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NOETHERIAN PROPERTIES OF RINGS OF DIFFERENTIAL OPERATORS OF AFFINE SEMIGROUP ALGEBRAS

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Abstract

We consider the Noetherian properties of the ring of differential operators of an affine semigroup algebra. First we show that it is always right Noetherian. Next we give a condition, based on the data of the difference between the semigroup and its scored closure, for the ring of differential operators being anti-isomorphic to another ring of differential operators. Using this, we prove that the ring of differential operators is left Noetherian if the condition is satisfied. Moreover we give some other conditions for the ring of differential operators being left Noetherian. Finally we conjecture necessary and sufficient conditions for the ring of differential operators being left Noetherian.

1. Introduction

Let K be an algebraically closed field of characteristic zero. Let $D(R)$ be the ring of differential operators of a finitely generated commutative K-algebra R as defined by Grothendieck [11]. We study the Noetherian properties of $D(R)$ when R is an affine semigroup algebra.

It is well known that, if R is a regular domain, then $D(R)$ is Noetherian, and the category of left $D(R)$-modules and that of right $D(R)$-modules are equivalent (see for example [3], [16]). Bernstein-Gel'fand-Gel'fand [2] showed that $D(R)$ is not Noetherian in general if we do not assume the regularity. However, $D(R)$ is known to be Noetherian for some families of interesting algebras; Muhasky [17] and Smith-Stafford [29] independently proved that $D(R)$ is Noetherian if R is an integral domain of Krull dimension one. Tripp [31] proved that the ring $D(K[\Delta])$ of differential operators of the Stanley-Reisner ring $K[\Delta]$ is right Noetherian, and gave a necessary and sufficient condition for $D(K[\Delta])$ to be left Noetherian.

Let $A := \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} \subset \mathbb{Z}^d$ be a finite subset. We denote by $\mathbb{N}A$ the monoid generated by A, and by $K[\mathbb{N}A]$ its semigroup algebra. We consider the ring $D(K[\mathbb{N}A])$ of differential operators of $K[\mathbb{N}A]$. We saw in [25, 26] that the algebra $D(K[\mathbb{N}A])$ is strongly related to A-hypergeometric systems (also known as GKZ hypergeometric systems), defined in [15] and [13], and systematically studied by Gel'fand and his

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collaborators (e.g. [7, 8, 9, 10]). The ring $D(K[\mathbb{N}A])$ was studied thoroughly when the affine semigroup algebra $K[\mathbb{N}A]$ is normal (e.g. [14] and [18, 20]). In particular, Jones [14] and Musson [20] independently proved that $D(K[\mathbb{N}A])$ is a Noetherian finitely generated K-algebra when $K[\mathbb{N}A]$ is normal. Traves and the first author [26, 27] proved that $D(K[\mathbb{N}A])$ is a finitely generated K-algebra in general, and that $D(K[\mathbb{N}A])$ is Noetherian if the semigroup $\mathbb{N}A$ is scored. The scoredness means that $K[\mathbb{N}A]$ satisfies Serre's (S_2) condition and it is geometrically unibranched. The question of Morita equivalence between $D(K[\mathbb{N}A])$ for a scored $K[\mathbb{N}A]$ and that for its normalization was studied in Smith-Stafford [29], Chamarie-Stafford [6], Hart-Smith [12], and Ben Zvi-Nevins [1].

Generally speaking, $D(R)$ is more apt to be right Noetherian than to be left Noetherian, and a prime of height more than one is often an obstacle for the left Noetherian property of $D(R)$ (see for example [5], [19], [29], and [31]). We observe this phenomenon as well.

As in [29], by using Robson's lemma (Lemma 5.1), we prove that $D(K[\mathbb{N}A])$ is right Noetherian (Theorem 5.10) for any A. To state conditions for the left Noetherian property, we need to introduce the standard expression of a semigroup $\mathbb{N}A$; let $S_c(\mathbb{N}A)$ be the *scored closure* of $\mathbb{N}A$, the smallest scored semigroup containing $\mathbb{N}A$ (see 6.4). There exist $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m \in S_c(\mathbb{N}A)$ and faces $\tau_1, \dots, \tau_m$ of $\mathbb{R}_{\geq 0}A$, the cone generated by A, such that

$$\mathbb{N}A = S_c(\mathbb{N}A) \setminus \bigcup_{i=1}^{m} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)). \tag{1.1}$$

Assuming that the expression (1.1) is irredundant, we see that the set $\{\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i) : i = 1, \dots, m\}$ is unique, and we call the expression (1.1) the *standard expression* of $\mathbb{N}A$.

One way to prove the left Noetherian property is to show the correspondence between left ideals and right ideals, and then to use the right Noetherian property. To show the correspondence, we define a set $\mathcal{B}$ based on the standard expression (see (6.1) for the definition of $\mathcal{B}$), and, when $\mathcal{B} \neq \emptyset$, we consider a right $D(K[\mathbb{N}A])$-module $K[\omega(\mathbb{N}A)]$, an analogue of the canonical module. Then we see that the category of left $D(K[\mathbb{N}A])$-modules and that of right $D(K[\omega(\mathbb{N}A)])$-modules are equivalent (Theorem 6.11), and hence we derive the left Noetherian property of $D(K[\mathbb{N}A])$ from the right Noetherian property of $D(K[\omega(\mathbb{N}A)])$ (Theorem 6.12). (For this reason and another technical reason, we prove the right Noetherian property not only of $D(K[\mathbb{N}A])$ but of a little more general algebras.) In this way, for example, we see that $D(K[\mathbb{N}A])$ is left Noetherian if $K[\mathbb{N}A]$ satisfies Serre's (S_2) condition.

Another way to prove the left Noetherian property is the way similar to the one used for showing the right Noetherian property. As in [29, Proposition 7.3], a sufficient condition for the left Noetherian property is given in this way (Theorem 7.3).

Finally we conjecture that $D(K[\mathbb{N}A])$ is left Noetherian if and only if for all i with codim $\tau_i > 1$

$$\left(\bigcap_{\tau_j \succ \tau_i, \boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j, \operatorname{codim} \tau_j = 1} \tau_j \right) = \tau_i.$$

When this condition is not satisfied, we construct a left ideal of $D(K[\mathbb{N}A])$ which is not finitely generated (Theorem 7.8).

This paper is organized as follows. In Section 2, we recall some fundamental facts about the rings of differential operators of semigroup algebras, and fix some notation. In Section 3, we recall the results on the finite generation in [26, 27], and generalize them suitably for our proof of the right Noetherian property. In Section 4, we introduce preorders, which indicate the D-module structures of $K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. In Section 5, we prove the right Noetherian property. In Section 6, we consider the correspondence between left D-modules and right D-modules. In Section 7, we consider the left Noetherian property; we give a sufficient condition in Subsection 7.1 and a necessary condition in Subsection 7.2.

2. Rings of differential operators

In this section, we briefly recall some fundamental facts about the rings of differential operators of semigroup algebras, and fix some notation.

