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SELF-MAPPING DEGREES OF TORUS BUNDLES AND TORUS SEMI-BUNDLES

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Abstract

Each closed oriented 3-manifold M is naturally associated with a set of integers $D(M)$, the degrees of all self-maps on M. $D(M)$ is determined for each torus bundle and semi-bundle M. The structure of torus semi-bundle is studied in detail. The paper is a part of a project to determine $D(M)$ for all 3-manifolds in Thurston's picture.

Contents

1. Introduction

1.1. Background. Each closed oriented n-manifold M is naturally associated with a set of integers, the degrees of all self-maps on M, denoted as $D(M) = \{\deg(f) \mid f\colon M \to M\}$.

Indeed the calculation of $D(M)$ is a classical topic appeared in many literatures. The result is simple and well-known for dimension $n = 1, 2$, and for dimension $n > 3$, there are many interesting special results (see [2] and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

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The case of dimension 3 becomes attractive in the topic and it is possible to calculate $D(M)$ for any closed oriented 3-manifold M. Since Thurston's geometrization conjecture, which seems to be confirmed, implies that closed oriented 3-manifolds can be classified in reasonable sense.

Thurston's geometrization conjecture claims that the each Jaco–Shalen–Johanson decomposition piece of a prime 3-manifold supports one of the eight geometries which are H^3, $\widetilde{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3, S^3 and $S^2 \times E^1$ (for details see [11] and [10]). Call a closed orientable 3-manifold M is *geometrizable* if each prime factor of M meets Thurston's geometrization conjecture.

A known rather general fact about $D(M)$ for geometrizable 3-manifolds is the following:

Theorem 1.1 ([12], Corollary 4.3). *Suppose M is a geometrizable 3-manifold. Then M admits a self-map of degree larger than 1 if and only if M is either*

(1) *covered by a torus bundle over the circle, or*
(2) *covered by an $F \times S^1$ for some compact surface F with $\chi(F) < 0$, or*
(3) *each prime factor of M is covered by S^3 or $S^2 \times E^1$.*

The proof of the "only if" part in Theorem 1.1 is based on the theory of simplicial volume, and various results on 3-manifold topology and group theory. The proof of "if" part in Theorem 1.1 is a sequence of elementary constructions, which were essentially known before.

Hence for any M not listed in Theorem 1.1, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether M admits a self map of degree -1 or not. To determine $D(M)$ for geometrizable 3-manifolds listed in Theorem 1.1, let's have a close look of those 3-manifolds from geometric and topological aspects.

Among Thurston's eight geometries, six of them belong to the list in Theorem 1.1. 3-manifolds in (1) are exactly those supporting either E^3, or Sol or Nil geometries. E^3 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundles or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert spaces having Euclidean orbifolds with three singular points. 3-manifolds in (2) are exactly those support $H^2 \times E^1$ geometry; 3-manifolds supporting S^3 or $S^2 \times E^1$ geometries form a proper subset of (3).

For 3-manifold M with S^3-geometry, $D(M)$ has been presented recently in [1] in term of the orders of $\pi_1(M)$ and its elements (and determined earlier in [5] when the maps induce automorphisms on π_1). Note an algorithm is given to calculate the degree set of maps between S^3-manifolds in term of their Seifert invariants [8].

To determine $D(M)$ for the remaining geometrizable 3-manifolds M, the main task is to solve the question for the following three groups ($D(M)$ is rather easy to determine for Seifert manifold M supporting $H^2 \times E^1$ or $S^2 \times E^1$ geometry):

(a) torus bundles and semi-bundles;
(b) Nil Seifert manifolds not in (a);

(c) connected sums of 3-manifolds in (3) do not supporting S^3 or $S^2 \times E^1$ geometries.

Indeed $D(M)$ for M in (a) will be determined in this paper (hopefully all the remaining cases will be solved in a forthcoming paper by the authors and Hao Zheng).

1.2. Main result. In this paper we calculate $D(M)$ for 3-manifold M which is either a torus bundle or semi-bundle. To do this, we need first to coordinate torus bundles and semi-bundles by integer matrices in Propositions 1.3 and 1.5, then state the results of $D(M)$ in term of those matrices in Theorems 1.6 and 1.7.

CONVENTION. (1) To simplify notions, for a diffeomorphism ϕ on torus T, we also use ϕ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

(2) Each 3-manifold M is oriented, and each 3-submanifold of M and its boundary have induced orientations.

(3) Suppose S (resp. P) is a properly embedded surface (resp. an embedded 3-manifold) in a 3-manifold M. We use $M \setminus S$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along S (resp. removing int P, the interior of P).

DEFINITION 1.2. A *torus bundle* is $M_\phi = T \times I/(x, 1) \sim (\phi(x), 0)$ where ϕ is a self-diffeomorphism of the torus T and I is the interval $[0, 1]$.

For a torus bundle M_ϕ, we can isotopic ϕ to be a linear diffeomorphism, which means $\phi \in GL_2(\mathbb{Z})$ while not changing M_ϕ. Since we consider the orientable case only, ϕ must be in the special linear group $SL_2(\mathbb{Z})$.

Proposition 1.3. (1) *M_ϕ admits E^3 geometry if and only if ϕ is periodical, or equivalently ϕ is conjugate to one of the following matrices* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ *of finite order* 1, 2, 3, 4 *and* 6 *respectively*;

(2) *M_ϕ admits Nil geometry if and only if ϕ is reducible, or equivalently ϕ is conjugate to* $\pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ *where* $n \neq 0$;

(3) *M_ϕ admits Sol geometry if and only if ϕ is Anosov or equivalently ϕ is conjugate to* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *where* $|a + d| > 2$, $ad - bc = 1$.

Proof. See [4]. □

DEFINITION 1.4. Let K be the Klein bottle and $N = K \tilde{\times} I$ be the twisted I-bundle over K. A *torus semi-bundle* $N_\phi = N \cup_\phi N$ is obtained by gluing two copies along their torus boundary ∂N via a diffeomorphism ϕ. Note N_ϕ is foliated by tori parallel to ∂N with a Klein bottle at the core of each copy of N.

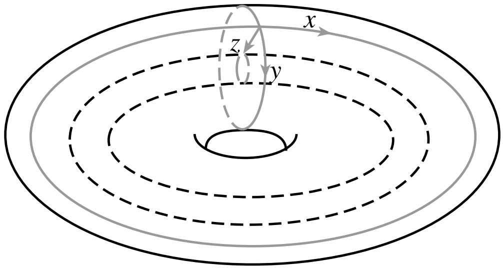


Fig. 1. Coordinates of $S^1 \times S^1 \times I$.

Let (x, y, z) be the coordinate of $S^1 \times S^1 \times I$. Then $N = S^1 \times S^1 \times I/\tau$, where τ is an orientation preserving involution such that $\tau(x, y, z) = (x + \pi, -y, 1 - z)$, and we have the double covering $p\colon S^1 \times S^1 \times I \to N$. Let C_x and C_y be the two circles on $S^1 \times S^1 \times \{1\}$ defined by y to be constant and x to be constant, see Fig. 1. Denote by $l_0 = p(C_x)$ (0 slope) and $l_\infty = p(C_y)$ (∞ slope) on ∂N. *A canonical coordinate* is an orientation of $l_0 \cup l_\infty$, hence there are four choices of canonical coordinate on ∂N. Once canonical coordinates on each ∂N are chosen, ϕ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi(l_0, l_\infty) = (l_0, l_\infty)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proposition 1.5. *With suitable choice of canonical coordinates of ∂N, we have*:
(1) *N_ϕ admits E^3 geometry if and only if $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$*;
(2) *N_ϕ admits Nil geometry if and only if $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ where $z \neq 0$*;
(3) *N_ϕ admits Sol geometry if and only if $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0$, $ad - bc = 1$.*

Moreover a torus semi-bundle N_ϕ is also a torus bundle if and only if $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ under suitable choice of canonical coordinates.

We will prove Proposition 1.5 in Section 2.

Theorem 1.6. *Using matrix coordinates given by* Proposition 1.3, *$D(M_\phi)$ is listed in* Table 1 *for torus bundle M_ϕ, where $\delta(3) = \delta(6) = 1$, $\delta(4) = 0$.*

Theorem 1.7. *Using matrix coordinates given by* Proposition 1.5, *$D(N_\phi)$ is listed in* Table 2 *for torus semi-bundle N_ϕ, where $\delta(a, d) = ad/\gcd(a, d)^2$.*

Table 1. Degrees of self maps of orientable torus bundles.

