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COMPARING HYPERBOLIC DISTANCE WITH KRA'S DISTANCE ON THE UNIT DISK

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Abstract

In this paper, Kra's distance d_K and the hyperbolic distance $d_{\mathbb{D}}$ are compared on the unit disk $\mathbb{D}$. It is shown that $2d_K < d_{\mathbb{D}} < (\pi^2/8)\exp d_K$ on $\mathbb{D}\times\mathbb{D}\setminus\{\text{diagonal}\}$, where the constants 2 and $\pi^2/8$ are sharp. As a consequence, this result gives a negative answer to a question posed by Martin [7] in a stronger sense.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{|z|<1\}$ in the complex plane $\mathbb{C}$ and let $\rho(z)|dz|$ denote the hyperbolic metric, i.e.,

$$\rho(z)|dz| = \frac{1}{1-|z|^2}|dz|, \quad z\in\mathbb{D}.$$

Then the hyperbolic distance $d_{\mathbb{D}}(z_1, z_2)$ between two points z_1, z_2 induced by $\rho(z)$ is

$$d_{\mathbb{D}}(z_1, z_2) = \frac{1}{2}\log\frac{1+|(z_1-z_2)/(1-\bar{z}_1 z_2)|}{1-|(z_1-z_2)/(1-\bar{z}_1 z_2)|}.$$

Let R be a hyperbolic Riemann surface covered by $\mathbb{D}$. Let $\omega\colon\mathbb{D}\to R$ be the canonical holomorphic universal covering of R. Then $d_{\mathbb{D}}$ induces a quotient hyperbolic distance d_R on R that satisfies

$$d_R(\omega(a), q) = \min\{d_{\mathbb{D}}(z, a)\colon \omega(z) = q\}$$

for all $a\in\mathbb{D}$ and $q\in R$.

A Teichmüller shift mapping on R is the uniquely extremal quasiconformal mapping T_{p_1,p_2} which sends p_1 to p_2 and is homotopic to the identity mapping modulo the ideal boundary ∂R. It is a Teichmüller mapping with Beltrami coefficient μ_{p_1,p_2} such that, for $p_1 = p_2$, $\mu_{p_1,p_2} = 0$, while for $p_1 \neq p_2$, $\mu_{p_1,p_2} = k_{p_1,p_2}|\phi_{p_1,p_2}|/\phi_{p_1,p_2}$,

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where $k_{p_1,p_2} \in (0, 1)$ is a constant and ϕ_{p_1,p_2} is a holomorphic quadratic differential in $R - \{p_1\}$, which has a first order pole at p_1 and has unit L^1-norm.

When studying the self-maps of Riemann surfaces and the geometry of Teichmüller spaces, Kra [4] introduced a distance d_K on every hyperbolic Riemann surface R by the Teichmüller shift mapping, which is defined as follows: for any two points p_1 and p_2 in R,

$$d_K(p_1, p_2) = \frac{1}{2} \log \frac{1 + k_{p_1,p_2}}{1 - k_{p_1,p_2}}.$$

Kra [4] compared d_R with d_K for certain Riemann surfaces:

Theorem A. *When R is of analytic finite type and is not conformally equivalent to $\mathbb{C} \setminus \{0, 1\}$, there exists a universal constant $c > 0$ such that*

$$cd_R < d_K < d_R, \tag{1.1}$$

on $R \times R \setminus \{diagonal\}$.

Earle and Lakic [2] proved

Theorem B. *If R is not conformally equivalent to $\mathbb{C} \setminus \{0,1\}$, then the identity map $id\colon (R, d_R) \to (R, d_K)$ is not an isometry, moreover, $d_K < d_R$ on $R \times R \setminus \{diagonal\}$.*

REMARK. Liu [5] proved Theorem B for all hyperbolic Riemann surfaces with three exceptions: $\mathbb{D}$, $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, or an annulus.

In this paper, we compare d_R with d_K on the unit disk and give sharp inequalities between them.