Let K be an algebraically closed field of characteristic zero, and R a commutative K-algebra. For R-modules M and N, we inductively define the space of K-linear differential operators from M to N of order at most k by

$$D^k(M, N) := \{P \in \operatorname{Hom}_K(M, N) \colon Pr - rP \in D^{k-1}(M, N) \text{ for all } r \in R\}.$$

Set $D(M, N) := \bigcup_{k=0}^{\infty} D^k(M, N)$, and $D(M) := D(M, M)$. Then, by the natural composition, $D(M)$ is a K-algebra, and $D(M, N)$ is a $(D(N), D(M))$-bimodule. We call $D(M)$ the *ring of differential operators* of M. For the generalities of the ring of differential operators, see [11], [16], [30], etc.

Let

$$A := \{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n\} \tag{2.1}$$

be a finite set of vectors in $\mathbb{Z}^d$. Sometimes we identify A with the matrix of column vectors $(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$. Let $\mathbb{N}A$ and $\mathbb{Z}A$ denote the monoid and the group generated by A, respectively. Throughout this paper, we assume that $\mathbb{Z}A = \mathbb{Z}^d$ for simplicity. We also assume that the cone $\mathbb{R}_{\geq 0}A$ generated by A is strongly convex.

The (semi)group algebra of $\mathbb{Z}^d$ is the Laurent polynomial ring $K[\mathbb{Z}^d] = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. Its ring of differential operators is the ring of differential operators with Laurent polynomial coefficients

$$D(K[\mathbb{Z}^d]) = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]\langle \partial_1, \dots, \partial_d \rangle,$$

where $[\partial_i, t_j] = \delta_{ij}$, $[\partial_i, t_j^{-1}] = -\delta_{ij} t_j^{-2}$, and the other pairs of generators commute. Here $[\ ,\ ]$ denotes the commutator, and δ_{ij} is 1 if $i = j$ and 0 otherwise.

The semigroup algebra $K[\mathbb{N}A] = \bigoplus_{\boldsymbol{a} \in \mathbb{N}A} Kt^{\boldsymbol{a}}$ is the ring of regular functions on the affine toric variety defined by A, where $t^{\boldsymbol{a}} = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\boldsymbol{a} = {}^t(a_1, a_2, \dots, a_d)$, the transpose of the row vector $(a_1, \dots, a_d)$. We say that $S \subseteq \mathbb{Z}^d$ is an $\mathbb{N}A$*-set* if $S + \mathbb{N}A \subseteq S$. Then $K[S] := \bigoplus_{\boldsymbol{a} \in S} Kt^{\boldsymbol{a}}$ is a $K[\mathbb{N}A]$-module. Let $S, S' \subseteq \mathbb{Z}^d$ be $\mathbb{N}A$-sets. Throughout this paper, we simply write $D(S, S')$ and $D(S)$ for $D(K[S], K[S'])$ and $D(K[S])$, respectively. Then $D(S, S')$ can be realized as a submodule of the ring $D(\mathbb{Z}^d) = D(K[\mathbb{Z}^d])$ as follows:

$$D(S, S') = \{P \in D(\mathbb{Z}^d) \colon P(K[S]) \subset K[S']\}.$$

(See [4, p. 31], [17, Proposition 1.10], and [29, Lemma 2.7].)

Put $s_j := t_j \partial_j$ for $j = 1, 2, \dots, d$. We introduce a $\mathbb{Z}^d$-grading on the ring $D(\mathbb{Z}^d)$ as follows: For $\boldsymbol{a} = {}^t(a_1, a_2, \dots, a_d) \in \mathbb{Z}^d$, set

$$D(\mathbb{Z}^d)_{\boldsymbol{a}} := \{P \in D(\mathbb{Z}^d) \colon [s_j, P] = a_j P \text{ for } j = 1, 2, \dots, d\}.$$

Then the algebra $D(\mathbb{Z}^d)$ is $\mathbb{Z}^d$-graded; $D(\mathbb{Z}^d) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(\mathbb{Z}^d)_{\boldsymbol{a}}$. Let $S, S' \subseteq \mathbb{Z}^d$ be $\mathbb{N}A$-sets. For $\boldsymbol{a} \in \mathbb{Z}^d$, set $D(S, S')_{\boldsymbol{a}} := D(S, S') \cap D(\mathbb{Z}^d)_{\boldsymbol{a}}$ and $D(S)_{\boldsymbol{a}} := D(S) \cap D(\mathbb{Z}^d)_{\boldsymbol{a}}$. Then $D(S) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(S)_{\boldsymbol{a}}$ is a $\mathbb{Z}^d$-graded algebra, and $D(S, S') = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(S, S')_{\boldsymbol{a}}$ is a $\mathbb{Z}^d$-graded $(D(S'), D(S))$-bimodule. We can describe $D(S, S')_{\boldsymbol{a}}$ explicitly as in [18, Theorem 2.3]. For $\boldsymbol{a} \in \mathbb{Z}^d$, we define a subset $\Omega_{S,S'}(\boldsymbol{a})$ of $\mathbb{Z}^d$ by

$$\Omega_{S,S'}(\boldsymbol{a}) = \{\boldsymbol{d} \in S \colon \boldsymbol{d} + \boldsymbol{a} \notin S'\} = S \setminus (-\boldsymbol{a} + S').$$

We simply write $\Omega_S(\boldsymbol{a})$ for $\Omega_{S,S}(\boldsymbol{a})$. We regard the set $\Omega_{S,S'}(\boldsymbol{a})$ as a subset in K^d.

Proposition 2.1.

$$D(S, S')_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega_{S,S'}(\boldsymbol{a})),$$

where

$$\mathbb{I}(\Omega_{S,S'}(\boldsymbol{a})) := \{f(s) \in K[s] := K[s_1, \dots, s_d] \colon f \textit{ vanishes on } \Omega_{S,S'}(\boldsymbol{a})\}.$$

3. Finite generation

In [26] and [27], we proved that $D(\mathbb{N}A)$ is finitely generated as a K-algebra, and that $\mathrm{Gr}(D(\mathbb{N}A))$ is Noetherian if $\mathbb{N}A$ is scored, where $\mathrm{Gr}(D(\mathbb{N}A))$ is the graded ring associated with the order filtration of $D(\mathbb{N}A)$. In Section 5, we prove that $D(\mathbb{N}A)$ is right Noetherian (Theorem 5.10) for any A, by using Robson's lemma (Lemma 5.1). To this end, we need to generalize the results in [26] and [27] in a wider situation. This section is devoted to this purpose.

Let us recall the primitive integral support function of a facet (face of codimension one) of the cone $\mathbb{R}_{\geq 0}A$. We denote by $\mathcal{F}$ the set of facets of the cone $\mathbb{R}_{\geq 0}A$. Given $\sigma \in \mathcal{F}$, we denote by F_σ the *primitive integral support function* of σ, i.e., F_σ is a uniquely determined linear form on $\mathbb{R}^d$ satisfying
(1) $F_\sigma(\mathbb{R}_{\geq 0}A) \geq 0$,
(2) $F_\sigma(\sigma) = 0$,
(3) $F_\sigma(\mathbb{Z}^d) = \mathbb{Z}$.