M_ϕ	ϕ	$D(M_\phi)$
E^3	finite order $k = 1, 2$	$\mathbb{Z}$
E^3	finite order $k = 3, 4, 6$	$\{(kt+1)(p^2 - \delta(k)pq + q^2) \mid t,\ p,\ q \in \mathbb{Z}\}$
Nil	$\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},\ n \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ \lvert a+d\rvert > 2$	$\{p^2 + (d-a)pr/c - br^2/c \mid$ $p, r \in \mathbb{Z},$ either $br/c,\ (d-a)r/c \in \mathbb{Z}$ or $(p(d-a) - br)/c \in \mathbb{Z}\}$

Table 2. Degrees of self maps of torus semi-bundles.

N_ϕ	ϕ	$D(N_\phi)$
E^3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathbb{Z}$
E^3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\{2l+1 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},\ z \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},\ z \neq 0$	$\{(2l+1)^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix},\ abcd \neq 0,\ ad - bc = 1$	$\{(2l+1)^2 \mid l \in \mathbb{Z}\}$, if $\delta(a,d)$ is even or $\{(2l+1)^2 \mid l \in \mathbb{Z}\} \cup \{(2l+1)^2 \cdot \delta(a,d) \mid l \in \mathbb{Z}\}$, if $\delta(a,d)$ is odd

1.3. Remark on orientation reversing homeomorphisms. Suppose M is a torus bundle or semi-bundle. Then any non-zero degree map is homotopic to a covering ([12] Corollary 0.4). Hence if $-1 \in D(M)$ (which is computable by Theorems 1.6 and 1.7), then M admits an orientation reversing self homeomorphism.

If M is a torus semi-bundle, or M supports the geometry of either E^3 or Nil, then when M admits an orientation reversing self homeomorphism is explicitly presented in the following:

Corollary 1.8. (1) *A torus semi-bundle N_ϕ admits an orientation reversing homeomorphism if and only if ϕ is either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ where $abc \neq 0$.*
(2) *A torus bundle M_ϕ supporting E^3 geometry admits an orientation reversing homeomorphism if and only if ϕ is either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.*
(3) *If M supports Nil geometry, then M admits no orientation reversing homeomorphism.*

For torus bundle with given Anosov monodromy, even we can calculate whether $-1 \in D(M_\phi)$, but there seems no simple description as in Corollary 1.8. (The referee informed us that there is a convenient description of when $-1 \in D(M_\phi)$, see Lemma 1.7, [9].)

EXAMPLE 1.9. For the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $-1 \in D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $|a+d| = 3$, then $-1 \in D(M_\phi)$. Since $p^2 + (d-a)pr/c - br^2/c = -1$ has solution $p = 1-d$, $r = c$ when $a+d = 3$, and solution $p = -1-d$, $r = c$ when $a+d = -3$.

EXAMPLE 1.10. For the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$, $-1 \notin D(M_\phi)$. Indeed for $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $a+d \pm 2$ has prime decomposition $p_1^{e_1} \cdots p_n^{e_n}$ such that $p_i = 4l+3$ and $e_i = 2m+1$ for some i, then $-1 \notin D(M_\phi)$. Since if the equation $p^2 + (d-a)pr/c - br^2/c = -1$ has integer solution, $(((a+d)^2 - 4)r^2 - 4c^2)/c^2$ should be a square of rational number. That is $((a+d)^2 - 4)r^2 - 4c^2 = s^2$ for some integer s. Therefore $(a+d+2)(a+d-2)r^2$ is a sum of two squares. By a fact in elementary number theory, neither $a+d+2$ nor $a+d-2$ has $4k+3$ type prime factor with odd power (see p. 279, [7]).

EXAMPLE 1.11. Note if $-1 \in D(M)$, then $k \in D(M)$ implies $-k \in D(M)$. For the torus bundle M_ϕ, $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers > 0, exactly 1, 4, 5, 9, 11, 16, 19, $20 \in D(M_\phi)$.

1.4. Organization of the paper. Theorems 1.6 and 1.7 will be proved in Sections 3 and 4 respectively. To prove these theorems, we need have a careful look of the structures of torus bundle and semi-bundles. This is carried out in Section 2.

We explain more about Section 2. The most convenient and useful reference for us is "Notes on basic 3-manifold topology" by Hatcher [4], which is not formally published, but widely circulated (see `http://www.math.cornell.edu/~hatcher/`). In particular Chapter 2 of [4] is devoted to the study of torus bundles and semi-bundles. Theorems 2.3 and 2.4 about classifications of torus bundles and semi-bundles are quoted from [4] directly. It seems that the proof of Theorem 2.4 in [4] missed an existed and rather complicated case, so we rewrite a proof for it (most parts still follow that in [4]). Lemma 2.6 studies incompressible surfaces in torus semi-bundle, which relies on the proof of Theorem 2.4. Then Proposition 1.5 is proved by using Theorem 2.4, Lemma 2.6, and Lemma 2.8 which presents the relation between gluing maps of a torus semi-bundles and its torus bundle double covers. Finally, Theorem 2.9 studies lifting of maps between torus semi-bundles to their torus bundle double covers.

2. Structures of orientable torus bundles and semi-bundles

2.1. Some elementary facts. All facts in this sub-section are known, and one can find them in [6], or more directly in [4].

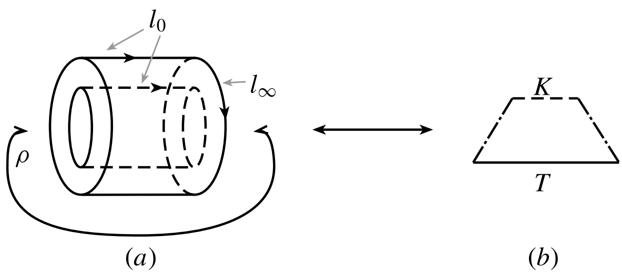


Fig. 2. Coordinates of ∂N.

DEFINITION 2.1. Suppose an oriented 3-manifold M' is a circle bundle with a given section F, where F is a compact surface with boundary components $c_1, \ldots, c_n, \ldots, c_{n+b}$ with $n > 0$. On each boundary component of M', orient c_i and the circle fiber l_i so that the product of their orientations match with the induced orientation of M'. Now attaching n solid tori S_i to the first n boundary tori of M' so that the meridian of S_i is identified with slope $r_i = a_i c_i + b_i l_i$ with $a_i > 0$. Denote the resulting manifold by M which has the Seifert fiber structure extended from the circle bundle structure of M'.

We will denote this Seifert fibering of M by $M(\pm g, b; r_1, \ldots, r_s)$ where g is the genus of the section F of M, with the sign $+$ if F is orientable and $-$ if F is nonorientable, here 'genus' of nonorientable surfaces means the number of RP^2 connected summands. When $b = 0$, call $e(M) = \sum_1^s r_i$ the Euler number of the Seifert fiberation.

Another view of N described in Fig. 2 (a): N is obtained from $S^1 \times I \times I$ by identifying $S^1 \times I \times \{0\}$ with $S^1 \times I \times \{1\}$ via a diffeomorphism ρ which reflects both the S^1 and I factors. Fig. 2 (b) is a schematic picture of N which will be used in the paper.

We list some properties of N as:

Lemma 2.2. (1) *N has two types of Seifert fiber structures*:

I: $M(0, 1; 1/2, -1/2)$ *in which l_0 on ∂N is a regular fiber and l_∞ is the boundary of the section defining the Seifert invariant.*

II: $M(-1, 1;\ )$ *in which l_∞ on ∂N is a regular fiber and l_0 is the boundary of the section defining the Seifert invariant.*

(2) *N has three types of essential* (*orientable, incompressible, ∂-incompressible*) *surfaces*:

I. *A torus parallel to ∂N.*

II. *An annulus whose boundary is l_∞ in ∂N* (Fig. 3 (a)) *which does not separate N.*

III. *An annulus whose boundary is l_0 in ∂N* (Fig. 3 (b)) *which separates N.*

(3) *Suppose M is a torus bundle or semi-bundle and F is a closed incompressible surface in M, then F is union of parallel tori.*

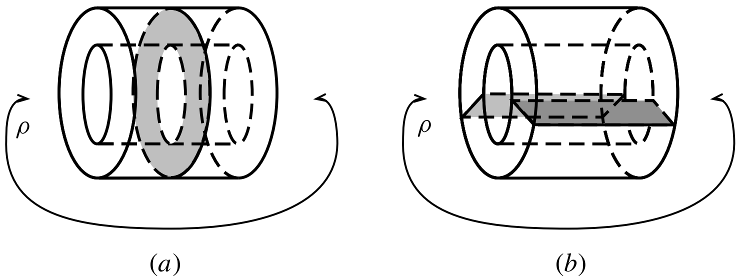


Fig. 3. Essential surface in N.