Theorem 1. *For the unit disk $\mathbb{D}$, the hyperbolic distance $d_{\mathbb{D}}$ and Kra's distance satisfy*

$$2d_K < d_{\mathbb{D}} < \frac{\pi^2}{8} \exp d_K \tag{1.2}$$

on $\mathbb{D} \times \mathbb{D} \setminus \{diagonal\}$, where the constants 2 and $\pi^2/8$ are sharp.

We now introduce a basic concept. A sense preserving homeomorphism f of a domain $\Omega \subset \mathbb{C}$ is called K-quasiconformal ($1 \leq K < \infty$), if f is an L^2-solution of the equation

$$\bar{\partial} f = \mu \, \partial f,$$

where μ is a measurable function with

$$\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1.$$

There is a classical result of Teichmüller's concerning the distortion of normalized quasiconformal mappings [9]. We state Teichmüller's theorem as follows.

Theorem C. *Let $\rho(z, w)$ denote the hyperbolic metric of constant curvature -4 in the three punctured sphere $\mathbb{C} \setminus \{0, 1\}$. We have*
(a) *if f is a K-quasiconformal mapping of the Riemann sphere fixing 0, 1 and ∞, then for any $z \in \mathbb{C} \setminus \{0, 1\}$,*

$$\rho(z, f(z)) \leq \log K, \tag{1.3}$$

(b) *if $z, w \in \mathbb{C} \setminus \{0, 1\}$ satisfy $\rho(z, w) \leq \log K$, then there is a K-quasiconformal map of the Riemann sphere fixing 0, 1 and ∞ such that $w = f(z)$.*

In [7], Martin used holomorphic motions to extend the (b) part of Teichmüller's theorem to any planar domain. He obtained the following theorem.

Theorem D. *Let Ω be a planar domain with at least three boundary points and let $\rho_\Omega(z, w)$ be the hyperbolic metric of Ω with constant curvature -1. Suppose $z, w \in \Omega$ and*

$$\rho_\Omega(z, w) \leq \log K.$$

Then there is a K-quasiconformal self-homeomorphism f of Ω such that
(1) $f(\zeta) = \zeta$ *for all* $\zeta \in \partial\Omega$,
(2) $f(z) = w$.

Martin also asked if the (a) part of the theorem can be extended likewise. His question is precisely described as follows.

Let R be a planar domain with at least three boundary points and suppose that f is a K-quasiconformal mapping of R such that $f(\zeta) = \zeta$ for all $\zeta \in \partial R$. Does it follow that $2d_R(z, f(z)) \leq \log K$ for all $z \in R$? (Notice that the curvature of the hyperbolic metric determined by d_R is -4.)

In [3], Huang and Cho gave a negative answer to this question for any planar simply-connected domain. Actually, Martin's question can be reduced to whether $d_R \leq d_K$ holds on $R \times R$. Evidently it has a negative answer by Theorem B. When $R = \mathbb{D}$, our Theorem 1 implies a negative answer in a stronger sense.

Theorem 2. *For any given $c > 0$ and $z \in \mathbb{D}$, there exists a K-quasiconformal mapping f of $\mathbb{D}$ fixing all boundary points of $\mathbb{D}$ such that*

$$d_{\mathbb{D}}(z, f(z)) > c \log K, \tag{1.4}$$

where K depends only on c.

We note that it might be hard, but would be very interesting to compare $d_{\mathbb{D}^*}$ and d_K on $\mathbb{D}^*$.

2. $2d_K < d_{\mathbb{D}}$

In fact, on the unit disk, we have the following exact formula:

$$\log \frac{\exp d_K + 1}{\exp d_K - 1} = \mu\left(\frac{\exp(2d_{\mathbb{D}}) - 1}{\exp(2d_{\mathbb{D}}) + 1}\right), \tag{2.1}$$

where $\mu(r)$ is the conformal module of the Grötzsch ring domain whose boundary components are the unit circle and the line segment $\{x : 0 \le x \le r\}$. Since d_K and $d_{\mathbb{D}}$ are invariant under Möbius transformations, we only need to prove that

$$2d_K(0, r) < d_{\mathbb{D}}(0, r)$$

for $r \in (0, 1)$.

