THE TWIST SUBGROUP OF
THE MAPPING CLASS GROUP OF
A NONORIENTABLE SURFACE

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Abstract

Let $\mathcal{T}(N)$ be the subgroup of the mapping class group of a nonorientable surface N (possibly with punctures and/or boundary components) generated by twists about two-sided circles. We obtain a simple generating set for $\mathcal{T}(N)$. As an application we compute the first homology group (abelianization) of $\mathcal{T}(N)$.

1. Introduction

Let $N_{g,s}^n$ be a smooth, nonorientable, compact surface of genus g with s boundary components and n punctures. If s and/or n is zero then we omit it from the notation. If we do not want to emphasise the numbers g, s, n, we simply write N for a surface $N_{g,s}^n$. Recall that N_g is a connected sum of g projective planes and $N_{g,s}^n$ is obtained from N_g by removing s open disks and specifying the set $\Sigma = \{z_1, \ldots, z_n\}$ of n distinguished points in the interior of N_g.

Let $\text{Diff}(N)$ be the group of all diffeomorphisms $h: N \to N$ such that h is the identity on each boundary component and $h(\Sigma) = \Sigma$. By $\mathcal{M}(N)$ we denote the quotient group of $\text{Diff}(N)$ by the subgroup consisting of maps isotopic to the identity, where we assume that isotopies fix Σ and are the identity on each boundary component. $\mathcal{M}(N)$ is called the *mapping class group* of N. The mapping class group of an orientable surface is defined analogously, but we consider only orientation preserving maps.

If we assume that maps and isotopies fix the set Σ pointwise then we obtain the so-called *pure mapping class group* $\mathcal{PM}(N_{g,s}^n)$. If we also require that maps preserve the local orientation around each of the punctures then we obtain the group $\mathcal{PM}^+(N_{g,s}^n)$. It is an easy observation that $\mathcal{PM}(N_{g,s}^n)$ is the subgroup of index $n!$ in $\mathcal{M}(N_{g,s}^n)$ and $\mathcal{PM}^+(N_{g,s}^n)$ is of index 2^n in $\mathcal{PM}(N_{g,s}^n)$.

Define also $\mathcal{T}(N)$ to be the *twist subgroup* of $\mathcal{M}(N)$, that is the subgroup generated by Dehn twists about two-sided circles. The main goal of the paper is to establish

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basic algebraic properties of the group $\mathcal{T}(N)$. Namely, we find a simple generating set for this group and compute its abelianization.

By abuse of notation we will use the same letter for a map and its isotopy class and we will use the functional notation for the composition of diffeomorphisms.

1.1.   Background.   The study of algebraic properties of mapping class groups of an orientable surface goes back to the work of Dehn [3] and Nielsen [14, 15, 16]. Probably the best modern exposition of this reach theory is a survey article by Ivanov [6]. On the other hand, the nonorientable case has not been studied much. The first significant result is due to Lickorish [11], who proved that the twist subgroup $\mathcal{T}(N_g)$ is a subgroup of index 2 in the mapping class group $\mathcal{M}(N_g)$. Moreover, $\mathcal{M}(N_g)$ is generated by Dehn twists and a so-called "crosscap slide" (or a "Y-homeomorphism"). Later Chillingworth [2] found a finite generating set for the groups $\mathcal{T}(N_g)$ and $\mathcal{M}(N_g)$, and Birman and Chillingworth [1] showed how these generating sets can be derived from the known properties of the mapping class group of the orientable double cover of N_g.

These studies were continued much later by Korkmaz [10], who found finite generating sets for the groups $\mathcal{M}(N_g^n)$ and $\mathcal{PM}(N_g^n)$. Korkmaz [9, 10] also computed the first integral homology group of $\mathcal{M}(N_g^n)$, and under additional assumption $g \geq 7$, of $\mathcal{PM}(N_g^n)$. He also showed [10] that $\mathcal{T}(N_g^n)$ is of index $2^{n+1}n!$ in $\mathcal{M}(N_g^n)$, provided $g \geq 7$.

Recently [17], we extended some of the above results to arbitrary mapping class groups $\mathcal{M}(N_{g,s}^n)$ and $\mathcal{PM}(N_{g,s}^n)$, provided $g \geq 3$. In particular we obtained simple generating sets for these groups and we computed their abelianizations.

Finally, let us mention that recently Wahl [20] proved some stability theorems for the homology of mapping class groups of nonorientable surfaces and using the ideas of Madsen and Weiss [13] she managed to identify the stable rational cohomology of $\mathcal{M}(N)$.

Another very promising project is the work of Szepietowski [19] who showed a method to obtain a presentation for the group $\mathcal{M}(N_{g,s}^n)$. Using this technique he managed [18] to derive a presentation of the group $\mathcal{M}(N_4)$.

1.2.   Main results.   The present paper is a natural continuation of the results mentioned above, especially it should be thought as a continuation of [17]. Namely, we study basic algebraic properties of the twist subgroup of the mapping class group of a nonorientable surface of genus $g \geq 3$. The crucial observation which makes such a study possible is that $\mathcal{T}(N_{g,s}^n)$ is a subgroup of index 2 in $\mathcal{PM}^+(N_{g,s}^n)$ (hence of index $2^{n+1}n!$ in $\mathcal{M}(N_{g,s}^n)$—see Corollaries 6.3 and 6.4). Using this observation, we obtain surprisingly simple generating set for the twist subgroup—see Theorem 6.2. Moreover, we compute the first integral homology group (abelianization) of this subgroup—see Theorem 8.1.

The reason for the assumption $g \geq 3$ is the exceptional (and non-trivial) nature of the cases $g = 1$ and $g = 2$. Most of our analysis make no sense in these cases, hence

we leave them for future consideration.

2. Preliminaries

By a *circle* on N we mean an unoriented simple closed curve on $N \setminus \Sigma$, which is disjoint from the boundary of N. Usually we identify a circle with its image. Moreover, as in the case of diffeomorphisms, we will use the same letter for a circle and its isotopy class. According to whether a regular neighbourhood of a circle is an annulus or a Möbius strip, we call the circle *two-sided* or *one-sided* respectively. We say that a circle is *generic* if it bounds neither a disk with less than two punctures nor a Möbius strip disjoint from Σ.

Let a be a two-sided circle. By definition, a regular neighbourhood of a is an annulus, so if we fix one of its two possible orientations, we can define the *right Dehn twist* t_a about a in the usual way. We emphasise that since we are dealing with nonorientable surfaces, there is no canonical way to choose the orientation of S_a. Therefore by a twist about a we always mean one of two possible twists about a (the second one is then its inverse). By a *boundary twist* we mean a twist about a circle isotopic to a boundary component. It is known that if a is not generic then the Dehn twist t_a is trivial. In particular a Dehn twist about the boundary of a Möbius strip is trivial—see Theorem 3.4 of [4].

Other important examples of diffeomorphisms of a nonorientable surface are the *crosscap slide* and the *puncture slide*. They are defined as a slide of a crosscap and of a puncture respectively, along a loop. The general convention is that one considers only crosscap slides along one—sided simple loops (in such a form it was introduced by Lickorish [11]; for precise definitions and properties see [10]).

The following two propositions follow immediately from the above definitions.

Proposition 2.1. *Let N_a be an oriented regular neighbourhood of a two-sided circle a in a surface N, and let $f : N \to N$ be any diffeomorphism. Then $f t_a f^{-1} = t_{f(a)}$, where the orientation of a regular neighbourhood of $f(a)$ is induced by the orientation of $f(N_a)$.*

Proposition 2.2. *Let K be a subsurface of N which is a Klein bottle with one boundary component ξ, and let y be a crosscap slide on K such that $y^2 = t_\xi$. Then for any diffeomorphism $f : N \to N$, $f y f^{-1}$ is a crosscap slide on $f(K)$ such that*

$$(f y f^{-1})^2 = t_{f(\xi)},$$

where the orientation of a regular neighbourhood of $f(\xi)$ is induced via f by the orientation of a regular neighbourhood of ξ.

