FINITE ORBITS OF HURWITZ ACTIONS ON BRAID SYSTEMS

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Abstract
There are natural actions of the braid group B_n on B_m^n, the n-fold product of the braid group B_m, called the Hurwitz action. We first study the roots of centralizers in the braid groups. By using the structure of the roots, we provide a criterion for the Hurwitz orbit to be finite and give an upper bound of the size for a finite orbit in $n = 2$ or $m = 3$ case.

1. Introduction

Let S_n be the degree n symmetric group and B_n be the braid group of n-strands, defined by the presentation

$$B_n = \left\langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \ \middle| \ \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i, & |i - j| \geq 2 \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, & |i - j| = 1 \end{array} \right\rangle.$$

The pure braid group P_n is defined as the kernel of the natural projection $\pi \colon B_n \to S_n$, defined by $\sigma_i \mapsto (i, i + 1)$. For a braid $\beta = \sigma_{i_1}^{e_1} \sigma_{i_2}^{e_2} \cdots \in B_n$, the exponent sum of β is defined by the integer $e_1 + e_2 + \cdots$ and denoted by $e(\beta)$.

A *braid system* of *degree* m and *length* n is, by definition, an element of the n-fold product of the braid group B_m. The *Hurwitz action* is an action of B_n on the set of length n, degree m braid systems B_m^n, defined by

$$(\beta_1, \beta_2, \ldots, \beta_n) \cdot \sigma_i = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \beta_i^{\beta_{i+1}}, \beta_{i+2}, \ldots, \beta_n)$$

where we denote $\beta_{i+1}^{-1} \beta_i \beta_{i+1}$ by $\beta_i^{\beta_{i+1}}$.

Diagrammatically, the definition of the Hurwitz action can be understood by Fig. 1. More generally, we can define the action of the braid group B_n on the n-fold product of groups or racks in a similar way [3].

For a braid system $\mathcal{S}$, we denote the orbit of $\mathcal{S}$ under the Hurwitz action by $\mathcal{S} \cdot B_n$ and call it the *Hurwitz orbit*. The main object studied in this paper is finite Hurwitz orbit. Although the definition of the Hurwitz action is simple, a computation of a Hurwitz orbit is not easy. Some interesting calculations for Hurwitz orbits for Artin groups are done in [8]. We study the structure of a finite Hurwitz orbit for general
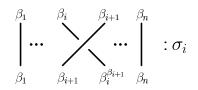

Fig. 1. Diagrammatic description of the Hurwitz action.

braid systems, and provide an upper bound of finite Hurwitz orbit for length 2 or degree 3 braid systems.

To study finite Hurwitz orbit, we first study the roots of centralizers of braids. We denote by $Z(\beta)$ the centralizer of an n-braid β. The following results use the structure theorem of centralizers in [7], which is based on the classification of surface automorphisms due to Thurston [5].

Theorem 1. *Let α, $\beta \in B_m$ and suppose $\alpha \in Z(\beta^s)$ for some $s > 0$.*
1. *If β is periodic, then $\alpha \in Z(\beta^{m \cdot (m-1)})$.*
2. *If β is pseudo-Anosov, then $\alpha \in Z(\beta)$.*
3. *If β is reducible, then $\alpha \in Z(\beta^{(m-1)!})$.*

This result is interesting in its own right. This theorem implies, for two n-braids α and β, if α^M and β^M commute for some non-zero integer M, then $\alpha^{n!}$ and $\beta^{n!}$ always commute.

Now we return to consider finite Hurwitz orbit. To state our results, we introduce a notion of a reducible braid system. We say a length n braid system $\mathcal{S} = (\beta_1, \beta_2, \ldots, \beta_n)$ is *reducible* if there exists a non-trivial partition $I \bigsqcup J$ of the set $\{1, 2, \ldots, n\}$ such that $\beta_i \beta_j = \beta_j \beta_i$ for all $i \in I$, $j \in J$. For a reducible braid system $\mathcal{S}$, let us define $\mathcal{S}' = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_l})$, where $i_p \in I$, $i_p < i_{p+1}$ and $\mathcal{S}'' = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_m})$, where $j_p \in J$, $j_p < j_{p+1}$.

As is easily checked, if a reducible braid system $\mathcal{S}$ has finite Hurwitz orbit, then Hurwitz orbits of $\mathcal{S}'$ and $\mathcal{S}''$ are also finite, and the inequality

$$\#(\mathcal{S} \cdot B_n) \leq \binom{n}{l} \#(\mathcal{S}' \cdot B_l) \cdot \#(\mathcal{S}'' \cdot B_{n-l})$$

holds. So in this paper we mainly focus on irreducible braid systems. Our main results are the following.

Theorem 2 (Finiteness theorem for length 2 braid systems). *Let $\mathcal{S}$ be a degree m, length two braid system having finite Hurwitz orbit.*
1. *If $m = 3$, then $\#(\mathcal{S} \cdot B_2) \leq 6$.*
2. *If $m \geq 4$, then $\#(\mathcal{S} \cdot B_2) \leq 2 \cdot (m - 1)!$.*

Theorem 3 (Finiteness theorem for degree 3 braid systems). *Let $\mathcal{S}$ be a degree* 3, *length n braid system having finite Hurwitz orbit.*

1. *If $n = 2$, then $\#\mathcal{S} \cdot B_n \leq 6$.*
2. *If $n \geq 3$, then $\#\mathcal{S} \cdot B_n \leq 27 \cdot n!$.*
3. *If $n \geq 5$, then $\mathcal{S}$ is reducible.*

2. Roots of centralizers

2.1. Structure of the centralizers of braids. In this subsection we briefly review the results of [7], the structure of the centralizers of a braid. The braid group B_n is naturally identified with the relative mapping class group $MCG(D_n, \partial D_n)$ of the n-punctured disc D_n, which is the group of isotopy classes of homeomorphisms of D_n which fixes ∂D_n pointwise [1].

From the Nielsen–Thurston theory, each element of the braid group B_n is classified into the following three types, *periodic*, *reducible*, and *pseudo-Anosov* according to its dynamical property. See [5] for details of Nielsen–Thurston theory. In this paper we treat the trivial element of B_n as a periodic braid.

A *periodic braid* is a braid some of whose powers belong to the center of the braid group, which is an infinite cyclic group generated by the square of the Garside element

$$\Delta^2 = \{(\sigma_1\sigma_2\cdots\sigma_{n-1})(\sigma_1\cdots\sigma_{n-2})\cdots(\sigma_1\sigma_2)(\sigma_1)\}^2.$$

It is classically known [4] that each periodic n-braid is conjugate to either

$$(\sigma_1\sigma_2\cdots\sigma_{n-1})^m$$

or

$$(\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1)^m$$

for some integer m. This implies that the n-th or $(n-1)$-st powers of a periodic braid always belong to the center of B_n.

The centralizer of a periodic braid is simple, in some special case. From the above facts, we can write a periodic n-braid as

$$\gamma^{-1}(\sigma_1\sigma_2\cdots\sigma_{n-1})^k\gamma$$

or

$$\gamma^{-1}(\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1)^k\gamma.$$

In the former case, if k and n are coprime, then the centralizer $Z(\beta)$ is an infinite cyclic group generated by $\gamma^{-1}(\sigma_1\sigma_2\cdots\sigma_{n-1})\gamma$. Similarly, in the latter case, if k and $n-1$ are coprime, then the centralizer $Z(\beta)$ is an infinite cyclic group generated by

$\gamma^{-1}(\sigma_1\sigma_2\cdots\sigma_{n-1}\sigma_1)\gamma$ [7, Proposition 3.3]. If k and n (or $n-1$) are not coprime, then the centralizer of periodic braids are isomorphic to the braid group of annulus [7, Corollary 3.6].

A *pseudo-Anosov braid* is a braid which is represented by a pseudo-Anosov homeomorphism. A pseudo-Anosov homeomorphism f is a homeomorphism which has the two invariant measured foliations $(\mathcal{F}^s, \mu^s)$, $(\mathcal{F}^u, \mu^u)$ called the *stable* and *unstable foliation* and the real number $\lambda > 1$ called the *dilatation*. They satisfy the condition $f(\mathcal{F}^s, \mu^s) = (\mathcal{F}^s, \lambda^{-1}\mu^s)$ and $f(\mathcal{F}^u, \mu^u) = (\mathcal{F}^u, \lambda\mu^u)$.