By the result in [6], $\mu(r)$ satisfies

$$\log \frac{(1 + \sqrt{1 - r^2})^2}{r} < \mu(r) < \log \frac{4}{r}. \tag{2.2}$$

Therefore, $\mu(r)$ has the asymptotic behavior: as $r \to 0$,

$$\mu(r) = \log \frac{4}{r} + s(r), \tag{2.3}$$

where

$$0 > s(r) > \log \frac{(1 + \sqrt{1 - r^2})^2}{r} - \log \frac{4}{r} > -\frac{r^2}{2} + o(r^3). \tag{2.4}$$

Thus, we obtain the asymptotic behavior of $d_K(0, r)$:

$$\begin{aligned} d_K(0, r) &= \log \frac{\exp \mu(r) + 1}{\exp \mu(r) - 1} = \log \frac{(4/r) \exp s(r) + 1}{(4/r) \exp s(r) - 1} \\ &= \log \frac{\exp s(r) + r/4}{\exp s(r) - r/4} = \log \frac{1 + r/4 + s(r) + o(r^3)}{1 - r/4 + s(r) + o(r^3)} \end{aligned}$$

$$
\begin{aligned}
&= \left[\left(\frac{r}{4}+s(r)\right)-\frac{1}{2}\left(\frac{r}{4}+s(r)\right)^2+\frac{1}{3}\left(\frac{r}{4}+s(r)\right)^3+o(r^3)\right] \\
&\quad -\left[\left(-\frac{r}{4}+s(r)\right)-\frac{1}{2}\left(-\frac{r}{4}+s(r)\right)^2+\frac{1}{3}\left(-\frac{r}{4}+s(r)\right)^3+o(r^3)\right] \\
&= \frac{r}{2}-\frac{r}{2}s(r)+\frac{r^3}{96}+o(r^3), \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Using (2.4), we obtain

$$d_K(0,r)=\frac{r}{2}+O(r^3), \quad \text{as} \quad r\to 0.$$

On the other hand, it is easy to check that

$$d_{\mathbb{D}}(0,r)=\frac{1}{2}\log\frac{1+r}{1-r}=r+\frac{r^3}{3}+o(r^3), \quad \text{as} \quad r\to 0. \tag{2.5}$$

Thus, we have

$$\lim_{r\to 0^+}\frac{d_K(0,r)}{d_{\mathbb{D}}(0,r)}=\frac{1}{2}. \tag{2.6}$$

So, for any given $c>1/2$, there exists some $r(c)\in(0,1)$ such that

$$d_K(0,r)<cd_{\mathbb{D}}(0,r) \tag{2.7}$$

holds whenever $r\in(0,r(c))$. Now, we show that (2.7) holds for all $r\in(0,1)$. Let OA denote the line segment $\{x\colon 0\le x\le r\}$ in $\mathbb{D}$, where O is the origin $z=0$ and A is the endpoint $z=r$. Choose orderly $n+1$ (sufficiently large) points $A_0, A_1, \ldots, A_n$ in OA from O to A such that $O=A_0$, $A=A_n$ and

$$d_{\mathbb{D}}(A_k, A_{k+1})<d_{\mathbb{D}}(0,r(c)) \tag{2.8}$$

for $k=0,1,\ldots,n-1$. By the invariance of d_K and $d_{\mathbb{D}}$ under Möbius transformations and inequality (2.7), we have

$$d_K(A_K, A_{K+1})<cd_{\mathbb{D}}(A_K, A_{K+1}).$$

Thus,

$$
\begin{aligned}
d_K(0,r)=d_K(O,A)&\le\sum_{k=0}^{n-1}d_K(A_k,A_{k+1}) \\
&< c\sum_{k=0}^{n-1}d_{\mathbb{D}}(A_k,A_{k+1})=cd_{\mathbb{D}}(O,A)=cd_{\mathbb{D}}(0,r).
\end{aligned}
$$

Since c is arbitrarily chosen in $(1/2, \infty)$, we conclude that

$$2d_K(0, r) \leq d_{\mathbb{D}}(0, r). \tag{2.9}$$

Observe that

$$\begin{aligned} 2d_K(0, r) &\leq 2d_K(0, r') + 2d_K(r', r) \\ &\leq d_{\mathbb{D}}(0, r') + d_{\mathbb{D}}(r', r) = d_{\mathbb{D}}(0, r). \end{aligned}$$