2.2. Classifications of torus bundles and semi-bundles. Orientable torus bundles and semi-bundles are classified by two theorems below.

Theorem 2.3 ([3]; [4], Theorem 2.6). *An orientable torus bundle M_ϕ is diffeomorphic to M_ψ if and only if ϕ conjugates to $\psi^{\pm 1}$ in $GL_2(\mathbb{Z})$.*

Theorem 2.4 ([4], Theorem 2.8). *The torus semi-bundle N_ϕ is diffeomorphic to N_ψ if and only if $\phi = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \psi^{\pm 1} \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ in $GL_2(\mathbb{Z})$, with independent choices of signs understood.*

Proof. (We start the proof as that in [4].) Suppose $f\colon N_\phi \to N_\psi$ is a diffeomorphism and T, T' are the torus fibers of N_ϕ, N_ψ respectively. $N_\psi \setminus T' = N_1 \cup N_2$ where N_1, N_2 are homeomorphic to N.

Since f is a diffeomorphism, two components of $N_\psi \setminus f(T)$ are both homeomorphic to N. We can isotope f, such that every component of $f(T) \cap N_i$ is an essential surface in N_i, $i = 1, 2$. So $f(T) \cap N_i$ is in the three types listed in Lemma 2.2 (2). Thus either $f(T)$ is parallel to T', or ψ takes l_0 or l_∞ to l_0 or l_∞.

Suppose $f(T)$ is parallel to T'. We can assume $f(T) = T'$. Then ϕ must be obtained from ψ by composing on the left and right homeomorphisms of ∂N which extend to homeomorphisms of N. Such homeomorphisms must preserve both l_0 and l_∞ (may reverse the directions), since l_0 is the unique slopes of the boundaries of essential separating annulus and l_∞ is the unique slopes of the boundaries of essential non-separating annulus in N. Theorem 2.4 is proved in this situation.

Suppose ψ takes l_0 or l_∞ to l_0 or l_∞. Then there are three cases as below:

CASE (1) ψ takes l_∞ to l_0 (if ψ takes l_0 to l_∞, then we consider ψ^{-1}).

CASE (2) ψ takes l_∞ to l_∞.

CASE (3) ψ takes l_0 to l_0.

(The proof in [4] claims that only Case (3) is possible, while we show below that only Case (2) is impossible.)

Case (1). Now $\psi = \begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}$, and $N_\psi = M(-1, 0; 1/2, -1/2, z)$, and $e(M) = z$. Note:

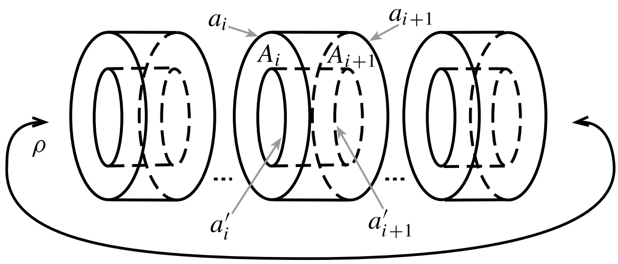


Fig. 4. Cut N_i through type (II) surfaces.

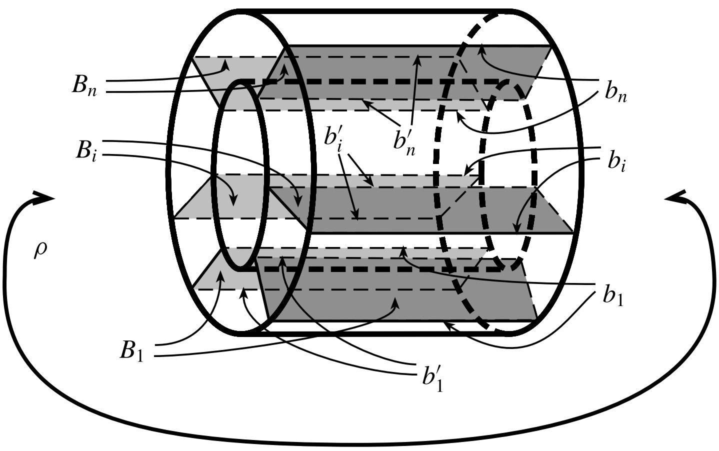
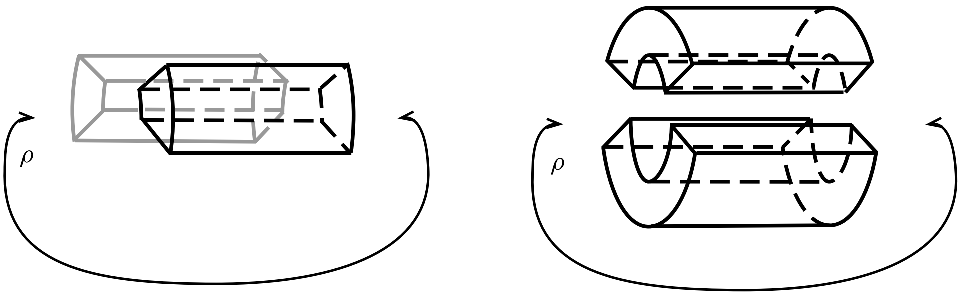


Fig. 5. Cut N_i through type (III) surfaces.

(i) $f(T) \cap N_1$ are n parallel annuli $A_1, \dots, A_n$ of type (II) (see Fig. 4), which are located in a cyclic order in N. Set $\partial A_i = a_i \cup a'_i$, then $2n$ circles $a_1, \dots, a_n, a'_1, \dots, a'_n$ are located in cyclic order in ∂N_1.
(ii) $f(T) \cap N_2$ are annuli $B_1, \dots, B_n$ of type (III) (see Fig. 5), where B_{i+1} is next to B_i, $i = 1, \dots, n-1$ in N_2. Set $\partial B_i = b_i \cup b'_i$ then $2n$ circles $b_1, \dots, b_n, b'_n, \dots, b'_1$ are located in cyclic order in ∂N_2.

If $n = 1$, we can check that ψ pastes A_1 and B_1 to a Klein bottle, which contradicts the fact that $f(T)$ is torus. When $n > 1$, we can assume ψ pastes a_1 to b_1 and pastes a_2 to b_2, after reindexing A_i if necessary. By the orders of sequences of $a_1, \dots, a_n, a'_1, \dots, a'_n$ and $b_1, \dots, b_n, b'_n, \dots, b'_1$ on ∂N_1 and ∂N_2, we have a_i is pasted to b_i, and a'_i pasted to b'_{n-i}, $i = 1, \dots, n$. So A_i, A_{n-i}, B_i, B_{n-i} are pasted to one component of $f(T)$ in N_ψ, and $f(T)$ has $[(n+1)/2]$ components. Since $f(T)$ is connected, we have $n = 2$.

Now $N_1 \setminus f(T)$ can be presented as two I-bundles over annulus: $I \times A_1$ and $I \times A_2$, where $f(T) \cap N_1 = A_1 \cup A_2$, as in Fig. 4. $N_2 \setminus f(T)$ can be presented as an I-bundle



(a) (b)

Fig. 6.

over annulus $I \times B$ as in Fig. 6 (a) and two solid tori P_1 and P_2 with the core of $P_i \cap \partial N_2$ to be the (2, 1) curve of ∂P_i as in Fig. 6 (b).

If we glue those five pieces along ∂N, we get two components of $N_\psi \setminus f(T)$ which are $N_1' = P_1 \cup_{\partial N} I \times A_1 \cup_{\partial N} P_2$ and $N_2' = I \times A_2 \cup_{\partial N} I \times B$ (re-index A_i if needed), each of them is a copy of N. Moreover under the inherited Seifert structure of N_ψ, $N_1' = M(0, 1; 1/2, -1/2)$ and $N_2' = M(-1, 1;\ \ )$.

If we consider that $M(-1, 0; 1/2, -1/2, z)$ is obtained by identifying N_1' and N_2' along $f(T)$, we get a new semi-bundle structure so that $f(T)$ become a fiber torus. Since the Euler number of the Seifert structure is z, the new gluing map must be $\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}^{\pm 1}$. This reduces us to the situation that $f(T)$ is parallel to T'.