The centralizer $Z(\beta)$ of a pseudo-Anosov braid β is also simple. The centralizer $Z(\beta)$ is isomorphic to the rank two free abelian group generated by one pseudo-Anosov element and one periodic element, both of which preserve the invariant foliations of β [7, Proposition 4.1]. In particular, all braids in $Z(\beta)$ are irreducible.

A *reducible braid* is a braid which preserves a non-empty essential submanifold $\mathcal{C}$ of D_n. In this paper we adapt the convention that every reducible braid is non-periodic. By taking an appropriate conjugation, each reducible braid β can be converted to the following simple form, called a *standard form*.

Regard $\mathcal{C}$ as a set of essential circles. A collection of essential circles $\mathcal{C}$ is called a *standard curve system* if $\mathcal{C}$ satisfies the following two conditions.
1. The center of each circle in $\mathcal{C}$ lies on x-axis.
2. For any two distinct circles C and C' in $\mathcal{C}$, C does not enclose C'.

By taking an appropriate conjugation, we can always assume that a reducible braid β preserves a standard curve system $\mathcal{C}$. The braid β acts on the set $\mathcal{C}$ as a permutation of circles. Let us denote the orbit decomposition of $\mathcal{C}$ by $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_l$, where $\mathcal{C}_i = \{C_{i,1},\ldots,C_{i,r_i}\}$. We choose the numbering $C_{i,j}$ so that $\beta(C_{i,j}) = C_{i,j+1}$ (modulo r_i) holds.

Let us denote the number of punctures in the circle $C_{i,j}$, which is independent of j, by c_i. Then the orbit decomposition defines the weighted partition $\mathbf{n}$ of an integer n, $\mathbf{n}$: $n = c_1r_1 + c_2r_2 + \cdots + c_kr_k$.

In this situation, we can write the reducible braid β as a composition of two parts. The first part is the *tubular braid*, which is a braiding of tubes corresponding to the permutation of the circles. Each tube contains some numbers of parallel strands (possibly one) which are not braided inside the tube. The other part is the *interior braids* $\beta_{i,j}$, which are braids inside the tube sending the circle $C_{i,j-1}$ to $C_{i,j}$. We denote the braid obtained by regarding each tube of the tubular braid as one strand by β_{ext} and call it the *exterior braid*. The interior braids $\beta_{i,j}$ and the exterior braid β_{ext} are chosen so that they are non-reducible.

Using the above notions, we denote the reducible braid β as

$$\beta = \beta_{ext}(\beta_{1,1} \oplus \beta_{1,2} \oplus \cdots \oplus \beta_{k,r_k})_\mathbf{n}$$

and call such a form of the braid the *standard form*. See Fig. 2.

We can make a reducible braid in standard form much simpler by taking a further conjugation so that the following hold.
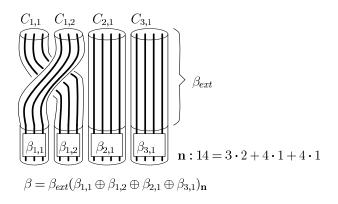
$$\mathbf{n} : 14 = 3 \cdot 2 + 4 \cdot 1 + 4 \cdot 1$$

$$\beta = \beta_{ext}(\beta_{1,1} \oplus \beta_{1,2} \oplus \beta_{2,1} \oplus \beta_{3,1})_{\mathbf{n}}$$

Fig. 2. Standard form of reducible braids.

1. Each interior braid $\beta_{i,j}$ is a trivial braid unless $j = 1$.
2. $\beta_{i,1}$ and $\beta_{j,1}$ are non-conjugate unless $\beta_{i,1} = \beta_{j,1}$.

   After this modification, we denote the non-trivial interior braids $\beta_{[i,1]}$ simply by $\beta_{[i]}$. Now the whole braid β is written as

$$\beta = \beta_{ext} \cdot (\beta_{[1]} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{r_1 - 1} \oplus \beta_{[2]} \oplus \cdots \oplus \beta_{[k]} \oplus \underbrace{1 \oplus \cdots \oplus 1}_{r_k - 1})_{\mathbf{n}}.$$

We denote this special standard form of a reducible braid by

$$\beta = \beta_{ext} \cdot (\beta_{[1]}, \beta_{[2]}, \ldots, \beta_{[k]})_{\mathbf{n}}.$$

and call it the *normal form*.

Let $\beta = \beta_{ext} \cdot (\beta_{[1]}, \beta_{[2]}, \ldots, \beta_{[m]})_{\mathbf{n}}$ be a normal form of a reducible braid which preserves a standard curve system $\mathcal{C}$. Then the centralizer of β is described as follows.

Every $\alpha \in Z(\beta)$ preserves $\mathcal{C}$, hence α is written as a standard form. In particular, the exterior part α_{ext} of α also induces the permutation of circles in $\mathcal{C}$. We say α_{ext} is *consistent with* β_{ext} if $\alpha_{ext}(C_{i,k}) = C_{j,l}$ then $\beta_{[i]} = \beta_{[j]}$ holds. Let $Z_0(\beta_{ext})$ be a subgroup of $Z(\beta_{ext})$ defined by

$$Z_0(\beta_{ext}) = \{\alpha_{ext} \in Z(\beta_{ext}) \mid \alpha_{ext} \text{ is consistent with } \beta_{ext}\}.$$

Then $Z(\beta)$ is described by the following split exact sequence [7, Theorem 1.1].

$$1 \to Z(\beta_{[1]}) \times Z(\beta_{[2]}) \times \cdots \times Z(\beta_{[k]}) \xrightarrow{i} Z(\beta) \xrightarrow{j} Z_0(\beta_{ext}) \to 1.$$

The map i is defined by

$$i(\alpha_{[1]}, \alpha_{[2]}, \ldots, \alpha_{[k]}) = 1 \cdot (\underbrace{\alpha_{[1]} \oplus \cdots \oplus \alpha_{[1]}}_{r_1} \oplus \cdots \oplus \underbrace{\alpha_{[k]} \oplus \cdots \oplus \alpha_{[k]}}_{r_k})_{\mathbf{n}}$$

and the map j is defined by

$$j(\alpha_{ext} \cdot (\alpha_{1,1} \oplus \cdots \oplus \alpha_{k,r_k})_{\mathbf{n}}) = \alpha_{ext}.$$

The splitting s of the above exact sequence is given by

$$s(\alpha_{ext}) = \alpha_{ext}(1 \oplus \cdots \oplus 1)_{\mathbf{n}}.$$

Therefore, for each $\alpha \in Z(\beta)$, we can write α as

$$\alpha = \alpha_{ext} \cdot \left(\alpha_{[1]}^{\oplus r_1} \oplus \alpha_{[2]}^{\oplus r_2} \oplus \cdots \oplus \alpha_{[k]}^{\oplus r_k} \right)_{\mathbf{n}}.$$

2.2. Proof of Theorem 1. Now we are ready to prove Theorem 1.

Proof of Theorem 1.   The assertion 1 is immediate because for a periodic braid $\beta \in B_m$, β^m or β^{m-1} belongs to $Z(B_m)$. The proof of the assertion 2 is also easy. Suppose β is pseudo-Anosov and let $\mathcal{F}^s$, $\mathcal{F}^u$ and λ be the stable, unstable measured foliation and the dilatation of β. Since α belongs to the center of the pseudo-Anosov braid β^s, whose invariant measured foliations are also $\mathcal{F}^s$ and $\mathcal{F}^u$, α also preserves both $\mathcal{F}^s$ and $\mathcal{F}^u$. Now the braid $\alpha\beta\alpha^{-1}\beta^{-1}$ preserves the measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ has the dilatation 1. This implies that the braid $\alpha\beta\alpha^{-1}\beta^{-1}$ is periodic. Since the exponent sum of $\alpha\beta\alpha^{-1}\beta^{-1}$is zero, we conclude that $\alpha\beta\alpha^{-1}\beta^{-1} = 1$. Therefore we obtain $\alpha \in Z(\beta)$.