One of our main tools in studying properties of mapping class groups is the so-called *lantern relation*. The proof can be found in Section 4 of [7].
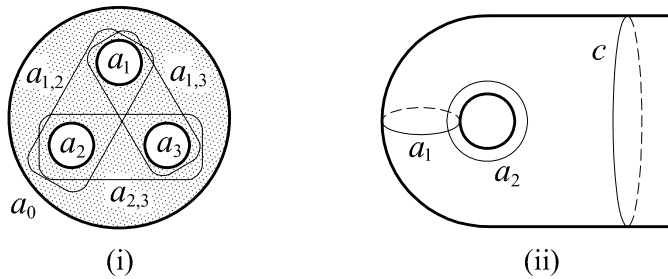
Fig. 1. Circles of the lantern and torus with a hole relations.

Proposition 2.3. *Let S be a sphere with four holes embedded in a surface $N \setminus \Sigma$ and let a_0, a_1, a_2, a_3 be the boundary circles of S. Define also $a_{1,2}$, $a_{2,3}$, $a_{1,3}$ as in Fig. 1 (i) and assume that the orientations of regular neighbourhoods of these seven circles are induced from the orientation of S. Then*

$$t_{a_0} t_{a_1} t_{a_2} t_{a_3} = t_{a_{1,2}} t_{a_{2,3}} t_{a_{1,3}}.$$

Finally, recall the *torus with a hole relation*—see Lemma 3 of [12]. It can be also thought as an instance of the so-called *star* relation [5].

Proposition 2.4. *Let S be a torus with one boundary component c embedded in a surface $N \setminus \Sigma$ and let a_1 and a_2 be two two-sided circles as in Fig. 1 (ii). If the orientations of regular neighbourhoods of a_1, a_2 and c are induced from the orientation of S then*

$$(t_{a_1} t_{a_2})^6 = t_c.$$

3. Two models for a nonorientable surface

Let $g = 2r + 1$ for g odd and $g = 2r + 2$ for g even. Represent the surface $N = N_{g,s}^n$ as a connected sum of an orientable surface of genus r and one or two projective planes (one for g odd and two for g even). Figs. 2 and 3 show this model of N—in these figures the big shaded disks represent crosscaps, hence their interiors are to be removed and then the antipodal points on each boundary component are to be identified. The smaller shaded disks represent components of ∂N (we will call them *holes*).

It is well known that $N = N_{g,s}^n$ can be also represented as a sphere with n punctures, s holes and g crosscaps—cf. Fig. 4. In order to distinguish this model from the previous one (provided by Figs. 2 and 3), let us denote it by $\tilde{N}$. The goal of this section is to construct an explicit homeomorphism $\Phi \colon N \to \tilde{N}$.

Let $a_1, \ldots, a_g$ and $u_1, \ldots, u_s$ be two-sided circles on N as in Figs. 2 and 3 (u_i are the boundary circles). Define also $v_1, \ldots, v_{n+s}$ to be the arcs as in Fig. 5 (i) and
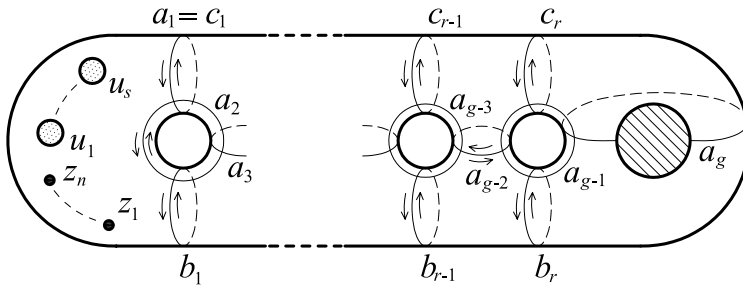
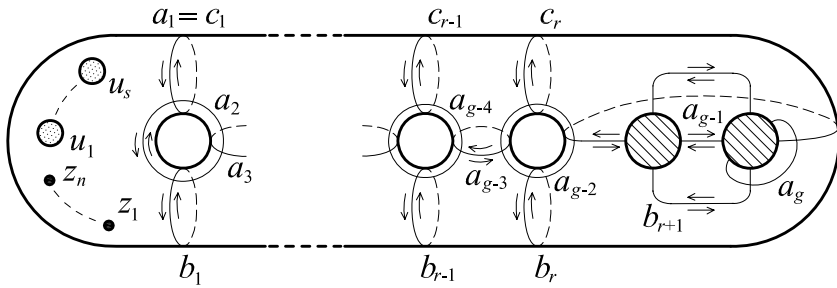Fig. 2. Circles a_i, b_i, c_i and u_i for $g = 2r + 1$.

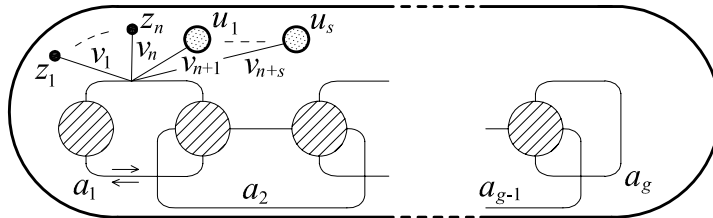Fig. 3. Circles a_i, b_i, c_i and u_i for $g = 2r + 2$.

Fig. 4. Surface $\tilde{N}$ as sphere with crosscaps.
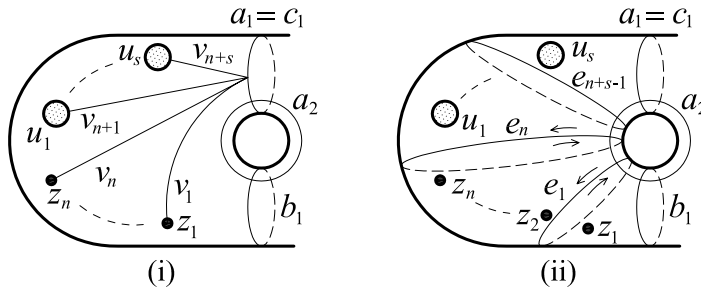

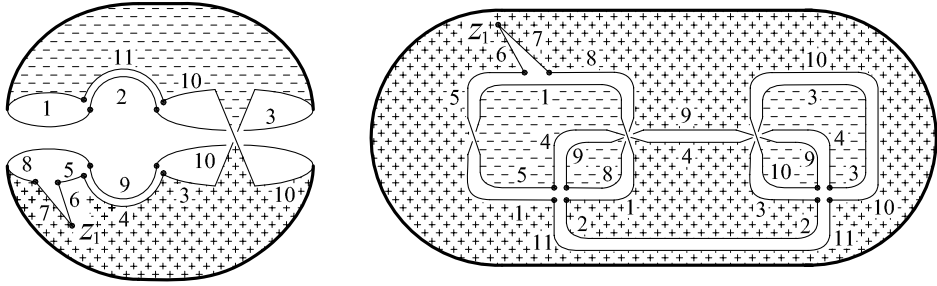Fig. 5. Circles e_i and arcs v_i.

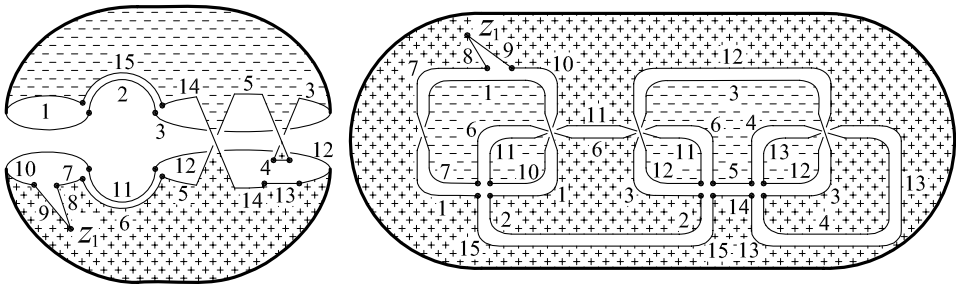Fig. 6. Cutting N and $\tilde{N}$ for $(g, s, n) = (3, 0, 1)$.

