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ORBIFOLD LENS SPACES THAT ARE ISOSPECTRAL BUT NOT ISOMETRIC

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Abstract

We answer Mark Kac's famous question [13], "can one hear the shape of a drum?" in the negative for orbifolds that are spherical space forms. This is done by extending the techniques developed by A. Ikeda on lens spaces to the orbifold setting. Several results are proved to show that with certain restrictions on the dimensionalities of orbifold lens spaces we can obtain infinitely many pairs of isospectral non-isometric lens spaces. These results are then generalized to show that for any dimension greater than 8 we can have pairs of isospectral non-isometric orbifold lens spaces.

Contents

1. Introduction

Given a closed Riemannian manifold (M, g), the eigenvalue spectrum of the associated Laplace Beltrami operator will be referred to as the spectrum of (M, g). The inverse spectral problem asks the extent to which the spectrum encodes the geometry of (M, g). While various geometric invariants such as dimension, volume and total scalar curvature are spectrally determined, numerous examples of isospectral Riemannian manifolds, i.e., manifolds with the same spectrum, show that the spectrum does not fully encode the geometry. Not surprisingly, the earliest examples of isospectral manifolds were manifolds of constant curvature including flat tori ([14]), hyperbolic manifolds ([22]), and spherical space forms ([10], [11] and [8]). In particular, lens spaces

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are quotients of round spheres by cyclic groups of orthogonal transformations that act freely on the sphere. Lens spaces have provided a rich source of isospectral manifolds with interesting properties. In addition to the work of Ikeda cited above, see the recent results of Gornet and McGowan [9].

In this paper we generalize this theme to the category of Riemannian orbifolds.

A smooth *orbifold* is a topological space that is locally modelled on an orbit space of $\mathbf{R}^n$ under the action of a finite group of diffeomorphisms. *Riemannian* orbifolds are spaces that are locally modelled on quotients of Riemannian manifolds by finite groups of isometries. Orbifolds have wide applicability, for example, in the study of 3-manifolds and in string theory.

The tools of spectral geometry can be transferred to the setting of Riemannian orbifolds by using their well-behaved local structure (see [4], [20] and [21]). As in the manifold setting, the spectrum of the Laplace operator of a compact Riemannian orbifold is a sequence $0 \le \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \uparrow \infty$ where each eigenvalue is repeated according to its finite multiplicity. We say that two orbifolds are isospectral if their Laplace spectra agree.

The literature on inverse spectral problems on orbifolds is less developed than that for manifolds. Examples of isospectral orbifolds include pairs with boundary ([1] and [3]); isospectral flat 2-orbifolds ([6]); arbitrarily large finite families of isospectral orbifolds ([19]); isospectral orbifolds with different maximal isotropy orders ([16]); and isospectral deformation of metrics on an orbifold quotient of a nilmanifold ([15]).

In this article, we study the spectrum of orbifold lens spaces, i.e., quotients of round spheres by cyclic groups of orthogonal transformations that have fixed points on the sphere. Generalizing the work of Ikeda (see [10], [11] and [12]) we construct the generating function for the spectrum and systematically construct isospectral orbifold lens spaces. Section two introduces the orbifold lens spaces and their generating functions. In section 3, we will develop the proofs of our main theorems. We will first prove:

Theorem 3.1.6. (i) *Let $p \geq 5$ (alt. $p \geq 3$) be an odd prime and let $m \geq 2$ (alt. $m \geq 3$) be any positive integer. Let $q = p^m$. Then there exist at least two $(q-6)$-dimensional orbifold lens spaces with fundamental groups of order p^m which are isospectral but not isometric.*
(ii) *Let p_1, p_2 be odd primes such that $q = p_1 \cdot p_2 \geq 3$. Then there exist at least two $(q-6)$-dimensional orbifold lens spaces with fundamental groups of order $p_1 \cdot p_2$ which are isospectral but not isometric.*
(iii) *Let $q = 2^m$ where $m \geq 6$ is any positive integer. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces with fundamental groups of order 2^m which are isospectral but not isometric.*
(iv) *Let $q = 2p$, where p is an odd prime and $p \geq 7$. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces with fundamental groups of order $2p$ which*

are isospectral but not isometric.

To prove these results we proceed as follows:
(1) Depending on the number of p^i (alt. p_1, p_2) divisors of $q = p^m$ (alt. $q = p_1 \cdot p_2$), we reformulate the generating function in terms of rational polynomial functions.
(2) Then we classify the number of generating functions that we will get by imposing different conditions on the domain values of these polynomial functions.
(3) We prove sufficiency conditions on the number of generating functions that would guarantee isospectrality for non-isometric orbifold lens spaces.

The techniques used to prove these results parallel similar techniques from the manifold lens space setting used in [10].

Generalizing this technique, we will get our second set of main results:

Theorem 3.2.5. *Let* $W \in \{0, 1, 2, \dots\}$.
(i) *Let* $P \geq 5$ (*alt.* $P \geq 3$) *be any odd prime and let* $m \geq 2$ (*alt.* $m \geq 3$) *be any positive integer. Let* $q = P^m$. *Then there exist at least two* $(q+W-6)$*-dimensional orbifold lens spaces with fundamental groups of order* P^m *which are isospectral but not isometric.*
(ii) *Let* P_1, P_2 *be two odd primes such that* $q = P_1 \cdot P_2 \geq 33$. *Then there exist at least two* $(q + W - 6)$*-dimensional orbifold lens spaces with fundamental groups of order* $P_1 \cdot P_2$ *which are isospectral but not isometric.*
(iii) *Let* $q = 2^m$ *where* $m \geq 6$ *is any positive integer. Then there exist at least two* $(q + W - 5)$*-dimensional orbifold lens spaces with fundamental groups of order* 2^m *which are isospectral but not isometric.*
(iv) *Let* $q = 2P$, *where* $P \geq 7$ *is an odd prime. Then there exist at least two* $(q + W - 5)$*-dimensional orbifold lens spaces with fundamental groups of order* $2P$ *which are isospectral but not isometric.*

A consequence of this theorem is that for every integer $x \geq 9$, we can find a pair of isospectral non-isometric orbifold lens spaces of dimension x.

In Section 4 we look at specific examples that show what the different generating functions would look like and the types of orbifold lens spaces that correspond to each generating function.

2. Orbifold lens spaces

In this section we will generalize the idea of manifold lens spaces to orbifold lens spaces. Manifold lens spaces are spherical space forms where the n-dimensional sphere S^n of constant curvature 1 is acted upon by a cyclic group of fixed point free isometries on S^n. We will generalize this notion to orbifolds by allowing the cyclic group of isometries to have fixed points. For a more general definition of orbifolds see Satake [17] and Scott [18]. For details of spectral geometry on orbifolds, see Stanhope [20] and E. Dryden, C. Gordon, S. Greenwald and D. Webb in [5]).

To obtain our main results we will focus on a special subfamily of lens spaces. Our technique will parallel Ikeda's technique as developed in [10].

2.1. Preliminaries. Let q be a positive integer that is not prime. Set

$$q_0 = \begin{cases} \dfrac{q-1}{2} & \text{if } q \text{ is odd,} \\ \dfrac{q}{2} & \text{if } q \text{ is even.} \end{cases}$$

Throughout this article we assume that $q_0 \geq 4$ and that q is not prime.

For any positive integer n with $2 \leq n \leq q_0 - 2$, we denote by $\tilde{I}(q, n)$ the set of n-tuples $(p_1, \dots, p_n)$ of integers. We define a subset $\tilde{I}_0(q, n)$ of $\tilde{I}(q, n)$ as follows:

$$\tilde{I}_0(q, n) = \{(p_1, \dots, p_n) \in \tilde{I}(q, n) \mid p_i \not\equiv \pm p_j \pmod{q},\ 1 \leq i < j \leq n,$$
$$g.c.d.(p_1, \dots, p_n, q) = 1\}.$$

We introduce an equivalence relation in $\tilde{I}(q,n)$ as follows: $(p_1,\dots, p_n)$ is equivalent to $(s_1, \dots, s_n)$ if and only if there is a number l prime to q and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \dots, p_n)$ is a permutation of $(e_1 l s_1, \dots, e_n l s_n) \pmod{q}$. This equivalence relation also defines an equivalence relation on $\tilde{I}_0(q, n)$.

We set $I(q, n) = \tilde{I}(q, n)/\sim$ and $I_0(q, n) = \tilde{I}_0(q, n)/\sim$.

Let $k = q_0 - n$. We define a map w of $I_0(q, n)$ into $I_0(q, k)$ as follows:

For any element $(p_1,\dots, p_n) \in \tilde{I}_0(q,n)$, we choose an element $(q_1,\dots,q_k) \in \tilde{I}_0(q,k)$ such that the set of integers

$$\{p_1, -p_1, \dots, p_n, -p_n, q_1, -q_1, \dots, q_k, -q_k\}$$

forms a complete set of incongruent residues $\pmod{q}$. Then we define

$$w([p_1, \dots, p_n]) = [q_1, \dots, q_k].$$

It is easy to see that this map is a well defined bijection.

The following proposition is similar to a result in [10]:

Proposition 2.1.1. *Let $I_0(q, n)$ be as above. Then,*

$$|I_0(q, n)| \geq \frac{1}{q_0} \binom{q_0}{n},$$

where $\binom{q_0}{n} = 1$ if $q_0 n = 0$, and

$$\binom{q_0}{n} = \frac{q_0!}{n!\,(q_0 - n)!} \quad \textit{otherwise.}$$

Proof. Let $I_0(q,n)$ be as above. Consider a subset $\tilde{I}_0'(q,n)$ of $\tilde{I}_0(q,n)$ as follows:

$$\tilde{I}_0'(q,n) = \{(p_1,\dots,p_n) \in \tilde{I}_0(q,n) \mid \text{at least one of the } p_i \text{ is co-prime to } q\}.$$

It is easy to see that the equivalence relation on $\tilde{I}_0(q,n)$ induces an equivalence relation on $\tilde{I}_0'(q,n)$. Since we eliminate classes where none of the p_i's is co-prime to q, we get

$$|I_0(q,n)| \geq |I_0'(q,n)|,$$

where $I_0'(q,n) = \tilde{I}_0'(q,n)/\sim$. Now consider a subset $\tilde{I}_0''(q,n)$ of $\tilde{I}_0'(q,n)$ as follows:

$$\tilde{I}_0''(q,n) = \{(p_1,\dots,p_n) \in \tilde{I}_0'(q,n) \mid 1 = p_1 < \cdots < p_n \leq q_0\}.$$

Then it is easy to see that any element of $\tilde{I}_0'(q,n)$ has an equivalent element in $\tilde{I}_0''(q,n)$. On the other hand, for any equivalence class in $I_0'(q,n)$, the number of elements in $\tilde{I}_0''(q,n)$ which belong to that class is at most n. Hence we have:

$$|I_0(q,n)| \geq |I_0'(q,n)| \geq \frac{1}{n}|\tilde{I}_0''(q,n)| = \frac{1}{n}\binom{q_0-1}{n-1} = \frac{1}{q_0}\binom{q_0}{n}.$$

This proves the proposition. $\square$

Lemma 2.1.2. *Let $q = p^m$ or $q = p_1 \cdot p_2$, where p, p_1, p_2 are primes. Let D be the set of all non-zero integers mod q that are not co-prime to q. Then $|D|$ is even if q is odd and $|D|$ is odd if q is even.*

Proof. For $q = p^m$.

If q is odd, then p is an odd prime. $q/p = p^{m-1}$ which is an odd number. Therefore the number of elements in D, $(p^{m-1}-1)$ is even.

If q is even, then $p = 2$. $q/p = 2^{m-1}$ is even. So the number of elements in D, $(2^{m-1}-1)$, is odd.

For $q = p_1 \cdot p_2$ $(p_1 \neq p_2)$.

If q is odd, then both p_1 and p_2 are odd primes. The number of elements in D is $(q/p_1 + q/p_2 - 2) = (p_2 + p_1 - 2)$ which is even since $p_1 + p_2$ is even.

If q is even, then one of the p_i's is 2 and the other is an odd prime. Assume $p_1 = 2$. So, the number of elements in D is $(q/p_1 + q/p_2 - 2) = (p_2 + p_1 - 2) = (p_2 + 2 - 2) = p_2$, which is odd.

This proves the lemma. $\square$

We will say that $|D| = 2r$ if $|D|$ is even; and $|D| = 2r-1$ if $|D|$ is odd, where r is some positive integer. It is easy to see that if $|D|$ is even, then exactly r members of D are less than q_0. If $|D|$ is odd, then $r-1$ members of D are strictly less than

q_0 and one member of D is equal to q_0 (recall that for even q, we set $q_0 = q/2$, and for odd q, we set $q_0 = (q-1)/2$).

With these results we now obtain a better lower bound for $|I_0(q, n)|$.

Proposition 2.1.3. *Let $I_0(q, n)$, $I_0'(q, n)$, $\tilde{I}_0'(q, n)$ and $\tilde{I}_0''(q, n)$ be as in* Proposition 2.2.1. *Let $k = q_0 - n$. Then*

$$|I_0(q, n)| \geq \sum_{t=u}^{r} \frac{1}{n-t} \binom{q_0 - 1 - r}{n - 1 - t} \binom{r}{t},$$

where $u = r - k$ if $r > k$ and $u = 0$ if $r \leq k$, and r is as defined above.

Proof. The number of ways in which we can assign values to the p_i's in $(1 = p_1, p_2, \ldots, p_n) \in \tilde{I}_0''(q, n)$ such that t of the p_i's are *not* co-prime to q is

$$\binom{q_0 - 1 - r}{n - 1 - t} \binom{r}{t}.$$

On the other hand for any equivalence class in $I_0'(q, n)$ with t of the p_i's not being co-prime to q, the number of elements which belong to that class is at most $n - t$. So the number of such possible classes is at least

$$\frac{1}{n-t} \binom{q_0 - 1 - r}{n - 1 - t} \binom{r}{t}.$$

Now if $r > k$, this would mean that $n > q_0 - r$, or $n - 1 > q_0 - 1 - r$. This means that t cannot take any values less than $r - k$, since that would mean that we are choosing $(n - 1 - t)$, a number larger than $(q_0 - 1 - r)$ from $q_0 - 1 - r$ and that is not possible. So, the smallest value for t in this case can be $r - k$.

On the other hand, if $r \leq k$, then $n \leq q_0 - r$, or $n - 1 \leq q_0 - 1 - r$. This means that it is possible for us to choose n-tuples in $\tilde{I}_0''(q, n)$ with all values being co-prime to q. Thus, the smallest value for t would be 0 in this case.