If the equality in (2.9) holds for some $r \in (0, 1)$, then

$$2d_K(0, x) = d_{\mathbb{D}}(0, x) \tag{2.10}$$

for all $x \in (0, r]$. This gives

$$\mu(x) = \log \frac{\sqrt[4]{1+x} + \sqrt[4]{1-x}}{\sqrt[4]{1+x} - \sqrt[4]{1-x}}, \quad x \in (0, r]$$

in terms of (2.1). However, it is impossible because the representation of $\mu(r)$ is not an elementary function in $(0, r)$. Thus, we obtain $2_K < d_{\mathbb{D}}$ on $\mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\}$. Finally, it follows that the constant 2 is sharp from (2.6).

Examining the argument above carefully, we actually prove that the hyperbolic distance has the maximal property in the following sense.

Theorem 3. *Let $d(\cdot, \cdot)$ be a distance function defined on $\mathbb{D} \times \mathbb{D}$. If $d(\cdot, \cdot)$ is invariant under Möbius transformations of $\mathbb{D}$ and satisfies*

$$\limsup_{r \to 0^+} \frac{d(0, r)}{d_{\mathbb{D}}(0, r)} = \lambda > 0, \tag{2.11}$$

then

$$d(z, w) \leq \lambda d_{\mathbb{D}}(z, w), \tag{2.12}$$

for all $(z, w) \in \mathbb{D} \times \mathbb{D}$.

3. $d_{\mathbb{D}} < (\pi^2/8) \exp d_K$

It suffices to show that

$$d_{\mathbb{D}}(0, r) < \frac{\pi^2}{8} \exp d_K(0, r) \tag{3.1}$$

for $r \in (0, 1)$.

We need two lemmas.

Lemma 1. *$g(r) = \mu(r)\, d_{\mathbb{D}}(0, r)$ is an increasing function from $(0, 1)$ onto $(0, \pi^2/4)$.*

Proof. Observe $g(r) = \mu(r)\log((1+r)/(1-r))/2$. Theorem 11.21 in [1] indicates that $g(r)$ satisfies the desired condition. □

Lemma 2. *$h(r) = 1/(\mu(r)\exp d_K(0, r))$ is an increasing function from $(0, 1)$ onto $(0, 1/2)$.*

Proof. Observe

$$h(r) = \frac{1}{\mu(r)}\frac{\exp \mu(r) - 1}{\exp \mu(r) + 1}.$$

Consider two auxiliary functions $x = \mu(r)$ and

$$\tilde{h}(x) = \frac{1}{x}\frac{\exp x - 1}{\exp x + 1}, \quad x \in (0, \infty).$$

We have

$$\tilde{h}'(x) = \frac{1 + 2x \exp x - \exp(2x)}{(x + x \exp x)^2}.$$

It is not difficult to verify that

$$1 + 2x \exp x - \exp(2x) < 0, \quad x \in (0, \infty),$$

and hence $h(x)$ is a decreasing function in $(0, \infty)$. On the other hand, it is well-known that $x = \mu(r)$ is a decreasing function from $(0, 1)$ onto $(0, \infty)$. Thus, $h(r)$ is an increasing function in $(0, 1)$. In addition,

$$\lim_{r\downarrow 0} h(r) = \lim_{x\uparrow\infty} \tilde{h}(x) = 0$$

and

$$\lim_{r\uparrow 1} h(r) = \lim_{x\downarrow 0} \tilde{h}(x) = \frac{1}{2}.$$

This completes the proof of this lemma. □

Combining Lemmas 1 and 2, we get

Theorem 4. *$F(r) = g(r)h(r) = d_{\mathbb{D}}(0, r)/\exp d_K(0, r)$ is an increasing function from $(0, 1)$ onto $(0, \pi^2/8)$.*

Now, we obtain $d_{\mathbb{D}} < (\pi^2/8) \exp d_K$ on $\mathbb{D} \times \mathbb{D} \setminus \{\text{diagonal}\}$, where $\pi^2/8$ is sharp. Moreover, Theorem 2 is naturally derived from Theorem 4 and the definition of Teichmüller shift mapping.

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