Fig. 7. Cutting N and $\tilde{N}$ for $(g, s, n) = (4, 0, 1)$.

observe that if we cut N along the circles $a_1, \ldots, a_g$ and arcs $v_1, \ldots, v_{n+s}$ we obtain a polygon Δ with sides

$$a_1, \ldots, a_{g-1}, a_g, a_{g-1}, \ldots, a_1, v_1, v_1, \ldots, v_n, v_n,$$

$$v_{n+1}, u_1, v_{n+1}, \ldots, v_{n+s}, u_s, v_{n+s}, a_1, \ldots, a_{g-1}, a_g, a_{g-1}, \ldots, a_2$$

where the labels indicate from which circle/arc the edge came from. Identical polygon can be obtained by cutting the surface $\tilde{N}$ along the circles $a_1, \ldots, a_g$ and arcs $v_1, \ldots, v_{n+s}$ indicated in Fig. 4. Moreover, it is not difficult to see that the identification patterns required to reconstruct N and $\tilde{N}$ form Δ are identical (cf. Figs. 6 and 7). This implies that there exists a homeomorphism $\Phi: N \to \tilde{N}$ which maps circles $a_1, \ldots, a_g, u_1, \ldots, u_s$ and arcs $v_1, \ldots, v_{n+s}$ in Figs. 2, 3, 5 (ii) to the circles/arcs with the same labels in Fig. 4.

The above geometric description of Φ is very convenient because it provides a simple method for transferring circles between two models of N. In fact, if c is a circle on N then c becomes a collection of arcs in Δ. Moreover, up to isotopy we can assume that c does not pass through any of the vertices of Δ. Since Δ is simply connected each of this arcs is uniquely determined by the position of its endpoints. To obtain the image $\Phi(c)$ it is enough to reconstruct the surface $\tilde{N}$ from Δ keeping track
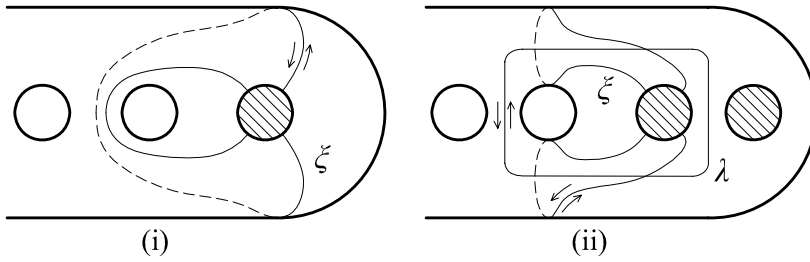
Fig. 8. Circles ξ and λ.

of the collection of arcs composing c. In practise this can be easily done using pictures like Figs. 6 and 7. Moreover, it is not difficult to see that we can transfer not only the circles but also the orientations of their neighbourhoods (if it exists)—small plus and minus signs in Figs. 6 and 7 indicate our choice of the orientation of Δ. Of course the above procedure works as well in the other direction.

Keeping in mind the above description, from now on we will transfer circles form N to $\tilde{N}$ and vice versa without further comments.

4. Generators for the group $\mathcal{PM}^+(N_{g,s}^n)$

Let

$$\mathcal{C} = \{a_2, \ldots, a_{g-1}, b_1, \ldots, b_r, c_1, \ldots, c_r, e_1, \ldots, e_{n+s-1}, u_1, \ldots, u_s\}$$

for g odd, and

$$\mathcal{C} = \{a_2, \ldots, a_{g-1}, b_1, \ldots, b_{r+1}, c_1, \ldots, c_r, e_1, \ldots, e_{n+s-1}, u_1, \ldots, u_s\}$$

for g even, where the circles a_i, b_i, c_i, u_i are as in Figs. 2 and 3 and e_i are as in Fig. 5 (ii). Moreover, these figures indicate the orientations of local neighbourhoods of circles in $\mathcal{C}$. We did it by indicating the direction of twists about these circles. Therefore by a twist about one of the circles in $\mathcal{C}$ we will always mean the twist determined by this particular choice of orientation (the general rule is that we consider *right Dehn twists*, that is if we approach the circle of twisting we turn to the right. Define also y to be a crosscap slide supported on a Klein bottle cut of by the circle ξ indicated in Fig. 8. To be more precise, in terms of the model $\tilde{N}$, let C_{g-1} and C_g be crosscaps as in Fig. 9. The same figure shows the circle ξ—it cuts off these two crosscaps. Hence we can define y to be a slide of the crosscap C_g along the path μ indicated in that figure. In particular $y^2 = t_\xi$.

Now we are ready to state the main theorem of this section, which is a simplification of the known generating set for the group $\mathcal{PM}^+(N_{g,s}^n)$.
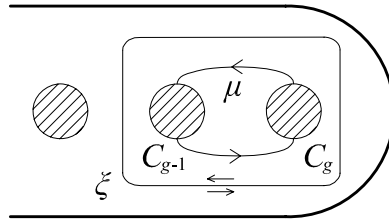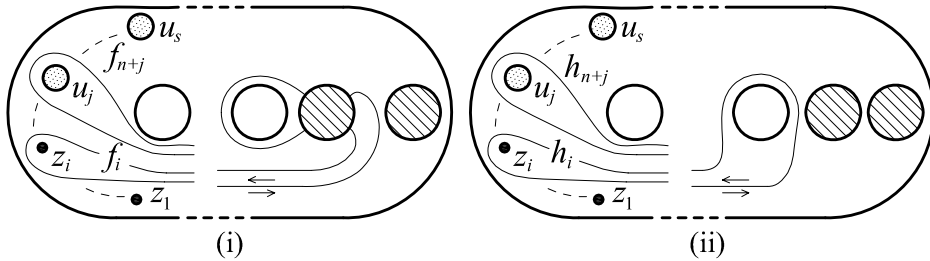
Fig. 9. Circle ξ.

Fig. 10. Circles f_i and h_i.

Theorem 4.1. *Let $g \geq 3$. Then the mapping class group $\mathcal{PM}^+(N_{g,s}^n)$ is generated by $\{t_l, y \mid l \in \mathcal{C}\}$.*

Proof. By Theorem 5.2 of [17] and by Propositions 2.1 and 2.2, the group $\mathcal{PM}^+(N_{g,s}^n)$ is generated by

- $\{t_l, y \mid l \in \mathcal{C}'\}$ if g is odd,
- $\{t_l, (t_{a_{g-2}}t_{a_{g-1}})^{-1} y(t_{a_{g-2}}t_{a_{g-1}}), t_{b_r}^{-1}t_\lambda t_{b_r} \mid l \in \mathcal{C}'\}$ if g is even,

where λ is as in Fig. 8 (ii) and

$$\mathcal{C}' = \mathcal{C} \cup \{f_1, \ldots, f_{n+s}\}$$

for $f_1, \ldots, f_{n+s}$ as in Fig. 10 (i) (Fig. 10 (i) defines f_i for g even; for g odd just forget the second crosscap).

Therefore to complete the proof it is enough to show that if $G = \langle t_l, y \mid l \in \mathcal{C} \rangle$ then $t_{f_1}, \ldots, t_{f_{n+s}} \in G$, and $t_\lambda \in G$ for g even.

Let $h_1, \ldots, h_{n+s}$ be circles as in Fig. 10 (ii) (as before, for g odd forget the second crosscap). We claim that

- $f_i = t_{c_r}^{-1} t_{a_{g-1}}^{-2} y^{-1} t_{b_r}(h_i)$ if g is odd,
- $f_i = y^{-1} t_{b_{r+1}}^{-1} t_{a_{g-1}}^{-1} t_\lambda t_{a_{g-2}}^{-1} t_{a_{g-1}}^{-1} t_{b_r} t_{a_{g-2}}^2 t_{b_r}(h_i)$ if g is even.