It is obvious that the maximum value t can take is r since $(1, p_2, \ldots, p_n)$ cannot have more than r values that are not co-prime to q. Now, adding up all the degrees for different values of t we get

$$|I_0(q, n)| \geq |I_0'(q, n)| \geq \sum_{t=u}^{r} \frac{1}{n-t} \binom{q_0 - 1 - r}{n - 1 - t} \binom{r}{t},$$

where $u = 0$ if $r \leq k$ and $u = r - k$ if $r > k$.

This proves the proposition. $\square$

DEFINITION 2.1.4. (i) Let q be a positive integer and γ a primitive q-th root of 1. We denote by $\mathbf{Q}(\gamma)$ the q-th cyclotomic field over the rational number field $\mathbf{Q}$ and denote by $\Phi_q(z)$ the q-th cyclotomic polynomial

$$\Phi_q(z) = \sum_{t=0}^{q-1} z^t.$$

Let A be the set of residues mod q that are co-prime to q. We define a map $\psi_{q,k}$ of $I_0(q, k)$ into $\mathbf{Q}(\gamma)[z]$ as follows:

For any equivalence class in $I_0(q, k)$, we take an element $(q_1, \dots, q_k)$ of $\tilde{I}_0(q, k)$ which belongs to that class. We define

$$\psi_{q,k}([q_1, \dots, q_k])(z) = \sum_{l \in A} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

This polynomial in $\mathbf{Q}(\gamma)[z]$ is independent of the choice of elements which belong to the class $[q_1, \dots, q_k]$. Therefore, the map is well-defined.
(ii) Given $q = p^m$, we define

$$B_j = \{x \pmod q) \in \mathbf{Z}^+ : \ p^j \mid x, \ \ p^{j+1} \nmid x\}.$$

We define the maps $\alpha^{(j)}_{q,k}$ of $I_0(q, k)$ into $\mathbf{Q}(\gamma)[z]$ as follows:

$$\alpha^{(j)}_{q,k}([q_1, \dots, q_k])(z) = \sum_{l \in B_j} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

(iii) Now assume $q = p_1 \cdot p_2$. We define the following sets of numbers that are not co-prime to q.

$$B = \{x p_1 \mid x = 1, 2, \dots, (p_2 - 1)\}$$

and

$$C = \{x p_2 \mid x = 1, 2, \dots, (p_1 - 1)\}.$$

We define maps $\alpha_{q,k}$ and $\beta_{q,k}$ as follows:

$$\alpha_{q,k}([q_1, \dots, q_k])(z) = \sum_{l \in B} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l})$$

and

$$\beta_{q,k}([q_1, \dots, q_k])(z) = \sum_{l \in C} \prod_{i=1}^{k} (z - \gamma^{q_i l})(z - \gamma^{-q_i l}).$$

Since $(z-\gamma^{q_i l})(z-\gamma^{-q_i l}) = (\gamma^{q_i l}z-1)(\gamma^{-q_i l}z-1)$, the following proposition is easy to see.

Proposition 2.1.5. *If we put*

$$\psi_{q,k}([q_1,\ldots,q_k])(z) = \sum_{i=0}^{2k}(-1)^i a_i z^{2k-i},$$

$$\alpha^{(j)}_{q,k}([q_1,\ldots,q_k])(z) = \sum_{i=0}^{2k}(-1)^i b_{i,j} z^{2k-i},$$

$$\alpha_{q,k}([q_1,\ldots,q_k])(z) = \sum_{i=0}^{2k}(-1)^i b_i z^{2k-i},$$

$$\beta_{q,k}([q_1,\ldots,q_k])(z) = \sum_{i=0}^{2k}(-1)^i c_i z^{2k-i},$$

then we have
(i) $a_i = a_{2k-i}$, $b_{i,j} = b_{(2k-i),j}$, $b_i = b_{2k-i}$ *and* $c_i = c_{2k-i}$,
(ii) $a_0 = |A|$, $b_{0,j} = |B_j|$, $b_0 = |B|$ *and* $c_0 = |C|$.

2.2. Orbifold lens spaces and their generating functions. Let q be a positive integer and $p_1,\ldots,p_n$ be n integers mod q such that $g.c.d.(p_1,\ldots,p_n,q) = 1$. We denote by g the orthogonal matrix given by

$$g = \begin{pmatrix} R(p_1/q) & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & R(p_n/q) \end{pmatrix},$$

where $R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix}$. Then g generates the cyclic subgroup $G = \{g^l\}_{l=1}^{q}$ of order q of the orthogonal group $O(2n)$.

We define a lens space $L(q: p_1,\ldots,p_n)$ as follows:

$$L(q: p_1,\ldots,p_n) = S^{2n-1}/G.$$

$L(q: p_1,\ldots,p_n)$ is a good smooth orbifold with S^{2n-1} as its covering manifold. Let π be the covering projection of S^{2n-1} onto S^{2n-1}/G

$$\pi : S^{2n-1} \to S^{2n-1}/G.$$

Since the round metric of constant curvature one on S^{2n-1} is G-invariant, it induces a Riemannian metric on S^{2n-1}/G.

We emphasize that, in contrast to the classical notion of lens space, we do not require that the integers p_i be co-prime to q, and thus we allow the lens spaces to have singular points. In particular they are good orbifolds. Henceforth, the term "lens space" will refer to this generalized definition.

Proposition 2.2.1. *Let $L = L(q: p_1, \dots, p_n)$ and $L' = L(q: s_1', \dots, s_n')$ be lens spaces. Then L is isometric to L' if and only if there is a number l co-prime with q and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \dots, p_n)$ is a permutation of $(e_1 l s_1, \dots, e_n l s_n)$* (mod q).

Proof. Assume L is isometric to L'. Let ϕ be an isometry between L and L'. Then ϕ lifts to an isometry $\tilde{\phi}$ of S^{2n-1}. In other words, ϕ is an orthogonal transformation that conjugates G and G' where $L = S^{2n-1}/G$ and $L' = S^{2n-1}/G'$.

So $\tilde{\phi}$ takes g, a generator of G, to g'^l, a generator of G'. This means that the eigenvalues of g and g'^l are the same. This means that each p_i is equivalent to some ls_j or $-ls_j$ (mod q). That means $(p_1, \dots, p_n)$ is a permutation of $(e_1 l s_1, \dots, e_n l s_n)$ (mod q), for $e_i \in \{-1, 1\}$, $(i = 1, \dots, n)$.

Conversely, assume that there exists an integer l co-prime with q and numbers $e_i \in \{-1, 1\}$ $(i = 1, \dots, n)$ such that $(p_1, \dots, p_n)$ is a permutation of $(e_1 l s_1, \dots, e_n l s_n)$ (mod q).

Note that the isometry of S^{2n-1} onto S^{2n-1} defined by the map

$$(z_1, \dots, z_i, \dots, z_n) \mapsto (z_{\sigma(0)}, \dots, \bar{z}_{\sigma(i)}, \dots, z_{\sigma(n)}),$$

where σ is a permutation, induces an isometry of $L(q: s_1, \dots, s_n)$ onto $L(q: s_{\sigma(1)}, \dots, s_{\sigma(i)}, \dots, s_{\sigma(n)})$. Since g'^l is also a generator of G', the lens space $L(q: ls_1, \dots, ls_n)$ is identical to $L(q: s_1, \dots, s_n)$.

Now the above isometry induces an isometry of $L(q: s_1, \dots, s_n)$ onto $L(q: e_1 l s_1, \dots, e_n l s_n)$. This means that $L' = L(q: s_1, \dots, s_n)$ is isometric to $L(q: e_1 l s_1, \dots, e_n l s_n)$. But $(e_1 l s_1, \dots, e_n l s_n)$ is simply a permutation of $(p_1, \dots, p_n)$ (mod q). Therefore $L(q: p_1, \dots, p_n)$ is isometric to $L(q: e_1 l s_1, \dots, e_n l s_n)$, which, in turn, is isometric to $L(q: s_1, \dots, s_n)$. This proves the converse. □

For any $f \in C^\infty(L(q: p_1, \dots, p_n))$, we define the Laplacian on the lense space as $\tilde{\Delta}(\pi^* f) = \pi^*(\Delta f)$. We construct the generating function associated with the Laplacian on $L(q: p_1, \dots, p_n)$ analogous to the construction in the manifold case (see [10], [11] and [12]).

Let $\tilde{\Delta}$, Δ and Δ_0 denote the Laplacian of S^{2n-1}, $L(q: p_1, \dots, p_n)$ and $\mathbf{R}^{2n}$, respectively.

DEFINITION 2.2.2. For any non-negative real number λ, we define the *eigenspaces* $\tilde{E}_\lambda$ and E_λ as follows:

$$\tilde{E}_\lambda = \{ f \in C^\infty(S^{2n-1}) \mid \tilde{\Delta} f = \lambda f \},$$
$$E_\lambda = \{ f \in C^\infty(L(q\colon p_1, \dots, p_n)) \mid \Delta f = \lambda f \}.$$

The following lemma follows from the definitions of Δ and smooth function.

Lemma 2.2.3. *Let $(\tilde{E}_\lambda)_G$ be the space of all G-invariant functions of $\tilde{E}_\lambda$. Then* $\dim(E_\lambda) = \dim(\tilde{E})_G$.

Let Δ_0 be the Laplacian on $\mathbf{R}^{2n}$ with respect to the flat Kähler metric. Set $r^2 = \sum_{i=1}^{2n} x_i^2$, where $(x_1, x_2, \dots, x_{2n})$ is the standard coordinate system on $\mathbf{R}^{2n}$.

For $k \geq 0$, let P^k denote the space of complex valued homogeneous polynomials of degree k on $\mathbf{R}^{2n}$. Let H^k be the subspace of P^k consisting of harmonic polynomials on $\mathbf{R}^{2n}$,

$$H^k = \{ f \in P^k \mid \Delta_0 f = 0 \}.$$

Each orthogonal transformation of $\mathbf{R}^{2n}$ induces canonically a linear isomorphism of P^k.

Proposition 2.2.4. *The space H^k is $O(2n)$-invariant, and P^k has the direct sum decomposition: $P^k = H^k \oplus r^2 P^{k-2}$. The injection map $i\colon S^{2n-1} \to \mathbf{R}^{2n}$ induces a linear map $i^*\colon C^\infty(\mathbf{R}^{2n}) \to C^\infty(S^{2n-1})$. We denote $i^*(H^k)$ by $\mathcal{H}^k$.*

Proposition 2.2.5. *$\mathcal{H}^k$ is an eigenspace of $\tilde{\Delta}$ on S^{2n-1} with eigenvalue $k(k+2n-2)$ and $\sum_{k=0}^\infty \mathcal{H}^k$ is dense in $C^\infty(S^{2n-1})$ in the uniform convergence topology. Moreover, $\mathcal{H}^k$ is isomorphic to H^k. That is, $i^*\colon H^k \xrightarrow{\simeq} \mathcal{H}^k$.*

For proofs of these propositions, see [2].

Now by Lemma 2.2.3 and Proposition 2.2.5, we get

Corollary 2.2.6. *Let*

$$L(q\colon p_1, \dots, p_n) = S^{2n-1}/G$$

be a lens space and $\mathcal{H}_G^k$ be the space of all G-invariant functions in $\mathcal{H}^k$. Then

$$\dim E_{k(k+2n-2)} = \dim \mathcal{H}_G^k.$$

Moreover, for any integer k such that $\dim \mathcal{H}_G^k \neq 0$, $k(k+2n-2)$ is an eigenvalue of Δ on $L(q\colon p_1, \dots, p_n)$ with multiplicity equal to $\dim \mathcal{H}_G^k$, and no other eigenvalues appear in the spectrum of Δ.

DEFINITION 2.2.7. The *generating function* $F_q(z\colon p_1,\ldots,p_n)$ associated to the spectrum of the Laplacian on $L(q\colon p_1,\ldots,p_n)$ is the generating function associated to the infinite sequence $\{\dim \mathcal{H}^k_G\}_{k=0}^{\infty}$, i.e.,

$$F_q(z\colon p_1,\ldots,p_n)=\sum_{k=0}^{\infty}(\dim \mathcal{H}^k_G)z^k.$$

By Corollary 2.2.6 we know that the generating function determines the spectrum of $L(q\colon p_1,\ldots,p_n)$. This fact gives us the following proposition:

Proposition 2.2.8. *Let $L=L(q\colon p_1,\ldots,p_n)$ and $L'=L(q'\colon s_1,\ldots,s_n)$ be two lens spaces. Let $F_q(z\colon p_1,\ldots,p_n)$ and $F_{q'}(z\colon s_1,\ldots,s_n)$ be the generating functions associated to the spectrum of L and L', respectively. Then L is isospectral to L' if and only if $F_q(z\colon p_1,\ldots,p_n)=F_{q'}(z\colon s_1,\ldots,s_n)$.*

The following theorem ([10] and [11]) holds true for the orbifold lens spaces as well.

Theorem 2.2.9. *Let $L(q\colon p_1,\ldots,p_n)$ be a lens space and $F_q(z\colon p_1,\ldots,p_n)$ the generating function associated to the spectrum of $L(q\colon p_1,\ldots,p_n)$. Then, on the domain $\{z\in\mathbb{C}\mid |z|<1\}$,*

$$F_q(z\colon p_1,\ldots,p_n)=\frac{1}{q}\sum_{l=1}^{q}\frac{1-z^2}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}.$$

Corollary 2.2.10. *Let $L(q\colon p_1,\ldots,p_n)$ be isospectral to $L(q'\colon s_1,\ldots,s_n)$. Then $q=q'$.*

Now let $\tilde{\mathcal{L}}(q,n)$ be the family of all $(2n-1)$-dimensional lens spaces with fundamental groups of order q, and let $\tilde{\mathcal{L}}_0(q,n)$ be the subfamily of $\tilde{\mathcal{L}}(q,n)$ defined by:

$$\tilde{\mathcal{L}}_0(q,n)=\{L(q\colon p_1,\ldots,p_n)\in\tilde{\mathcal{L}}(q,n)\mid p_i\not\equiv \pm p_j \pmod q,\ 1\le i<j\le n\}.$$

The set of isometry classes of $\tilde{\mathcal{L}}(q,n)$ is denoted by $\mathcal{L}(q,n)$, and the set of isometry classes of $\tilde{\mathcal{L}}_0(q,n)$ is denoted by $\mathcal{L}_0(q,n)$.