REMARK 3.1. Let $\sigma \in \mathcal{F}$. By the definition of F_σ, there exists $m \in \mathbb{N}$ such that $F_\sigma(\mathbb{N}A) \supseteq m + \mathbb{N}$. Accordingly, for an $\mathbb{N}A$-set S, there exists $m \in \mathbb{N}$ such that $F_\sigma(S) \supseteq m + \mathbb{N}$.

Let S_c be a scored $\mathbb{N}A$-set, i.e., by definition,

$$S_c = \bigcap_{\sigma \in \mathcal{F}} \{\boldsymbol{a} \in \mathbb{Z}^d : F_\sigma(\boldsymbol{a}) \in F_\sigma(S_c)\}.$$

Lemma 3.2. *Let S_c be a scored $\mathbb{N}A$-set. Then S_c is a finitely generated $\mathbb{N}A$-set if and only if $F_\sigma(S_c)$ is a finitely generated $F_\sigma(\mathbb{N}A)$-set for each facet σ.*

Proof. The only-if direction is obvious.

Suppose that $F_\sigma(S_c)$ is a finitely generated $F_\sigma(\mathbb{N}A)$-set for each facet σ. Let $M_\sigma := \max(\mathbb{N} \setminus F_\sigma(S_c))$ (cf. Remark 3.1), and $F_\sigma(S_c)_+ := \{m \in F_\sigma(S_c) \colon m < M_\sigma\} \cup \{\infty\}$. Then $F_\sigma(S_c)_+$ is a finite set. For a map ν which assigns a facet σ to an element of $F_\sigma(S_c)_+$, we define a subset $S_c(\nu)$ of S_c by

$$S_c(\nu) = \{\boldsymbol{a} \in S_c : F_\sigma(\boldsymbol{a}) = \nu(\sigma) \text{ for all facets } \sigma\}.$$

Here we agree that $F_\sigma(\boldsymbol{a}) = \infty$ means $F_\sigma(\boldsymbol{a}) \geq M_\sigma + 1$. Then $S_c = \bigcup_\nu S_c(\nu)$. We also set

$$S_c(\nu)_{\mathbb{R}} = \{\boldsymbol{a} \in \mathbb{R}^d : F_\sigma(\boldsymbol{a}) = \nu(\sigma) \text{ for all facets } \sigma\}.$$

Since S_c is scored, $S_c(\nu) = S_c(\nu)_{\mathbb{R}} \cap \mathbb{Z}^d$.

For each ray (1-dimensional face) ρ of $\mathbb{R}_{\geq 0}A$, fix $\boldsymbol{d}_\rho \in \mathbb{N}A \cap \rho$. Set

$$F(\nu) = \{\boldsymbol{d}_\rho : \rho \preceq \sigma \text{ for all facets } \sigma \text{ with } \nu(\sigma) \neq \infty\}.$$

Since any strongly convex cone is generated by its 1-dimensional faces,

$$\left\{\boldsymbol{a} \in \mathbb{R}^d : \begin{array}{ll} F_\sigma(\boldsymbol{a}) = 0 & \text{for all facets } \sigma \text{ with } \nu(\sigma) \neq \infty, \\ F_\sigma(\boldsymbol{a}) \geq 0 & \text{for all facets } \sigma \text{ with } \nu(\sigma) = \infty \end{array}\right\} = \mathbb{R}_{\geq 0} F(\nu).$$

Hence $\mathbb{R}_{\geq 0}F(\nu)$ is the characteristic cone of $S_c(\nu)_{\mathbb{R}}$, and there exists a polytope $P(\nu)$ such that $S_c(\nu)_{\mathbb{R}} = P(\nu)+\mathbb{R}_{\geq 0}F(\nu)$ (see [28, §8.9]). Clearly there exists a finite set $G(\nu)$ such that $\mathbb{R}_{\geq 0}F(\nu) \cap \mathbb{Z}^d = G(\nu) + \sum_{\boldsymbol{d}_\rho \in F(\nu)} \mathbb{N}\boldsymbol{d}_\rho$. Thus $S_c(\nu)$ is generated by $(P(\nu) \cap \mathbb{Z}^d) \cup G(\nu)$ as an $\mathbb{N}F(\nu)$-set. Therefore S_c is finitely generated as an $\mathbb{N}A$-set. □

Let S_c be a scored finitely generated $\mathbb{N}A$-set. Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m \in S_c$, and let $\tau_1, \dots, \tau_m$ be faces of $\mathbb{R}_{\geq 0}A$. Let

$$S := S_c \setminus \bigcup_{i=1}^{m} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)) \tag{3.1}$$

satisfy $F_\sigma(S) = F_\sigma(S_c)$ for all facets σ. We say that S_c is the *scored closure* of S. We assume that the expression (3.1) is irredundant. Then $\{\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i) \colon i = 1, \dots, m\}$ is unique, and we call (3.1) the *standard expression* of S. We do not assume that S is an $\mathbb{N}A$-set, unless we state the contrary. In the remainder of this section and the next section, S_c and S are fixed as above.

We define a ring of differential operators by

$$D(S) := \{P \in D(\mathbb{Z}^d) \colon P(K[S]) \subseteq K[S]\}.$$

First we consider the $\mathbb{Z}^d$-graded structure of $D(S)$. Put $D(S)_{\boldsymbol{a}} := D(S) \cap D(\mathbb{Z}^d)_{\boldsymbol{a}}$. Similarly to Proposition 2.1, we can write

$$D(S) = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^d} D(S)_{\boldsymbol{a}}, \quad D(S)_{\boldsymbol{a}} = t^{\boldsymbol{a}} \mathbb{I}(\Omega_S(\boldsymbol{a})),$$

where $\Omega_S(\boldsymbol{a}) = S \setminus (-\boldsymbol{a} + S)$.

Proposition 3.3. *Let* ZC *stand for Zariski closure in* K^d. *We regard* F_σ *as a linear map from* K^d *to* K.

(1)

$$\mathrm{ZC}(\Omega_{S_c}(\boldsymbol{d})) = \bigcup_{\sigma \in \mathcal{F}} \bigcup_{k \in F_\sigma(S_c) \setminus (-F_\sigma(\boldsymbol{d}) + F_\sigma(S_c))} F_\sigma^{-1}(k).$$

(2)

$$\mathrm{ZC}(\Omega_S(\boldsymbol{d})) = \mathrm{ZC}(\Omega_{S_c}(\boldsymbol{d})) \cup \bigcup_{\boldsymbol{b}_i - \boldsymbol{d} \in S + \mathbb{Z}(A \cap \tau_i)} (\boldsymbol{b}_i - \boldsymbol{d} + K\tau_i).$$

In particular, $D(S)$ *is a subalgebra of* $D(S_c)$.