In fact, using the procedure of transferring circles between two models N and $\tilde{N}$ described in Section 3, it is not difficult to check that Fig. 11 (i) shows the circle $t_{b_r}(h_i)$
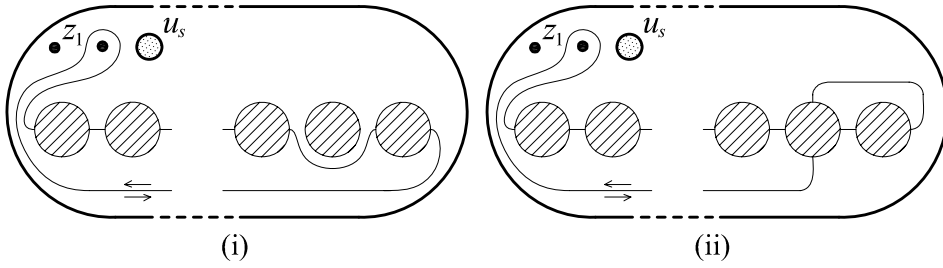
Fig. 11. Circles $t_{b_r}(h_i)$ and $y^{-1}t_{b_r}(h_i)$ for g odd.

on $\tilde{N}$ and Fig. 11 (ii) shows $y^{-1}(t_{b_r}(h_i))$. Then transferring this circle back to N easily leads to the first of the above relations. The second one can be proved analogously.

By Lemma 3.3 of [17], $t_{h_i} \in G$ for $i = 1, \dots, n + s$. Hence by Proposition 2.1, $t_{f_i} \in G$ provided $t_\lambda \in G$. But this follows from the relation

$$\lambda = t_{b_{r+1}} y(a_{g-2}). \qquad \square$$

5. The action of $\mathcal{M}(N_{g,s}^n)$ on $H_1(N_{g,s}^n, \mathbb{R})$

It is known that $H_1(N, \mathbb{R})$ has a basis in which every linear map $f_*: H_1(N, \mathbb{R}) \to H_1(N, \mathbb{R})$, induced by a diffeomorphism $f: N \to N$, has a matrix with integral co-efficients. Therefore we can define the *determinant homomorphism* $D: \mathcal{PM}^+(N) \to \mathbb{Z}_2$ as follows: $D(f) = \det(f_*)$ (we use the multiplicative notation for the group $\mathbb{Z}_2$).

Lemma 5.1. *Let c be a two-sided nonseparating circle on N. Then $D(t_c) = 1$.*

Proof. It is an easy topological fact, that if c_1 and c_2 are two-sided, nonseparating circles in N such that both $N \setminus c_1$ and $N \setminus c_2$ are either both orientable or both non-orientable, then $f(c_1) = c_2$ for some diffeomorphism of N. Let us stress the fact that f may not be the identity on ∂N, hence we can not assume that f induces an element of the group $\mathcal{M}(N)$, however this is of no importance to what follows. In particular, $(t_{c_1})_*$ and $(t_{c_2})_*$ are conjugate in $\mathrm{Aut}(H_1(N, \mathbb{R}))$, hence $D(t_{c_1}) = D(t_{c_2})$. Moreover, if g is odd then there is no nonseparating two-sided circle c on N such that $N \setminus c$ is orientable. Therefore, to prove the lemma it is enough to show that $D(t_{a_1}) = 1$ and that $D(t_{b_{r+1}}) = 1$ if g is even. This can be easily done—we skip the computations. $\square$

Lemma 5.2. *Let c be a two-sided separating circle on N. Then $(t_c)_*: H_1(N, \mathbb{R}) \to H_1(N, \mathbb{R})$ is the identity map.*

Proof. Since c is two-sided, we can fix an orientation of a regular neighbourhood of c. Therefore, for any circle a transverse to c, we can define the algebraic intersection number $I(c, a)$ in a usual way (we do not claim that $I(c, a)$ has any particular

properties—it just counts the points $c \cap a$ with appropriate signs). By the definition of a twist, it is obvious that

$$[t_c(a)] = [a] \pm I(c, a)[c]$$

where $[x]$ denotes the homology class of x. Moreover, it is clear that if c is separating then $I(c, a) = 0$ for any circle a. Therefore

$$[t_c(a)] = [a]$$

and the lemma follows by the fact that homology classes of circles span $H_1(N, \mathbb{R})$. $\square$

By Lemmas 5.1 and 5.2 we obtain

Proposition 5.3. *Let c be a two-sided circle on N. Then $D(t_c) = 1$.*

The explicit definition of y made in the previous section easily leads to the following

Proposition 5.4. *Let $D: \mathcal{PM}^+(N) \to \mathbb{Z}_2$ as above. Then $D(y) = -1$.*

6.   Generators for the twist subgroup

The main goal of this section is to find a simple generating set for the twist subgroup $\mathcal{T}(N)$. Our main tool will be the following well known fact from combinatorial group theory—see for example Chapter 9 of [8].

Proposition 6.1. *Let X be a generating set for a group G and let U be a left transversal for a subgroup H. Then H is generated by the set*

$$\{ux\overline{ux}^{-1} : u \in U, \ x \in X, \ ux \notin U\},$$

where $\{\overline{g}\} = gH \cap U$ for $g \in G$.

Let $\tilde{\mathcal{T}}(N)$ denote the kernel of the homomorphism

$$D: \mathcal{PM}^+(N) \to \mathbb{Z}_2$$

defined in Section 5. The reason for our choice of notation will become apparent after Corollary 6.3 below, where we will prove that in fact $\tilde{\mathcal{T}}(N) = \mathcal{T}(N)$ is the twist subgroup.
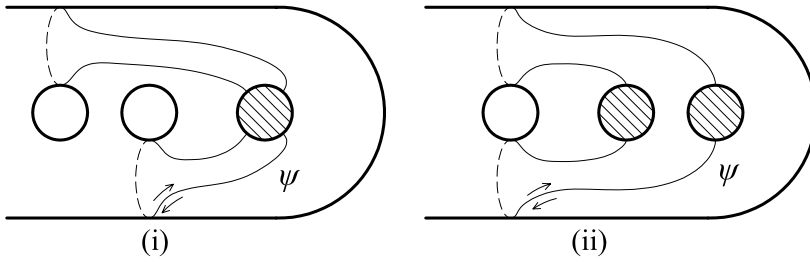
Fig. 12. Circle ψ for g odd and even.

Theorem 6.2. *Let $g \geq 3$. Then the group $\tilde{\mathcal{T}}(N)$ is generated by*

- $\{t_l \mid l \in \mathcal{C} \cup \{f_1, \ldots, f_{n+s-1}, \xi\}\}$ *if $g = 3$,*
- $\{t_l \mid l \in \mathcal{C} \cup \{\psi, \xi\}\}$ *if $g \geq 5$ odd,*
- $\{t_l \mid l \in \mathcal{C} \cup \{\lambda, \psi, \xi\}\}$ *if g even,*

where ψ is the two-sided circle indicated in Fig. 12, $f_1, \ldots, f_{n+s-1}$ *are as in* Fig. 10 (i), λ *is as in* Fig. 8 (ii) *and ξ is as in* Fig. 8.