By Proposition 2.2.1, the map $L(q\colon p_1,\ldots,p_n)\mapsto(p_1,\ldots,p_n)$ of $\tilde{\mathcal{L}}_0(q,n)$ (resp. $\tilde{\mathcal{L}}(q,n)$) onto $\tilde{I}_0(q,n)$ (resp. $\tilde{I}(q,n)$) induces a one-to-one map between $\mathcal{L}_0(q,n)$ and $I_0(q,n)$ (resp. $\mathcal{L}(q,n)$ and $I(q,n)$).

The above fact, together with Proposition 2.1.3, gives us the following:

Proposition 2.2.11. *Retaining the notations as above, we get*

$$|\mathcal{L}_0(q,n)|\ge\sum_{t=u}^{r}\frac{1}{n-t}\binom{q_0-1-r}{n-1-t}\binom{r}{t},$$

where $u = r - k$ *if* $r > k$, *and* $u = 0$ *if* $r \leq k$; r *is the number of residues* mod q *that are not co-prime to* q *and are less than or equal to* q_0.

Note that by Proposition 2.2.1, we also get that

$$|\mathcal{L}_0(q, n)| \geq \frac{1}{q_0}\binom{q_0}{n}.$$

Next, we will re-formulate the generating function $F_q(z\colon p_1, \dots, p_n)$ in a form that will help us find isospectral pairs that are non-isometric.

Proposition 2.2.12. *Let* $L(q\colon p_1, \dots, p_n)$ *be a lens space belonging to* $\tilde{\mathcal{L}}_0(q, n)$, $k = q_0 - n$, *and let* w *be the map of* $I_0(q, n)$ *onto* $I_0(q, k)$ *defined in* Section 2.1. *Then*
(i) *If* $q = P^m$, *where* P *is a prime, we have*

$$F_q(z\colon p_1, \dots, p_n) = \frac{1}{q}\left\{\frac{(1-z^2)}{(1-z)^{2n}} + \frac{\psi_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{\Phi_q(z)}\right.$$
$$\left. + \sum_{j=1}^{m-1} \frac{\alpha_{q,k}^{(j)}(w([p_1, \dots, p_n]))(z)(1-z^2)}{(\Phi_{P^{m-j}}(z))^{P^j}(1-z)^{P^j-1}}\right\},$$

(ii) *If* $q = P_1 \cdot P_2$, *where* P_1 *and* P_2 *are primes, we have*

$$F_q(z\colon p_1, \dots, p_n) = \frac{1}{q}\left\{\frac{(1-z^2)}{(1-z)^{2n}} + \frac{\psi_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{\Phi_q(z)}\right.$$
$$+ \frac{\alpha_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{(\Phi_{P_2}(z))^{P_1}(1-z)^{P_1-1}}$$
$$\left. + \frac{\beta_{q,k}(w([p_1, \dots, p_n]))(z)(1-z^2)}{(\Phi_{P_1}(z))^{P_2}(1-z)^{P_2-1}}\right\},$$

where $\psi_{q,k}$, $\alpha_{q,k}^{(j)}$, $\alpha_{q,k}$ *and* $\beta_{q,k}$ *are as defined in* Definition 2.1.4 *and* $\Phi_t(z) = \sum_{v=0}^{t-1} z^v$.

Proof. We choose integers $q_1, \dots, q_k$ such that the set of integers

$$\{p_1, -p_1, \dots, p_n, -p_n, q_1, -q_1, \dots, q_k, -q_k\}$$

forms a complete set of residues mod q.
(i) We write

$$F_q(z\colon p_1, \dots, p_n) = \frac{1}{q}\left[\sum_{l \in A} \frac{(1-z^2)}{\prod_{i=1}^n (z - \gamma^{p_i l})(z - \gamma^{-p_i l})}\right.$$
$$\left. + \sum_{j=1}^{m-1} \sum_{l \in B_j} \frac{(1-z^2)}{\prod_{i=1}^n (z - \gamma^{p_i l})(z - \gamma^{-p_i l})}\right].$$

Now, for any $l \in A$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l})}{\Phi_q(z)}.$$

For $l \in B_j$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l})}{(\Phi_{P^{m-j}}(z))^{P^j}(1-z)^{P^j-1}},$$

where $\Phi_t(z) = \sum_{v=0}^{t-1} z^v$.

Now, (i) follows from these facts.

(ii) We write

$$\begin{aligned} F_q(z\colon p_1,\dots,p_n) = \frac{1}{q}\Bigg[&\sum_{l\in A}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} \\ &+\sum_{l\in B}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} \\ &+\sum_{l\in C}\frac{(1-z^2)}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}\Bigg]. \end{aligned}$$

Again, for $l \in A$,

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l})}{\Phi_q(z)}.$$

For $l \in B$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l})}{(\Phi_{P_2}(z))^{P_1}(1-z)^{P_1-1}}.$$

For $l \in C$, we have

$$\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} = \frac{\prod_{i=1}^{k}(z-\gamma^{q_i l})(z-\gamma^{-q_i l})}{(\Phi_{P_1}(z))^{P_2}(1-z)^{P_2-1}}.$$

Now, (ii) follows from these facts. $\square$

From Proposition 2.2.8 and Proposition 2.2.12, we get the following proposition

Proposition 2.2.13. *Let $L = L(q\colon p_1,\dots,p_n)$ and $L' = L(q\colon s_1,\dots,s_n)$ be lens spaces belonging to $\mathcal{L}_0(q,n)$. Let $k = q_0 - n$.*

(i) *If* $q = P^m$*, then* L *is isospectral to* L' *if*

$$\psi_{q,k}(w([p_1, \dots, p_n])) = \psi_{q,k}(w([s_1, \dots, s_n]))$$

and

$$\alpha^{(j)}_{q,k}(w([p_1, \dots, p_n])) = \alpha^{(j)}_{q,k}(w([s_1, \dots, s_n]))$$

for $j = 1, \dots, m-1$.

(ii) *If* $q = P_1 \cdot P_2$*, then* L *is isospectral to* L' *if*

$$\psi_{q,k}(w([p_1, \dots, p_n])) = \psi_{q,k}(w([s_1, \dots, s_n])),$$
$$\alpha_{q,k}(w([p_1, \dots, p_n])) = \alpha_{q,k}(w([s_1, \dots, s_n]))$$

and

$$\beta_{q,k}(w([p_1, \dots, p_n])) = \beta_{q,k}(w([s_1, \dots, s_n])).$$

3. Isospectral non-isometric lens spaces

By applying Proposition 2.2.11 and Proposition 2.2.13 we will obtain our main theorem in this section for odd-dimensional lens spaces. Next, we will extend the results to obtain even-dimensional pairs of lens spaces corresponding to every pair of odd-dimensional lens spaces.

3.1. Odd-dimensional lens spaces. From the results in Sections 2 and 3 we get the following diagrams: For $q = P^m$,

$$\mathcal{L}_0(q, n) \overset{\sim}{\rightarrow} I_0(q, n) \underset{w}{\overset{\sim}{\rightarrow}} I_0(q, k) \xrightarrow[\tau^{(m)}_{q,k}]{} Q^m(\gamma)[z], \tag{3.1}$$

where $\tau^{(m)}_{q,k} = (\psi_{q,k}, \alpha^{(1)}_{q,k}, \dots, \alpha^{(m-1)}_{q,k})$, and $Q^m(\gamma)[z]$ denotes m copies of the field of rational polynomials $Q(\gamma)[z]$. For $q = P_1 \cdot P_2$,

$$\mathcal{L}_0(q, n) \overset{\sim}{\rightarrow} I_0(q, n) \underset{w}{\overset{\sim}{\rightarrow}} I_0(q, k) \xrightarrow[\mathcal{S}^{(3)}_{q,k}]{} Q^3(\gamma)[z], \tag{3.2}$$

where $\mathcal{S}^{(3)}_{q,k} = (\psi_{q,k}, \alpha_{q,k}, \beta_{q,k})$.

Now, from Proposition 2.2.13, if $\tau^{(m)}_{q,k}$ (resp. $\mathcal{S}^{(3)}_{q,k}$) is not one-to-one, then we will have non-isometric lens spaces having the same generating function. This would give us our desired results.

We first calculate the values for the required coefficients of $\psi_{q,2}$, $\alpha^{(j)}_{q,2}$, $\alpha_{q,2}$ and $\beta_{q,2}$.

Using Proposition 2.1.5 we can calculate the values of the various coefficients of $\psi_{q,k}$, $\alpha^{(j)}_{q,k}$, $\alpha_{q,k}$ and $\beta_{q,k}$.

First we will find coefficients for z and z^2 for any given k, and from that we can find the values for $k = 2$.

From the definitions of $\psi_{q,k}([q_1, \dots, q_k])$, in the notation of Proposition 2.1.5, it is easy to see that

$$a_1 = \sum_{i=1}^{k} \sum_{l \in A} \gamma^{q_i l} + \sum_{i=1}^{k} \sum_{l \in A} \gamma^{-q_i l} = 2 \sum_{i=1}^{k} \sum_{l \in A} \gamma^{q_i l}.$$

Similarly,

$$b_{1,j} = 2 \sum_{i=1}^{k} \sum_{l \in B_j} \gamma^{q_i l}, \quad b_1 = 2 \sum_{i=1}^{k} \sum_{l \in B} \gamma^{q_i l}, \quad c_1 = 2 \sum_{i=1}^{k} \sum_{l \in C} \gamma^{q_i l}.$$

Also,

$$\begin{aligned} a_2 &= \sum_{l \in A} \left[k + \sum_{1 \le i < t \le k} \gamma^{(q_i + q_t)l} + \sum_{1 \le i < t \le k} \gamma^{-(q_i + q_t)l} + \sum_{1 \le i < t \le k} \gamma^{(q_i - q_t)l} + \sum_{1 \le i < t \le k} \gamma^{-(q_i - q_t)l} \right] \\ &= k|A| + 2 \sum_{l \in A} \sum_{1 \le i < t \le k} \gamma^{(q_i + q_t)l} + 2 \sum_{l \in A} \sum_{1 \le i < t \le k} \gamma^{(q_i - q_t)l}. \end{aligned}$$

Similarly,

$$\begin{aligned} b_{2,j} &= k|B_j| + 2 \sum_{l \in B_j} \sum_{1 \le i < t \le k} \gamma^{(q_i + q_t)l} + 2 \sum_{l \in B_j} \sum_{1 \le i < t \le k} \gamma^{(q_i - q_t)l}, \\ b_2 &= k|B| + 2 \sum_{l \in B} \sum_{1 \le i < t \le k} \gamma^{(q_i + q_t)l} + 2 \sum_{l \in B} \sum_{1 \le i < t \le k} \gamma^{(q_i - q_t)l}, \\ c_2 &= k|C| + 2 \sum_{l \in C} \sum_{1 \le i < t \le k} \gamma^{(q_i + q_t)l} + 2 \sum_{l \in C} \sum_{1 \le i < t \le k} \gamma^{(q_i - q_t)l}, \end{aligned}$$

where $|A|$, $|B_j|$, $|B|$ and $|C|$ are cardinalities of A, B_j, B and C—as defined in Definition 2.1.4—respectively.

Proposition 3.1.1. *Let p be an odd prime and let $q = p^m$ where m is an integer greater than* 1. *Let $q_0 = (q-1)/2$. Let $k = 2$ and $n = q_0 - 2$. Then the maps $\tau_{q,k}^{(m)}$ and $\mathcal{S}_{q,k}^{(3)}$ as defined in* (3.1) *and* (3.2) (*and hence the generating function*) *are dependent only on where the various q_i's and their sums and differences reside.*

In a similar fashion we can find values of coefficients of higher powers of z when $k > 2$. These coefficients will contain terms that include higher sums and differences of the various q_i's in the powers of γ.

We will prove two propositions that will give us upper bounds on the number of expressions for $\tau_{q,k}^{(j)}$ and $\mathcal{S}_{q,k}^{(3)}$, respectively, where $k = 2$.

Proposition 3.1.2. *Let p be an odd prime and let $q = p^m$ where m is an integer greater than* 1. *Let $q_0 = (q-1)/2$. Let $k = 2$ and $n = q_0 - 2$. Then the number of expressions that $\tau_{q,2}^{(j)}$ can have is at most m^2.*

Proof. We will find the number of $\tau_{q,2}^{(j)}$ by considering the following cases:

CASE 1: $q_1, q_2 \in B_j$ $(j = 1, 2, \ldots, (m-1))$ where $B_j = \{x \pmod q) \in \mathbf{Z}^+ : p^j \mid x,\ p^{j+1} \nmid x\}$. If $q_1, q_2 \in B_j$, then either both of $q_1 \pm q_2$ lie in B_j or else one lies in B_j and the other in some B_k with $j < k \leq m-1$. Thus there are $m - j$ possibilities. As j varies from 1 to $m-1$, we thus obtain $(m-1) + (m-2) + \cdots + 1 = m(m-1)/2$ expressions.

CASE 2: $q_1 \in B_j$ and $q_2 \in B_t$, $B_j \neq B_t$. We may assume $j < t$. It follows that $q_1 \pm q_2$ both lie in B_j. Thus as j and t vary, we obtain $\binom{m-1}{2} = (m-1)(m-2)/2$ expressions.

CASE 3: $q_1 \in B_j$ and $q_2 \in A$, or vice versa. Here we note that $q_1 \pm q_2$ always belongs to A. Therefore, in this case we will get $(m-1)$ possible expressions for $\tau_{q,2}^{(j)}$, one each for the case where $q_1 \in A$ and $q_2 \in B_j$ $(j = 1, 2, \ldots, (m-1))$, or vice versa.

CASE 4: $q_1, q_2 \in A$. We will get 1 possible expression if $q_1 \pm q_2 \in A$. Then we will get 1 possible expression each for the case when $q_1 + q_2 \in A$ and $q_1 - q_2 \in B_j$ (or vice versa) for $j = 1, 2, \ldots, (m-1)$. There are no other possibilities in this case. So the maximum number of possible expressions for $\tau_{q,2}^{(j)}$ in this case will be $m - 1 + 1 = m$.

Case 1 though Case 4 are the only possible cases that occur for $k = 2$. Adding up the numbers of all possible expressions for $\tau_{q,2}^{(j)}$ from each case we get the maximum number of possible expressions that $\tau_{q,2}^{(j)}$ can have:

$$\frac{m(m-1)}{2} + \frac{(m-1)(m-2)}{2} + (m-1) + m$$
$$= \frac{m^2 - m + m^2 - 3m + 2 + 2m - 2 + 2m}{2} = \frac{2m^2}{2} = m^2. \qquad \square$$

Proposition 3.1.3. *Let $q = p_1 \cdot p_2$, where p_1, p_2 are distinct odd primes. Let $q_0 = (q-1)/2$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions for $\mathcal{S}_{q,2}^{(3)}$ is at most* 11.