Case (2). Both $f(T) \cap N_i$ are type (II) surfaces, for $i = 1, 2$ (Fig. 4). Hence $f(T) \cap N_1$ is exactly as that in Case (1) (i). Similarly, $f(T) \cap N_2$ are n parallel annulus $B_1, \dots, B_n$ located in a cyclic order in N. Set $\partial B_i = b_i \cup b_i'$, then $2n$ circles $b_1, \dots, b_n, b_1', \dots, b_n'$ are located in cyclic order in ∂N_2.

We can assume ψ paste a_1 to b_1 and paste a_2 to b_2 (re-index $\{B_i\}$ if needed). Then we have a_i is pasted to b_i, and a_i' pasted to b_i', $i = 1, \dots, n$. So A_i and B_i are pasted to one component of $f(T)$ in N_ψ. Since $f(T)$ is connected, $n = 1$. But here $f(T)$ does not separate N_ψ, it is impossible.

Case (3). (We copy the proof of [4] for this case.) Now $\psi = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, and $N_\psi = M(0, 0; 1/2, 1/2, -1/2, -1/2, z)$, $e(N_\psi) = z$. (Both $f(T) \cap N_i$ are type (III).)

We may assume that $f(T)$ has been isotoped to be either vertical or horizontal in this Seifert fibering. Since a connected horizontal essential surface is not separating, $f(T)$ must be vertical. Then $f(T)$ must separate $M(0, 0; 1/2, 1/2, -1/2, -1/2, z)$ into two copies of N both having the inherited Seifert structure $M(0, 1; 1/2, -1/2)$. We can rechoose the semi-bundle structure so that $f(T)$ become a fiber torus. Then for the new torus semi-bundle structure the gluing map must also be $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$. This reduces us to the situation that $f(T)$ is parallel to T'. $\square$

2.3. Incompressible surfaces.

Lemma 2.5 ([4], Lemma 2.7). *For a torus bundle M_ϕ, if ϕ is not conjugate to $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$, then any essential closed surface in M_ϕ is isotopic to a union of torus fibers.*

Lemma 2.6. *If a torus semi-bundle N_ϕ has no torus bundle structure, then any essential closed surface in N_ϕ is isotopic to copies of torus fibers of a torus semi-bundle structure on N_ϕ, which is isomorphic to N_ϕ.*

Proof. Let F be an essential close surface in $N_\phi = N_1 \cup N_2$. By Lemma 2.2 (3), F is a union of parallel tori. For our purpose we may assume that F is a torus. Isotope F so that $F \cap N_i$ is essential in N_i. Then each component of $F \cap N_i$ must be in one of the three types listed in Lemma 2.2.

If $F \cap N_i$ is of type (I), then the proof is finished.

There are two cases remaining:

(a) Both $F \cap N_i$ are of type (II) for $i = 1, 2$ (Fig. 4). Then $N_i \setminus F$ are I-bundles over $N_i \cap F$. Gluing those two I-bundles along ∂N will get an I-bundle over F and N_ϕ is obtained from this I-bundle by identifying its top and bottom, which provides a torus bundle structure of N_ϕ.

(b) Some $F \cap N_i$ is of type (III), say $i = 2$ (Fig. 5). Then F is the same as $f(T)$ either in Case (1) or Case (3) of the proof of Theorem 2.4, depends on $F \cap N_1$ is of type (III) or type (II).

As indicated in the proof of Theorem 2.4, we can rechoose the new torus semi-bundle structure N_ψ so that F become a fiber torus; moreover if choosing suitable coordinates, we can make ψ to be ϕ. □

2.4. Coordinates of torus semi-bundles. Call a map $g\colon (M, \partial M) \to (M', \partial M')$ is *proper* if $g^{-1}(\partial M') \subset \partial M$.

Lemma 2.7. *If $V = T \times I$ with the two boundaries T^+, T^- and $g\colon (V, T^+, T^-) \to (N, \partial N)$ is a proper map, then $(g|_{T^+})_* = \tau_* \cdot (g|_{T^-})_*$, where $\tau_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.*

Proof. Let $p\colon T \times I \to N$ be the double covering and τ be the deck transformation map.

Since $g_*(\pi_1(V)) = (g|_{T^+})_*(\pi_1(T^+)) \subset \pi_1(\partial N) \subset \pi_1(N)$, thus g can be lifted to a map $\tilde{g}\colon V \to T \times I$.

$$\begin{tikzcd}
& (T \times I, T \times \{0\}, T \times \{1\}) \arrow[r, "\tau"] \arrow[d, "p"] & (T \times I, T \times \{1\}, T \times \{0\}) \arrow[dl, "p"] \\
(V, T^+, T^-) \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & (N, \partial N) &
\end{tikzcd}$$

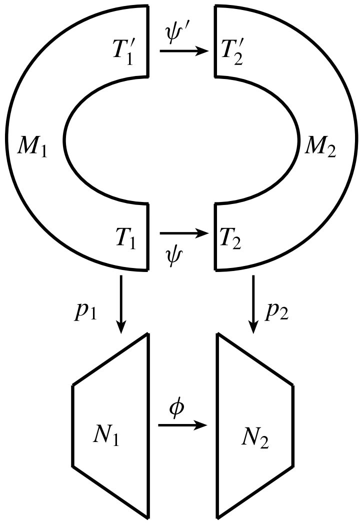


Fig. 7. N_ϕ is double covered by $M_{\tau\phi\tau\phi^{-1}}$.

From the commuted diagram above, we have:

$$\begin{cases} g|_{T^-} = p|_{T\times\{1\}} \circ \tilde{g}|_{T^-}, \\ g|_{T^+} = p|_{T\times\{1\}} \circ \tau|_{T\times\{0\}} \circ \tilde{g}|_{T^+}. \end{cases}$$

We can choose coordinate on $(T \times I,\, T \times \{0\},\, T \times \{1\})$, such that $p|_{T\times\{1\}} = id$.

When considering fundamental group, we have $(\tilde{g}|_{T^-})_* = (\tilde{g}|_{T^+})_*$. Thus by the above equation:

$$(g|_{T^+})_* = \tau_* \cdot (g|_{T^-})_*$$

where $\tau_* = (\gamma|_{T\times\{0\}})_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. □

Lemma 2.8. *A torus semi-bundle N_ϕ is doubly covered by a torus bundle $M_{\tau\phi\tau\phi^{-1}}$ where $\tau(x, y) = (x + \pi, -y)$ with suitable choice of coordinate (x, y) on the torus.*

Proof. Let $N_\phi = N_1 \cup_\phi N_2$ with $\partial N_1 = \partial N_2 = T$. Let $p\colon M \to N_\phi$ be the double cover, where M is a torus bundle, $p^{-1}(N_i) = M_i$ is homeomorphic to $T \times I$, $p^{-1}(T) = T_1 \cup T_2$. Cut M along T_1, T_2, get $M \setminus T_1 \cup T_2$. The two boundaries of M_i are denoted by T_i and T_i', T_1 is pasted to T_2 by ψ, T_1' is pasted to T_2' by ψ'. Let $p_i = p|_{M_i}$. All of these are shown in Fig. 7.

We can choose coordinate on T_1, T_2, such that $(p_i|_{T_i})_* = id$. Since T_i' is parallel to T_i, we can identify $\pi_1(T_i')$ with $\pi_1(T_i)$. By Lemma 2.7, we have $(p_i|_{T_i'})_* = \tau_* \cdot (p_i|_{T_i})_*$.

From Fig. 7, we know that

$$\begin{cases}(p_2|_{T_2})_* \circ \psi = \phi \circ (p_1|_{T_1})_*,\\ (p_2|_{T_2'})_* \circ \psi' = \phi \circ (p_1|_{T_1'})_*.\end{cases}$$

Then we get

$$\begin{cases}\psi = \phi,\\ \psi' = \tau \circ \phi \circ \tau.\end{cases}$$

Thus M has the torus bundle structure $M_{\psi'\psi^{-1}} = M_{\tau\phi\tau\phi^{-1}}$. □

By Theorem 2.4, and the fact that $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}^{-1} = \begin{pmatrix} -z & 1 \\ 1 & 0 \end{pmatrix}$, with suitable choice of canonical coordinates of ∂N, we can set ϕ is one of the four matrices: $\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$, $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0$, $ad - bc = 1$.