Proof. By Proposition 5.4, D is onto, hence

$$[\mathcal{PM}^+(N) : \tilde{\mathcal{T}}(N)] = 2$$

and as a transversal for $\tilde{\mathcal{T}}(N)$ we can take $U = \{1, y\}$. By Theorem 4.1 and Propositions 6.1, 5.3, 5.4, $\tilde{\mathcal{T}}(N)$ is generated by

$$\{t_l, y t_l y^{-1}, y^2 \mid l \in \mathcal{C}\}.$$

Clearly the circles

- $u_1, \ldots, u_s$ if $g = 3$,
- $a_2, \ldots, a_{g-3}, b_1, \ldots, b_{r-1}, c_1, \ldots, c_{r-1}, e_1, \ldots, e_{n+s-1}, u_1, \ldots, u_s$ if $g \geq 5$ odd,
- $a_2, \ldots, a_{g-3}, b_1, \ldots, b_r, c_1, \ldots, c_r, e_1, \ldots, e_{n+s-1}, u_1, \ldots, u_s$ if g even,

are disjoint from the support of y (cf. Figs. 2, 3 and 8). Since $y t_l y^{-1} = t_{y(l)}^{\pm}$ and $y^2 = t_\xi$, $\tilde{\mathcal{T}}(N)$ is generated by

- $\{t_l \mid l \in \mathcal{C} \cup \{y(a_2), y(b_1), y(c_1), y(e_1), \ldots, y(e_{n+s-1}), \xi\}\}$ if $g = 3$,
- $\{t_l \mid l \in \mathcal{C} \cup \{y(a_{g-2}), y(a_{g-1}), y(b_r), y(c_r), \xi\}\}$ if $g \geq 5$ odd,
- $\{t_l \mid l \in \mathcal{C} \cup \{y(a_{g-2}), y(a_{g-1}), y(b_{r+1}), \xi\}\}$ if g even.

We defined y using $\tilde{N}$, so it is convenient to transfer circles to that model. Once this is done, it is not difficult to check that we have relations:

- $y(a_2) = a_2$, $y(c_1) = b_1$, $y^{-1}(e_1) = \tau_1, \ldots, y^{-1}(e_{n+s-1}) = \tau_{n+s-1}$ if $g = 3$, where τ_i is as in Fig. 13,
- $y(a_{g-2}) = \psi$, $y(a_{g-1}) = a_{g-1}$, $y(c_r) = b_r$ if $g \geq 5$ odd,
- $y(a_{g-2}) = t_{b_{r+1}}^{-1}(\lambda)$, $y(a_{g-1}) = a_{g-1}$, $y^{-1}(b_{r+1}) = \psi$ if g even.
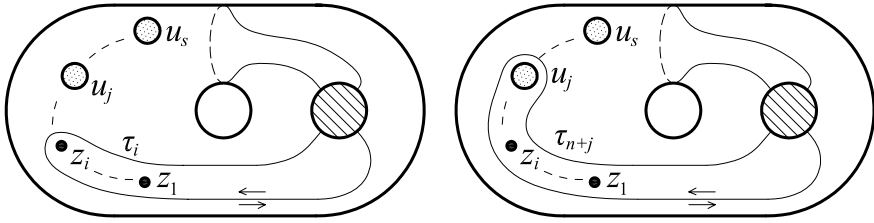
Fig. 13. Circles $\tau_1, \ldots, \tau_{n+s-1}$ for $g = 3$.

To complete the proof it is enough to show that the twists $t_{\tau_1}, \ldots, t_{\tau_{n+s-1}}$ can be replaced by twists $t_{f_1}, \ldots, t_{f_{n+s-1}}$ in case $g = 3$. In order to prove this, let $\varrho_i = t_{c_1}^{-1} t_{a_2}^{-1}(\tau_i)$. Then it is straightforward to check that

$$\tau_i = t_{\varrho_{i-1}} t_{a_2}^2 t_{c_1}(f_i) \quad \text{for} \quad i = 2, \ldots, n + s - 1.$$

Therefore, using the relation $\tau_1 = f_1$, we can inductively replace each t_{τ_i} by t_{f_i}.   □

Corollary 6.3. *Let $g \geq 3$. Then the kernel of the determinant homomorphism $D \colon \mathcal{PM}^+(N) \to \mathbb{Z}_2$ is the twist subgroup of $\mathcal{M}(N)$, that is $\tilde{\mathcal{T}}(N) = \mathcal{T}(N)$.*

Proof.   Clearly $\mathcal{T}(N) \leq \mathcal{PM}^+(N)$ and by Proposition 5.3,

$$\mathcal{T}(N) \leq \ker D = \tilde{\mathcal{T}}(N).$$

On the other hand, by Theorem 6.2, $\tilde{\mathcal{T}}(N) \leq \mathcal{T}(N)$.   □

Corollary 6.4. *Let $g \geq 3$. Then the twist subgroup $\mathcal{T}(N)$ has index $2^{n+1} n!$ in $\mathcal{M}(N_{g,s}^n)$.*

Proof.   By the previous corollary, $\mathcal{T}(N) = \ker D$, so the conclusion follows from the obvious equality

$$[\mathcal{M}(N) : \ker D]$$
$$= [\mathcal{M}(N) : \mathcal{PM}(N)] \cdot [\mathcal{PM}(N) : \mathcal{PM}^+(N)] \cdot [\mathcal{PM}^+(N) : \ker D] = n! \cdot 2^n \cdot 2.   □$$

In the case of a closed surface N_g, the above corollary was first proved by Lickorish [11]. Later Korkmaz [10] proved it for a punctured surface N_g^n under additional assumption $g \geq 7$.

7. Homological results for the twist subgroup

Recall that for a group G, the first homology group $H_1(G, \mathbb{Z})$ coincides with the abelianization $G/[G, G]$. For the rest of the paper, for any $f \in \mathcal{T}(N)$ let $[f]$ denotes
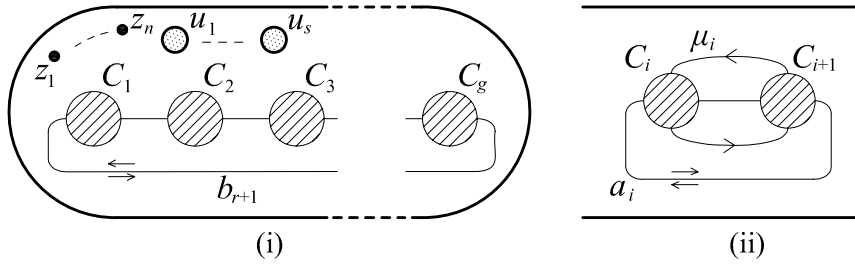
Fig. 14. Circle b_{r+1} and crosscap slide y_i, Lemma 7.3.

the homology class of f in $H_1(\mathcal{T}(N), \mathbb{Z})$. Moreover, we will use the additive notation in $H_1(\mathcal{T}(N), \mathbb{Z})$.

7.1. Homology classes of non-peripheral twists.

Lemma 7.1. *Let a and b be two two-sided circles on N such that $I(a, b) = 1$ and the orientations of regular neighbourhoods of a and b are induced from the orientation of $a \cup b$. Then t_a and t_b are conjugate in $\mathcal{T}(N)$. In particular $[t_a] = [t_b]$ in $H_1(\mathcal{T}(N))$.*

Proof. The lemma follows form the braid relation

$$t_b = (t_a t_b) t_a (t_a t_b)^{-1}. \qquad \square$$

Lemma 7.2. *Assume that $g \geq 3$ and let a and b be two nonseparating two-sided circles on N such that $N \setminus a$ and $N \setminus b$ are nonorientable. Then t_a is conjugate in $\mathcal{T}(N)$ either to t_b or to t_b^{-1}.*

Proof. By Lemma 7.1, it is enough to prove that there exits a sequence of two-sided circles $p_1, \ldots, p_k$ such that $p_1 = a$, $p_k = b$ and $I(p_i, p_{i+1}) = 1$ for $i = 1, \ldots, k-1$. In other words, using the terminology of [10], we have to prove that a and b are dually equivalent. For a nonorientable surface with punctures this was proved in Theorem 3.1 of [10]. It is straightforward to check that the same proof applies to the case of a surface with boundary. $\qquad \square$

Lemma 7.3. *Let $g = 2r + 2 \geq 4$. Then $t_{b_{r+1}}$ is conjugate in $\mathcal{T}(N)$ to t_ψ^{-1}, where $\psi = y^{-1}(b_{r+1})$ is as in Theorem 6.2.*

Proof. Fig. 14 (i) shows the circle b_{r+1} as a circle on $\tilde{N}$. Using the notation from this figure, let y_i for $i = 1, \ldots, g-1$, be a slide of the crosscap C_{i+1} along the loop μ_i as shown in Fig. 14 (ii). In particular $y_{g-1} = y$. It is straightforward to check that

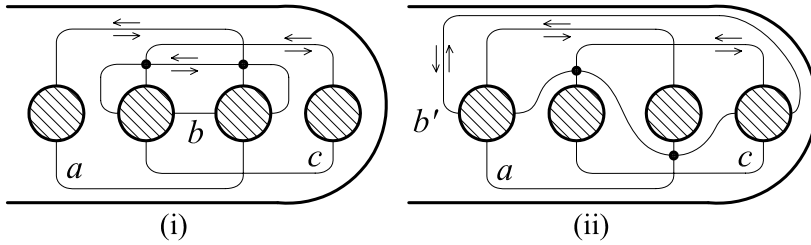$$y_{g-2} t_{a_{g-2}} y_{g-3} t_{a_{g-3}} \cdots y_2 t_{a_2} y_1 (b_{r+1}) = y_{g-1}^{-1}(b_{r+1}) = \psi.$$

Fig. 15. Circles a, b, b' and c, Lemma 7.4.