Proof. As in the previous proposition, we prove this result by considering all the possible cases for q_1 and q_2 (where $q_1 \pm q_2$ is not congruent to 0 (mod q)).

Case 1: $q_1, q_2 \in B$ or $(q_1, q_2 \in C)$, where $B = \{xp_1 \mid x = 1, \ldots, (p_2 - 1)\}$ and $C = \{xp_2 \mid x = 1, \ldots, (p_1 - 1)\}$. Then $q_1 \pm q_2 \in B$ or $(q_1 \pm q_2 \in C$, respectively). There are no other possibilities for this case.

CASE 2: $q_1 \in B$ and $q_2 \in C$ (or vice versa). We have just one possible expression in this case, when $q_1 \pm q_2 \in A$.

CASE 3: $q_1 \in A, q_2 \in B$ or $q_1 \in A, q_2 \in C$ (or vice versa). We will get one expression each when $q_1 \pm q_2 \in A$. Then we will get one possible expression for the case when $q_1 \in A$, $q_2 \in B$, and $q_1 + q_2 \in A$, $q_1 - q_2 \in C$ (or vice versa). We will get one more possible expression for the case when $q_1 \in A$, $q_2 \in C$, and $q_1 + q_2 \in A$, $q_1 - q_2 \in B$ (or vice versa). So, in this case we get a possible 4 expressions for $\mathcal{S}_{q,2}^{(3)}$.

CASE 4: $q_1, q_2 \in A$. We will get one possible expression where $q_1 \pm q_2 \in A$. We get another possible expression where $q_1 + q_2 \in A$ and $q_1 - q_2 \in B$ (or vice versa). We get a third possible expression where $q_1 + q_2 \in A$ and $q_1 - q_2 \in C$ (or vice versa). We get a fourth possible expression where $q_1 + q_2 \in B$ and $q_1 - q_2 \in C$ (or vice versa). So, we get a total of 4 possible expressions for $\mathcal{S}^{(3)}_{q,2}$ in this case.

Case 1 through Case 4 are the only possible cases than can occur for $k = 2$. Adding up the number of all possible expressions for $\mathcal{S}^{(3)}_{q,2}$ from each case we get the maximum number of possible expressions for $\mathcal{S}^{(3)}_{q,2}$:

$$2 + 1 + 4 + 4 = 11. \qquad \square$$

It is important to note that in the above propositions the number of possible expressions is the *maximum* number of expressions that can happen. It is possible that for a given $q = p^m$ or $q = p_1 \cdot p_2$ not all the expressions will occur. We will see this in an example later.

We now prove two similar propositions for even q of the form 2^m and $2p$, where m is a positive integer and p is a prime.

Proposition 3.1.4. *Let $q = 2^m$ where $m \geq 3$. Let $q_0 = q/2$, i.e., $q_0 = 2^{m-1}$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions that $\tau^{(j)}_{q,2}$ can have is at most $(m-1)^2$.*

Proof. We proceed as in the previous propositions.

CASE 1: $q_1, q_2 \in B_j$ $(j = 1, 2, \ldots, (m-3))$, where $B_j = \{x \pmod q) \in \mathbf{Z}^+ \colon 2^j \mid x, 2^{j+1} \nmid x\}$.

We first note that the cases where $q_1, q_2 \in B_{m-2}$ or B_{m-1} will not occur. Now when $q_1, q_2 \in B_j$, then one of the $q_1 + q_2$ or $q_1 - q_2$ will belong to B_{j+1} and the other will belong to B_t for $t > j + 1$.

Now, with $q_1, q_2 \in B_j$ (where $j < m-2$), we get $(m-2-j)$ possible expressions for $\tau^{(j)}_{q,2}$.

So, in this case, the total number of possible expressions for $\tau^{(j)}_{q,2}$ are:

$$(m-3) + (m-4) + \cdots + 3 + 2 + 1 = \frac{(m-2)(m-3)}{2}.$$

CASE 2: $q_1 \in B_j$ and $q_2 \in B_t$, where $B_j \neq B_t$. We can assume that $j < t$. This would mean that $q_1 \pm q_2 \in B_j$ always. So, as in Case 2 of Proposition 3.1.2, we get that the total number of expressions for $\tau^{(j)}_{q,2}$ will be $(m-1)(m-2)/2$.

CASE 3: $q_1 \in B_j$ and $q_2 \in A$ (or vice versa). We notice that $q_1 \pm q_2 \in A$ always. So, just like in Case 3 of Proposition 3.1.2, we will get that the total number of possible expressions for $\tau^{(j)}_{q,2}$ will be $(m-1)$.

CASE 4: $q_1, q_2 \in A$. In this case one of $q_1 + q_2$ or $q_1 - q_2$ will belong to B_1 and the other will belong to one of the B_j for $j > 1$. Therefore, for this case we will get $(m-2)$ possible expressions for $\tau^{(j)}_{q,2}$, one each for the case when $q_1 + q_2 \in B_1$ (alt. $q_1 - q_2 \in B_1$) and $q_1 - q_2 \in B_t$ (alt. $q_1 + q_2 \in B_t$) for $t = 2, 3, \dots, m-1$.

Now, adding up all the possible expressions from the four cases above we get the maximum number of possible expressions for $\tau^{(j)}_{q,2}$:

$$\begin{aligned} &\frac{(m-2)(m-3)}{2} + \frac{(m-1)(m-2)}{2} + (m-1) + (m-2) \\ &= \frac{m^2 - 5m + 6 + m^2 - 3m + 2 + 2m - 2 + 2m - 4}{2} \\ &= m^2 - 2m + 1 = (m-1)^2. \end{aligned}$$ □

Our next proposition gives us the maximum number of expressions for $\mathcal{S}^{(3)}_{q,2}$ when $q = 2p$ for some prime p.

Proposition 3.1.5. *Let $q = 2p$ where p is an odd prime. Let $q_0 = q/2 = p$. Let $k = 2$ and $n = q_0 - 2$. Then the number of possible expressions for $\mathcal{S}^{(3)}_{q,2}$ is at most* 6.

Proof. As before we will analyze the different possible cases. Note that in this situation we have $B = \{2, 4, 6, \dots, 2(p-1)\}$ and $C = \{p\}$.

CASE 1: $q_1, q_2 \in B$. We will have $q_1 \pm q_2 \in B$ always. Notice that in this case q_1, q_2 cannot belong to C since C has only one element. So we get 1 possible expression in this case for $\mathcal{S}^{(3)}_{q,2}$.

CASE 2: $q_1 \in B$, $q_2 \in C$. In this case $q_1 \pm q_2 \in A$ always. So, we get 1 possible expression in this case for $\mathcal{S}^{(3)}_{q,2}$.

CASE 3: $q_1 \in A$, $q_2 \in B$ or $q_1 \in A$, $q_2 \in C$. When $q_1 \in A$ and $q_2 \in C$, then $q_1 \pm q_2 \in B$ always. So, we get 1 possible expression for $\mathcal{S}^{(3)}_{q,2}$. When $q_1 \in A$, $q_2 \in B$, we will get 1 possible expression for the situation when $q_1 \pm q_2 \in A$. We will get another possible expression for $\mathcal{S}^{(3)}_{q,2}$ where $q_1 + q_2 \in A$ (alt. $q_1 - q_2 \in A$) and $q_1 - q_2 \in C$ (alt. $q_1 + q_2 \in C$).

So, there are a total of 3 possible expressions for $\mathcal{S}^{(3)}_{q,2}$ in this case.

CASE 4: $q_1, q_2 \in A$. Then $q_1 \pm q_2 \in B$ always. So, we get 1 possible expression for this case.

Now, adding up all the possible expressions from the above four cases we get the maximum number of possible expressions for $\mathcal{S}^{(3)}_{q,2}$ to be $1 + 1 + 3 + 1 = 6$. □

With these four propositions, we are now ready for our first main theorem.

Theorem 3.1.6. (i) *Let $p \geq 5$ (alt. $p \geq 3$) be an odd prime and let $m \geq 2$ (alt. $m \geq 3$) be any positive integer. Let $q = p^m$. Then there exist at least two $(q-6)$-dimensional orbifold lens spaces with fundamental groups of order p^m which are isospectral but not isometric.*
(ii) *Let p_1, p_2 be odd primes such that $q = p_1 \cdot p_2 \geq 33$. Then there exists at least two $(q-6)$-dimensional orbifold lens spaces with fundamental groups of order $p_1 \cdot p_2$ which are isospectral but not isometric.*
(iii) *Let $q = 2^m$ where $m \geq 6$ be any positive integer. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces with fundamental groups of order 2^m which are isospectral but not isometric.*
(iv) *Let $q = 2p$, where $p \geq 7$ is an odd prime. Then there exist at least two $(q-5)$-dimensional orbifold lens spaces with fundamental groups of order $2p$ which are isospectral but not isometric.*

Proof. We first recall from Proposition 2.2.11 that

$$|\mathcal{L}_0(q,n)| \geq \sum_{t=r-2}^{r} \frac{1}{n-t}\binom{q_0-1-r}{n-1-t}\binom{r}{t}$$

for $k=2$ and $r>2$. Thus for $k=2$ and $r>2$ we have

(3.3)

$$\begin{aligned}
|\mathcal{L}_0(q,n)| &\geq \frac{1}{n-(r-2)}\binom{q_0-r-1}{n-1-(r-2)}\binom{r}{r-2} \\
&\quad + \frac{1}{n-(r-1)}\binom{q_0-r-1}{(n-1)-(r-1)}\binom{r}{r-1} + \frac{1}{n-r}\binom{q_0-r-1}{n-r-1}\binom{r}{r} \\
&= \frac{1}{q_0-2-r+2}\binom{q_0-r-1}{q_0-2-1-r+2}\binom{r}{r-2} \\
&\quad + \frac{1}{q_0-2-r+1}\binom{q_0-r-1}{q_0-2-1-r+1}\binom{r}{r-1} \\
&\quad + \frac{1}{q_0-2-r}\binom{q_0-r-1}{q_0-2-r-1} \quad \text{since} \quad n = q_0-2 \\
&= \frac{1}{q_0-r}\binom{q_0-r-1}{q_0-r-1}\binom{r}{r-2} + \frac{1}{q_0-r-1}\binom{q_0-r-1}{q_0-r-2}\binom{r}{r-1} \\
&\quad + \frac{1}{q_0-r-2}\binom{q_0-r-1}{q_0-r-3}\binom{r}{r} \\
&= \frac{1}{q_0-r}\cdot 1\cdot \frac{r(r-1)}{2} + \frac{1}{(q_0-r-1)}\cdot (q_0-r-1)\cdot r \\
&\quad + \frac{1}{(q_0-r-2)}\cdot \frac{(q_0-r-1)(q_0-r-2)}{2}\cdot 1 \\
&= \frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2}.
\end{aligned}$$

It is sufficient to show that the final expression in (3.3) is greater than the number of possible expressions for the generating functions computed in Propositions 3.1.2–3.1.5 in order to establish the existence of isospectral pairs of non-isometric lens spaces.
(i) For $q = p^m$, we have a total of m^2 possible expressions for $\tau^{(j)}_{q,2}$ from Proposition 3.1.2. So, we will have isospectrality when (3.3) is greater than or equal to m^2+1. That is

$$\begin{aligned}
&\frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2} \geq m^2+1\\
&\Rightarrow r(r-1) + 2r(q_0-r) + (q_0-r)(q_0-r-1) \geq 2(q_0-r)(m^2+1)\\
&\Rightarrow r^2 - r + (q_0-r)[2r + q_0 - r - 1 - 2m^2 - 2] \geq 0\\
&\Rightarrow (r^2-r) + (q_0-r)[q_0 + r - 2m^2 - 3] \geq 0\\
&\Rightarrow r^2 - r + q_0^2 + q_0 r - q_0 2m^2 - 3q_0 - q_0 r - r^2 + 2m^2 r + 3r \geq 0\\
&\Rightarrow q_0(q_0 - 2m^2 - 3) + 2r(m^2+1) \geq 0\\
&\Rightarrow -q_0[(2m^2+3) - q_0] \geq -2r(m^2+1)
\end{aligned}$$

$$\Rightarrow q_0[(2m^2+3) - q_0] \leq 2r(m^2+1). \tag{3.4}$$

So for any given m, we can choose p big enough so that $2m^2+3 \leq q_0$. This would guarantee isospectrality. We can calculate r by $r = (p^{m-1}-1)/2$ in this case. Now if $p \geq 5$, $q_0 \geq (5^m-1)/2 > 2m^2+3$ for all $m \geq 2$. This is easy to see since $5^m > 4m^2+7$ for $m \geq 2$ as the left hand side grows exponentially greater than the right hand side. So, for all $p \geq 5$ and all $m \geq 2$, (3.4) will be true and we will get isospectral pairs of dimension $(q-6) = 2n-1$. Now for $q = 3^m$, we have $3^m > 4m^2+7$ for $m \geq 4$. So we will have isospectrality. We check cases $m = 2$ and $m = 3$.

When $m = 2$, $q = 9$, $r = 1$, $q_0 = 4$. So L.H.S. of (3.4) gives $4[2(4)+3-4] = 4(7) = 28$ and R.H.S. of (3.4) gives $2(1)(4+1) = 10$. So the sufficiency condition is not satisfied.

When $m = 3$, $q = 27$, $r = 4$, $q_0 = 13$. L.H.S. of (3.4) gives $13[2(9)+3-13] = 13[8] = 104$ and R.H.S. of (3.4) gives $2(4)[9+1] = 8(10) = 80$. So the sufficiency condition is not satisfied.

However, when we check individually all the possible expressions for these cases we realize that they are less than m^2.

For $q = 3^2$, the only two expressions are for the cases when $q_1 \in A$, $q_2 \in B_1$, $q_1 \pm q_2 \in A$, and $q_1, q_2 \in A$, $q_1+q_2 \in A$, $q_1-q_2 \in B_1$. No other possible expressions exist.