Now we proceed to the most difficult case, reducible case. By taking a conjugate of β, we may assume β is a normal form

$$\beta = \beta_{ext} \cdot (\beta_{[1]}, \beta_{[2]}, \ldots, \beta_{[k]})_{\mathbf{n}}$$

where $\mathbf{n}: m = c_1 \cdot r_1 + \cdots + c_k \cdot r_k$ is an associated weighted partition of m. Let us define integers a_i by $a_i = (m-1)!/r_i$. Since the exterior part of $\beta^{(m-1)!}$ is a pure braid $\beta_{ext}^{(m-1)!}$, so $\beta^{(m-1)!}$ is written as a normal form

$$\beta^{(m-1)!} = \beta_{ext}^{(m-1)!} \cdot \big(\underbrace{\beta_{[1]}^{a_1}, \ldots, \beta_{[1]}^{a_1}}_{r_1}, \ldots, \underbrace{\beta_{[k]}^{a_k}, \ldots, \beta_{[k]}^{a_k}}_{r_k} \big)_{\mathbf{n}^*}.$$

where $\mathbf{n}^*$ is a weighted partition defined by

$$\mathbf{n}^*: m = \underbrace{c_1 \cdot 1 + \cdots + c_1 \cdot 1}_{r_1} + \cdots + \underbrace{c_k \cdot 1 + \cdots + c_k \cdot 1}_{r_k}.$$

Let $\alpha \in Z(\beta^s)$. Then $\alpha \in Z(\beta^{(m-1)! \cdot s})$. From the normal form of $\beta^{(m-1)! \cdot s}$, α can be written as a standard form

$$\alpha = \alpha_{ext} \cdot (\alpha_{1,1} \oplus \alpha_{1,2} \oplus \cdots \oplus \alpha_{k,r_k})_{\mathbf{n}^*}.$$

Since the interior braids $\alpha_{i,j}$ are irreducible c_i-braid and $\alpha_{i,j} \in Z(\beta_{[i]}^{a_i \cdot s})$, from the assertion 1 and 2 we obtain $\alpha_{i,j} \in \beta_{[i]}^{c_i \cdot (c_i - 1)}$.

Now observe that $a_i / c_i \cdot (c_i - 1) = (m-1)!/r_i c_i (c_1 - 1)$ is an integer. Therefore we conclude that $\alpha_{i,j} \in Z(\beta_{[i]}^{a_i})$. By the same argument, we also obtain $\alpha_{ext} \in Z(\beta_{ext}^{(m-1)!})$.

If α_{ext} is not consistent with $\beta_{ext}^{(m-1)!}$, then there exist pairs (i,k) and (j,l) such that $\alpha_{ext}(C_{[i,k]}) = C_{[j,l]}$ but $(\beta^{(m-1)!})_{i,k} = \beta_{[i]}^{a_i} \neq \beta_{[j]}^{a_j} = (\beta^{(m-1)!})_{j,l}$ holds. On the other hand, $\alpha \in Z(\beta^{(m-1)! \cdot s})$ implies that α_{ext} is consistent with $\beta_{ext}^{(m-1)! \cdot s}$. Therefore $(\beta^{(m-1)! \cdot s})_{i,k} = \beta_{[i]}^{a_i \cdot s} = \beta_{[j]}^{a_j \cdot s} = (\beta^{(m-1)! \cdot s})_{j,l}$ holds.

It is known that the root of a braid is unique up to conjugacy [6]. Therefore the above equality means that $\beta_{[i]}^{a_i}$ and $\beta_{[j]}^{a_j}$ are conjugate. Since $\beta^{(m-1)!}$ is a normal form, we conclude that $\beta_{[i]}^{a_i} = \beta_{[j]}^{a_j}$, which is a contradiction. Thus we conclude that $\alpha_{ext} \in Z_0(\beta_{ext}^{(m-1)!})$, so $\alpha \in Z(\beta^{(m-1)!})$. $\square$

We remark that our value $(m-1)!$ for reducible braids case is not optimal. Only the properties of the number $(m-1)!$ we used in the proof is that the number $a_i / c_i \cdot (c_i - 1) = (m-1)!/r_i c_i (c_1 - 1)$ is an integer and that $\beta_{ext}^{(m-1)!}$ is a pure braid. By considering these two properties more carefully, we can easily decrease our value $(m-1)!$. We give the smallest value for small m for later use.

Proposition 1. *Let* $\alpha, \beta \in B_m$ *and suppose* $\alpha \in Z(\beta^s)$ *for some* $s > 0$ *and* β *is reducible.*
1. *If* $m = 3$, *then* $\alpha \in Z(\beta)$ *and* $Z(\beta)$ *is a free abelian group of rank two.*
2. *If* $m = 4$, *then* $\alpha \in Z(\beta^s)$ *for some* $s \leq 3$.

Proof. If $m = 3$, then we may assume that by taking an appropriate conjugate, the reducible braid β can be written by $\beta = \sigma_1^{2p}(\sigma_1^k \oplus 1)_{(2,1)}$. Thus the centralizer of β is the free abelian group of rank two generated by $\sigma_1^2(1 \oplus 1)_{(2,1)}$ and $1(\sigma_1 \oplus 1)_{(2,1)}$. Thus if $\alpha \in Z(\beta^s)$ for some $s \geq 1$, then $\alpha \in Z(\beta)$ holds.

The proof of $m = 4$ case is also a direct calculation of the centralizers. By taking an appropriate conjugation, we may assume that the braid β has one of the following forms.
1. $\beta = \sigma_1^p(\sigma_1^q \oplus \sigma_1^r)_{(2,2)}$.
2. $\beta = \sigma_1^{2p}(\beta_{int} \oplus 1)_{(3,1)}$ where $\beta_{int} \in B_3$.
3. $\beta = \beta_{ext}(\sigma_1^p \oplus 1 \oplus 1)_{(2,1,1)}$.
In the first case we obtain $\alpha \in Z(\beta^2)$. In the second and the third case, $\alpha \in Z(\beta^2)$ or $\alpha \in Z(\beta^3)$ holds. $\square$

3. Some computations of Hurwitz actions

Now we begin our study of the Hurwitz action. In this section we do some calculations, which will be used later. For two braid systems $\mathcal{S} = (\beta_1, \ldots, \beta_n)$ and $\mathcal{S}' =$

$(\beta'_1, \ldots, \beta'_n)$ having the same degree and length, we say $\mathcal{S}$ and $\mathcal{S}'$ are conjugate if $\beta'_i = \alpha^{-1}\beta_i\alpha$ for some braid α and all $i = 1, 2, \ldots, n$. Then there is a one-to-one correspondence between two Hurwitz orbits $\mathcal{S} \cdot B_n$ and $\mathcal{S}' \cdot B_n$ if two braid systems $\mathcal{S}$ and $\mathcal{S}'$ are conjugate. So we try to take a conjugate of braid systems so that computations are easier.

Since the pure braid group P_n has finite index $n!$ in B_n, to classify the finite orbits of B_n, it is sufficient to consider the orbits of pure braid group P_n. For $i = 1, 2, \ldots, n-1$, let c_i be the pure braid defined by

$$c_i = (\sigma_1^{-1} \cdots \sigma_{i-1}^{-1})\sigma_i^2(\sigma_{i-1} \cdots \sigma_1)$$

and F_{n-1} be a subgroup of P_n generated by $\{c_1, c_2, \ldots, c_{n-1}\}$. It is known that F_{n-1} is a free group of rank $n-1$ and there exists a split exact sequence

$$1 \to F_{n-1} \to P_n \to P_{n-1} \to 1.$$

Hence the pure braid group P_n can be described as a semi-direct products of free groups,

$$P_n = P_{n-1} \ltimes F_{n-1} = F_1 \ltimes F_2 \ltimes \cdots \ltimes F_{n-1}.$$

See [1] for details. Thus, to classify or estimate the size of finite Hurwitz orbit, it is sufficient to consider the F_n actions.

Now we compute some actions of element of F_n.