When ϕ is in the first three matrices, N_ϕ is a Seifert manifold with Euler number z. N_ϕ is E^3 manifold if $z = 0$ and is Nil manifold if $z \neq 0$. Now suppose $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $abcd \neq 0$, $ad - bc = 1$. Then by Lemma 2.8, N_ϕ is double covered by $M_{\tau\phi\tau\phi^{-1}}$. Since

$$(\tau\phi\tau\phi^{-1})_* = \tau_* \cdot \phi_* \cdot \tau_* \cdot \phi_*^{-1} = \begin{pmatrix} ad + bc & -2ab \\ -2cd & ad + bc \end{pmatrix},$$

we have

$$|\mathrm{Trace}((\tau\phi\tau\phi^{-1})_*)| = 2|ad + bc| = 2|ad - bc + 2bc| = 2|2bc + 1| > 2.$$

By Proposition 1.3, $M_{\tau\phi\tau\phi^{-1}}$ admits Sol geometry, thus N_ϕ admits Sol geometry. The first part of Proposition 1.5 is proved.

If N_ϕ also has torus bundle structure, it must have non-separating essential torus. Recall the proof of Lemma 2.6, an essential torus in N_ϕ can be non-separating only if case (a) is happened, and in this case $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ under suitable choice of canonical coordinates, and N_ϕ does have torus bundle structure. This finishes the "moreover" part of Proposition 1.5.

2.5. Lifting automorphism from semi-bundle to bundle.

Theorem 2.9. *Suppose $f\colon N_\phi \to N_\psi$ is a non-zero degree map and $f^{-1}(T')$ is a union of copies of T, where T, T' are the torus fiber of N_ϕ, N_ψ respectively. Then we* *have commute diagram*

$$\begin{tikzcd} M \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & M' \arrow[d, "p'"] \\ N_\phi \arrow[r, "f"] & N_\psi \end{tikzcd}$$

where M, M' are the torus bundle which are double covers of N_ϕ, N_ψ respectively and $\tilde{f}: M \to M'$ is a lift of f.

Proof. We only have to check $f_*(p_*(\pi_1(M))) \subset p'_*(\pi_1(M'))$.

Let $\tilde{T}$, $\tilde{T}'$ be one of the lifting of T, T' in M, M' respectively. In torus bundle M, we have the exact sequence:

$$1 \to \pi_1(\tilde{T}) \to \pi_1(M) \to \pi_1(S^1) \to 1.$$

In torus semi-bundle N_ϕ, we have another exact sequence:

$$1 \to \pi_1(T) \to \pi_1(N_\phi) \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1.$$

Since $f^{-1}(T')$ is a union of copies of T, we can assume $f(T) = T'$. Then we have the commuted diagram (every row is exact):

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(\tilde{T}) \arrow[r, "\tilde{i}_1"] \arrow[d, "(p|)_*"'] & \pi_1(M) \arrow[r, "\tilde{j}_1"] \arrow[d, "p_*"'] & \pi_1(S^1) \arrow[r] \arrow[d, "\bar{p}_*"'] & 1 \\
1 \arrow[r] & \pi_1(T) \arrow[r, "i_1"] \arrow[d, "(f|)_*"'] & \pi_1(N_\phi) \arrow[r, "j_1"] \arrow[d, "f_*"'] & \mathbb{Z}_2 * \mathbb{Z}_2 \arrow[r] \arrow[d, "\bar{f}_*"'] & 1 \\
1 \arrow[r] & \pi_1(T') \arrow[r, "i_2"] \arrow[d, "(p'|)_*"'] & \pi_1(N_\psi) \arrow[r, "j_2"] \arrow[d, "p'_*"'] & \mathbb{Z}_2 * \mathbb{Z}_2 \arrow[r] \arrow[d, "\bar{p}'_*"'] & 1 \\
1 \arrow[r] & \pi_1(\tilde{T}') \arrow[r, "\tilde{i}_2"] & \pi_1(M') \arrow[r, "\tilde{j}_2"] & \pi_1(S^1) \arrow[r] & 1,
\end{tikzcd}$$

here $\bar{p}_*$, $\bar{p}'_*$, $\bar{f}_*$ are the maps among the fundamental groups of the base spaces of fiber bundles induced by the maps among the fundamental groups of the total spaces.

We present the group $\mathbb{Z}_2 * \mathbb{Z}_2$ by $\langle a, b \mid a^2 = b^2 = 1 \rangle$ and choose the generator a, b such that $\bar{p}_*(1) = ab$, $\bar{p}'_*(1) = ab$ (here 1 is the generator of $\pi_1(S^1)$).

Since $a^2 = b^2 = 1$, so $\bar{f}_*(a)^2 = \bar{f}_*(b)^2 = 1$, then $\bar{f}_*(a), \bar{f}_*(b)$ must be of the form $ab \cdots ba$ or $ba \cdots ab$, and $\bar{f}_*(ab) = (ab)^k$ or $(ba)^k = (ab)^{-k}$. So $\bar{f}_*(\bar{p}_*(\pi_1(S^1))) \subset \bar{p}'_*(\pi_1(S^1))$.

For any $\alpha \in \pi_1(M)$, let $\beta = f_*(p_*(\alpha))$. Since $j_2(\beta) = \bar{f}_*(\bar{p}_*(\tilde{j}_1(\alpha))) \in \bar{p}'_*(\pi_1(S^1))$, and there is $\gamma \in \pi_1(M')$ such that $\bar{p}'_*(\tilde{j}_2(\gamma)) = j_2(\beta)$, so

$$j_2(p'_*(\gamma) \cdot \beta^{-1}) = \bar{p}'_*(\tilde{j}_2(\gamma)) \cdot j_2(\beta^{-1}) = j_2(\beta) \cdot j_2(\beta^{-1}) = 1.$$

Since $(p'|)_*$ is an isomorphism, there is $\delta \in \pi_1(\tilde{T}')$ such that $i_2((p'|)_*(\delta)) = p'_*(\gamma) \cdot \beta^{-1}$. We have

$$p'_*(\tilde{i}_2(\delta^{-1}) \cdot \gamma) = i_2((p'|)_*(\delta^{-1})) \cdot p'_*(\gamma) = (p'_*(\gamma) \cdot \beta^{-1})^{-1} \cdot p'_*(\gamma) = \beta.$$

So $f_*(p_*(\pi_1(M))) \subset p'_*(\pi_1(M'))$, thus $\tilde{f}$ exists. $\square$

3. The degrees of self maps of torus bundles

We are going to prove Theorem 1.6 (ref. Proposition 1.3). There are two cases to consider:

CASE 1: ϕ is conjugated to $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$. Now M_ϕ is a Seifert manifold whose Euler number of Seifert fibering $e(M_\phi)$ is equal to n.

(1.I) If $n = 0$, M_ϕ is T^3 or $S^1 \tilde{\times} S^1 \tilde{\times} S^1$. Here $\phi = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, any 2×2 integer matrix A commutes with ϕ, so M_ϕ admits self maps of any degrees.

(1.II) If $n \neq 0$, for a none zero degree map $f \colon M_\phi \to M_\phi$, by [12, Corollary 0.4], f is homotopic to a covering map $g \colon M_\phi \to M_\phi$. We can choose a suitable Seifert fibering of M_ϕ such that g is a fiber preserving map. Denote the orbifold of M_ϕ by $O(M_\phi)$. By [10, Lemma 3.5], we have:

$$\begin{cases} e(M_\phi) = e(M_\phi) \cdot \dfrac{l}{m}, \\ \deg(g) = l \cdot m, \end{cases} \tag{3.1}$$

where l is the covering degree of $O(M_\phi) \to O(M_\phi)$ and m is the fiber degree.

Since $e(M_\phi) \neq 0$, from equation (3.1) we get $l = m$. Thus $\deg(f) = \deg(g)$ is a square number. Conversely, given a square number l^2, it is easy to construct a covering map $f \colon M_\phi \to M_\phi$ of degree l^2.