However, there are only two classes in $\mathcal{L}_0(q,n)$, i.e., $|\mathcal{L}_0(q,n)| = 2$. The two classes are

$$\begin{aligned}
&[1,2] = \{(p_1, p_2) \in \tilde{\mathcal{L}}_0(q,2) \mid p_1, p_2 \in A\},\\
&[1,3] = \{(p_1, p_2) \in \tilde{\mathcal{L}}_0(q,2) \mid p_1 \in A,\ p_2 \in B_1\ (\text{alt. } p_1 \in B_1,\ p_2 \in A)\},
\end{aligned}$$

where $n = 2$, $A = \{1, 2, 4, 5, 7, 8\}$ and $B_1 = \{3, 6\}$.

Therefore, we do not obtain isospectral pairs.

For $q = 3^3$, there are 7 expressions (instead of $3^2 = 9$ possible expressions). The case where $q_1, q_2, q_1 \pm q_2 \in B_1$ and the case where $q_1, q_2 \in B_2$ do not occur. This gives us 2 less expressions than the estimated number of 9. But the number of classes is greater than or equal to

$$\frac{4(4-1)}{2(13-4)} + 4 + \frac{13-4-1}{2} = \frac{2}{3} + 4 + 4 = 8\frac{2}{3} > 7 \quad \text{(from (3.3))}.$$

This means we will have non-isometric isospectral lens spaces. This gives us our result that for $p \geq 3$ and $m \geq 3$, we will get isospectral pairs that are non-isometric.

(ii) For $q = p_1 \cdot p_2$, $r = (p_1 + p_2 - 2)/2$. From (3.3) and Proposition 3.1.3 we get the following sufficiency condition:

$$\begin{aligned} &\frac{r(r-1)}{2(q_0 - r)} + r + \frac{(q_0 - r - 1)}{2} \geq 12 \\ &\Rightarrow q_0(25 - q_0) \leq 24r. \end{aligned} \tag{3.5}$$

From this we get that for $q_0 \geq 25$, we will always find non-isometric, isospectral lens spaces because (3.5) will always be satisfied. We now check for cases where $q = 2q_0 + 1 < 51$.

For $q < 51$, and $q = p_1 \cdot p_2$ with p_1, p_2 being odd primes, there are only the following possibilities:

(a) $q = 3 \cdot 7 = 21$; $B = \{3, 6, 9, 12, 15, 18\}$, $C = \{7, 14\}$. In this case we have 9 instead of 11 possible expressions. The case where $q_1, q_2 \in C = \{7, 14\}$ is not possible, and the case where $q_1, q_2 \in A$ and $q_1 \pm q_2 \in A$ is also not possible since then $q_2 \equiv -q_1 \pmod q$. Therefore, we get 2 less expressions. now for isospectrality we use (3.3):

$$\frac{4(4-1)}{2(10-4)} + 4 + \frac{(10-4-1)}{2} = 7\frac{1}{2},$$

which is not greater than 9. So the isospectrality condition is not met.

(b) $q = 3 \cdot 5 = 15$. In this case we have 7 instead of 11 expressions. Here $B = \{3, 6, 9, 12\}$ and $C = \{5, 10\}$. In this case, the following cases do not occur: $q_1, q_2 \in C$; $q_1 \in A$, $q_2 \in C$, $q_1 \pm q_2 \in A$; $q_1, q_2, q_1 \pm q_2 \in A$; $q_1, q_2 \in A$, $q_1 + q_2 \in A$, $q_1 - q_2 \in C$. So we get 4 less expressions than 11. To check for isospectrality we use (3.3) and get $3(3-1)/(2(7-3)) + 3 + (7-3-1)/2 = 5(1/4)$, which is less than 7. So the isospectrality condition is not satisfied. For (a) and (b) it can be easily shown that $|\mathcal{L}_0(q, n)|$ is equal to 9 and 7 respectively. This means that there are no isospectral pairs in these cases.

(c) $q = 3 \cdot 11 = 33$. $B = \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30\}$ and $C = \{11, 22\}$. Here $q_0 = 16$ and $r = 6$. We check for isospectrality using (3.5):

$$q_0(25 - q_0) = 16(25 - 16) = 144,$$
$$24r = 24(6) = 144.$$

So (3.5) is satisfied.
(d) $q = 5\cdot 7 = 35$, $B = \{5, 10, 15, 20, 25, 30\}$ and $C = \{7, 14, 21, 28\}$. Here $q_0 = 17$ and $r = 5$. Using (3.5) we get

$$q_0(25 - q_0) = 17(25 - 17) = 138,$$
$$24r = 24(5) = 120.$$

So (3.5) is not satisfied. However, we notice that in this case the actual number of expressions is 10 instead of 11. So, we use (3.3) to check for isospectrality. Plugging in $r = 5$ and $q_0 = 17$ into (3.3) we get

$$\frac{5(4)}{2(12)} + 5 + \frac{11}{2} = 11\frac{1}{3} > 10.$$

This implies that $\mathcal{S}^{(3)}_{q,2}$ is not one-one and therefore, we will have non-isometric isospectral lens spaces in this case.
(e) Finally, we check $q = 3 \cdot 13 = 39$. Here $q_0 = 19$ and $r = 7$. Using (3.5) we see

$$q_0(25 - q_0) = 19(25 - 19) = 114,$$
$$24r = 24(7) = 168.$$

So (3.5) is satisfied and we will have isospectral pairs in this case. (a)–(e) are all the cases of $q = p_1 \cdot p_2 < 51$, where p_1, p_2 are odd primes. Combining these results with the fact that for $q \geq 51$, (3.5) will always be satisfied, we prove (iii).
(iii) Let $q = 2^m$. We use Proposition 3.1.4 and (3.3) to get a sufficiency condition for isospectrality:

$$\frac{r(r-1)}{2(q_0 - r)} + r + \frac{(q_0 - r - 1)}{2} \geq (m-1)^2 + 1.$$

Here $q_0 = 2^m/2 = 2^{m-1}$ and $2r = 2^{m-1}$. Therefore, $q_0 = 2r$ in this case. Simplifying the above inequality, we get

$$q_0[(2m^2 - 4m + 5) - q_0] \leq 2r(m^2 - 2m + 2).$$

But since $q_0 = 2r$, we get

$$(m^2 - 2m + 3) \leq q_0. \tag{3.6}$$

If $m \geq 6$, then $m^2 - 2m + 3 < 2^{m-1}$. Further, the right hand side of (3.6) grows exponentially bigger than the left hand side as m grows. For $m = 3, 4$ and 5, the actual number of expressions for $\tau^{(j)}_{q,2}$ are 4, 9 and 16 respectively. Further, it can be easily shown that for $m = 3, 4$ and 5, $|\mathcal{L}_0(q, n)|$ is 4, 9 and 16 respectively. Therefore, for $m = 3, 4$ and 5 we do not get isospectrality. This gives us (iii).
(iv) Using Proposition 3.1.5 and (3.3) we get the sufficiency condition for isospectrality for $q = 2p$, where p is an odd prime ≥ 7. Note that in this case $q_0 = q/2 = p$ and $r = (q + 2)/4$. Now for isospectrality we should have

$$\begin{aligned}
&\frac{r(r-1)}{2(q_0-r)} + r + \frac{(q_0-r-1)}{2} \geq 7 \\
&\Rightarrow q_0(15-q_0) \leq 14r \\
&\Rightarrow p(15-p) \leq 7(p+1) \\
&\Rightarrow 0 \leq p^2 - 8p + 7 \quad \text{or} \quad (p-1)(p-7) \geq 0.
\end{aligned}$$

Since p is positive, whenever $p \geq 7$, we will have isospectrality. When $q = 2 \cdot 5 = 10$, then $|\mathcal{L}_0(q, n)| = 6 =$ number of expressions for $\mathcal{S}^{(3)}_{q,2}$. So, we do not get isospectral pairs when $p = 5$. This proves (iv). $\square$

3.2. Lens spaces for general integers.

Let

$$L = L(q : p_1, \ldots, p_n) = S^{2n-1}/G$$

and

$$L' = L(q : p_1, \ldots, p_n) = S^{2n-1}/G'$$

be two isospectral non-isometric orbifold lens spaces as obtained in Section 3.1 where $G = \langle g \rangle$, $G' = \langle g' \rangle$.

$$g = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix}$$

and

$$g' = \begin{pmatrix} R(s_1/q) & & 0 \\ & \ddots & \\ 0 & & R(s_n/q) \end{pmatrix}.$$

We define

$$\tilde{g}_{W+} = \begin{pmatrix} R(p_1/q) & & & 0 \\ & \ddots & & \\ & & R(p_n/q) & \\ 0 & & & I_W \end{pmatrix}$$

and

$$\tilde{g}'_{W+} = \begin{pmatrix} R(s_1/q) & & & 0 \\ & \ddots & & \\ & & R(s_n/q) & \\ 0 & & & I_W \end{pmatrix},$$

where I_W is the $W \times W$ identity matrix for some integer W. We can define $\tilde{G}_{W+} = \langle \tilde{g}_{W+} \rangle$ and $\tilde{G}'_{W+} = \langle \tilde{g}'_{W+} \rangle$. Then $\tilde{G}_{W+}$ and $\tilde{G}'_{W+}$ are cyclic groups of order q. We define lens spaces $\tilde{L}_{W+} = S^{2n+W-1}/\tilde{G}_{W+}$ and $\tilde{L}'_{W+} = S^{2n+W-1}/\tilde{G}'_{W+}$. Then, like Proposition 2.2.1, we get:

Proposition 3.2.1. *Let $\tilde{L}_{W+}$ and $\tilde{L}'_{W+}$ be as defined above. Then $\tilde{L}_{W+}$ is isometric to $\tilde{L}'_{W+}$ iff there is a number l co-prime with q and there are numbers $e_i \in \{-1, 1\}$ such that $(p_1, \dots, p_n)$ is a permutation of $(e_1 l s_1, \dots, e_n l s_n) \pmod q$.*

The following lemma follows directly from this proposition.

Lemma 3.2.2. *Let L, L', $\tilde{L}_{W+}$ and $\tilde{L}'_{W+}$ be as defined above. Then L is isometric to L' iff $\tilde{L}_{W+}$ is isometric to $\tilde{L}'_{W+}$.*

We get the following theorem:

Theorem 3.2.3. *Let $\tilde{\mathcal{L}}_0^{W+}(q, n, 0)$ be the family of all $(2n + W - 1)$-dimensional orbifold lens spaces with fundamental groups of order q that are obtained in the manner described above. Let $\tilde{L}_{W+} \in \mathcal{L}_0^{W+}(q, n, 0)$ (where $\mathcal{L}_0^{W+}(q, n, 0)$ denotes the set of isometry classes of $\tilde{\mathcal{L}}_0^{W+}(q, n, 0)$). Let $F_q^{W+}(z\colon p_1, \dots, p_n, 0)$ be the generating function associated to the spectrum of $\tilde{L}_{W+}$. Then on the domain $\{z \in \mathbf{C} \mid |z| < 1\}$,*

$$F_q^{W+}(z\colon p_1, \dots, p_n, 0) = \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(z - \gamma^{p_i l})(z - \gamma^{-p_i l})}.$$

Proof. Recall the definitions of Δ_0, r^2, P^k, H^k, $\mathcal{H}^k$ and $\mathcal{H}^k_G$ from Section 2. We extend the definitions for $\mathbf{R}^{2n+W}$. That is, let Δ_0 be the Laplacian on $\mathbf{R}^{2n+W}$

with respect to the flat Riemannian metric; $r^2 = \sum_{i=1}^{2n+W} x_i^2$, where $(x_1, \ldots, x_{2n+W})$ is the standard coordinate system on $\mathbf{R}^{2n+W}$; for $k \geq 0$, P^k is the space of complex valued homogeneous polynomials of degree k in $\mathbf{R}^{2n+W}$; H^k is the subspace of P^k consisting of harmonic polynomials on $\mathbf{R}^{2n+W}$; $\mathcal{H}^k$ is the image of $i^*\colon C^\infty(\mathbf{R}^{2n+W}) \to C^\infty(S^{2n+W-1})$ where $i\colon S^{2n+W-1} \to \mathbf{R}^{2n+W}$ is the natural injection; and $\mathcal{H}^k_{\tilde{G}}$ is the space of all $\tilde{G}$-invariant functions of $\mathcal{H}^k$.

Then from Proposition 2.2.4 and Proposition 2.2.5, we get that H^k is $O(2n+W)$-invariant; P^k has the direct sum decomposition $P^k = H^k \oplus r^2 P^{k-2}$; $\mathcal{H}^k$ is an eigenspace of $\tilde{\Delta}$ on S^{2n+W-1} with eigenvalue $k(k+2n+W-2)$; $\sum_{k=0}^{\infty} \mathcal{H}^k$ is dense in $C^\infty(S^{2n+W-1})$ in the uniform convergence topology and $\mathcal{H}^k$ is isomorphic to H^k.

This along with the results in Corollary 2.2.6, where $\dim E_{k(k+2n+W-1)} = \dim \mathcal{H}^k_{\tilde{G}_{W+}}$, we get

$$F_q(z\colon p_1, \ldots, p_n, 0) = \sum_{k=0}^{\infty} \big(\dim \mathcal{H}^k_{\tilde{G}_{W+}}\big) z^k.$$

Now $\tilde{G}_{W+}$ is contained in $SO(2n+W)$.

Let χ_k and $\tilde{\chi}_k$ be the characters of the natural representations of $SO(2n+W)$ on H^k and P^k respectively. Then

$$\dim \mathcal{H}^k_{\tilde{G}_{W+}} = \frac{1}{|\tilde{G}_{W+}|} \sum_{\tilde{g}_{W+} \in \tilde{G}} \chi_k(\tilde{g}_{W+}) = \frac{1}{q} \sum_{l=1}^{q} \chi_k(\tilde{g}^l_{W+}). \tag{3.7}$$

Proposition 2.2.4 gives

$$\chi_k(\tilde{g}^l_{W+}) = \tilde{\chi}_k(\tilde{g}^l_{W+}) - \tilde{\chi}_{k-2}(\tilde{g}^l_{W+}), \tag{3.8}$$

where $\tilde{\chi}_t = 0$ for $t > 0$.