Proof. (1) is easy, and (2) follows from Lemma 3.5 below. See also [27, Proposition 5.1]. □

Fix $M \in \mathbb{N}$ so that

(3.2) $$M > \max F_\sigma(S_c)^c \cup \{F_\sigma(\boldsymbol{b}_i) : i\} - \min F_\sigma(S_c) \cup \{F_\sigma(\boldsymbol{b}_i) : i\} \quad \text{for all facets} \quad \sigma,$$

where $F_\sigma(S_c)^c = \mathbb{Z} \setminus F_\sigma(S_c)$. Then, for $N \geq M$,

$$N + F_\sigma(S_c) \subseteq F_\sigma(S_c) \quad \text{for all facets} \quad \sigma.$$

For a face τ, put

$$\mathbb{N}(\overset{\circ\circ}{A \cap \tau}) := \{\boldsymbol{a} \in \mathbb{N}(A \cap \tau) : F_\sigma(\boldsymbol{a}) \geq M \text{ for all } \sigma \in \mathcal{F} \text{ with } \sigma \not\succeq \tau\}.$$

Lemma 3.4. *Let τ be a face of $\mathbb{R}_{\geq 0}A$. Then*

$$S + \mathbb{N}(\overset{\circ\circ}{A \cap \tau}) \subseteq S.$$

Proof. Since S_c is an $\mathbb{N}A$-set, $S + \mathbb{N}(A \cap \tau) \subseteq S_c$. Let $\boldsymbol{a} \in \mathbb{N}(\overset{\circ\circ}{A \cap \tau})$, and let $\boldsymbol{b} \in S$. It remains to show that $\boldsymbol{b} + \boldsymbol{a} \notin \boldsymbol{b}_j + \mathbb{Z}(A \cap \tau_j)$ for any j. If $\tau_j \not\succeq \tau$, then there exists a facet σ such that $\sigma \succeq \tau_j$ and $\sigma \not\succeq \tau$. For such σ,

$$F_\sigma(\boldsymbol{a} + \boldsymbol{b}) \geq M + \min F_\sigma(S_c) > F_\sigma(\boldsymbol{b}_j).$$

Hence $\boldsymbol{a} + \boldsymbol{b} \notin \boldsymbol{b}_j + \mathbb{Z}(A \cap \tau_j)$.

If $\tau_j \succeq \tau$, then $\boldsymbol{a} + \boldsymbol{b} \notin \boldsymbol{b}_j + \mathbb{Z}(A \cap \tau_j)$ since $\boldsymbol{b} \notin \boldsymbol{b}_j + \mathbb{Z}(A \cap \tau_j)$. □

Lemma 3.5.

$$(\boldsymbol{d} + S) \cap S_c \cap (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)) \neq \emptyset \Leftrightarrow \boldsymbol{b}_i - \boldsymbol{d} \in S + \mathbb{Z}(A \cap \tau_i).$$

Proof. The implication $\Rightarrow$ is obvious.

For the implication $\Leftarrow$, let $\boldsymbol{a} \in S$ and $\boldsymbol{d} + \boldsymbol{a} \in \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)$. For $\sigma \in \mathcal{F}$ with $\sigma \succeq \tau_i$,

$$F_\sigma(\boldsymbol{d} + \boldsymbol{a}) = F_\sigma(\boldsymbol{b}_i) \in F_\sigma(S_c).$$

If $\sigma \not\succeq \tau_i$, then there exists $\boldsymbol{a}_\sigma \in \mathbb{N}(\overset{\circ\circ}{A \cap \tau_i})$ such that $F_\sigma(\boldsymbol{d} + \boldsymbol{a} + \boldsymbol{a}_\sigma) \in F_\sigma(S_c)$. By Lemma 3.4, $\boldsymbol{a} + \sum_{\sigma \not\succeq \tau_i} \boldsymbol{a}_\sigma \in S$. Hence $\boldsymbol{d} + \boldsymbol{a} + \sum_{\sigma \not\succeq \tau_i} \boldsymbol{a}_\sigma$ belongs to the set in the left hand side. □

By Proposition 3.3, $\mathbb{I}(\Omega_{S_c}(\boldsymbol{d})) = \langle p_{\boldsymbol{d}} \rangle$, where

(3.3) $$p_{\boldsymbol{d}}(s) = \prod_\sigma \prod_{k \in F_\sigma(S_c) \setminus (-F_\sigma(\boldsymbol{d}) + F_\sigma(S_c))} (F_\sigma(s) - k).$$

Theorem 3.6. *Let* $\mathrm{Gr}(D(S_c))$ *denote the graded ring associated with the order filtration of* $D(S_c)$. *Then the following hold*:

(1) $\mathrm{Gr}(D(S_c))$ *is finitely generated as a* K*-algebra*. $D(S_c)$ *is left and right Noetherian*.

(2) $D(S)$ *is finitely generated as a* K*-algebra*.

(3) $D(S_c)$ *is finitely generated as a right* $D(S)$*-module*.

Proof. For (1) and (2), the argument in [27, Sections 5 and 6] works with the new M (3.2) in place of the old M [27, (5)]. In [27], we did the argument when $S_c = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$.

For (3), we briefly recall some notation from [27]. For a ray ρ of the arrangement determined by A, i.e. $\mathbb{R}\rho$ is the intersection of some hyperplanes $(F_\sigma = 0)$, take a nonzero vector $\boldsymbol{d}_\rho$ from $\mathbb{Z}^d \cap \rho$ satisfying the conditions:

$$\begin{aligned} F_\sigma(\boldsymbol{d}_\rho) &\geq M \quad \text{if} \quad F_\sigma(\boldsymbol{d}_\rho) > 0, \\ F_\sigma(\boldsymbol{d}_\rho) &\leq -M \quad \text{if} \quad F_\sigma(\boldsymbol{d}_\rho) < 0, \end{aligned} \tag{3.4}$$

and

$$\boldsymbol{d}_\rho \in \mathbb{Z}(A \cap \tau) \cap \rho \quad \text{for all faces} \quad \tau \quad \text{of} \quad \mathbb{R}_{\geq 0}A \quad \text{satisfying} \quad \mathbb{R}\tau \supset \rho. \tag{3.5}$$

Let μ be a map from $\mathcal{F}$ to a set

$$\tilde{M} := \{-\infty\} \cup \{+\infty\} \cup \{m \in \mathbb{Z} \colon |m| < M\}.$$