Lemma 1. *Let $\mathcal{S} = (\beta_1, \beta_2, \ldots, \beta_n)$ be a length n braid system.*
1. *For all k and i,*

$$\mathcal{S} \cdot c_i^k = \left(\beta_1^{(\beta_1\beta_{i+1})^k}, \beta_2^{(\beta_{i+1}\beta_1)^{-k}(\beta_1\beta_{i+1})^k}, \ldots, \right.$$
$$\left. \beta_i^{(\beta_{i+1}\beta_1)^{-k}(\beta_1\beta_{i+1})^k}, \beta_{i+1}^{(\beta_1\beta_{i+1})^k}, \beta_{i+2}, \ldots, \beta_n \right).$$

2. *For $j > 2$,*

$$\mathcal{S} \cdot (c_1c_2 \cdots c_j)^k = \left(\beta_1^{C^k}, \beta_2^{(\beta_1^{-1}C)^{-k}C^k}, \ldots, \beta_{j+1}^{(\beta_1^{-1}C)^{-k}C^k}, \beta_{j+2}, \ldots, \beta_n \right)$$

where $C = \beta_1\beta_2 \cdots \beta_{j+1}$.
3. *Let $\Delta_{(i,j)} = (\sigma_i\sigma_{i+1} \cdots \sigma_j)(\sigma_i\sigma_{i+1} \cdots \sigma_{j-1}) \cdots (\sigma_i\sigma_{i+1})(\sigma_i)$. Then*

$$\mathcal{S} \cdot \Delta_{(i,j)}^{2p} = (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_i^{C^p}, \ldots, \beta_j^{C^p}, \beta_{j+1}, \ldots, \beta_n).$$

where $C = \beta_i\beta_{i+1} \cdots \beta_j$.

Proof.   Direct computation.   □

4. Partial Coxeter element

In this section, we provide a finiteness and infiniteness criterion of Hurwitz orbits for general degree and length by using the notion of (partial) Coxeter element. The partial Coxeter element argument provides a strong restriction for the finiteness of Hurwitz orbit and gives evidence that finite Hurwitz orbits with non-commutative entries are rare.

DEFINITION 1. For a braid system $\mathcal{S} = (\beta_1, \ldots, \beta_n) \in B_m^n$ and strictly increasing sequence of integers $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq m\}$, we define $C_I(\mathcal{S})$, the *partial Coxeter element* of $\mathcal{S}$ by $C_I(\mathcal{S}) = \beta_{i_1} \beta_{i_2} \cdots \beta_{i_k}$. For the sequence $I = \{1, 2, 3, \ldots, m\}$, we call $C_I(\mathcal{S})$ the *(full) Coxeter element* of $\mathcal{S}$ and denote it by $C(\mathcal{S})$.

From the definition of the Hurwitz action, the full Coxeter element $C(\mathcal{S})$ is invariant under the Hurwitz action, so it is an invariant of the Hurwitz orbit. On the other hand, the partial Coxeter element $C_I(\mathcal{S})$ might dramatically change by the Hurwitz action. Even the Nielsen–Thurston types might change. Now Lemma 1 and the knowledge of the centralizers provide the following criterion of finiteness.

Theorem 4 (Partial Coxeter element criterion). *Let $\mathcal{S} = (\beta_1, \beta_2, \ldots, \beta_n)$ be a braid system of degree m, length n having the finite Hurwitz orbit $\mathcal{S} \cdot B_n$ and $I = \{1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ be a strictly increasing sequence of integers of length $k \geq 2$.*
1. *If $C_I(\mathcal{S})$ is pseudo-Anosov, then $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}$ are irreducible and commutative.*
2. *If $C_I(\mathcal{S})$ is reducible, then $\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}$ preserves the same essential 1-submanifold. Especially, they are not pseudo-Anosov.*
3. *If $C_{\{1,2,\ldots,j\}}$ is periodic, then $\mathcal{S} \cdot (c_1 c_2 \cdots c_{j-1})^r = \mathcal{S}$ for some $1 \leq r \leq m!$.*

Proof. First we prove 1 and 2. By considering the action of an appropriate braid, there is a braid system $\mathcal{S}'$ in the Hurwitz orbit of $\mathcal{S}$, which is written as $\mathcal{S}' = (\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k}, \beta'_{k+1}, \ldots)$. From the assertion 3 of Lemma 1,

$$\mathcal{S}' \cdot \Delta_{(1,k)}^{2p} = \left(\beta_{i_1}^{c^p}, \beta_{i_2}^{c^p}, \ldots, \beta_{i_k}^{c^p}, \beta'_{k+1}, \ldots \right)$$

where $c = C_I(\mathcal{S})$. Since $\mathcal{S} \cdot B_n$ is finite, $\beta_{i_j} \in Z(c^p)$ for some $p > 0$. This means all of β_{i_j} are irreducible and commutative if c is pseudo-Anosov, and all of β_{i_j} preserve the same 1-submanifold if c is reducible.

Next we prove 3. Let $C = \beta_1 \beta_2 \cdots \beta_j$ be the partial Coxeter element and q be a period of C. From the assertion 2 of Lemma 1,

$$\mathcal{S} \cdot (c_1 c_2 \cdots c_{j-1})^p = \left(\beta_1^{C^p}, \beta_2^{(\beta_1^{-1}C)^{-p}C^p}, \ldots, \beta_j^{(\beta_1^{-1}C)^{-p}C^p}, \beta_{j+1}, \ldots, \beta_n \right).$$

Since $\mathcal{S} \cdot B_n$ is finite, we can find $0 < p$ satisfying $\mathcal{S} \cdot (c_1 c_2 \cdots c_{j-1})^p = \mathcal{S}$. Then

$$\mathcal{S} \cdot (c_1 c_2 \cdots c_{j-1})^{pq} = \left(\beta_1, \beta_2^{(\beta_1^{-1} C)^{-pq}}, \ldots, \beta_j^{(\beta_1^{-1} C)^{-pq}}, \beta_{j+1}, \ldots, \beta_n \right)$$
$$= (\beta_1, \beta_2, \beta_3, \ldots, \beta_n).$$

Thus all of $\beta_2, \beta_3, \ldots, \beta_j$ belong to the centralizer of $(\beta_1^{-1} C)^{pq}$. From Theorem 1, there exists $s \leq m!$ such that all of $\beta_2, \beta_3, \ldots, \beta_j \in Z((\beta_1 C^{-1})^s)$. Therefore, we conclude that $\mathcal{S} \cdot (c_1 c_2 \cdots c_{j-1})^r = \mathcal{S}$ for some $0 < r \leq m!$. $\square$

These result imply that each entry of a braid system with finite Hurwitz orbit must satisfy the following conditions.
• If its full Coxeter element is pseudo-Anosov, then all of its entries must be commutative.
• If its full Coxeter element is reducible, then all of its entries must not be pseudo-Anosov and preserve the same 1-submanifold $\mathcal{C}$.

Using this condition, sometimes we can easily check whether the Hurwitz orbit is finite or not.

EXAMPLE 1.   Now we give some examples.
1.   Let $\mathcal{S} = (\sigma_1, \sigma_2^2, \sigma_1)$. Each entry of $\mathcal{S}$ is reducible and the full Coxeter element is also reducible. However, σ_1 and σ_2 do not preserve the same essential 1-submanifolds, so we conclude that $\mathcal{S}$ has infinite Hurwitz orbit.
2.   Let $\mathcal{S} = (\sigma_1, \sigma_1, \sigma_1, \sigma_1, \sigma_2)$. It is easily checked that braid systems (σ_1, σ_2), $(\sigma_1, \sigma_1, \sigma_2)$ and $(\sigma_1, \sigma_1, \sigma_1, \sigma_2)$ have finite Hurwitz orbits. However, the Hurwitz orbit of $\mathcal{S}$ is infinite because the full Coxeter element is pseudo-Anosov but σ_1 is reducible.

As these examples suggest, a braid system might have infinite Hurwitz orbit even if its entries have simple relations.

5.   Classification of finite Hurwitz orbits

Now we begin a classification of finite Hurwitz orbits.