If W is even, then we can view the space P^k as having a basis consisting of all monomials of the form:

$$z^I \cdot \bar{z}^J = (z_1)^{i_1} \cdots (z_{n+v})^{i_{n+v}} \cdot (\bar{z}_1)^{j_1} \cdots (\bar{z}_{n+v})^{j_{n+v}},$$

where $W = 2v$ and where $I_{n+v} + J_{n+v} = i_1 + \cdots + i_{n+v} + j_1 + \cdots + j_{n+v} = k$ and $i_1, \ldots, i_{n+v}, j_1, \ldots, j_{n+v} \geq 0$. Then,

$$\tilde{g}^l_{W+}(z^I \cdot \bar{z}^J) = \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l}(z^I \cdot \bar{z}^J).$$

If W is odd, (say $W = 2u+1$), then we get for basis of P^k

$$z^I \cdot \bar{z}^J \cdot z^t_{n+2u+1} = (z_1)^{i_1} \cdots (z_{n+u})^{i_{n+u}} \cdot (\bar{z}_1)^{j_1} \cdots (\bar{z}_{n+u})^{j_{n+u}} \cdot (z_{n+2u+1})^t,$$

where $z_{n+2u+1} = x_{n+W}$ where $(x_1, y_1, \ldots, x_{n+W-1}, y_{n+W-1}, x_{n+W})$ is the standard euclidean coordinate system on $\mathbf{R}^{2n+W}$ with $z_i = x_i + iy_i$ for $i = 1, 2, \ldots, n+W-1$, and

$i_1, \dots, i_{n+u}$, $j_1, \dots, j_{n+u}$, $t \geq 0$ and $i_1 + \cdots + i_{n+u} + j_1 + \cdots + j_{n+u} + t = k = I_{n+u} + J_{n+u} + t$. So, in that case

$$\tilde{g}^l_{W+}(z^I \cdot \bar{z}^J \cdot z_{n+2u+1}) = \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l}(z^I \cdot \bar{z}^J \cdot z_{n+2u+1}).$$

So, for W, even case, we will get

$$\begin{aligned}
&F_q^{W+}(z\colon p_1, \dots, p_n, 0)\\
&= \frac{1}{q}\sum_{k=0}^{\infty}\sum_{l=1}^{q} \chi_k(\tilde{g}^l_{W+}) z^k\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\sum_{k=0}^{\infty} \tilde{\chi}_k(\tilde{g}^l_{W+}) z^k\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\sum_{k=0}^{\infty}\sum_{I_{n+v}+J_{n+v}=k} \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} z^k\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\sum_{k=0}^{\infty}\sum_{I_{n+v}+J_{n+v}=k} (\gamma^{p_1 l} z)^{i_1} \cdots (\gamma^{p_n l} z)^{i_n} (\gamma^{-p_1 l} z)^{j_1} \cdots\\
&\qquad\qquad \times (\gamma^{-p_n l} z)^{j_n} \cdot z^{i_{n+1} + \cdots + i_{n+v} + j_{n+1} + \cdots + j_{n+v}}\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\prod_{i=1}^{n}(1 + \gamma^{p_i l} z + \gamma^{2 p_i l} z^2 + \cdots)(1 + \gamma^{-p_i l} z + \gamma^{-2 p_i l} z^2 + \cdots)\\
&\qquad \times (1 + z + z^2 + \cdots)^W\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i l} z)(1-\gamma^{-p_i l} z)(1-z)^W} \quad \text{on} \quad \{z \in \mathbf{C} \mid |z| < 1\}\\
&= \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q}\sum_{l=1}^{q}\frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})}.
\end{aligned}$$

For W odd case, we get by similar calculations,

$$\begin{aligned}
&F_q^{W+}(z\colon p_1, \dots, p_n)\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\sum_{k=0}^{\infty}\sum_{I_{n+u}+J_{n+u}+t=k} \gamma^{i_1 p_1 l + \cdots + i_n p_n l - j_1 p_1 l - \cdots - j_n p_n l} z^k\\
&= \frac{(1-z^2)}{q}\sum_{l=1}^{q}\sum_{k=0}^{\infty}\sum_{I_{n+u}+J_{n+u}+t=k} (\gamma^{p_1 l} z)^{i_1} \cdots (\gamma^{p_n l} z)^{i_n} (\gamma^{-p_1 l} z)^{j_1} \cdots\\
&\qquad\qquad \times (\gamma^{-p_n l} z)^{j_n} \cdot z^{i_{n+1} + \cdots + i_{n+u} + j_{n+1} + \cdots + j_{n+u} + t}
\end{aligned}$$

$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \prod_{i=1}^{n} (1+\gamma^{p_i l} z + \gamma^{2p_i l} z^2 + \cdots)(1+\gamma^{-p_i l} z + \gamma^{-2p_i l} z^2 + \cdots)$$
$$\times (1+z+z^2+\cdots)^W$$
$$= \frac{(1-z^2)}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(1-\gamma^{p_i l} z)(1-\gamma^{-p_i l} z)(1-z)^W} \quad \text{on} \quad \{z \in \mathbf{C} \mid |z| < 1\}$$
$$= \frac{(1+z)}{(1-z)^{W-1}} \cdot \frac{1}{q} \sum_{l=1}^{q} \frac{1}{\prod_{i=1}^{n}(z-\gamma^{p_i l})(z-\gamma^{-p_i l})} \quad \text{as before.} \qquad \square$$

Corollary 3.2.4. *When $L(q\colon p_1, \ldots, p_n)$ and $L(q\colon s_1, \ldots, s_n)$ have the same generating function, then $\tilde{L}_{W+}$ and $\tilde{L}'_{W+}$ (as defined above) also have the same generating function.*

Proof. This follows from the fact that

$$F_q{}^{W+}(z\colon p_1, \ldots, p_n, 0) = \frac{1}{(1-z)^W} F_q(z\colon p_1, \ldots, p_n). \qquad \square$$

The above results give us the following theorem.

Theorem 3.2.5. (i) *Let $P \geq 5$ (alt. $P \geq 3$) be any odd prime and let $m \geq 2$ (alt. $m \geq 3$) be any positive integer. Let $q = P^m$. Then there exist at least two $(q + W - 6)$-dimensional orbifold lens spaces with fundamental groups of order P^m which are isospectral but not isometric.*
(ii) *Let P_1, P_2 be two odd primes such that $q = P_1 \cdot P_2 \geq 33$. Then there exist at least two $(q + W - 6)$-dimensional orbifold lens spaces with fundamental groups of order $P_1 \cdot P_2$ which are isospectral but not isometric.*
(iii) *Let $q = 2^m$ where $m \geq 6$ is any positive integer. Then there exist at least two $(q + W - 5)$-dimensional orbifold lens spaces with fundamental groups of order 2^m which are isospectral but not isometric.*
(iv) *Let $q = 2P$, where $P \geq 7$ is an odd prime. Then there exist at least two $(q + W - 5)$-dimensional orbifold lens spaces with fundamental groups of order $2P$ which are isospectral but not isometric.*

Corollary 3.2.6. (i) *Let $x \geq 19$ be any integer. Then there exist at least two x-dimensional orbifold lens spaces with fundamental groups of order 25 which are isospectral but not isometric.*
(ii) *Let $x \geq 27$ be any integer. Then there exist at least two x-dimensional orbifold lens spaces with fundamental group of order 33 which are isospectral but not isometric.*
(iii) *Let $x \geq 59$ be any integer. Then there exist at least two x dimensional orbifold lens spaces with fundamental group of order 64 which are isospectral but not isometric.*

(iv) *Let $x \geq 9$ be any integer. Then there exist at least two x dimensional orbifold lens spaces with fundamental group of order 14 which are isospectral but not isometric.*

Proof. (i) Let $q = 25$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (i) of the theorem.
(ii) Let $q = 33$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (ii) of the theorem.
(iii) Let $q = 64$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (iii) of the theorem.
(iv) Let $q = 14$ and $W \in \{0, 1, 2, 3, \ldots\}$ in (iv) of the theorem. □

When W is an odd number, we get even dimensional orbifold lens spaces that are isospectral but not isometric.

4. Examples

In this section we will look at some examples of isospectral non-isometric orbifold lens spaces by calculating their generating functions. We will also look at an example that will suggest that our technique can be generalized for higher values of $k = q_0 - n$. Recall that $q_0 = (q-1)/2$ for odd q and $q_0 = q/2$ for even q. In the previous sections we assumed $k = 2$. The technique for getting examples for higher values of k is similar, but as we shall see, the calculations for the different types of generating functions becomes more difficult as k increases.

In all the examples we will denote a lens space by $L(q : p_1, \ldots, p_n) = S^{2n-1}/G$, where G is the cyclic group generated by $g = \begin{pmatrix} R(p_1/q) & & 0 \\ & \ddots & \\ 0 & & R(p_n/q) \end{pmatrix}$. We will write $G = \langle g \rangle$.

4.1. Examples for $k = 2$.

EXAMPLE 4.1.1. Let $q = 5^2 = 25$, $q_0 = (q-1)/2 = 12$, $n = 10$, $k = 2$, $A = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$, $B_1 = \{5, 10, 15, 20\}$. Let $w([p_1, \ldots, p_{10}]) = [q_1, q_2]$. Let $\gamma = e^{2\pi i/25}$ and $\lambda = e^{2\pi i/5}$. $a_0 = |A| = 20$, $b_{0,1} = |B_1| = 4$, $\sum_{l \in A} \gamma^l = 0$, $\sum_{l \in B_1} \gamma^l = -1$, $\sum_{l \in A} \lambda^l = -5$ and $\sum_{l \in B_1} \lambda^l = 4$ (from (4.1)).

CASE 1: $q_1, q_2 \in B_1$ and $q_1 \pm q_2 \in B_1$. So,

$$\psi_{25,2}([q_1, q_2])(z) = 20z^4 + 20z^3 + 20z^2 + 20z + 20,$$
$$\alpha^{(1)}_{25,2}([q_1, q_2])(z) = 4z^4 - 16z^3 + 24z^2 - 16z + 4.$$

This corresponds to the case where $[p_1, \ldots, p_{10}] = [1, 2, 3, 4, 6, 7, 8, 9, 11, 12]$ which corresponds to a manifold lens spaces.

CASE 2: Since there is only one B_1 this case does not occur.

CASE 3: $q_1 \in B_1$ and $q_2 \in A$ (alt.$q_1 \in A$, $q_2 \in B_1$). $q_1 \pm q_2 \in A$ always. So,

$$\psi_{25,2}([q_1, q_2])(z) = 20z^4 + 10z^3 + 40z^2 + 10z + 20,$$
$$\alpha^{(1)}_{25,2}([q_1, q_2])(z) = 4z^4 - 6z^3 + 4z^2 - 6z + 4$$

corresponding to

$$[p_1, \dots, p_{10}] = [1, 2, 3, 4, 5, 6, 7, 8, 9, 11],$$

and to

$$[s_1, \dots, s_{10}] = [1, 2, 3, 4, 6, 7, 8, 9, 10, 11],$$

and

$$[p_1, \dots, p_{10}] \neq [s_1, \dots, s_{10}].$$

So, we get two isospectral non-isometric orbifolds: $L_1 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 9, 11)$ and $L_2 = (25\colon 1, 2, 3, 4, 6, 7, 8, 9, 10, 11)$. We denote by Σ_i the singular set of L_i. Then $\Sigma_1 = \{(0, 0, \dots, x_9, x_{10}, 0, 0, \dots, 0) \in S^{19} \mid x_9^2 + x_{10}^2 = 1\}$ and $\Sigma_2 = \{(0, 0, \dots, x_{17}, x_{18}, 0, 0) \in S^{19} \mid x_{17}^2 + x_{18}^2 = 1\}$ with isotropy groups $\langle g_1^5\rangle$ and $\langle g_2^5\rangle$ where

$$g_1^5 = \begin{pmatrix} R(5p_1/25) & & 0 \\ & \ddots & \\ 0 & & R(5p_{10}/25) \end{pmatrix} = \begin{pmatrix} R(p_1/5) & & 0 \\ & \ddots & \\ 0 & & R(p_{10}/5) \end{pmatrix}$$

and

$$g_2^5 = \begin{pmatrix} R(s_1/5) & & 0 \\ & \ddots & \\ 0 & & R(s_{10}/5) \end{pmatrix},$$

where g_1 and g_2 are generators of G_1 and G_2, respectively with $L_1 = S^{19}/G_1$ and $L_2 = S^{19}/G_2$. Σ_1 and Σ_2 are homeomorphic to S^1. We denote the two isotropy groups by $H_1 = \langle g_1^5\rangle$ and $H_2 = \langle g_2^5\rangle$.

CASE 4: (a) $q_1, q_2 \in A$ and $q_1 \pm q_2 \in A$. So,

$$\psi_{25,2}([q_1, q_2])(z) = 20z^4 + 40z^2 + 20,$$
$$\alpha^{(1)}_{25,2}([q_1, q_2])(z) = 4z^4 + 4z^3 + 4z^2 + 4z + 4$$

corresponding to

$$L_3 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 9, 10) = S^{19}/G_3, \quad \text{where} \quad G_3 = \langle g_3\rangle,$$
$$L_4 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 10, 11) = S^{19}/G_4, \quad \text{where} \quad G_4 = \langle g_4\rangle,$$

and

$$L_5 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 10, 11, 12) = S^{19}/G_5, \quad \text{where} \quad G_5 = \langle g_5\rangle.$$

The isotropy groups for L_3, L_4 and L_5 are $\langle g_3^5\rangle$, $\langle g_4^5\rangle$ and $\langle g_5^5\rangle$, respectively. Σ_3, Σ_4 and Σ_5 are all homeomorphic to S^3. So, here we get 3 isospectral orbifold lens spaces that are non-isometric.

(b) $q_1, q_2 \in A$ and $q_1+q_2 \in B_1$, $q_1-q_2 \in A$ (alt. $q_1+q_2 \in A$, $q_1-q_2 \in B_1$). So,

$$\psi_{25,2}([q_1, q_2])(z) = 20z^4 + 30z^2 + 20,$$
$$\alpha^{(1)}_{25,2}([q_1, q_2])(z) = 4z^4 + 4z^3 + 14z^2 + 4z + 4$$

corresponding to

$$L_6 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 9, 10, 11) = S^{19}/G_6, \quad \text{where} \quad G_6 = \langle g_6\rangle$$

and

$$L_7 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 10, 11) = S^{19}/G_7, \quad \text{where} \quad G_7 = \langle g_7\rangle.$$

Then, again, Σ_6 and Σ_7 are homeomorphic to S^3, and L_6 and L_7 have isotropy groups $\langle g_6^5\rangle$ and $\langle g_7^5\rangle$.