Define a subset $S_c(\mu)$ of $\mathbb{Z}^d$ by

$$S_c(\mu) := \{\boldsymbol{d} \in \mathbb{Z}^d \colon F_\sigma(\boldsymbol{d}) = \mu(\sigma) \text{ for all } \sigma \in \mathcal{F}\}, \tag{3.6}$$

where we agree that $F_\sigma(\boldsymbol{d}) = +\infty$ ($-\infty$, respectively) mean $F_\sigma(\boldsymbol{d}) \geq M$ ($\leq -M$, respectively). We also define

$$F(\mu)_{\mathbb{R}} := \bigcap_{\sigma \in \mathcal{F}} \left\{ \boldsymbol{d} \in \mathbb{R}^d \colon \begin{array}{ll} F_\sigma(\boldsymbol{d}) = 0 & \text{if} \quad \mu(\sigma) \neq \pm\infty, \\ F_\sigma(\boldsymbol{d}) \geq 0 & \text{if} \quad \mu(\sigma) = +\infty, \\ F_\sigma(\boldsymbol{d}) \leq 0 & \text{if} \quad \mu(\sigma) = -\infty \end{array} \right\}, \tag{3.7}$$

In the case of (3), for $\boldsymbol{d}_1 \in S_c(\mu)$ and $\rho \subset F(\mu)_{\mathbb{R}}$, the deficiency ideal (cf. [27, Definition 5.4]) is the same as in the case of (2), i.e.,

$$D(S_c)_{\boldsymbol{d}_1} \cdot D(S)_{\boldsymbol{d}_\rho} = D(S_c)_{\boldsymbol{d}_1 + \boldsymbol{d}_\rho} \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{d}_\rho \in S + \mathbb{Z}(A \cap \tau_i)} (\boldsymbol{b}_i - \boldsymbol{d}_\rho + K(A \cap \tau_i)) \right).$$

Hence the same argument in the proof of [27, Theorem 5.14] works as well. □

4. Partial preorders

In this section, we keep situation (3.1), and introduce preorders, which indicate the $D(S)$-submodule structure of $K[\mathbb{Z}^d] = K[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

For an ideal I of $K[s]$ and a vector $\boldsymbol{c} \in K^d$, we define a new ideal $I + \boldsymbol{c}$ by

$$I + \boldsymbol{c} := \{ f(s - \boldsymbol{c}) \colon f(s) \in I \}.$$

The following lemma is immediate from the definition.

Lemma 4.1. *For a subset V of K^d and a point $\boldsymbol{a} \in K^d$, let $\mathbb{I}(V)$ denote the ideal of polynomials vanishing on V, and $\mathfrak{m}_{\boldsymbol{a}} := \mathbb{I}(\{\boldsymbol{a}\})$ the maximal ideal at $\boldsymbol{a}$. Then the following hold*:

(1) $\mathbb{I}(V) + \boldsymbol{c} = \mathbb{I}(V + \boldsymbol{c})$.
(2) $\mathfrak{m}_{\boldsymbol{a}} + (\boldsymbol{b} - \boldsymbol{a}) = \mathfrak{m}_{\boldsymbol{b}}$.
(3) *If $\mathfrak{p}$ is prime, then so is $\mathfrak{p} + \boldsymbol{c}$.*

Let $\mathfrak{p}$ be a prime ideal of $K[s]$. In the set $\{\mathfrak{p} + \boldsymbol{a} \colon \boldsymbol{a} \in \mathbb{Z}^d\}$, we define $\preceq_S$ by

$$\mathfrak{p} \preceq_S \mathfrak{p} + \boldsymbol{a} \overset{\text{def.}}{\Leftrightarrow} \mathbb{I}(\Omega_S(\boldsymbol{a})) \not\subseteq \mathfrak{p}. \tag{4.1}$$

REMARK 4.2. When $S = \mathbb{N}A$, $\mathfrak{m}_{\boldsymbol{a}} \preceq_S \mathfrak{m}_{\boldsymbol{b}}$ if and only if $\boldsymbol{a} \preceq \boldsymbol{b}$ in the sense of [26], which was also considered in [21].

Lemma 4.3. *$\preceq_S$ is a partial preorder.*

Proof. First, since $\mathbb{I}(\Omega_S(\boldsymbol{0})) = (1) \not\subseteq \mathfrak{p}$, we have $\mathfrak{p} \preceq_S \mathfrak{p}$.

Second, let $\mathfrak{p} \preceq_S \mathfrak{p} + \boldsymbol{a}$ and $\mathfrak{p} + \boldsymbol{a} \preceq_S \mathfrak{p} + \boldsymbol{a} + \boldsymbol{b}$. Then we have $\mathbb{I}(\Omega_S(\boldsymbol{a})) \not\subseteq \mathfrak{p}$ and $\mathbb{I}(\Omega_S(\boldsymbol{b})) \not\subseteq \mathfrak{p} + \boldsymbol{a}$. The latter is equivalent to $\mathbb{I}(\Omega_S(\boldsymbol{b}) - \boldsymbol{a}) \not\subseteq \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we have

$$\mathbb{I}(\Omega(\boldsymbol{a}))\mathbb{I}(\Omega(\boldsymbol{b}) - \boldsymbol{a}) \not\subseteq \mathfrak{p}.$$

From the inclusion $D(S)_{\boldsymbol{b}} D(S)_{\boldsymbol{a}} \subseteq D(S)_{\boldsymbol{a}+\boldsymbol{b}}$, we obtain

$$\mathbb{I}(\Omega_S(\boldsymbol{a}))\mathbb{I}(\Omega_S(\boldsymbol{b}) - \boldsymbol{a}) \subseteq \mathbb{I}(\Omega_S(\boldsymbol{a} + \boldsymbol{b})).$$

Hence we have

$$\mathbb{I}(\Omega_S(\boldsymbol{a} + \boldsymbol{b})) \not\subseteq \mathfrak{p},$$

or equivalently

$$\mathfrak{p} \preceq_S \mathfrak{p} + \boldsymbol{a} + \boldsymbol{b}. \qquad \square$$

Lemma 4.4. *Let $\boldsymbol{\alpha} \in K^d$, and let $\boldsymbol{a} \in \mathbb{Z}^d$. Then $\mathbb{I}(\tau) + \boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau) + \boldsymbol{\alpha} + \boldsymbol{a}$ if and only if, for all facets $\sigma \succeq \tau$, $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(S_c)$ implies $F_\sigma(\boldsymbol{\alpha} + \boldsymbol{a}) \in F_\sigma(S_c)$, and, for all faces $\tau_i \succeq \tau$, $\boldsymbol{\alpha} + \boldsymbol{a} - \boldsymbol{b}_i \in K\tau_i$ implies $\boldsymbol{b}_i - \boldsymbol{a} \notin S + \mathbb{Z}(A \cap \tau_i)$.*

Proof. By definition, $\mathbb{I}(\tau) + \boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau) + \boldsymbol{\alpha} + \boldsymbol{a}$ means $\mathrm{ZC}(\Omega_S(\boldsymbol{a})) \not\supseteq \boldsymbol{\alpha} + K\tau$. By Proposition 3.3, the latter condition means that, for all facets $\sigma \succeq \tau$, $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(S_c)$ implies $F_\sigma(\boldsymbol{\alpha} + \boldsymbol{a}) \in F_\sigma(S_c)$, and, for all faces $\tau_i \succeq \tau$, $\boldsymbol{\alpha} + \boldsymbol{a} - \boldsymbol{b}_i \in K\tau_i$ implies $\boldsymbol{b}_i - \boldsymbol{a} \notin S + \mathbb{Z}(A \cap \tau_i)$. □