Moreover, $y_{g-2}t_{a_{g-2}} \cdots y_2 t_{a_2} y_1$, as a product of twists and even number of crosscap slides, is in the kernel of the determinant homomorphism $D: \mathcal{PM}^+(N) \to \mathbb{Z}_2$ defined in Section 5. Therefore, by Corollary 6.3, $t_{b_{r+1}}$ is conjugate in $\mathcal{T}(N)$ to either t_ψ or t_ψ^{-1}. Careful examination of the orientations of local neighbourhoods of b_{r+1} and ψ shows that in fact $t_{b_{r+1}}$ is conjugate to t_ψ^{-1}.   □

Lemma 7.4. *Let* $g \geq 4$. *Then* $[t_{a_1}^2] = 0$ *in* $\mathrm{H}_1(\mathcal{T}(N))$.

Proof.   Fig. 15 (i) shows three two-sided nonseparating circles a, b and c such that $I(a, b) = 1$, $I(b, c) = 1$ and the complement of each of these circles in N is nonorientable. Hence by Lemma 7.1, all three twists t_a, t_b and t_c are conjugate in $\mathcal{T}(N)$. Similarly, Fig. 15 (ii) shows that t_a, $t_{b'}$ and t_c^{-1} are conjugate. Hence t_c and t_c^{-1} are conjugate in $\mathcal{T}(N)$. By Lemma 7.2, the same is true for t_{a_1}.   □

REMARK 7.5.   Observe that if a and b are two-sided nonseparating circles on N such that $N \setminus a$ and $N \setminus b$ are nonorientable, then Lemma 7.2 gives us no hint if t_a is conjugate to t_b or to t_b^{-1}. However, by Lemma 7.4, as long as $g \geq 4$ and we are concern with homology classes, the above ambiguity is inessential.

Now let recall some results form [17].

Lemma 7.6 (Lemma 6.7 of [17]).   *Let* $g \geq 7$. *Then* $[t_{a_1}] = 0$.

Lemma 7.7 (Lemma 6.6 of [17]).   *Assume* $g = 2r + 2 \geq 6$. *Then* $[t_{b_{r+1}}] = 0$.

Lemma 7.8 (Lemma 6.12 of [17]).   *Assume* $g \geq 5$. *Then the homology classes of the boundary twists* $t_{u_1}, \ldots, t_{u_s}$ *are trivial in* $\mathrm{H}_1(\mathcal{T}(N))$.

Lemma 7.9 (Lemma 6.14 of [17]).   *Assume that* $g \geq 3$ *and let* κ *be a two-sided separating circle on* N *such that one of the components of* $N \setminus \kappa$ *is a disk* Δ *containing*
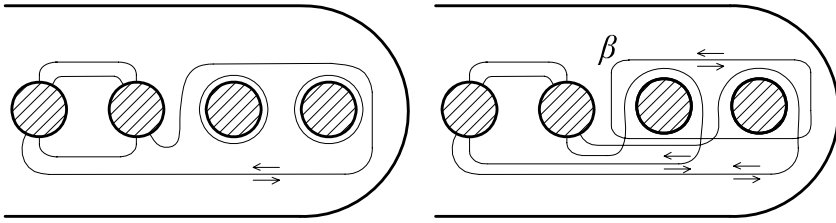
Fig. 16. Lantern relation, Lemma 7.10.

the punctures $z_1, \ldots, z_n$ and the holes $u_1, \ldots, u_s$. Moreover, assume that the orientation of Δ agrees with the orientations of neighbourhoods of $u_1, \ldots, u_s$ and κ. Then

$$[t_\kappa] = [t_{u_1} \cdots t_{u_s}] = [t_{u_1}] + \cdots + [t_{u_s}].$$

Although each of the above lemmas was originally stated in terms of the group $\mathcal{PM}^+(N^n_{g,s})$, by Lemmas 7.1, 7.4 and by Remark 7.5, their proofs work as well in the case of the twist subgroup $\mathcal{T}(N)$.

Lemma 7.10. *Let $g \geq 4$ and let β be a separating circle on N such that one component of $N \setminus \beta$ is Klein bottle with one hole and the second component is non-orientable. Then $[t_\beta] = 0$. In particular*

$$[y^2] = [t_\xi] = 0.$$

Proof. Fig. 16 shows that there is a lantern configuration with one circle β and all other twists either trivial or conjugate to t_{a_1}. Hence

$$[t_{a_1} t_{a_1}] = [t_\beta t_{a_1} t_{a_1}]. \qquad \square$$

Lemma 7.11. *Let $g \geq 4$ and let γ be a circle on N such that one of the components of $N \setminus \gamma$ is a nonorientable surface of genus 3 with one hole. Then $[t_\gamma] = 0$.*

Proof. Fig. 17 shows that there is a lantern configuration with one circle γ and all other circles either bounding Möbius strips or satisfying the assumptions of Lemma 7.10.
$\qquad \square$

7.2. Homology classes of boundary twists.

Lemma 7.12. *Let $g = 4$. Then*

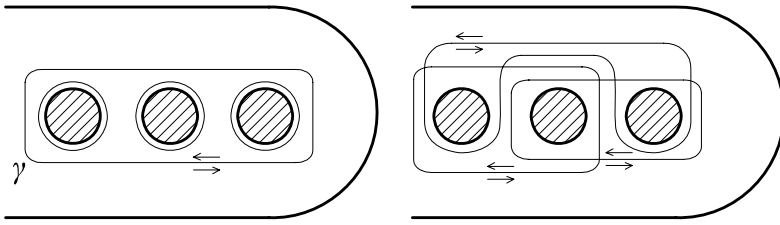$$[t_{u_1}] + [t_{u_2}] + \cdots + [t_{u_s}] = 0.$$
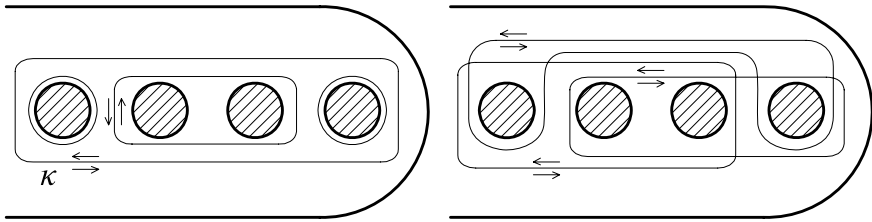
Fig. 17. Lantern relation, Lemma 7.11.

Fig. 18. Lantern relation, Lemma 7.12.