5.1.   Length two braid systems.   First of all, we prove Theorem 2.

Proof of Theorem 2.   From the assertion 1 of Lemma 1, $(\beta_1, \beta_2)\sigma_1^{2p} = \left(\beta_1^{(\beta_1 \beta_2)^p}, \beta_2^{(\beta_1 \beta_2)^p} \right)$ holds. Since the Hurwitz orbit of $\mathcal{S}$ is finite, $\beta_1, \beta_2 \in Z((\beta_1 \beta_2)^p)$ for some $p > 0$. From Theorem 1, $p \leq \max\{(m-1)!, m\}$, so the conclusion holds. $\square$

As in the remark after Theorem 1, this upper bound is not sharp for general m. For $m = 3, 4$, we give an accurate upper bound.

Corollary 1. *Let $\mathcal{S}$ be a degree m, length 2 braid system having finite Hurwitz orbit.*
1. *If $m = 3$, $\#(\mathcal{S} \cdot B_2) \leq 6$.*
2. *If $m = 4$, $\#(\mathcal{S} \cdot B_2) \leq 8$.*

The above upper bounds are exact. $\#((\sigma_1^{-1}, \sigma_1^2 \sigma_2) \cdot B_2) = 6$ and $\#((\sigma_1, \sigma_2 \sigma_3) \cdot B_2) = 8$. We remark that there is no universal bound for $\#(\beta_1, \beta_2) \cdot B_2$ if we do not fix the degree m. For $m \geq 4$, the size of the Hurwitz orbit of the braid system $(\sigma_1, \sigma_2 \sigma_3 \cdots \sigma_{m-1})$ is $2m$.

5.2. Normal form of periodic 3-braids.

Next we study degree 3 braid systems, where difficulties due to the fact B_3 is not abelian arise.

Recall that the centralizer of a 3-braid β is abelian unless β is central in B_3. Our classification result relies on this special feature of B_3. In this subsection, we briefly summarize the dual Garside structure of B_3 and the left normal forms and prepare some lemmas which will be used. See [2] for details.

Let $a_{1,2} = \sigma_1$, $a_{2,3} = \sigma_2$, $a_{1,3} = \sigma_2^{-1} \sigma_1 \sigma_2$ and $\delta = a_{1,2} a_{2,3} = a_{2,3} a_{1,3} = a_{1,3} a_{1,2}$. Using the braids $\{a_{1,2}, a_{2,3}, a_{1,3}\}$, the braid group B_3 is presented by

$$B_3 = \langle a_{1,2}, a_{2,3}, a_{1,3} \mid a_{1,2} a_{2,3} = a_{2,3} a_{1,3} = a_{1,3} a_{1,2} \rangle$$

Each 3-braid $\beta \in B_3$ has the one of the following unique word representative $N(\beta)$, called the (left-greedy) normal form.

$$N(\beta) = \begin{cases} \delta^m a_{1,2}^{p_1} a_{1,3}^{p_2} a_{2,3}^{p_3} a_{1,2}^{p_4} \cdots a_{*,*}^{p_k}, \\ \delta^m a_{1,3}^{p_1} a_{2,3}^{p_2} a_{1,2}^{p_3} a_{1,3}^{p_4} \cdots a_{*,*}^{p_k}, \\ \delta^m a_{2,3}^{p_1} a_{1,2}^{p_2} a_{1,3}^{p_3} a_{2,3}^{p_4} \cdots a_{*,*}^{p_k} \end{cases}$$

where p_i is a positive integer. In the normal form, the integer m is called the *supremum* of β and denoted by $\sup(\beta)$. We define $d(\beta)$, the *depth* of β, by $d(\beta) = k$.

Lemma 2. *For a periodic 3-braid β, if $d(\beta) \neq 0$, $d(\beta) + \sup(\beta) \equiv 2 \pmod{3}$.*

Proof. Let β be a periodic 3-braid and $s = \sup(\beta)$, $d = d(\beta)$. We only prove $s \equiv 0 \pmod 3$ case. Other cases are similar. Assume that $d \not\equiv 2 \pmod 3$. Then by taking a conjugation by δ, we can assume that the normal form of β is either

$$N(\beta) = \begin{cases} \delta^{3s'} a_{1,2}^{p_1} \cdots a_{2,3}^{p_d} & \text{or} \\ \delta^{3s'} a_{1,2}^{p_1} \cdots a_{1,2}^{p_d}. \end{cases}$$

In either case, the normal form of β^6 is given by

$$N(\beta^6) = \begin{cases} \delta^{18s'} \left(a_{1,2}^{p_1} \cdots a_{2,3}^{p_d} \right) \left(a_{1,2}^{p_1} \cdots a_{2,3}^{p_d} \right) \cdots \left(a_{1,2}^{p_1} \cdots a_{2,3}^{p_d} \right) & \text{or} \\ \delta^{18s'} \left(a_{1,2}^{p_1} \cdots a_{1,2}^{p_d+p_1} \right) \left(a_{1,3}^{p_2} \cdots a_{2,3}^{p_d+p_1} \right) \cdots \left(a_{1,3}^{p_2} \cdots a_{2,3}^{p_d} \right). \end{cases}$$

Therefore, β is not periodic. $\square$

Now we prove the key lemma which plays an important role in proving our finiteness results for degree 3 braid systems.

Lemma 3. *Let α be a periodic 3-braid whose period is 3. Then for $\beta, \gamma \in B_3$, not all of $\beta\gamma$, $\beta^\alpha\gamma$, $\beta^{\alpha^2}\gamma$ are periodic unless either β or γ belongs to $Z(\alpha)$.*

Proof. Assume that both β and γ do not belong to $Z(\alpha)$. By considering a conjugate of the braid system, we can assume that $\alpha = \delta^p$ and the normal form of γ is written as

$$N(\gamma) = \delta^g a_{1,2}^{p_1} a_{1,3}^{p_2} \cdots a_{*,*}^{p_k}.$$

Since both γ and β do not commute with $\alpha = \delta^p$, we obtain $d(\gamma) \neq 0$ and $d(\beta) \neq 0$. Now let us denote the normal form of β by

$$N(\beta) = \delta^b \cdots a_{i,j}^q.$$

Then for some distinct $e, f \in \{1,2,3\}$, the normal forms of $\beta^{\delta^{e+g}}$ and $\beta^{\delta^{f+g}}$ are given by

$$N(\beta^{\delta^{e+g}}) = \delta^b \cdots a_{1,2}^q, \quad N(\beta^{\delta^{f+g}}) = \delta^b \cdots a_{2,3}^q.$$

Now the normal forms of $\beta^{\delta^e}\gamma$ and $\beta^{\delta^f}\gamma$ are written as

$$N(\beta^{\delta^e}\gamma) = \delta^{b+g} \cdots a_{1,2}^{q+p_1} a_{1,3}^{p_2} \cdots a_{*,*}^{p_k}, \quad N(\beta^{\delta^f}\gamma) = \delta^{b+g} \cdots a_{2,3}^q a_{1,2}^{p_1} a_{1,3}^{p_2} \cdots a_{*,*}^{p_k}.$$

Thus, $\sup(\beta^{\delta^e}\gamma) + d(\beta^{\delta^e}\gamma) = b + g + q + k - 1$ and $\sup(\beta^{\delta^f}\gamma) + d(\beta^{\delta^f}\gamma) = b + g + q + k$. By Lemma 2, we conclude that not both of $\beta^{\delta^e}\gamma$ and $\beta^{\delta^f}\gamma$ are periodic. $\square$

5.3.   Exponent sum restriction.   In this subsection, we study the exponent sum of the entries of braid systems having finite Hurwitz orbit. We observe the following simple, but crucial lemma about degree 3 braid systems having finite Hurwitz orbits.

Lemma 4. *Let $\mathcal{S} = (\beta_1, \ldots, \beta_l)$ be a degree 3 braid system having finite Hurwitz orbit and assume that all of β_i are not central in B_3. If $e(\beta_{i_1}) + e(\beta_{i_2}) + \cdots + e(\beta_{i_k}) \not\equiv \pm 2, 3 \pmod 6$ for some $1 \leq i_1 < i_2 < \cdots < i_k \leq l$ $(1 < k < l)$, then all of its entry β_i are mutually commutative.*

Proof. With no loss of generality, we can assume that $e(\beta_1) + e(\beta_2) + \cdots + e(\beta_k) \not\equiv \pm 2, 3 \pmod 6$. First we show that β_k commutes with β_{k+1}. Let $C = \beta_1 \beta_2 \cdots \beta_{k-1}$.