CASE 2: $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is not conjugated to $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

Theorem 3.1. *Suppose ϕ is not conjugated to $\pm\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ M_ϕ admits a self map of degree $l \neq 0$ if and only if there exist a 2×2 nondegenerate integer matrix A and a positive integer k such that $l = k \cdot \epsilon \cdot \det(A)$ and $A \cdot \phi_* = (\phi^\epsilon)^k_* \cdot A$ where $\epsilon = \pm 1$.*

Proof. For a torus fiber $T \in M_\phi$, T is incompressible. Suppose $f \colon M_\phi \to M_\phi$ is a self-map of degree $l \neq 0$. By [6, Lemma 6.5], f is homotopic to $g \colon M_\phi \to M_\phi$

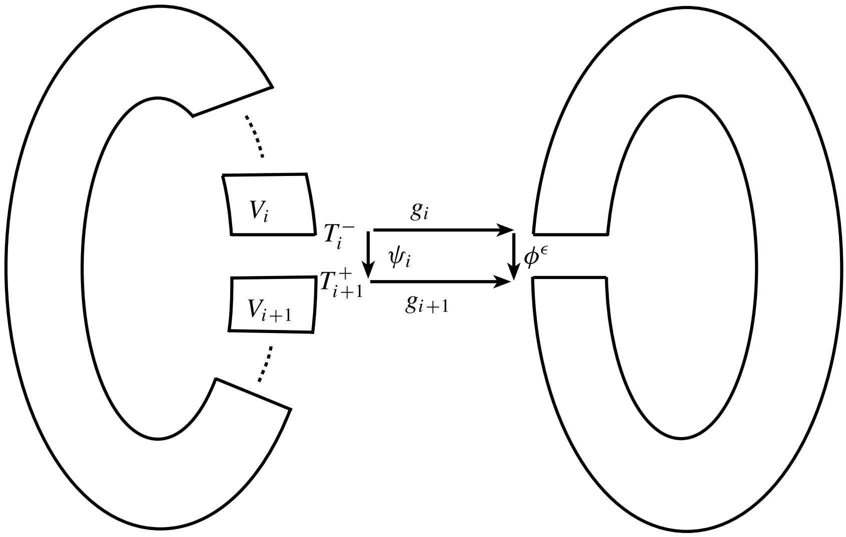


Fig. 8. Non-zero degree self-map of M_ϕ.

such that $g^{-1}(T)$ is an incompressible surface of M_ϕ. Thus by Lemma 2.5, $g^{-1}(T)$ is isotopic to a union of torus fibers.

Suppose $M_\phi \setminus g^{-1}(T)$ has k components $V_1, \ldots, V_k$. Each V_i is a $T \times I$. Denote two torus boundary components of V_i by T_i^+ and T_i^-, and the homeomorphism gluing T_i^- to T_{i+1}^+ by ψ_i see Fig. 8. Then $M_{\psi_k \circ \cdots \circ \psi_1} = M_\phi$. By choosing suitable coordinate on the torus fiber, we have $\psi_k \circ \cdots \circ \psi_0 = \phi^\epsilon$, $\epsilon = \pm 1$ according to Theorem 2.3. Below we assume $\psi_k \circ \cdots \circ \psi_0 = \phi$ (replace ϕ by ϕ^{-1} if needed). Let $\tilde{g}\colon M_\phi \setminus g^{-1}(T) \to M_\phi \setminus T$ be the map induced by g. We have the following commuted diagram:

$$\begin{array}{ccc} M_\phi \setminus g^{-1}(T) & \xrightarrow{\tilde{g}} & M_\phi \setminus T \\ \Big\downarrow{\scriptstyle \bigcup \psi_i} & & \Big\downarrow{\scriptstyle \phi^\epsilon} \\ M_\phi & \xrightarrow[g]{} & M_\phi. \end{array} \tag{3.2}$$

Denote the restriction of $\tilde{g}$ to V_i by g_i. From the commuted diagram in Fig. 8, we have:

$$g_{i+1}|_{T_{i+1}^+} \circ \psi_i = \phi^\epsilon \circ g_i|_{T_i^-}, \tag{3.3}$$

where $\epsilon = \pm 1$, $i = 1, \ldots, k$ and if $i = k$ then $i + 1$ is 1.

Since T_i^- is parallel to T_i^+, we can identify $\pi_1(T_i^-)$ with $\pi_1(T_i^+)$. Thus $(g_i|_{T_i^-})_* = (g_i|_{T_i^+})_*$ and $(\psi_k)_* \cdots (\psi_1)_* = \phi_*$ on fundamental group. The identity (3.3) deduces that:

$$\begin{aligned}
(g_1|_{T_1^+})_* \cdot \phi_* &= (g_1|_{T_1^+})_* \cdot (\psi_k)_* \cdots (\psi_1)_* \\
&= (g_{k+1}|_{T_{k+1}^+})_* \cdot (\psi_k)_* \cdots (\psi_1)_* \\
&= \phi_*^\epsilon \cdot (g_k|_{T_k^-})_* \cdot (\psi_{k-1})_* \cdots (\psi_1)_* \\
&= \phi_*^\epsilon \cdot (g_k|_{T_k^+})_* \cdot (\psi_{k-1})_* \cdots (\psi_1)_* \\
&= \cdots \\
&= (\phi^\epsilon)_*^k \cdot (g_1|_{T_1^+})_*.
\end{aligned}$$

Set $A = (g_1|_{T_1^+})_*$ and get:

$$A \cdot \phi_* = (\phi^\epsilon)_*^k \cdot A. \tag{3.4}$$

Clearly $|\deg(g)| = k|\det(A)|$. The sign of $\deg(g)$ is decided by ϵ and the sign of $\det(A)$. Thus $l = \deg(f) = \deg(g) = k \cdot \epsilon \cdot \det(A)$.

Conversely, we set $\psi_1 = \cdots = \psi_{k-1} = id$, $\psi_k = \phi$ and construct the map $\tilde{g}\colon M_\phi \setminus g^{-1}(T) \to M_\phi \setminus T$ such that $\tilde{g}|_{V_i} = (\phi^{\epsilon\cdot(i-1)} \circ A) \times id\colon T \times I \to T \times I$ for $i = 1, \ldots, k$. This construction fits the commuted diagram (3.2). Thus we get the quotient $g\colon M_\phi \to M_\phi$ whose degree is equal to $k \cdot \epsilon \cdot \det(A)$. $\square$

Suppose $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ where $p, q, r, s \in \mathbb{Z}$. We use equation (3.4) to solve p, q, r, s and then can determine l by Theorem 3.1.

(2.I) If ϕ is Anosov which means the absolute value of one eigenvalue of ϕ is larger than 1 while the other is less than 1. In this case, the k in the equation (3.4) must be equal to 1. We have:

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\epsilon \cdot \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Solve this matrix equation and get:

$$A = \begin{cases} \begin{pmatrix} p & \dfrac{br}{c} \\ r & \dfrac{cp + (d-a)r}{c} \end{pmatrix} & (\epsilon = 1), \\ \begin{pmatrix} p & \dfrac{p(d-a) - br}{c} \\ r & -p \end{pmatrix} & (\epsilon = -1) \end{cases}$$

where br/c, $(d-a)r/c$, $(p(d-a) - br)/c \in \mathbb{Z}$.

By Theorem 3.1, we have:

$$l = p^2 + \frac{(d-a)}{c} \cdot pr - \frac{b}{c} \cdot r^2.$$

(2.II) If ϕ is periodic, may assume ϕ is either $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

(A) If $\phi = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ (ϕ has order 3), the equation (3.4) means:

$$A \cdot \phi_* = \begin{cases} A & (k \equiv 0 \bmod 3), \\ \phi_*^{\epsilon} \cdot A & (k \equiv 1 \bmod 3), \\ \phi_*^{2\epsilon} \cdot A & (k \equiv 2 \bmod 3). \end{cases}$$

After solving all the above possible cases, we get:

$$A = \begin{cases} \begin{pmatrix} p & q \\ -q & p-q \end{pmatrix} & (k \equiv 1 \bmod 3,\ \epsilon = 1), \\ \begin{pmatrix} p & q \\ q-p & -p \end{pmatrix} & (k \equiv 1 \bmod 3,\ \epsilon = -1), \\ \begin{pmatrix} p & q \\ q-p & -p \end{pmatrix} & (k \equiv 2 \bmod 3,\ \epsilon = 1), \\ \begin{pmatrix} p & q \\ -q & p-q \end{pmatrix} & (k \equiv 2 \bmod 3,\ \epsilon = -1). \end{cases}$$

If $k \equiv 0 \bmod 3$, we have $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, which induces degree 0 map.