Proof. By Lemma 7.9,

$$[t_{u_1}] + [t_{u_2}] + \cdots + [t_{u_s}] = [t_\kappa],$$

where κ is a circle on N bounding all the punctures and boundary circles. On the other hand, Fig. 18 shows that there is a lantern configuration with one circle γ and all other circles either bounding Möbius strips or satisfying the assumptions of Lemmas 7.10 or 7.11. □

Lemma 7.13. *Let $g \geq 3$. Then $2[t_{u_j}] = [t_{u_j}^2] = 0$ for $j = 1, \ldots, s$.*

Proof. Consider the lantern relation indicated in Fig. 19. Using Lemma 7.1, it is easy to prove that all four twists about nonseparating circles in this figure are conjugate to t_{a_1}, hence we have

$$[t_{u_j} t_{a_1} t_{a_1}^{-1}] = [t_{\eta_j} t_{a_1}^{-1} t_{a_1}].$$

Therefore it is enough to show that $f t_{\eta_j} f^{-1} = t_{\eta_j}^{-1}$ for some $f \in \mathcal{T}(N)$.

The circle η_j divides the surface N into a projective plane N' with two holes and a nonorientable surface N''. Let $h \colon N \to N$ be a diffeomorphism obtained as follows. On each of N' and N'', h is a slide of η_j along the core of a crosscap such that h is $-id$ on η_j. Clearly $h \in \mathcal{PM}^+(N_{g,s}^n)$ and $h t_{\eta_j} h^{-1} = t_{\eta_j}^{-1}$. By Proposition 5.4, $D(y) = -1$ where $D \colon \mathcal{PM}^+(N) \to \mathbb{Z}_2$ is the determinant homomorphism. Moreover $y t_{\eta_j} y^{-1} = t_{\eta_j}$, hence by Corollary 6.3, either $f = h$ or $f = hy$ is the required diffeomorphism. □
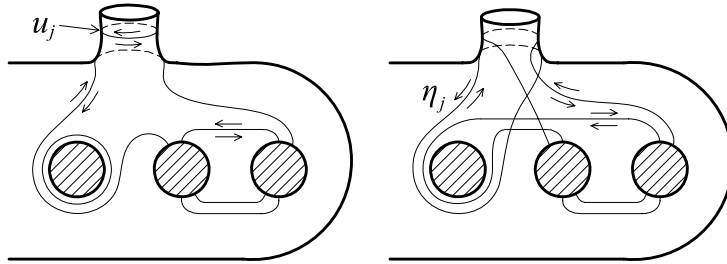
Fig. 19. Lantern relation $[t_{u_j} t_{a_1} t_{a_1}^{-1}] = [t_{\eta_j} t_{a_1}^{-1} t_{a_1}]$.
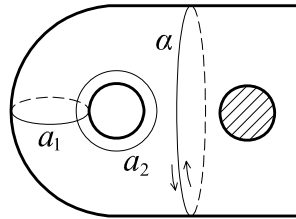

Fig. 20. Relation $(t_{a_1} t_{a_2})^6 = t_\alpha$.

7.3. Homology classes of twists for $g = 3$.

Lemma 7.14. *Assume that $g = 3$. Then*

$$12[t_{a_1}] = [t_\xi] = [t_{u_1}] + [t_{u_2}] + \cdots + [t_{u_s}].$$

In particular $12[t_{a_1}] = [t_\xi] = 0$ if $s = 0$.

Proof. Applying Lemma 2.4 to the configuration shown in Fig. 20, we obtain

$$(t_{a_1} t_{a_2})^6 = t_\alpha.$$

Hence by Lemmas 7.1 and 7.2,

(7.1) $$12[t_{a_1}] = [t_\alpha].$$

Using Lemma 7.1, one can check that all twists about nonseparating essential circles indicated in Fig. 21 are conjugate to t_{a_1}. Therefore we have a lantern relation

$$[t_{a_1} t_{a_1} t_\kappa] = [t_{a_1} t_{a_1} t_\alpha],$$

where κ is as in Lemma 7.9. Together with (7.1) this yields
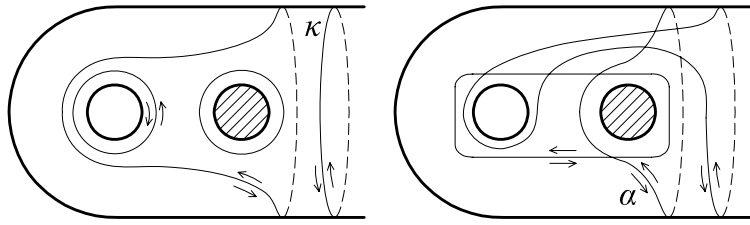
(7.2) $$12[t_{a_1}] = [t_\kappa].$$

Fig. 21. Relation $[t_{a_1} t_{a_1} t_\kappa] = [t_{a_1} t_{a_1} t_\alpha]$.

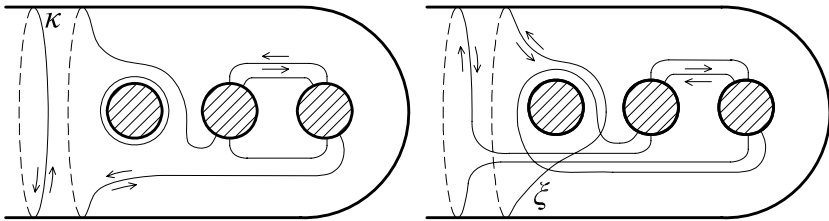Fig. 22. Relation $[t_\kappa t_{a_1} t_{a_1}^{-1}] = [t_\xi t_{a_1}^{-1} t_{a_1}]$.
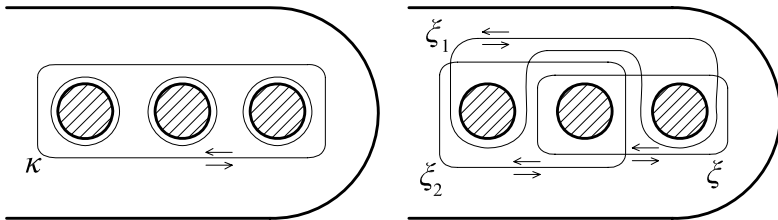
Fig. 23. Lantern relation $t_\kappa = t_\xi t_{\xi_1} t_{\xi_2}$.

On the other hand, by the lantern relation provided by Fig. 22, we have

$$[t_\kappa t_{a_1} t_{a_1}^{-1}] = [t_\xi t_{a_1}^{-1} t_{a_1}].$$

By (7.2), this gives $12[t_{a_1}] = [t_\kappa] = [t_\xi]$. Moreover, by Lemma 7.9,

$$[t_\kappa] = [t_{u_1}] + [t_{u_2}] + \cdots + [t_{u_s}]. \qquad \square$$

Lemma 7.15. *Assume that* $g = 3$. *Then* $24[t_{a_1}] = 0$.

Proof. Fig. 23 shows that there is a lantern relation

$$t_\kappa = t_\xi t_{\xi_1} t_{\xi_2},$$

where κ is as in Lemma 7.14. Moreover by the proof of that lemma, $[t_\xi] = [t_{\xi_1}] = [t_{\xi_2}] = [t_\kappa]$. Hence

$$[t_\xi] = 3[t_\xi].$$

Using once again Lemma 7.14, we have $24[t_{a_1}] = 2[t_\xi] = 0.$ $\qquad\square$

7.4. Some special cases.

Proposition 7.16. *Let $N = N_{g,s}^n$ be a nonorientable surface of genus g with s holes and n punctures. Then*

$$H_1(\mathcal{T}(N), \mathbb{Z}) = \begin{cases} \langle [t_{a_1}] \rangle \cong \mathbb{Z}_{12} & \text{for} \quad (g, s, n) = (3, 0, 0), \\ \langle [t_{a_1}] \rangle \cong \mathbb{Z}_{24} & \text{for} \quad (g, s, n) = (3, 1, 0), \\ \langle [t_{a_1}], [t_{b_2}] \rangle \cong \mathbb{Z}_2 \times \mathbb{Z} & \text{for} \quad (g, s, n) = (4, 0, 0). \end{cases}$$