For $\boldsymbol{\alpha} \in K^d$ and a face τ, set

$$E(S)_\tau(\boldsymbol{\alpha}) := \{\boldsymbol{\lambda} \in K\tau/\mathbb{Z}(A \cap \tau) : \boldsymbol{\alpha} - \boldsymbol{\lambda} \in S + \mathbb{Z}(A \cap \tau)\}. \tag{4.2}$$

Define another partial preorder $\preceq_{S,\tau}$ by

$$\boldsymbol{\alpha} \preceq_{S,\tau} \boldsymbol{\beta} \overset{\text{def.}}{\Leftrightarrow} E(S)_{\tau'}(\boldsymbol{\alpha}) \subseteq E(S)_{\tau'}(\boldsymbol{\beta}) \quad \text{for all faces} \quad \tau' \quad \text{with} \quad \tau' \succeq \tau. \tag{4.3}$$

We denote by $\boldsymbol{\alpha} \sim_{S,\tau} \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \preceq_{S,\tau} \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \succeq_{S,\tau} \boldsymbol{\beta}$, or equivalently, if

$$E(S)_{\tau'}(\boldsymbol{\alpha}) = E(S)_{\tau'}(\boldsymbol{\beta}) \quad \text{for all faces} \quad \tau' \quad \text{with} \quad \tau' \succeq \tau.$$

When $S = \mathbb{N}A$, the set $E(S)_\tau(\boldsymbol{\alpha})$ was considered in [24]. As in the case when $S = \mathbb{N}A$, we have the following lemma.

Lemma 4.5. (1) *$E(S)_\tau(\boldsymbol{\alpha})$ is a finite set.*
(2) $E(S)_{\mathbb{R}_{\geq 0}A}(\boldsymbol{\alpha}) = \{\boldsymbol{\alpha} \bmod \mathbb{Z}^d\}$.
(3) *For a facet $\sigma \in \mathcal{F}$, $E(S)_\sigma(\boldsymbol{\alpha}) \neq \emptyset$ if and only if $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(S) = F_\sigma(S_c)$.*

Proof. The proofs are the same as in the case when $S = \mathbb{N}A$. See [24, Propositions 2.2 and 2.3]. □

By Lemma 4.5 (2), $\boldsymbol{\alpha} + \mathbb{Z}^d = \boldsymbol{\beta} + \mathbb{Z}^d$ if $\boldsymbol{\alpha} \preceq_{S,\tau} \boldsymbol{\beta}$.

Lemma 4.6. *For any $\boldsymbol{\alpha} \in K^d$, $\boldsymbol{\alpha} + \mathbb{Z}^d$ has only finitely many equivalence classes with respect to $\sim_{S,\tau}$.*

Proof. Let $\boldsymbol{\beta} \in \boldsymbol{\alpha} + \mathbb{Z}^d$. If there exists $\boldsymbol{\lambda} \in K\tau'$ such that $\boldsymbol{\alpha} - \boldsymbol{\lambda} \in \mathbb{Z}^d$, then $\boldsymbol{\beta} - \boldsymbol{\lambda} \in \mathbb{Z}^d$, and $E(S)_{\tau'}(\boldsymbol{\alpha})$ and $E(S)_{\tau'}(\boldsymbol{\beta})$ are contained in the finite set $\boldsymbol{\lambda} + \mathbb{Q}(\tau' \cap \mathbb{Z}^d)/\mathbb{Z}(A \cap \tau')$. If there exists no such $\boldsymbol{\lambda} \in K\tau'$, then $E(S)_{\tau'}(\boldsymbol{\alpha})$ and $E(S)_{\tau'}(\boldsymbol{\beta})$ are empty. Hence the number of equivalence classes is finite. □

Next we compare two preorders $\preceq_S$ and $\preceq_{S,\tau}$. By Lemmas 4.4 and 4.5,

$$\mathbb{I}(\tau)+\boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau)+\boldsymbol{\alpha}+\boldsymbol{a} \tag{4.4}$$
$$\Leftrightarrow \begin{cases} \text{for all facets} \quad \sigma \succeq \tau,\ E(S)_\sigma(\boldsymbol{\alpha}) \neq \emptyset \Rightarrow E(S)_\sigma(\boldsymbol{\alpha}+\boldsymbol{a}) \neq \emptyset, \\ \text{for all faces} \quad \tau_i \succeq \tau,\ \boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{b}_i \notin E(S)_{\tau_i}(\boldsymbol{\alpha}). \end{cases}$$

Note that $\boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{b}_i \notin E(S)_{\tau_i}(\boldsymbol{\alpha}+\boldsymbol{a})$ is automatic.

Hence we have proved the following proposition.

Proposition 4.7.

$$\boldsymbol{\alpha} \preceq_{S,\tau} \boldsymbol{\beta} \Rightarrow \mathbb{I}(\tau)+\boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau)+\boldsymbol{\beta}.$$

We denote by $\mathbb{I}(\tau)+\boldsymbol{\alpha} \sim_S \mathbb{I}(\tau)+\boldsymbol{\beta}$ if $\mathbb{I}(\tau)+\boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau)+\boldsymbol{\beta}$ and $\mathbb{I}(\tau)+\boldsymbol{\alpha} \succeq_S \mathbb{I}(\tau)+\boldsymbol{\beta}$.

Corollary 4.8.

$$\boldsymbol{\alpha} \sim_{S,\tau} \boldsymbol{\beta} \Rightarrow \mathbb{I}(\tau)+\boldsymbol{\alpha} \sim_S \mathbb{I}(\tau)+\boldsymbol{\beta}.$$

Similarly to [27, Lemma 3.6], the following holds.