Using the result of Eilenberg [4] alluded to above and the hypothesis on the exponent sum, the partial Coxeter element $\beta_1 \beta_2 \cdots \beta_k = C\beta_k$ is non-periodic or central

in B_3. Thus β_k belongs to $Z(C)$. Similarly, by considering the partial Coxeter element of $\mathcal{S} \cdot \sigma_k^2$ we obtain that $\beta_k^{\beta_{k+1}}$ also belongs to $Z(C)$.

First we consider the case C is periodic. Since we have assumed that β_k is non-central, so C is also non-central. This implies $Z(C)$ is an infinite cyclic group generated by an element having non-zero exponent sum. Thus we conclude $\beta_k = \beta_k^{\beta_{k+1}}$, so β_k and β_{k+1} commute.

If C is pseudo-Anosov, then $\beta_k^{-1}\beta_k^{\beta_{k+1}}$ has the dilatation 1 and zero exponent sum, hence $\beta_k^{-1}\beta_k^{\beta_{k+1}} = 1$.

Finally, if C is reducible, then β_{k+1} and C preserve the same essential submanifold because β_k and C preserve the same essential submanifold. In B_3, this implies that β_{k+1} also belongs to $Z(C)$. Thus, β_k and β_{k+1} commute.

For each $i < k < j$, there exists a braid $\alpha \in B_k \times B_{n-k} \subset B_n$ such that $\mathcal{S} \cdot \alpha = (\beta_1', \ldots, \beta_{k-1}', \beta_i, \beta_j, \ldots, \beta_l')$. So from the above argument, β_i commutes with β_j. Therefore all entries of $\mathcal{S}$ commute. $\square$

This lemma imposes a strong restriction on the exponent sums (modulo 6) for non-commutative braid systems having finite Hurwitz orbit.

Proposition 2. *There are no irreducible braid systems with degree 3, length ≥ 5 having finite Hurwitz orbit.*

Proof. For a braid system $\mathcal{S} = (\beta_1, \beta_2, \ldots, \beta_l)$, having the length $l \geq 5$, we can always find a sequence of integers $1 \leq i_1 < i_2 < \cdots < i_k \leq l$ ($1 < k \leq l$) such that $e(\beta_{i_1}) + \cdots + e(\beta_{i_k}) \not\equiv \pm 2, 3 \pmod{6}$. By Lemma 4, this implies all entries of $\mathcal{S}$ commute, so $\mathcal{S}$ is reducible. $\square$

This proves the assertion 3 of Theorem 3.

5.4. Degree 3, length 3 braid system. Let $\mathcal{S} = (\beta_1, \beta_2, \beta_3)$ be a length 3, degree 3 irreducible braid system having finite Hurwitz orbit. We denote the full Coxeter element $\beta_1\beta_2\beta_3$ by C. As is described in Section 3, we consider the action of the rank two free group $F = F_2$ generated by $c_1 = \sigma_1^2$ and $c_2 = \sigma_1^{-1}\sigma_2^2\sigma_1$.

To treat degree 3 braid systems, it is convenient to consider the quotient group $B_3' = B_3/\langle \Delta^2 \rangle$ because the centralizer $Z(\beta)$ of a non-trivial element $[\beta] \in B_3'$ is a cyclic group. For $\alpha, \beta \in B_3$, we denote by $\alpha \equiv \beta$ if α and β defines the same elements in B_3'.

5.4.1. Orbit graphs. The Hurwitz orbit $\mathcal{S} \cdot F$ is described by an oriented, labeled graph G, which we call the *orbit graph* of $\mathcal{S}$. The set of vertices of G consists of the set of orbits $\mathcal{S} \cdot F$. Two vertices $\mathcal{S}$ and $\mathcal{S}'$ are connected by an edge oriented from $\mathcal{S}$ to $\mathcal{S}'$ labeled by 1 (resp. 2) if $\mathcal{S} \cdot c_1 = \mathcal{S}'$ (resp. $\mathcal{S} \cdot c_2 = \mathcal{S}'$). We will classify the orbit graphs of irreducible braid systems of the degree 3 and the length 3.
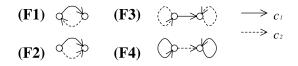
(F1)   **(F3)** $\longrightarrow c_1$

                                                $\dashrightarrow c_2$

(F2)   **(F4)**

Fig. 3. Forbidden graphs.

A *simple vertex* of G is defined as a vertex $\mathcal{S}$ such that $\mathcal{S} \cdot c_i = \mathcal{S}$ holds for some $i = 1, 2$. An *i-path* is an edge path of G having the same label i ($i = 1, 2$). An *alternate path* is an edge-path whose labels alternate. We call a closed i-path of length 3 a *triangle*. A triangle is *special* if all vertices of the triangle are non-simple.

First of all, we study the fundamental properties of orbit graphs.

Lemma 5. *Let $\mathcal{S} = (\beta_1, \beta_2, \beta_3)$ be an irreducible braid system having finite Hurwitz orbit. Then the orbit graph G of $\mathcal{S}$ has the following properties.*
1. *Every closed i-path in G has the length at most* 3, *and the length* 2 *closed i-path and length* 3 *closed i-path does not occur simultaneously.*
2. *Every alternate path of length* 12 *must be a loop.*
3. *There exist no subgraphs of the form* (F1)–(F4).

Proof.   The assertion 1 follows from Proposition 1, and the assertion 2 follows from the assertion 3 of Theorem 4. If there exists a subgraph of the form (F1), then there exists a vertex $\mathcal{S}' = (\beta_1, \beta_2, \beta_3)$ such that $\mathcal{S}' \cdot (c_1 c_2) = \mathcal{S}'$ holds. However this implies β_1, β_2 and β_3 commute, hence it contradicts the assumption that $\mathcal{S}'$ is irreducible. The non-existence of the other subgraphs (F2), (F3) and (F4) are proved by the similar way. $\square$

We remark that the orbit graph G has a closed 1-path of the length 2 (resp. of the length 3) only if $e(\beta_1) + e(\beta_2) \equiv 3 \pmod 6$ (resp. $e(\beta_1) + e(\beta_2) \equiv \pm 2 \pmod 6$). Similarly, G has a closed 2-path of the length 2 (resp. of the length 3) only if $e(\beta_1) + e(\beta_3) \equiv 3 \pmod 6$ (resp. $e(\beta_1) + e(\beta_3) \equiv \pm 2 \pmod 6$).

To extract further restrictions of the orbit graph, we consider the exponent sums. For an irreducible braid system having finite Hurwitz orbit, from Lemma 4, all possibilities of the exponent sum modulo 6 are the following.

$$(e(\beta_1), e(\beta_2), e(\beta_3)) \equiv \begin{cases} (\pm 2, \pm 1, \pm 1) & \cdots (a) \\ (\pm 1, \pm 2, \pm 1) & \cdots (b) \\ (\pm 1, \pm 1, \pm 2) & \cdots (c) \\ (0, \pm 2, \pm 2), (\pm 2, 0, \pm 2), (\pm 2, \pm 2, 0), (\pm 1, \pm 1, \pm 1) & \cdots (d) \end{cases}$$

We call a braid system whose exponent sum is a pattern (a) a $(2, 2)$-*periodic system*. Similarly, we call a braid system whose exponent sum is a pattern (b), (c) and (d), $(2, 3)$-*periodic system*, $(3, 2)$-*periodic system*, and $(3, 3)$-*periodic system* respectively. Now we study each case separately.
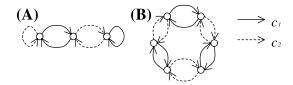
Fig. 4. Orbit graphs of (2, 2)-periodic systems.

5.4.2. (2, 2)-periodic systems.