Proof. By Theorem 3 of [1], the group $\mathcal{M}(N_3)$ has a presentation

$$\mathcal{M}(N_3) = \langle t_{a_1}, t_{a_2}, y \mid t_{a_1} t_{a_2} t_{a_1} = t_{a_2} t_{a_1} t_{a_2}, \, y t_{a_1} y^{-1} = t_{a_1}^{-1},$$

$$y t_{a_2} y^{-1} = t_{a_2}^{-1}, \, y^2 = 1, \, (t_{a_1} t_{a_2})^6 = 1 \rangle.$$

Using $U = \{1, y\}$ as a transversal for the subgroup $\mathcal{T}(N_3)$, it is straightforward to obtain that

$$\mathcal{T}(N_3) = \langle t_{a_1}, t_{a_2} \mid t_{a_1} t_{a_2} t_{a_1} = t_{a_2} t_{a_1} t_{a_2}, \, (t_{a_1} t_{a_2})^6 = 1 \rangle.$$

This implies that

$$H_1(\mathcal{T}(N_3)) \cong \langle t_{a_1} \mid t_{a_1}^{12} = 1 \rangle.$$

The reasoning for the surfaces $N_{3,1}$ and N_4 is similar, one has to use the known presentations for the groups $\mathcal{M}(N_{3,1})$ and $\mathcal{M}(N_4)$—see Theorem 7.16 of [19] and Theorem 2.1 of [18]. $\qquad\square$

8. Computing $H_1(\mathcal{T}(N), \mathbb{Z})$

Theorem 8.1. *Let $N = N_{g,s}^n$ be a nonorientable surface with n punctures and s holes. Then*

$$H_1(\mathcal{T}(N), \mathbb{Z}) = \begin{cases} \mathbb{Z}_{12} & \text{if} \quad g = 3 \quad \text{and} \quad s = 0, \\ \mathbb{Z}_{24} \times \mathbb{Z}_2^{s-1} & \text{if} \quad g = 3 \quad \text{and} \quad s \geq 1, \\ \mathbb{Z}_2 \times \mathbb{Z} & \text{if} \quad g = 4 \quad \text{and} \quad s = 0, \\ \mathbb{Z}_2^s \times \mathbb{Z} & \text{if} \quad g = 4 \quad \text{and} \quad s \geq 1, \\ \mathbb{Z}_2 & \text{if} \quad g = 5, 6, \\ 0 & \text{if} \quad g \geq 7. \end{cases}$$

Proof.   By Theorem 6.2, Corollary 6.3, Lemmas 7.2 and 7.3, $H_1(\mathcal{T}(N))$ is generated by

- $[t_{a_1}]$, $[t_\xi]$, $[t_{u_1}]$, . . . , $[t_{u_s}]$ if g is odd,
- $[t_{a_1}]$, $[t_{b_{r+1}}]$, $[t_\xi]$, $[t_{u_1}]$, . . . , $[t_{u_s}]$ if g is even.

Moreover

- $[t_{a_1}] = 0$ if $g \geq 7$ (Lemma 7.6),
- $[t_{b_{r+1}}] = 0$ if $g \geq 6$ even (Lemma 7.7),
- $[t_{u_1}] = \cdots = [t_{u_s}] = 0$ if $g \geq 5$ (Lemma 7.8),
- $[t_\xi] = 0$ if $g \geq 4$ (Lemma 7.10),
- $[t_{u_1}] + \cdots + [t_{u_s}] = 0$ if $g = 4$ (Lemma 7.12),
- $[t_\xi] = [t_{u_1}] + \cdots + [t_{u_s}] = 12[t_{a_1}]$ if $g = 3$ (Lemma 7.14).

Hence $H_1(\mathcal{T}(N))$ is generated by

- $[t_{a_1}]$, $[t_{u_1}]$, . . . , $[t_{u_{s-1}}]$ if $g = 3$,
- $[t_{a_1}]$, $[t_{b_2}]$, $[t_{u_1}]$, . . . , $[t_{u_{s-1}}]$ if $g = 4$,
- $[t_{a_1}]$ if $g = 5, 6$,

and $H_1(\mathcal{T}(N)) = 0$ if $g \geq 7$. In particular this concludes the proof in the case $g \geq 7$. Therefore in what follows we assume that $g \leq 6$.

We also know that

(8.1) $$2[t_{a_1}] = 0 \quad \text{if} \quad g \geq 4 \quad \text{(Lemma 7.4)},$$

(8.2) $$2[t_{u_1}] = \cdots = 2[t_{u_{s-1}}] = 0 \quad \text{if} \quad g \geq 3 \quad \text{(Lemma 7.13)},$$

(8.3) $$12[t_{a_1}] = 0 \quad \text{if} \quad g = 3 \quad \text{and} \quad s = 0 \quad \text{(Lemma 7.14)},$$

(8.4) $$24[t_{a_1}] = 0 \quad \text{if} \quad g = 3 \quad \text{(Lemma 7.15)}.$$

Hence it is enough to prove that every relation in the abelian group $H_1(\mathcal{T}(N))$ is a consequence of the relations (8.1)–(8.4) above.

Let $g = 3$ and suppose that

(8.5) $$\alpha[t_{a_1}] + \varepsilon_1[t_{u_1}] + \cdots + \varepsilon_{s-1}[t_{u_{s-1}}] = 0.$$

If $s > 0$ define N' to be a surface of genus 3 obtained from N by forgetting all the punctures and gluing a disk to each boundary component but one, say u_s. If $s = 0$ define N' by forgetting about the punctures in N. We have a homomorphism

$$\Phi\colon \mathcal{T}(N) \to \mathcal{T}(N').$$

Clearly $\Phi(t_{u_i}) = 0$ for $1 \leq i \leq s - 1$, hence equation (8.5) yields

$$\alpha[t_{a_1}] = 0 \quad \text{in} \quad H_1(\mathcal{T}(N')).$$

By Proposition 7.16, α must be a multiple of 24 if $s > 0$ and a multiple of 12 if $s = 0$.

Therefore by relations (8.3) and (8.4), equation (8.5) becomes

$$(8.6) \qquad \varepsilon_1[t_{u_1}] + \cdots + \varepsilon_{s-1}[t_{u_{s-1}}] = 0.$$

This concludes the proof if $s \leq 1$, hence assume that $s \geq 2$. Now let $\hat{N}_j$ for $j = 1, \ldots, s - 1$, be the surface obtained from N by forgetting the punctures, gluing a cylinder to the circles u_j and u_s and finally gluing a disk to each of the remaining boundary components. Then $\hat{N}_j$ is a closed nonorientable surface of genus 5. Let

$$\Upsilon_j \colon \mathcal{T}(N) \to \mathcal{M}(\hat{N}_j)$$

be the homomorphism induced by inclusion. Since $\Upsilon_j(t_{u_i}) = 0$ for $i \neq j$ and $i \neq s$, equation (8.6) gives us

$$\varepsilon_j[t_{u_j}] = 0 \quad \text{in} \quad H_1(\mathcal{M}(\hat{N}_j)).$$

Since u_j is a nonseparating two-sided circle on $\hat{N}_j$, by Theorem 1.1 of [9], ε_j is even. By relation (8.2), equation (8.6) becomes $0 = 0$, which completes the proof for $g = 3$.

The proof for $g = 4$ is analogous. If we assume that

$$\alpha[t_{a_1}] + \beta[t_{b_2}] + \varepsilon_1[t_{u_1}] + \cdots + \varepsilon_{s-1}[t_{u_{s-1}}] = 0$$

the we can show that $\alpha = \beta = 0$ in the same manner as in the case $g = 3$, namely by mapping N into a closed surface of genus 4 and using Proposition 7.16. Similarly, we can show that $\varepsilon_1 = \cdots = \varepsilon_{s-1} = 0$ by mapping N into a closed surface of genus 6 and using Theorem 1.1 of [9].

If $g = 5$ or $g = 6$ the proof is even simpler, it is enough to map N into a closed surface and use Theorem 1.1 of [9], we skip the details. $\qquad \square$

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