EXAMPLE 4.1.2. $q = 3^3 = 27$; $q_0 = 13$, $k = 2$, $n = 11$ and $A = \{1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26\}$, $B_1 = \{3, 6, 12, 15, 21, 24\}$, $B_2 = \{9, 18\}$. Let $w([p_1, \dots, p_{11}]) = [q_1, q_2]$. Let $\gamma = e^{2\pi i/27}$, $\lambda = e^{2\pi i/9}$ and $\delta = e^{2\pi i/3}$ be primitive 27^{th}, 9^{th} and 3^{rd} roots of unity, respectively. Here we get isospectral non-isometric pairs only in two cases:

CASE 1: $q_1 \in B_1$ and $q_2 \in A$ (alt. $q_1 \in A$ and $q_2 \in B_1$). $q_1 \pm q_2 \in A$ always. So we get,

$$\psi_{27,2}([q_1, q_2])(z) = 18z^4 + 36z^2 + 18,$$
$$\alpha^{(1)}_{27,2}([q_1, q_2])(z) = 6z^4 + 6z^3 + 12z^2 + 6z + 6,$$
$$\alpha^{(2)}_{27,2}([q_1, q_2])(z) = 2z^4 - 2z^3 - 2z + 2$$

corresponding to

$$L_1 = L(27\colon 1, 2, 4, 5, 6, 7, 9, 10, 11, 12, 13),$$
$$L_2 = L(27\colon 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13)$$

and

$$L_3 = L(27\colon 1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13).$$

If $G_1 = \langle g_1\rangle$, $G_2 = \langle g_2\rangle$, $G_3 = \langle g_3\rangle$ are such that $L_1 = S^{21}/G_1$, $L_2 = S^{21}/G_2$ and $L_3 = S^{21}/G_3$, then

$$\Sigma_1 = \{(0, 0, \dots, x_9, x_{10}, 0, x_{13}, x_{14}, 0, \dots, x_{19}, x_{20}, 0, 0) \in S^{21}\colon \text{isotropy group} = \langle g_1^9\rangle\}$$
$$\cup \{(0, 0, \dots, x_{13}, x_{14}, 0, \dots, 0) \in S^{21}\colon \text{isotropy group} = \langle g_1^3\rangle\},$$

$$\Sigma_2 = \{(0,0,\dots,x_7,x_8,0,\dots,x_{13},x_{14},\dots,0,x_{19},x_{20},0,0)\in S^{21}: \text{isotropy group} = \langle g_2^9\rangle\}$$
$$\cup\{(0,0,\dots,x_{13},x_{14},0,\dots,0)\in S^{21}: \text{isotropy group} = \langle g_2^3\rangle\},$$
$$\Sigma_3 = \{(0,0,\dots,x_7,x_8,0,\dots,x_{13},x_{14},\dots,x_{19},x_{20},0,0)\in S^{21}: \text{isotropy group} = \langle g_3^9\rangle\}$$
$$\cup\{(0,0,\dots,x_{13},x_{14},0,\dots,0)\in S^{21}: \text{isotropy group} = \langle g_3^3\rangle\}.$$

So all three orbifolds have the same isotropy type and all the singular sets are homeomorphic to S^5.

CASE 2: $q_1+q_2\in B_1$, $q_1-q_2\in A$ (alt. $q_1+q_2\in A$, $q_1-q_2\in B_1$). So we get,

$$\psi_{27,2}([q_1,q_2])(z) = 18z^4+36z^2+18,$$
$$\alpha^{(1)}_{27,2}([q_1,q_2])(z) = 6z^4+6z^2+6,$$
$$\alpha^{(2)}_{27,2}([q_1,q_2])(z) = 2z^4+4z^3+6z^2+4z+2$$

corresponding to

$$L_4 = L(27\colon 1,3,4,5,6,7,9,10,11,12,13) = S^{21}/G_4;\quad G_4=\langle g_4\rangle,$$
$$L_5 = L(27\colon 1,3,4,5,6,7,8,9,11,12,13) = S^{21}/G_5;\quad G_5=\langle g_5\rangle,$$
$$L_6 = L(27\colon 1,3,5,6,7,8,9,10,11,12,13) = S^{21}/G_6;\quad G_6=\langle g_6\rangle.$$

Then,

$$\Sigma_4 = \{(0,0,x_3,x_4,0,\dots,0,x_9,x_{10},0,0,x_{13},x_{14},0,\dots,x_{19},x_{20},0,0)\in S^{21}$$
$$\text{with isotropy group} = \langle g_4^9\rangle\}$$
$$\cup\{(0,0,\dots,0,x_{13},x_{14},0,\dots,0)\in S^{21} \text{ with isotropy group} = \langle g_4^3\rangle\},$$
$$\Sigma_5 = \{(0,0,x_3,x_4,0,\dots,0,x_9,x_{10},0,\dots,0,x_{15},x_{16},0,0,x_{19},x_{20},0,0)\in S^{21}$$
$$\text{with isotropy group} = \langle g_5^9\rangle\}$$
$$\cup\{(0,0,\dots,0,x_{15},x_{16},0,\dots,0)\in S^{21} \text{ with isotropy group} = \langle g_5^3\rangle\},$$
$$\Sigma_6 = \{(0,0,x_3,x_4,0,0,x_7,x_8,0,\dots,0,x_{13},x_{14},0,\dots,0,x_{19},x_{20},0,0)\in S^{21}$$
$$\text{with isotropy group} = \langle g_6^9\rangle\}$$
$$\cup\{(0,0,\dots,0,x_{13},x_{14},0,\dots,0)\in S^{21} \text{ with isotropy group} = \langle g_6^3\rangle\}.$$

So, the singular sets are all homeomorphic to S^7 and they all have the same isotropy types.

EXAMPLE 4.1.3. Let $q=5\cdot 7=35$, $q_0=(35-1)/2=17$, $k=2$, $n=15$. Here $A=\{1,2,3,4,6,8,9,11,12,13,16,17,18,19,22,23,24,26,27,29,31,32,33,34\}$, $B=\{5,10,15,20,25,30\}$ and $C=\{7,14,21,28\}$. Here we get isospectral non-isometric pairs in three cases:

CASE 1: $q_1 \in A$, $q_2 \in B$, $q_1 \pm q_2 \in A$. So we get,

$$\psi_{35,2}([q_1, q_2])(z) = 24z^4 + 6z^3 + 52z^2 + 6z + 24,$$
$$\alpha_{35,2}([q_1, q_2])(z) = 6z^4 + 4z^3 + 8z^2 + 4z + 6,$$
$$\beta_{35,2}([q_1, q_2])(z) = 4z^4 - 6z^3 + 4z^2 - 6z + 4$$

corresponding to

$$L_1 = L(35\colon 1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_1,$$
$$L_2 = L(35\colon 1, 2, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_2.$$

L_1 and L_2 are isospectral non-isometric orbifold lens spaces. $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$.

$$\Sigma_1 = \{(0, \dots, 0, x_7, x_8, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_1^7 \rangle\}$$
$$\cup \{(0, \dots, 0, x_{11}, x_{12}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_1^5 \rangle\},$$
$$\Sigma_2 = \{(0, \dots, 0, x_{15}, x_{16}, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_2^7 \rangle\}$$
$$\cup \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_2^5 \rangle\}.$$

Σ_1 and Σ_2 are both homeomorphic to $S^3 \times S^3$.

CASE 2: $q_1, q_2 \in A$.

(a) $q_1 \pm q_2 \in A$. So we get,

$$\psi_{35,2}([q_1, q_2])(z) = 24z^4 - 4z^3 + 52z^2 - 4z + 24,$$
$$\alpha_{35,2}([q_1, q_2])(z) = 6z^4 + 4z^3 + 8z^2 + 4z + 6,$$
$$\beta_{35,2}([q_1, q_2])(z) = 4z^4 + 4z^3 + 4z^2 + 4z + 4$$

corresponding to

$$L_3 = L(35\colon 1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_3$$

and

$$L_4 = L(35\colon 1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_4,$$

where $G_3 = \langle g_3 \rangle$ and $G_4 = \langle g_4 \rangle$. Thus,

$$\Sigma_3 = \{(0, \dots, 0, x_5, x_6, 0, \dots, 0, x_{15}, x_{16}, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29}$$
$$\text{with isotropy group} = \langle g_3^7 \rangle\}$$
$$\cup \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_3^5 \rangle\},$$
$$\Sigma_4 = \{(0, \dots, 0, x_7, x_8, 0, \dots, 0, x_{15}, x_{16}, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29}$$
$$\text{with isotropy group} = \langle g_4^7 \rangle\}$$
$$\cup \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_4^5 \rangle\}.$$

Σ_3 and Σ_4 are homeomorphic to $S^5 \times S^3$.

(b) $q_1 + q_2 \in A$, $q_1 - q_2 \in B$. So we get,

$$\psi_{35,2}([q_1, q_2])(z) = 24z^4 - 4z^3 + 42z^2 - 4z + 24,$$
$$\alpha_{35,2}([q_1, q_2])(z) = 6z^4 + 4z^3 + 8z^2 + 4z + 6,$$
$$\beta_{35,2}([q_1, q_2])(z) = 4z^4 + 4z^3 + 14z^2 + 4z + 4$$

corresponding to

$$L_5 = L(35\colon 1, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_5$$

and

$$L_6 = L(35\colon 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17) = S^{29}/G_6,$$

where $G_5 = \langle g_5 \rangle$ and $G_6 = \langle g_6 \rangle$.

$$\begin{aligned}\Sigma_5 = \{&(0, \dots, 0, x_7, x_8, 0, \dots, 0, x_{15}, x_{16}, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29} \\ &\text{with isotropy group} = \langle g_5^7 \rangle\} \\ &\cup \{(0, \dots, 0, x_{11}, x_{12}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_5^5 \rangle\}, \\ \Sigma_6 = \{&(0, \dots, 0, x_5, x_6, 0, \dots, 0, x_{15}, x_{16}, 0, \dots, 0, x_{25}, x_{26}, 0, \dots, 0) \in S^{29} \\ &\text{with isotropy group} = \langle g_6^7 \rangle\} \\ &\cup \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0, x_{23}, x_{24}, 0, \dots, 0) \in S^{29} \text{ with isotropy group} = \langle g_6^5 \rangle\}.\end{aligned}$$

Σ_5 and Σ_6 are homeomorphic to $S^5 \times S^3$.

Our final example for the case when $k = 2$ comes when q is even.

EXAMPLE 4.1.4. Let $q = 2 \cdot 7 = 14$, $q_0 = 14/2 = 7$, $k = 2$ and $n = 5$. Here $A = \{1, 3, 5, 9, 11, 13\}$, $B = \{2, 4, 6, 8, 10, 12\}$ and $C = \{7\}$. Here we get isospectral non-isometric pairs in only one case:

CASE 1: $q_1 \in A$, $q_2 \in B$, $q_1 \pm q_2 \in A$.

$$a_1 = 2 \sum_{l \in A} \gamma^l + 2 \sum_{l \in A} \lambda^l = 2(1) + 2(-1) = 0,$$
$$b_1 = 2 \sum_{l \in B} \gamma^l + 2 \sum_{l \in B} \lambda^l = 2(-1) + 2(-1) = -4,$$
$$c_1 = 2 \sum_{l \in C} \gamma^l + 2 \sum_{l \in C} \lambda^l = 2(-1) + 2(1) = 0,$$
$$a_2 = 2|A| + 4 \sum_{l \in A} \gamma^l = 2(6) + 4(1) = 12 + 4 = 16,$$
$$b_2 = 2|B| + 4 \sum_{l \in B} \gamma^l = 2(6) + 4(-1) = 12 - 4 = 8,$$

$$c_2 = 2|C| + 4\sum_{l\in C}\gamma^l = 2(1) + 4(-1) = 2 - 4 = -2.$$

So we get,

$$\psi_{14,2}([q_1, q_2])(z) = 6z^4 + 16z^2 + 6,$$
$$\alpha_{14,2}([q_1, q_2])(z) = 6z^4 + 4z^3 + 8z^2 + 4z + 6,$$
$$\beta_{14,2}([q_1, q_2])(z) = z^4 - 2z^2 + 1$$

corresponding to

$$L_1 = L(14\colon 1, 2, 4, 5, 7) = S^9/G_1$$

and

$$L_2 = L(14\colon 1, 4, 5, 6, 7) = S^9/G_2,$$

where $G_1 = \langle g_1\rangle$ and $G_2 = \langle g_2\rangle$.

$$\Sigma_1 = \{(0, 0, x_3, x_4, x_5, x_6, 0, 0, 0, 0) \in S^9 \text{ with isotropy group} = \langle g_1^7\rangle\}$$
$$\cup\, \{(0, \ldots, 0, x_9, x_{10}) \in S^9 \text{ with isotropy group} = \langle g_1^2\rangle\},$$
$$\Sigma_2 = \{(0, 0, x_3, x_4, 0, 0, x_7, x_8, 0, 0) \in S^9 \text{ with isotropy group} = \langle g_2^7\rangle\}$$
$$\cup\, \{(0, \ldots, 0, x_9, x_{10}) \in S^9 \text{ with isotropy group} = \langle g_2^2\rangle\}.$$

Σ_1 and Σ_2 are homeomorphic to $S^3 \times S^1$.

4.2. Example for $k = 3$.

EXAMPLE 4.2.1. Let $q = 5^2 = 25$, $q_0 = (25 - 1)/2 = 12$, $k = 3$, $n = 9$. Let $w([p_1, \ldots, p_9]) = [q_1, q_2, q_3]$. Here $A = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$ and $B_1 = \{5, 10, 15, 20\}$.

We will consider all the possible cases for the various possibilities of q_i's, $(q_i \pm q_j)$'s and $(q_1 \pm q_2 \pm q_3)$'s lying in A or B_1. Many of these possibilities will not occur in our present example of $q = 25$. However, these possibilities are stated because they may occur for higher values of q.

CASE 1: All the q_i's $\in B_1$. This case does not happen for $q = 25$ since at most 2 of the q_i's can be in B_1 at one time by the definition of $I_0(25, 3)$.

CASE 2: All the q_i's $\in A$. Since $k = 3$, we can't have more than 3 of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ belonging to B_1. Also, at most only one of the $q_1 \pm q_2 \pm q_3$ can be $\equiv 0 \pmod{25}$.

Further, if one of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25), then we can't have any other of the $q_1 \pm q_2 \pm q_3$ belong to B_1. Also, at most 1 of the $q_1 \pm q_2 \pm q_3$ can be in B_1.

Further, if one of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25), then at most 1 of the $q_i \pm q_j$ (for $1 \le i < j \le 3$) can be in B_1. Similarly, if one of the $q_1 \pm q_2 \pm q_3 \in B_1$, then at most 1 of the $q_i \pm q_j$ can be in B_1. We note that the above results hold true for all $q = P^2$, where P is any odd prime.

We now look at the various sub-cases for Case 2.