Lemma 4.9.

$$S+\mathbb{Z}(A\cap\tau) = [S_c+\mathbb{Z}(A\cap\tau)] \setminus \bigcup_{\tau_i \succeq \tau} (\boldsymbol{b}_i+\mathbb{Z}(A\cap\tau_i)).$$

Lemma 4.10.

$$\begin{aligned} S_c+\mathbb{Z}(A\cap\tau) &= \{\boldsymbol{a}\in\mathbb{Z}^d : F_\sigma(\boldsymbol{a}) \in F_\sigma(S) = F_\sigma(S_c) \text{ for all facets } \sigma \succeq \tau\} \\ &= \{\boldsymbol{a}\in\mathbb{Z}^d : E(S)_\sigma(\boldsymbol{a}) \neq \emptyset \text{ for all facets } \sigma \succeq \tau\}. \end{aligned}$$

Proof. This is immediate from the definitions and Lemma 4.5. □

Theorem 4.11.

$$\boldsymbol{\alpha} \sim_{S,\tau} \boldsymbol{\beta} \Leftrightarrow \mathbb{I}(\tau)+\boldsymbol{\alpha} \sim_S \mathbb{I}(\tau)+\boldsymbol{\beta}.$$

Proof. It is left to prove the implication $\Leftarrow$. We suppose that $\mathbb{I}(\tau)+\boldsymbol{\alpha} \preceq_S \mathbb{I}(\tau)+\boldsymbol{\alpha}+\boldsymbol{a}$. Hence we suppose the two conditions in (4.4). We show that $E(S)_{\tau'}(\boldsymbol{\alpha}) \subseteq E(S)_{\tau'}(\boldsymbol{\alpha}+\boldsymbol{a})$ for all $\tau' \succeq \tau$. We assume the contrary; we suppose that $\boldsymbol{\lambda} \in E(S)_{\tau'}(\boldsymbol{\alpha}) \setminus E(S)_{\tau'}(\boldsymbol{\alpha}+\boldsymbol{a})$. Then we have

$$\boldsymbol{\lambda} \in K\tau', \tag{4.5}$$

$$\boldsymbol{\alpha}-\boldsymbol{\lambda} \in S+\mathbb{Z}(A\cap\tau'), \tag{4.6}$$

$$\boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{\lambda} \notin S+\mathbb{Z}(A\cap\tau'). \tag{4.7}$$

By (4.5) and (4.6), $F_\sigma(\boldsymbol{\alpha}) \in F_\sigma(S_c)$ for all $\sigma \succeq \tau'$. Then by (4.5) and the first condition of (4.4) $F_\sigma(\boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{\lambda}) = F_\sigma(\boldsymbol{\alpha}+\boldsymbol{a}) \in F_\sigma(S_c)$ for all $\sigma \succeq \tau'$. Then by (4.7) and Lemma 4.9 there exists $\tau_i \succeq \tau'$ such that $\boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{\lambda} \in \boldsymbol{b}_i + \mathbb{Z}(A\cap\tau_i)$. Hence $\boldsymbol{\alpha}+\boldsymbol{a}-\boldsymbol{b}_i + \mathbb{Z}(A\cap\tau_i) = \boldsymbol{\lambda} + \mathbb{Z}(A\cap\tau_i) \in E(S)_{\tau_i}(\boldsymbol{\alpha})$. This contradicts the second condition of (4.4). □

5. Right Noetherian property

In this section, we assume that S_0 is a scored finitely generated $\mathbb{N}A$-set, that the expression $S_m = S_0 \setminus \bigcup_{i=1}^m(\boldsymbol{b}_i + \mathbb{Z}(A\cap\tau_i))$ with all $\boldsymbol{b}_i \in S_0$ is irredundant, and that $F_\sigma(S_m) = F_\sigma(S_0)$ for all facets σ. In the notation in (3.1), $S_0 = S_c$ and $S_m = S$. We prove that $D(S_m)$ is right Noetherian by the induction on m.

Let

$$D(S_0, S_m) := \{P \in D(\mathbb{Z}^d) : P(K[S_0]) \subseteq K[S_m]\},$$

and for $\boldsymbol{a} \in \mathbb{Z}^d$ let

$$D(S_0, S_m)_{\boldsymbol{a}} := D(S_0, S_m) \cap D(\mathbb{Z}^d)_{\boldsymbol{a}}.$$

Then $D(S_0, S_m)$ is a right ideal of $D(S_0)$, and a left ideal of $D(S_m)$. We have

$$D(S_0, S_m)_{\boldsymbol{a}} = t^{\boldsymbol{a}}\mathbb{I}(\Omega_{S_0,S_m}(\boldsymbol{a})), \quad \Omega_{S_0,S_m}(\boldsymbol{a}) = S_0 \setminus (-\boldsymbol{a}+S_m), \tag{5.1}$$

and

$$\mathrm{ZC}(\Omega_{S_0,S_m}(\boldsymbol{a})) = \mathrm{ZC}(\Omega_{S_0}(\boldsymbol{a})) \cup \bigcup_{\boldsymbol{b}_i-\boldsymbol{a}\in S_0+\mathbb{Z}(A\cap\tau_i);1\le i\le m} (\boldsymbol{b}_i - \boldsymbol{a} + K\tau_i). \tag{5.2}$$

We use the following Robson's lemma to prove the right Noetherian property of $D(S_m)$.

Lemma 5.1 (Proposition 2.3 in [23]). *Let A be a right ideal of a right Noetherian ring S. Let R be a subring of S containing A. Suppose that S is finitely generated as a right R-module, and that S/A is a right Noetherian R-module. Then the ring R is right Noetherian.*

By Lemma 5.1 and Theorem 3.6, we only need to show that $D(S_0)/D(S_0, S_m)$ is a Noetherian right $D(S_m)$-module.

Let $k \le m$, and let $S_k = S_0 \setminus \bigcup_{i=1}^k(\boldsymbol{b}_i + \mathbb{Z}(A\cap\tau_i))$. Since we know, by Theorem 3.6, that $D(S_0)$ is right Noetherian, and that $D(S_0)$ is a finitely generated right $D(S_k)$-module, the sequence of right ideals of $D(S_0)$

$$D(S_0, S_m) \subseteq D(S_0, S_{m-1}) \subseteq \cdots \subseteq D(S_0, S_1) \subseteq D(S_0) \tag{5.3}$$

is a sequence of finitely generated right $D(S_k)$-modules.

Set

$$M_k := D(S_0, S_{k-1})/D(S_0, S_k). \tag{5.4}$$

We want to show that each M_k is a Noetherian right $D(S_m)$-module.

Lemma 5.2.

$$\mathbb{I}(\Omega_{S_0,S_k}(\boldsymbol{a})) = \langle p_{\boldsymbol{a}} \rangle \cdot \bigcap_{\boldsymbol{b}_i - \boldsymbol{a} \in S_0 + \mathbb{Z}(A \cap \tau_i); i \le k} \mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i).$$

Proof. By (3.3) and (5.2),

$$\mathbb{I}(\Omega_{S_0,S_k}(\boldsymbol{a})) = \langle p_{\boldsymbol{a}} \rangle \cap \bigcap_{\boldsymbol{b}_i - \boldsymbol{a} \in S_0 + \mathbb{Z}(A \cap \tau_i); i \le k} \mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i).$$

Suppose that $\boldsymbol{b}_i - \boldsymbol{a} \in S_0 + \mathbb{Z}(A \cap \tau_i)$. If $\boldsymbol{b}_i - \boldsymbol{a} + K\tau_i \subseteq \mathrm{ZC}(\Omega_{S_0}(\boldsymbol{a}))$, then there exists a facet $\sigma \succeq \tau_i$ such that $F_\sigma(\boldsymbol{b}_i - \boldsymbol{a}) \in F_\sigma(S_0) \setminus (-F_\sigma(\boldsymbol{a}) + F_\sigma(S_0))$. This contradicts the fact that $F_\sigma(\boldsymbol{b}_i) \in F_\sigma(S_0)$. Hence $p_{\boldsymbol{a}} \notin \mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i)$. If $f p_{\boldsymbol{a}} \in \bigcap_i \mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i)$, then $f \in \bigcap_i \mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i)$ since $\mathbb{I}(\boldsymbol{b}_i - \boldsymbol{a} + \tau_i)$ is prime. We have thus proved the assertion. □