By Theorem 3.1:

$$l = \begin{cases} k \cdot (p^2 - pq + q^2) & (k \equiv 1 \bmod 3), \\ k \cdot (-p^2 + pq - q^2) & (k \equiv 2 \bmod 3). \end{cases}$$

It's easy to deduce that:

$$l = (3t+1)(p^2 - pq + q^2), \quad t,\ p, q \in \mathbb{Z}.$$

The same method is applied to the other two cases and we get:

(B) If $\phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then:

$$l = (4t+1)(p^2 + q^2), \quad t,\ p, q \in \mathbb{Z}.$$

(C) If $\phi = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, then:

$$l = (6t + 1)(p^2 - pq + q^2), \quad t,\ p,\ q \in \mathbb{Z}.$$

4. The degrees of self maps of torus semi-bundles

We are going to prove Theorem 1.7 (ref. Proposition 1.5). We will assume that torus semi-bundle N_ϕ considered in this section has no torus bundle structure, otherwise $D(N_\phi)$ is determined in Section 3.

Suppose the degree of $f\colon N_\phi \to N_\phi$ is $l \neq 0$ and T is a torus fiber of N_ϕ. By [6, Lemma 6.5], f is homotopic to $g\colon N_\phi \to N_\phi$ such that $g^{-1}(T)$ is incompressible in N_ϕ. Thus by Lemma 2.6 and its proof (also ref. the proof of Theorem 2.4), we have $g^{-1}(T)$ is isotopic to either a union of torus fibers, or a union of torus fibers of another semi-bundle structure which is isomorphic to the original one. Also the later case happen only if N_ψ is a Nil manifold. Note by Theorem 2.9 and the proof in Section 3 (1.II), Nil 3-manifolds admits no orientation reversing homeomorphism.

Suppose now $g^{-1}(T)$ has k connected components, then $N_\phi \setminus g^{-1}(T)$ has two copies of N, denoted by V_0 and V_k, and $k-1$ copies of $T \times I$, denoted by V_i, $i = 1, \ldots, k-1$. Denote the boundaries of V_0 and V_k by T_0^- and T_k^+, the boundaries of V_i by T_i^+ and T_i^-, $i = 1, \ldots, k-1$, and the gluing map from T_i^- to T_{i+1}^+ by ψ_i ($i = 0, \ldots, k-1$) see Fig. 9.

Then $N_{\psi_{k-1}\circ\cdots\circ\psi_0} = N_\phi$, and $\psi_{k-1} \circ \cdots \circ \psi_0 = \phi^\epsilon$, $\epsilon = \pm 1$ by Theorem 2.4 (with a suitable orientation of the canonical coordinate). Below we assume $\psi_{k-1} \circ \cdots \circ \psi_0 = \phi$ (replace ϕ by ϕ^{-1} if needed). Let $\tilde{g}\colon N_\phi \setminus g^{-1}(T) \to N_\phi \setminus T$ be the map induced by g, and we have commuted diagram:

$$\begin{tikzcd} N_\phi \setminus g^{-1}(T) \arrow[r, "\tilde{g}"] \arrow[d, "\cup \psi_i"'] & N_\phi \setminus T \arrow[d, "\phi^\epsilon"] \\ N_\phi \arrow[r, "g"'] & N_\phi. \end{tikzcd} \tag{4.1}$$

Since T_i^+ is parallel to T_i^-, we can identity $\pi_1(T_i^+)$ with $\pi_1(T_i^-)$ ($i = 0, \ldots, k-1$). Thus $(\psi_{k-1})_* \cdots (\psi_0)_* = \phi_*$ on fundamental group. Denote the restriction of $\tilde{g}$ on V_i by g_i. Then $g\colon V_i \to N_1$ if i even, and $g\colon V_i \to N_2$ if i odd.

Lemma 4.1. *Under the canonical basis* (l_0, l_∞), $(g_0|_{T_0^-})_*$ *is of the form* $\begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$ *where* $n \neq 0$, $m, n \in \mathbb{Z}$, *and so is* $(g_k|_{T_k^+})_*$.

Proof. Let $g\colon N \to N$ be a proper map, we argue that under the basis (l_0, l_∞), $(g|_{\partial N})_*$ is of the form $\begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$ where $n \neq 0$, $m, n \in \mathbb{Z}$.

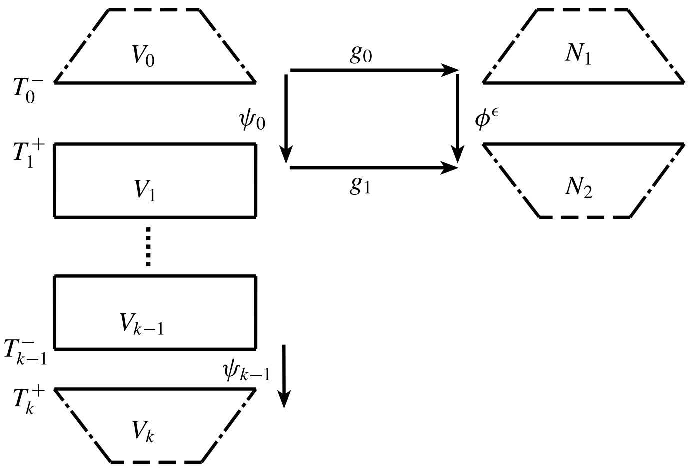


Fig. 9. Non-zero degree self-map of N_ϕ.

Choose a presentation $\pi_1(N) = \langle a, b \mid a = bab \rangle$ with $l_0 = a^2$ and $l_\infty = b$. Suppose $g_*(a) = a^{m'} b^q$, $g_*(b) = a^p b^n$. Since $g_*(a) = g_*(b) g_*(a) g_*(b)$, we get:

$$a^{m'} b^q = a^p b^n a^{m'} b^q a^p b^n = a^{m'+2p} b^{(-1)^{m'+p} \cdot n + (-1)^p \cdot q + n}.$$

Thus:

$$\begin{cases} m' = m' + 2p, \\ q = (-1)^{m'+p} \cdot n + (-1)^p \cdot q + n, \end{cases} \Longrightarrow \begin{cases} p = 0, \\ m' \text{ odd}, \end{cases} \quad \text{or} \quad \begin{cases} p = 0, \\ n = 0. \end{cases}$$

Abandon the case that $p = n = 0$ for g_0 is non-zero degree map and let $m' = 2m + 1$, we get: $g_*(a) = a^{2m+1} b^q$, $g_*(b) = b^n$.

Since $\pi_1(\partial N) = \langle a^2, b \mid [a^2, b] = 1 \rangle$ and $g_*(a^2) = a^{2m+1} b^q a^{2m+1} b^q = a^{4m+2}$, we have

$$(g|_{\partial N})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}. \qquad \square$$

Theorem 4.2. *If N_ϕ has no torus bundle structure, then N_ϕ admits a self map of degree $l \neq 0$ if and only if there exist a positive integer k and two integer matrices A_1, A_2 of form $\begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$, $m, n \in \mathbb{Z}$, $n \neq 0$, satisfying the following equation*:

$$A_2 \cdot \phi_* = \begin{cases} (\phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot \tau_*)^{s-1} \cdot \phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot A_1 & (k = 2s), \\ (\phi_*^{\epsilon} \cdot \tau_* \cdot \phi_*^{-\epsilon} \cdot \tau_*)^{s} \cdot \phi_*^{\epsilon} \cdot A_1 & (k = 2s+1), \end{cases}$$

such that $l = k \cdot \epsilon \cdot \det(A_1)$ where $\epsilon = \pm 1$.

Proof. From Fig. 9, we know that:

$$g_{i+1}|_{T^+_{i+1}} \circ \psi_i = \begin{cases} \phi^{\epsilon} \circ g_i|_{T^-_i} & (i \equiv 0 \bmod 2), \\ \phi^{-\epsilon} \circ g_i|_{T^-_i} & (i \equiv 1 \bmod 2), \end{cases} \tag{4.2}$$

where $\epsilon = \pm 1$, $i = 0, \dots, k-1$.