(a) All of the $q_1 \pm q_2 \pm q_3 \in A$ and all of the $q_i \pm q_j \in A$ $(1 \le i < j \le 3)$. This case does not occur for $q = 25$.

(b) All of the $q_1 \pm q_2 \pm q_3 \in A$ and exactly one of the $q_i \pm q_j \in B_1$ $(1 \le i < j \le 3)$. This case does not occur for $q = 25$.

(c) All of the $q_1 \pm q_2 \pm q_3 \in A$ and exactly two of the $q_i \pm q_j \in B_1$ $(1 \le i < j \le 3)$. Again, this case does not occur for $q = 25$.

(d) Exactly one of the $q_1 \pm q_2 \pm q_3 \in B_1$ and all of the $q_i \pm q_j \in A$ $(1 \le i < j \le 3)$. This case does not occur for $q = 25$.

(e) Exactly one of the $q_1 \pm q_2 \pm q_3 \equiv 0 \pmod{q}$ and all of $q_i \pm q_j \in A$ $(1 \le i < j \le 3)$. This case also does not occur for $q = 25$.

(f) All of the $q_1 \pm q_2 \pm q_3 \in A$ and exactly 3 of the $q_i \pm q_j \in B_1$ $(1 \le i < j \le 3)$. In this case, we get isospectral, non-isometric pairs since we get,

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 30z^4 + 30z^2 + 20,$$
$$\alpha^{(1)}_{25,3}([q_1, q_2, q_3])(z) = 4z^6 + 6z^5 + 30z^4 + 20z^3 + 30z^2 + 6z + 4$$

corresponding to

$$L_1 = L(25\colon 1, 4, 5, 6, 7, 8, 9, 10, 11) = S^{17}/G_1$$

and

$$L_2 = L(25\colon 1, 2, 3, 5, 7, 8, 9, 10, 12) = S^{17}/G_2,$$

where $G_1 = \langle g_1 \rangle$ and $G_2 = \langle g_2 \rangle$.

$$\Sigma_1 = \{(0, \dots, 0, x_5, x_6, 0, \dots, 0, x_{15}, x_{16}, 0, 0) \in S^{17} \text{ with isotropy group} = \langle g_1^5 \rangle\},$$
$$\Sigma_2 = \{(0, \dots, 0, x_7, x_8, 0, \dots, 0, x_{15}, x_{16}, 0, 0) \in S^{17} \text{ with isotropy group} = \langle g_2^5 \rangle\}.$$

(h) Exactly one of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25) and one of the $q_i \pm q_j$ (for $1 \le i < j \le 3$) is in B_1. We again get isospectral non-isometric pairs here since,

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 50z^4 + 10z^3 + 50z^2 + 20,$$
$$\alpha_{25,3}([q_1, q_2, q_3])(z) = 4z^6 + 6z^5 + 10z^4 + 10z^3 + 10z^2 + 6z + 4$$

corresponding to

$$L_3 = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 10) = S^{17}/G_3, \quad G_3 = \langle g_3 \rangle,$$

$$
\begin{aligned}
L_4 &= L(25\colon 1, 2, 3, 4, 5, 6, 7, 9, 10) = S^{17}/G_4, \quad G_4 = \langle g_4\rangle,\\
L_5 &= L(25\colon 1, 2, 3, 4, 5, 6, 8, 9, 10) = S^{17}/G_5, \quad G_5 = \langle g_5\rangle,\\
L_6 &= L(25\colon 1, 2, 3, 4, 5, 7, 8, 9, 10) = S^{17}/G_6, \quad G_6 = \langle g_6\rangle,\\
L_7 &= L(25\colon 1, 2, 4, 5, 6, 7, 8, 9, 10) = S^{17}/G_7, \quad G_7 = \langle g_7\rangle,\\
L_8 &= L(25\colon 1, 4, 5, 6, 7, 8, 9, 10, 12) = S^{17}/G_8, \quad G_8 = \langle g_8\rangle,\\
L_9 &= L(25\colon 1, 3, 5, 6, 7, 9, 10, 11, 12) = S^{17}/G_9, \quad G_9 = \langle g_9\rangle,\\
L_{10} &= L(25\colon 1, 2, 5, 6, 7, 8, 9, 10, 12) = S^{17}/G_{10}, \quad G_{10} = \langle g_{10}\rangle.
\end{aligned}
$$

So, in this case we get a family of 8 orbifold lens spaces that are isospectral but mutually non-isometric.

$$
\begin{aligned}
\Sigma_3 &= \{(0,\dots,0,x_9,x_{10},0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_3^5\rangle\},\\
\Sigma_4 &= \{(0,\dots,0,x_9,x_{10},0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_4^5\rangle\},\\
\Sigma_5 &= \{(0,\dots,0,x_9,x_{10},0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_5^5\rangle\},\\
\Sigma_6 &= \{(0,\dots,0,x_9,x_{10},0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_6^5\rangle\},\\
\Sigma_7 &= \{(0,\dots,0,x_7,x_8,0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_7^5\rangle\},\\
\Sigma_8 &= \{(0,\dots,0,x_5,x_6,0,\dots,0,x_{15},x_{16},0,0) \in S^{17} \text{ with isotropy group} = \langle g_8^5\rangle\},\\
\Sigma_9 &= \{(0,\dots,0,x_5,x_6,0,\dots,0,x_{13},x_{14},0,\dots,0) \in S^{17} \text{ with isotropy group} = \langle g_9^5\rangle\},\\
\Sigma_{10} &= \{(0,\dots,0,x_5,x_6,0,\dots,0,x_{15},x_{16},0,0) \in S^{17} \text{ with isotropy group} = \langle g_{10}^5\rangle\}.
\end{aligned}
$$

All of the Σ_i (for $i = 3, 4, \dots, 10$) are homeomorphic to S^3.

(h) Exactly one of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25) and one of the $q_i \pm q_j$ (for $1 \le i < j \le 3$) is in B_1. Here we get,

$$
\begin{aligned}
\psi_{25,3}([q_1, q_2, q_3])(z) &= 20z^6 + 50z^4 - 40z^3 + 50z^2 + 20,\\
\alpha_{25,3}([q_1, q_2, q_3])(z) &= 4z^6 + 6z^5 + 10z^4 + 10z^3 + 10z^2 + 6z + 4
\end{aligned}
$$

corresponding to

$$
\begin{aligned}
L_{11} &= L(25\colon 1,3,4,5,6,7,8,9,10) = S^{17}/G_{11}, \quad G_{11} = \langle g_{11}\rangle,\\
L_{12} &= L(25\colon 1,3,5,7,8,9,10,11,12) = S^{17}/G_{11}, \quad G_{12} = \langle g_{12}\rangle.
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{11} &= \{(0,\dots,0,x_7,x_8,0,\dots,0,x_{17},x_{18}) \in S^{17} \text{ with isotropy group} = \langle g_{11}^5\rangle\},\\
\Sigma_{12} &= \{(0,\dots,0,x_5,x_6,0,\dots,0,x_{13},x_{13},0,\dots,0) \in S^{17} \text{ with isotropy group} = \langle g_{12}^5\rangle\}.
\end{aligned}
$$

Σ_{11} and Σ_{12} are homeomorphic to S^3.

2 (a)–2 (h) are all of the possible cases when all the q_i's $\in A$.

CASE 3: Two of the q_i's $\in A$ and one of the q_i's $\in B_1$. In this case we will have at most one of the $q_1 \pm q_2 \pm q_3$ congruent to 0 (mod 25).

Also, we can have at most one of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ in B_1.

Further, it can be shown that at most two of the $q_1 \pm q_2 \pm q_3$ can be in B_1. Also, if one of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25), then at most one of the remaining $q_1 \pm q_2 \pm q_3$ can belong to B_1. In fact, it can be shown that exactly one of the remaining $q_1 \pm q_2 \pm q_3$ must belong to B_1.

All of these results can be shown to be true $q = P^2$, where P is any odd prime. Now we consider all the sub-cases for Case 3.

(a) If all the $q_1 \pm q_2 \pm q_3$ belong to A and exactly one of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ belongs to B_1. This case does not occur for $q = 25$.

(b) Exactly one of the $q_1 \pm q_2 \pm q_3$ belongs to B_1 and the remaining belong to A. This case does not occur for $q = 25$.

(c) If all of the $q_1 \pm q_2 \pm q_3$ belong to A and all of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ belong to A. Then we get

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 10z^5 + 60z^4 + 20z^3 + 60z^2 + 10z + 20,$$
$$\alpha^{(1)}_{25,3}([q_1, q_2, q_3])(z) = 4z^6 - 4z^5 - 4z + 4$$

corresponding to

$$L_{13} = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 9) = S^{17}/G_{13}, \quad G_{13} = \langle g_{13} \rangle,$$
$$L_{14} = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 11) = S^{17}/G_{14}, \quad G_{14} = \langle g_{14} \rangle,$$
$$L_{15} = L(25\colon 1, 2, 3, 4, 5, 6, 7, 9, 12) = S^{17}/G_{15}, \quad G_{15} = \langle g_{15} \rangle,$$
$$L_{16} = L(25\colon 1, 3, 5, 6, 7, 8, 9, 11, 12) = S^{17}/G_{16}, \quad G_{16} = \langle g_{16} \rangle,$$
$$L_{17} = L(25\colon 1, 2, 4, 5, 6, 7, 8, 9, 12) = S^{17}/G_{17}, \quad G_{17} = \langle g_{17} \rangle.$$

We get a family of 5 orbifold lens spaces that are non-isometric and isospectral.

$$\Sigma_{13} = \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0) \in S^{17} \text{ with isotropy group} = \langle g_{13}^5 \rangle\},$$
$$\Sigma_{14} = \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0) \in S^{17} \text{ with isotropy group} = \langle g_{14}^5 \rangle\},$$
$$\Sigma_{15} = \{(0, \dots, 0, x_9, x_{10}, 0, \dots, 0) \in S^{17} \text{ with isotropy group} = \langle g_{15}^5 \rangle\},$$
$$\Sigma_{16} = \{(0, \dots, 0, x_5, x_6, 0, \dots, 0) \in S^{17} \text{ with isotropy group} = \langle g_{16}^5 \rangle\},$$
$$\Sigma_{17} = \{(0, \dots, 0, x_7, x_8, 0, \dots, 0) \in S^{17} \text{ with isotropy group} = \langle g_{17}^5 \rangle\}.$$

All the Σ_i's $(i = 13, \dots, 17)$ are homeomorphic to S^1.

(d) Exactly two of the $q_1 \pm q_2 \pm q_3$ belong to B_1 and exactly one of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ belongs to B_1.

Here we get,

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 10z^5 + 50z^4 + 40z^3 + 50z^2 + 10z + 20,$$
$$\alpha^{(1)}_{25,3}([q_1, q_2, q_3])(z) = 4z^6 - 4z^5 + 10z^4 - 20z^3 + 10z^2 - 4z + 4$$

corresponding to

$$L_{18} = L(25\colon 1, 2, 3, 4, 5, 6, 7, 8, 12) = S^{17}/G_{18}, \quad G_{18} = \langle g_{18} \rangle,$$
$$L_{19} = L(25\colon 1, 2, 3, 4, 5, 6, 7, 9, 11) = S^{17}/G_{19}, \quad G_{19} = \langle g_{19} \rangle.$$

We have

$$\Sigma_{18} = \{(0, \ldots, 0, x_9, x_{10}, 0, \ldots, 0) \in S^{17} \text{ with isotropy group} = \langle g^5_{18} \rangle\},$$
$$\Sigma_{19} = \{(0, \ldots, 0, x_9, x_{10}, 0, \ldots, 0) \in S^{17} \text{ with isotropy group} = \langle g^5_{19} \rangle\}.$$

(e) One of the $q_1 \pm q_2 \pm q_3$ is congruent to 0 (mod 25), and one of the $q_1 \pm q_2 \pm q_3$ is in B_1, and exactly one of the $q_i \pm q_j$ $(1 \le i < j \le 3)$ is in B_1.

Here we get,

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 10z^5 + 50z^4 - 10z^3 + 50z^2 + 10z + 20,$$
$$\alpha^{(1)}_{25,3}([q_1, q_2, q_3])(z) = 4z^6 - 4z^5 + 10z^4 - 20z^3 + 10z^2 - 4z + 4$$

corresponding to

$$L_{20} = L(25\colon 1, 2, 3, 4, 6, 7, 8, 10, 12) = S^{17}/G_{20}, \quad G_{20} = \langle g_{20} \rangle,$$
$$L_{21} = L(25\colon 1, 4, 6, 7, 8, 9, 10, 11, 12) = S^{17}/G_{21}, \quad G_{21} = \langle g_{21} \rangle,$$
$$L_{22} = L(25\colon 1, 3, 4, 5, 6, 7, 9, 11, 12) = S^{17}/G_{22}, \quad G_{22} = \langle g_{22} \rangle.$$

We have

$$\Sigma_{20} = \{(0, \ldots, 0, x_{15}, x_{16}, 0, 0) \in S^{17} \text{ with isotropy group} = \langle g^5_{20} \rangle\},$$
$$\Sigma_{21} = \{(0, \ldots, 0, x_{13}, x_{14}, 0, \ldots, 0) \in S^{17} \text{ with isotropy group} = \langle g^5_{21} \rangle\},$$
$$\Sigma_{22} = \{(0, \ldots, 0, x_7, x_8, 0, \ldots, 0) \in S^{17} \text{ with isotropy group} = \langle g^5_{22} \rangle\}.$$

There are no other sub-cases for Case 3.

CASE 4: One of the q_i's $\in A$ and two of the q_i's $\in B_1$. In this case, $q_1 \pm q_2 \pm q_3$ will always belong to A, and exactly two of the $q_i \pm q_i$ $(1 \le i < j \le 3)$ will belong to B_1. There are no other variations that will occur in this case.

Here we get,

$$\psi_{25,3}([q_1, q_2, q_3])(z) = 20z^6 + 20z^5 + 40z^4 + 40z^3 + 40z^2 + 20z + 20,$$
$$\alpha^{(1)}_{25,3}([q_1, q_2, q_3])(z) = 4z^6 - 14z^5 + 20z^4 - 20z^3 + 20z^2 - 14z + 4$$

corresponding to $L(25\colon 1, 2, 3, 4, 6, 7, 8, 9, 11)$ and we do not get isospectral pairs. Note that this lens space is a manifold.

As this example illustrates, we can extend our technique for $k = 2$ to higher values of k and we will get many examples of isospectral non-isometric orbifold lens spaces. At the same time, the example also illustrates the difficulty in accounting for all the possible cases as the value of k is increased.

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