Lemma 6. *Let $\mathcal{S}$ be a (2, 2)-periodic system having finite Hurwitz orbit. Then $\mathcal{S}' \cdot (c_1 c_2)^3 = \mathcal{S}'$ holds for all $\mathcal{S}' \in \mathcal{S} \cdot F$. That is, every alternate path of length 6 must be a loop.*

Proof. Let $\mathcal{S}' = (\beta_1, \beta_2, \beta_3)$. Then its Coxeter element is a periodic braid with period 3 and $\beta_2, \beta_3 \in Z((\beta_2 \beta_3)^3)$. Therefore by Lemma 1 $\mathcal{S} \cdot (c_1 c_2)^3 = \mathcal{S}$. □

Proposition 3. *For a (2, 2)-periodic system $\mathcal{S}$ having finite Hurwitz orbit, the orbit graph G is either (A) or (B) in Fig. 4. Both (A) and (B) are realized as the orbit graph of a braid system.*
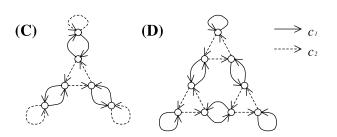
Proof. The orbit graphs of (2,2)-periodic systems have no triangles. By Lemma 5 and 6, if there are simple vertices in G, we obtain the graph (A). Similarly, if there are no simple vertices in G, then by Lemma 5 and 6, we obtain the graph (B). The graph (A) appears as the orbit graph of the braid system $(\sigma_1^2, \sigma_1, \sigma_2)$, and the graph (B) appears as the orbit graph of the braid system $(\sigma_1 \sigma_2, \sigma_1, \sigma_2)$. □

5.4.3. (2, 3)- and (3, 2)- periodic systems.

Next we consider (2, 3)- and (3, 2)-periodic systems. For simplicity, we consider (2, 3)-periodic systems. The orbit graphs of (3, 2)-periodic systems are the same except that the role of c_1 and c_2 are interchanged.

Lemma 7. *Let $\mathcal{S} = (\beta_1, \beta_2, \beta_3)$ be an irreducible (2, 3)-periodic system having finite Hurwitz orbit. Then*
1. $\mathcal{S} \cdot (c_1 c_2)^2 \neq \mathcal{S}$.
2. $\mathcal{S} \cdot (c_1 c_2)^2$ *is a simple vertex if and only if $\mathcal{S}$ is a simple vertex.*

Proof. Since $\mathcal{S}'$ is a (2,3)-periodic system, its Coxeter element C is periodic with period 3 and $\beta_2, \beta_3 \in Z((\beta_2 \beta_3)^2)$. Thus, $\mathcal{S} \cdot (c_1 c_2)^2 = (\beta_1^{C^2}, \beta_2^{C^2}, \beta_3^{C^2})$. So $\mathcal{S} \cdot (c_1 c_2)^2$ is a simple vertex if and only if $\mathcal{S}$ is a simple vertex. If $\mathcal{S} = \mathcal{S} \cdot (c_1 c_2)^2$, then β_1, β_2 and β_3 commute, hence it contradicts the assumption that $\mathcal{S}$ is irreducible. □

Fig. 5. Orbit graphs of (2, 3)-periodic systems.

Proposition 4. *If $\mathcal{S}$ is a* (2, 3)*-periodic system having finite Hurwitz orbit, then the orbit graph G is either* (C) *or* (D) *in* Fig. 5*. Both* (C) *and* (D) *are realized as an orbit graph.*

Proof.   First we consider the case that G has a special triangle. Let $(\beta_1, \beta_2, \beta_3)$ be a vertex of a special triangle. Then, $\beta_2\beta_1$, $\beta_2^{(\beta_1^{-1}\beta_3^{-1})}\beta_1$ and $\beta_2^{(\beta_1^{-1}\beta_3^{-1})^2}\beta_1$ are periodic. From Lemma 3, this implies that either β_2 or β_1 belongs to $Z(\beta_3\beta_1)$. Since β_1 and β_3 do not commute, we conclude that β_2 belongs to $Z(\beta_3\beta_1)$.

Since $\beta_3\beta_1$ is a periodic braid with period 3, by taking an conjugation of the braid system, we may assume that $\beta_2 \equiv \beta_3\beta_1 \equiv \delta^{\pm 1}$. Then the orbit graph of the braid system $(\beta_1, \delta^{\pm 1}, \delta^{\pm 1}\beta_1^{-1})$ is the graph (C). The graph (C) is realized as the orbit graph of the braid system $(\sigma_2, \sigma_1\sigma_2, \sigma_1)$.

Next we assume that G has no special triangles. Then by Lemma 5 and 7, the graph must be the form (D). The graph (D) is realized as the orbit graph of the braid system $(\sigma_1, \sigma_1^2, \sigma_2)$.   □

5.4.4.   (3, 3)-periodic systems.   Finally, we consider the orbit graph of (3, 3)-periodic systems.

Lemma 8. *Let $\mathcal{S}=(\beta_1, \beta_2, \beta_3)$ be a* (3,3)*-periodic system having finite Hurwitz orbit.*
1. $(e(\beta_1), e(\beta_2), e(\beta_3)) \equiv (\pm 1, \pm 1, \pm 1)$ (mod 6)*.*
2. $\mathcal{S} \cdot (c_1c_2)^3$ *is a simple vertex if and only if $\mathcal{S}$ is a simple vertex.*

Proof.   Assume that the exponent sum satisfies

$$(e(\beta_1), e(\beta_2), e(\beta_3)) \equiv (0, \pm 2, \pm 2), (\pm 2, 0, \pm 2), (\pm 2, \pm 2, 0) \quad (\text{mod } 6).$$

Then, the Coxeter element C of $\mathcal{S}$ is periodic with period 3, and $\beta_2, \beta_3 \in Z((\beta_2\beta_3)^3)$. So by Lemma 1, $\mathcal{S} \cdot (c_1c_2)^3 = \mathcal{S}$ holds.

First of all, we show that the orbit graphs of such (3, 3)-periodic systems have no special triangles. Assume that there exists a special triangle labeled by 2. Let $\mathcal{S} = (\beta_1, \beta_2, \beta_3)$ be a vertex of a special triangle. Then as in the proof of Proposition 4, we
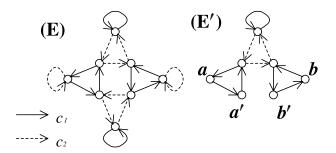
Fig. 6. Orbit graphs of $(3,3)$-periodic systems.

may assume that $\mathcal{S} = (\beta_1, \delta^{\pm 1}, \delta^{\pm 1}\beta_1^{-1})$ by taking a conjugation of the braid system. Let T be a triangle formed by the vertices $\mathcal{S}$, $\mathcal{S} \cdot c_1$ and $\mathcal{S} \cdot c_1^2$. Suppose that T is special. Then, $\beta_1(\delta^{\pm 1}\beta_1^{-1})$, $\beta_1^{(\beta_1\delta^{\pm 1})}(\delta^{\pm 1}\beta_1^{-1})$ and $\beta_1^{(\beta_1\delta^{\pm 1})^2}(\delta^{\pm 1}\beta_1^{-1})$ are periodic, so by Lemma 3, β_1 or $\delta^{\pm 1}\beta_1^{-1}$ commutes with $\beta_1\delta^{\pm 1}$. This implies β and δ commute. If T is non-special, then $\beta_1^{(\beta_1\delta^{\pm 1})}$ or $\beta_1^{(\beta_1\delta^{\pm 1})^2}$ commutes with $\delta^{\pm 1}\beta_1^{-1}$. Using the fact that $\beta_1\delta^{\pm 1}$ is a periodic braid with period 3, in either case, we obtain that β_1 commutes with δ. This contradicts the assumption that $\mathcal{S}$ is irreducible. The non-existence of special triangles labeled by 1 is similar.

Then it is impossible to construct an orbit graph G which satisfies all required properties

1. G satisfies the condition in Lemma 5. In particular, all closed i-paths in G have the length 3 or 1 ($i = 1, 2$).
2. G has no special triangles.
3. $\mathcal{S} \cdot (c_1 c_2)^3 = \mathcal{S}$ holds for all vertex $\mathcal{S}$ in G.

So irreducible braid systems having such exponent sums cannot have finite Hurwitz orbit. This proves 1.

