(K) - 2)/2$, we see that the equality does not hold by the first half of the proof of this theorem. We assume that $u(K) = (c(K) - 2)/2$. Now we prove $u_b(K) < c(K)/2$.

If K is the connected sum of a $(2,\, r)$-torus knot and a $(2,\, s)$-torus knot for some odd integers $r, s \neq \pm 1$, then it is easy to see that $u_b(K) \leq 2 < c(K)/2$. We assume that

K is not the connected sum of a $(2, r)$-torus knot and a $(2, s)$-torus knot for any odd integers $r, s \neq \pm 1$. Let D be a minimal crossing diagram of K. Then $u(D) = u(K)$ by Lemma 2.13. Therefore D is a positive or a negative 3-braid knot diagram by Theorem 2.12. By Lemma 4.2, there exists a crossing c such that one of the components of D_c, denoted by D_1, is a $(2, t)$-torus knot diagram for some odd integer t with $|t| \geq 3$ and the other component of D_c is the trivial knot diagram D_2. Now the following equality holds.

$$c(D) - 1 = t + c(D_1, D_2).$$

We change the over/under information of D so that D_2 is over (or under) than D_1 at the mutual crossings between D_1 and D_2. In this process, we need $c(D_1, D_2)/2$ crossing changes. There are three cases to consider:

CASE 1. $|t| \geq 5$.

Fig. 27 may help us understanding this process. Recall that $c(D_1, D_2)/2$ crossing changes are achieved by, at most, $(c(D_1, D_2)/2 + 1)$-band-moves. Therefore we obtain D_1 from D by, at most, $(c(D_1, D_2)/2 + 1)$-band-moves. Here D_1 represents the $(2, t)$-torus knot, whose band-unknotting number is one. Therefore we obtain

$$u_b(K) \leq \left(\frac{c(D_1, D_2)}{2} + 1\right) + 1 = \frac{c(D) + 3 - t}{2} \leq \frac{c(D) - 2}{2} < \frac{c(K)}{2}.$$

CASE 2. $|t| = 3$ and $c(D_1, D_2)/2$ is even.

Fig. 24 may help us understanding this process. Recall that $c(D_1, D_2)/2$ crossing changes are achieved by $c(D_1, D_2)/2$ band-moves. Therefore we obtain D_1 from D by $c(D_1, D_2)/2$ band-moves. Note that $c(D_1, D_2) = c(D) - 4$. Therefore we obtain

$$u_b(K) \leq \frac{c(D_1, D_2)}{2} + 1 = \frac{c(D)}{2} - 1 < \frac{c(K)}{2}.$$

CASE 3. $|t| = 3$ and $c(D_1, D_2)/2$ is odd.

Fig. 26 may help us understanding this process. We can deform D into the connected sum of D_1 and the Hopf link diagram by $c(D_1, D_2)/2$ band-moves (see the diagram D' in Fig. 26), which is deform into a diagram of the trivial knot by a single band-move. Therefore we obtain

$$u_b(K) \leq \frac{c(D_1, D_2)}{2} + 1 = \frac{c(D)}{2} - 1 < \frac{c(K)}{2}. \qquad \square$$

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