Thus if $k = 2s$ is even, then:

$$\begin{aligned} (g_k|_{T^+_k})_* \cdot \phi_* &= (g_k|_{T^+_k})_* \cdot (\psi_{k-1})_* \cdots (\psi_0)_* && \text{by Fig. 9} \\ &= \phi_*^{-\epsilon} \cdot (g_{k-1}|_{T^-_{k-1}})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* && \text{by (4.2)} \\ &= \phi_*^{-\epsilon} \cdot \tau_* \cdot (g_{k-1}|_{T^+_{k-1}})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* && \text{by Lemma 2.8} \\ &= \cdots \\ &= (\phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot \tau_*)^{s-1} \cdot \phi_*^{-\epsilon} \cdot \tau_* \cdot \phi_*^{\epsilon} \cdot (g_0|_{T^-_0})_*. \end{aligned} \tag{4.3}$$

If $k = 2s + 1$ is odd, then:

$$\begin{aligned} (g_k|_{T^+_k})_* \cdot \phi_* &= (g_k|_{T^+_k})_* \cdot (\psi_{k-1})_* \cdots (\psi_0)_* \\ &= \phi_*^{\epsilon} \cdot (g_{k-1}|_{T^-_{k-1}})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* \\ &= \phi_*^{\epsilon} \cdot \tau_* \cdot (g_{k-1}|_{T^+_{k-1}})_* \cdot (\psi_{k-2})_* \cdots (\psi_0)_* \\ &= \cdots \\ &= (\phi_*^{\epsilon} \cdot \tau_* \cdot \phi_*^{-\epsilon} \cdot \tau_*)^{s} \cdot \phi_*^{\epsilon} \cdot (g_0|_{T^-_0})_*. \end{aligned} \tag{4.4}$$

It is easy to see that $|\deg(g)| = k|\det(g_0|_{T^-_0})_*|$. The sign of $\deg(g)$ is decided by both ϵ and the sign of $\det(g_0|_{T^-_0})_*$. Thus $l = \deg(f) = \deg(g) = k \cdot \epsilon \cdot \det(g_0|_{T^-_0})_*$. Finally by applying Lemma 4.1, we finish the proof of one direction of Theorem 4.2.

Conversely, if given k, A_1, A_2, then we can easily construct the maps $g_0, g_k \colon N \to N$ such that $(g_0|_{T^-_0})_* = A_1$, $(g_k|_{T^+_k})_* = A_2$. Set $\psi_0 = \cdots = \psi_{k-2} = id$, $\psi_{k-1} = \phi$ and $g_i : T \times I \to N$ $(i = 1, \dots, k-1)$ is a map such that:

$$g_i|_{T^+_i} = \begin{cases} \phi^{\epsilon} \circ g_{i-1}|_{T^-_{i-1}} & (i \equiv 1 \bmod 2), \\ \phi^{-\epsilon} \circ g_{i-1}|_{T^-_{i-1}} & (i \equiv 0 \bmod 2). \end{cases}$$

Then $\tilde{g} = \bigcup g_i$ fits the commutative diagram (4.1). Thus we get the quotient map $g \colon N_\phi \to N_\phi$ of degree $k \cdot \epsilon \cdot \det(A_1)$. □

Given $\phi_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ and suppose $(g_0|_{T^-_0})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}$, $(g_k|_{T^+_k})_* = \begin{pmatrix} 2m'+1 & 0 \\ 0 & n' \end{pmatrix}$ where $m, n, m', n' \in \mathbb{Z}$.

CASE 1: $abcd \neq 0$, $ad - bc = 1$. (It should be noted that $(\tau\phi\tau\phi^{-1})_*$ is Anosov.)

Since $g: N_\phi \to N_\phi$ satisfies $g^{-1}(T)$ is copies of torus fiber, by Theorem 2.9 g can be lift to $g': M_{\tau\phi\tau\phi^{-1}} \to M_{\tau\phi\tau\phi^{-1}}$. By the argument of Anosov monodromy case in Section 3, the degree of g' in the S^1 direction is 1. So we have $k = 1$.

By equation (4.4), we have:

$$(g_1|_{T_1^+})_* \cdot \phi_* = \phi_*^\epsilon \cdot (g_0|_{T_0^-})_*.$$

If $\epsilon = 1$, then:

$$\begin{pmatrix} 2m'+1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation we have:

$$\begin{cases} n = 2m+1, \\ m' = m, \\ n' = 2m+1. \end{cases}$$

Thus $(g_0|_{T_0^-})_* = \begin{pmatrix} 2m+1 & 0 \\ 0 & 2m+1 \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = (2m+1)^2.$$

If $\epsilon = -1$, then:

$$\begin{pmatrix} 2m'+1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \begin{pmatrix} 2m+1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation we have:

$$\begin{cases} n = -(2m'+1), \\ (2m'+1) \cdot a = (2m+1) \cdot d, \\ n' = -(2m+1). \end{cases}$$

Suppose $(2m+1) = u \cdot a/\gcd(a, d)$, then both u and $a/\gcd(a, d)$ must be odd. Similarly, since $n = 2m'+1 = -u \cdot d/\gcd(a, d)$ is odd, then $d/\gcd(a, d)$ is odd also.

Thus $(g_0|_{T_0^-})_* = \begin{pmatrix} u \cdot a/\gcd(a, d) & 0 \\ 0 & -u \cdot d/\gcd(a, d) \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = u^2 \cdot \frac{ad}{\gcd(a, d)^2}.$$

This degree can be realized here if and only if $ad/\gcd(a, d)^2$ is odd.

CASE 2: $abcd = 0$. Then there are three subcases.

(2.I) $\phi_* = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$.

In this case N_ϕ is a torus bundle which has been discussed in Section 3.

(2.II) $\phi_* = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$, or equivalently $\begin{pmatrix} z & 1 \\ 1 & 0 \end{pmatrix}$.

When $z \neq 0$, we discuss the following four possible cases:

(A) If $\epsilon = 1$ and $k = 2s$ is even, then by equation (4.3), we have the following equation:

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & zk \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution.

(B) If $\epsilon = -1$ and $k = 2s$ is even, then by equation (4.3):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 1 & 0 \\ zk & -1 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution either.

(C) If $\epsilon = 1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} 0 & 1 \\ 1 & kz \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

Solving this matrix equation:

$$\begin{cases} n = (-1)^s (2m' + 1), \\ n' = (-1)^s (2m + 1), \\ n' = (-1)^s kn. \end{cases}$$

So $2m + 1 = kn$, thus k is odd, if k exists.

Then $(g_k|_{T_k^+})_* = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & k(2m' + 1) \end{pmatrix}$ which means:

$$\deg(g) = k \cdot \epsilon \cdot \det((g_0|_{T_0^-})_*) = k \cdot \epsilon \cdot \det((g_k|_{T_k^+})_*) = k^2 \cdot (2m' + 1)^2.$$

This degree is an odd square number. In another hand, when $k = 1$, all odd square number can be realized as a degree: $(g_k|_{T_k^+})_* = \begin{pmatrix} 2m' + 1 & 0 \\ 0 & 2m' + 1 \end{pmatrix}$.

(D) If $\epsilon = -1$ and $k = 2s + 1$ is odd, then by equation (4.4):

$$\begin{pmatrix} 2m' + 1 & 0 \\ 0 & n' \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix} = (-1)^s \begin{pmatrix} -zk & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2m + 1 & 0 \\ 0 & n \end{pmatrix}.$$

This equation has no solution.

When $z = 0$, the same method will show that $\deg(g)$ is odd, and all odd numbers can be realized.

(2.III) $\phi_* = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$.

In this case, $\deg(g)$ can be determined as in case (2.II).

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References

[1] X.M. Du: *On self-mapping degrees of S^3-geometry 3-manifolds*, to appear in Acta Math. Sin. (Engl. Ser.).
[2] H. Duan and S. Wang: *The degrees of maps between manifolds*, Math. Z. **244** (2003), 67–89.
[3] E. Ghys and V. Sergiescu: *Stabilité et conjugaison différentiable pour certains feuilletages*, Topology **19** (1980), 179–197.
[4] A. Hatcher: *Notes on basic 3-manifold topology*, `http://www.math.cornell.edu/~hatcher/`.
[5] C. Hayat-Legrand, E. Kudryavtseva, S.C. Wang and H. Zieschang: *Degrees of self-mappings of Seifert manifolds with finite fundamental groups*, Rend. Istit. Mat. Univ. Trieste **32** (2001), 131–147.
[6] J. Hempel: 3-Manifolds, Princeton Univ. Press, Princeton, N.J., 1976.
[7] K. Ireland and M. Rosen: A Classical Introduction to Modern Number Theory, second edition, Graduate Texts in Mathematics **84**, Springer, New York, 1990.
[8] S.V. Matveev and A.A. Perfil'ev: *Periodicity of degrees of mappings between Seifert manifolds*, Dokl. Akad. Nauk **395** (2004), 449–451.
[9] M. Sakuma: *Involutions on torus bundles over S^1*, Osaka J. Math. **22** (1985), 163–185.
[10] P. Scott: *The geometries of 3-manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
[11] W.P. Thurston: *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. (N.S.) **6** (1982), 357–381.
[12] S.C. Wang: *The π_1-injectivity of self-maps of nonzero degree on 3-manifolds*, Math. Ann. **297** (1993), 171–189.

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