Now, the Coxeter element C of $\mathcal{S}$ is periodic with period 2 and $e(\beta_2) + e(\beta_3) \equiv \pm 2$ (mod 6). Thus, $\mathcal{S} \cdot (c_1 c_2)^3 = (\beta_1^C, \beta_2^C, \beta_3^C)$ holds. So $\mathcal{S}$ is a simple vertex if and only if $\mathcal{S} \cdot (c_1 c_2)^3$ is a simple vertex. $\square$

Proposition 5. *If $\mathcal{S}$ is a $(3,3)$-periodic system having finite Hurwitz orbit, then the orbit graph G is the form* (E) *in* Fig. 6. *The graph* (E) *is realized as an orbit graph.*

Proof. If there exists a special triangle in the orbit graph, then as in the proof Lemma 8, either β_2 or β_3 is periodic. However, we have shown that in $e(\beta_2) \equiv e(\beta_3) \equiv \pm 1$ (mod 6) in Lemma 8, this is impossible. Thus, the orbit graph has no special triangles.

So by Lemma 5, the orbit graph must have a subgraph of the form (E') in Fig. 6. Non-existence of special triangles implies that either a or a' (resp. b or b') is a simple vertex. If a and b are simple, then we obtain a graph (E). The graph (E) is realized as

the orbit graph of the braid system $(\sigma_1, \sigma_2, \sigma_1)$. The other cases cannot occur, because it violates the condition in the assertion 2 of Lemma 8.   □

Now we have classified all orbit graphs of degree 3, length 3 irreducible braid systems. Summarizing, we obtain the following result.

Proposition 6. *Let $\mathcal{S}$ be an irreducible braid system of degree* 3, *length* 3 *which has finite Hurwitz orbit. Then* $\#(\mathcal{S} \cdot B_3) \leq 162$.

Proof. From our list of the orbit graphs, $\#(\mathcal{S} \cdot F) \leq 9$ holds for all irreducible braid system of degree 3, length 3 having finite Hurwitz orbit. Since $P_3 = F_1 \ltimes F_2 = \langle a_{2,3} \rangle \ltimes F$, $\#(\mathcal{S} \cdot P_3) \leq \#(\mathcal{S} \cdot \langle a_{2,3} \rangle) \cdot 9$ holds. Now $\#(\mathcal{S} \cdot \langle a_{2,3} \rangle) \leq 3$, so we conclude that $\#(\mathcal{S} \cdot B_3) \leq [B_3 : P_3] \cdot 3 \cdot 9 = 162$.   □

5.5. Completion of proof. Now we complete the proof of Theorem 3. The last step is to study length 4 braid systems.

Proposition 7. *Let $\mathcal{S}$ be a degree* 3, *length* 4 *irreducible braid system having finite Hurwitz orbit. Then* $\#(\mathcal{S} \cdot B_4) \leq 648$.

Proof. From Lemma 4, the possibility of the exponent sum modulo 6 for irreducible length 4 braid systems having finite Hurwitz orbit is $(\varepsilon, \varepsilon, \varepsilon, \varepsilon)$, $\varepsilon = \pm 1$. Let $\mathcal{S} = (\beta_1, \beta_2, \beta_3, \beta_4)$. We may assume that β_3 and β_4 do not commute hence $\beta_3 \beta_4$ is periodic. Moreover, since $(\beta_1, \beta_2, \beta_3)$ is a $(3,3)$-periodic system, so by the orbit graph (E) in Fig. 6, we may also assume that β_1 and β_2 commute. In particular, $\beta_1 \beta_2$ is non-periodic, and $\beta_1 \beta_2$ does not commute with $\beta_3 \beta_4$. Assume that the all partial Coxeter elements $C_{\{1,2,3\}}$ of $\mathcal{S}$, $\mathcal{S} \cdot \sigma_3^2$ and $\mathcal{S} \cdot \sigma_3^4$ are periodic. That is, $\beta_1 \beta_2 \beta_3$, $\beta_1 \beta_2 \beta_3^{(\beta_3 \beta_4)}$ and $\beta_1 \beta_2 \beta_3^{(\beta_3 \beta_4)^2}$ are periodic. Then, by Lemma 3, either β_3 or $\beta_1 \beta_2$ commute with $\beta_3 \beta_4$, which is a contradiction. Hence we may assume that $\beta_1 \beta_2 \beta_3$ is non-periodic, so β_1, β_2 and β_3 commute.

We consider in B_3'. Let us put $\beta_1 \equiv \beta^p$, $\beta_2 \equiv \beta^q$, $\beta_3 \equiv \beta^r$ and $\beta_4 \equiv \gamma$. Since $\mathcal{S}$ is irreducible, γ does not commute with β. Therefore, all of $\beta_2 \gamma$, $\beta_3 \gamma$, $\beta_2 \beta_3 \gamma$ are periodic. Then the exponent sum argument shows that their periods are $3, 3, 2$ respectively. Thus we have an equality $(\beta^q \gamma)^3 \equiv (\beta^r \gamma)^3 \equiv (\beta^{q+r} \gamma)^2 \equiv 1$. From this equality, we obtain $\beta^r \gamma \beta^r \equiv \gamma \beta^q \gamma$. Similar argument for β_1 and β_3 provide an equality $\beta^r \gamma \beta^r \equiv \gamma \beta^p \gamma$, hence we conclude $\beta^p \equiv \beta^q$. Similarly, by considering β_1, β_2, we obtain $\beta^q \equiv \beta^r$. Hence the equality $\beta_1 \equiv \beta_2 \equiv \beta_3$ holds.

Let G be a subgroup of B_3' generated by β^p and γ. Then the map $\tau : B_3' \to G$ defined by $\tau([\sigma_1]) = \beta^p$ and $\tau([\sigma_2]) = \gamma$ is a surjective homomorphism. Now the map τ induces a surjection between Hurwitz orbits $(\sigma_1, \sigma_1, \sigma_1, \sigma_2) \cdot P_4$ and $\mathcal{S} \cdot P_4$. Thus, we conclude that $\#(\mathcal{S} \cdot B_4) \leq 4! \#(\sigma_1, \sigma_1, \sigma_1, \sigma_2) \cdot P_4$. A direct calculation shows $\#(\sigma_1, \sigma_1, \sigma_1, \sigma_2) \cdot P_4 = 27$, hence we conclude $\#(\mathcal{S} \cdot B_4) \leq 648$.   □

REMARK 1. We remark that the upper bound 648 is achieved by the braid system $(\sigma_1, \Delta^2\sigma_1, \Delta^4\sigma_1, \Delta^6\sigma_2)$. The above proof implies that the orbit graph of an irreducible braid system of degree 3, length 4 with respect to the Hurwitz P_4-action is obtained as a quotient of the orbit graph of $(\sigma_1, \sigma_1, \sigma_1, \sigma_2)$. Since the possibilities of such graphs are finite, we can classify the whole patterns of the orbit graphs for P_4-action. This implies, theoretically we can list all the possibilities of the orbit graphs of finite Hurwitz orbits.

Proof of the assertion 2 of Theorem 3. The assertion 1 and 3 are already proved. Since we have already studied the irreducible case, we only need to consider the reducible case. Let $\mathcal{S} = (\beta_1, \ldots, \beta_n)$ be a reducible system having finite Hurwitz orbit and $I \sqcup J = \{1, 2, \ldots, n\}$ be the partition appeared in the definition of a reducible system.

Assume that β_i and $\beta_{i'}$ do not commute. Then we may assume that $i, i' \in I$. Now for $j \in J$, β_j commutes with both β_i and $\beta_{i'}$. Now β_i and $\beta_{i'}$ does not commute implies that $Z(\beta_i) \cap Z(\beta_{i'}) = Z(B_3)$, so $\beta_j \in Z(B_3)$ for all $j \in J$. Thus, we have one of

1. All the β_i commute with each other.
2. There exist $i_1 < i_2 < \cdots < i_k$ $(2 \leq k \leq 4)$ such that the braid system $(\beta_{i_1}, \beta_{i_2}, \ldots, \beta_{i_k})$ is irreducible braid system having finite Hurwitz orbit, and $\beta_j \in Z(B_3)$ for $j \notin \{i_1, i_2, \ldots, i_k\}$.

For the first case, we get $\#(\mathcal{S} \cdot B_n) \leq n!$. In the second case, we use the inequality of the size of finite Hurwitz orbit for reducible systems we mentioned at Section 1. By Proposition 6 and 7, we get $\#(\mathcal{S} \cdot B_n) \leq 27 \cdot n!$. □

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