Corollary 5.3. *If $(M_k)_{\boldsymbol{a}} \neq 0$, then $\boldsymbol{b}_k - \boldsymbol{a} \in S_0 + \mathbb{Z}(A \cap \tau_k)$, or equivalently, if $(M_k)_{\boldsymbol{b}_k - \boldsymbol{a}} \neq 0$, then $\boldsymbol{a} \in S_0 + \mathbb{Z}(A \cap \tau_k)$.*

Proof. This is immediate from Lemma 5.2 and the definition of M_k (5.4). □

Lemma 5.4. *If $(M_k)_{\boldsymbol{b}_k - \boldsymbol{a}} D(S_m)_{\boldsymbol{c}} \neq 0$, then $\boldsymbol{a} - \boldsymbol{c} \preceq_{S_m, \tau_k} \boldsymbol{a}$.*

Proof. We have

$$\begin{aligned} & \boldsymbol{a} - \boldsymbol{c} \not\preceq_{S_m, \tau_k} \boldsymbol{a} \\ & \Leftrightarrow \mathbb{I}(\tau_k) + \boldsymbol{a} - \boldsymbol{c} \not\preceq_{S_m} \mathbb{I}(\tau_k) + \boldsymbol{a} \\ & \Leftrightarrow \mathbb{I}(\Omega_{S_m}(\boldsymbol{c})) \subseteq \mathbb{I}(\tau_k + \boldsymbol{a} - \boldsymbol{c}) \\ & \Rightarrow (M_k)_{\boldsymbol{b}_k - \boldsymbol{a}} D(S_m)_{\boldsymbol{c}} = 0. \end{aligned} \tag{5.5}$$

Here the first equivalence is by Theorem 4.11; the second is by the definition of $\preceq_{S_m}$ (4.1). For the implication (5.5), let $X \in D(S_0, S_{k-1})_{\boldsymbol{b}_k - \boldsymbol{a}}$ and $t^{\boldsymbol{c}} f(s) \in D(S_m)_{\boldsymbol{c}}$. Since M_k is a right $D(S_m)$-module, $X t^{\boldsymbol{c}} f(s) \in t^{\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c}} \mathbb{I}(\Omega_{S_0, S_{k-1}}(\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c}))$. Since $f(s) \in \mathbb{I}(\Omega_{S_m}(\boldsymbol{c}))$, we have $f(s) \in \mathbb{I}(\tau_k + \boldsymbol{a} - \boldsymbol{c})$. Hence by Lemma 5.2

$$X t^{\boldsymbol{c}} f(s) \in t^{\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c}} \mathbb{I}(\Omega_{S_0, S_{k-1}}(\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c})) \cap \mathbb{I}(\tau_k + \boldsymbol{a} - \boldsymbol{c}) \subseteq t^{\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c}} \mathbb{I}(\Omega_{S_0, S_k}(\boldsymbol{b}_k - \boldsymbol{a} + \boldsymbol{c})).$$

Thus the implication (5.5) holds. □

The following proposition is immediate from Lemma 5.4.

Proposition 5.5. *Let C be a set of equivalence classes in $S_0 + \mathbb{Z}(A \cap \tau_k)$ with respect to $\sim_{S_m,\tau_k}$ such that $\boldsymbol{d} \preceq_{S_m,\tau_k} \boldsymbol{c}$ and $\boldsymbol{c} \in C$ imply $\boldsymbol{d} \in C$.*
Then $\bigoplus_{\boldsymbol{c}\in C}(M_k)_{\boldsymbol{b}_k-\boldsymbol{c}}$ is a right $D(S_m)$-submodule of $M_k = \bigoplus_{\boldsymbol{c}\in S_0+\mathbb{Z}(A\cap\tau_k)}(M_k)_{\boldsymbol{b}_k-\boldsymbol{c}}$.

For $1 \leq k \leq m$, set

$$\check{S}_k := S_0 \setminus \bigcup_{1\leq i\leq m; i\neq k} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)). \tag{5.6}$$

Then $D(\check{S}_k)$ is a K-subalgebra of $D(S_0)$ (Proposition 3.3), and we may assume that $D(\check{S}_k)$ is right Noetherian by the induction. Since M_k is a Noetherian right $D(S_0)$-module, and $D(S_0)$ is finitely generated as a right $D(\check{S}_k)$-module by Theorem 3.6, M_k is a Noetherian right $D(\check{S}_k)$-module.

The following lemma relates the Noetherian property as a right $D(\check{S}_k)$-module to that as a right $D(S_m)$-module.

Lemma 5.6. *Let $C_1 \subseteq C_2 \subseteq \boldsymbol{b}_k - S_0 + \mathbb{Z}(A\cap\tau_k)$, and suppose that $N_i := \bigoplus_{\boldsymbol{a}\in C_i}(M_k)_{\boldsymbol{a}}$ ($i = 1, 2$) are right $D(S_m)$-submodules of M_k. If, for any $\boldsymbol{a}, \boldsymbol{a}+\boldsymbol{c} \in C_2 \setminus C_1$, $X \in (M_k)_{\boldsymbol{a}}$, and $P \in D(\check{S}_k)_{\boldsymbol{c}}$, there exists $Q \in D(S_m)_{\boldsymbol{c}}$ such that $X.Q = X.P$, then N_2/N_1 is a Noetherian $\mathbb{Z}^d$-graded right $D(S_m)$-module.*

Proof. Let N be a $\mathbb{Z}^d$-graded right $D(S_m)$-submodule of M_k with $N_1 \subseteq N \subseteq N_2$. Put

$$\tilde{N} := \left(\bigoplus_{\boldsymbol{a}\in C_2\setminus C_1} N_{\boldsymbol{a}}\right) D(\check{S}_k).$$

Then $\tilde{N}$ is a right $D(\check{S}_k)$-submodule of M_k. By the assumption, $\tilde{N}_{\boldsymbol{a}} = N_{\boldsymbol{a}}$ for all $\boldsymbol{a} \in C_2 \setminus C_1$. Hence $(\tilde{N} \cap N_2 + N_1)/N_1 = N/N_1$. Therefore the Noetherian property of M_k as a right $D(\check{S}_k)$-module implies that of N_2/N_1 as a $\mathbb{Z}^d$-graded right $D(S_m)$-module. □

Replacing $\boldsymbol{a}$ by $\boldsymbol{b}_k - \boldsymbol{a}$ in Lemma 5.6, we have the following corollary.

Corollary 5.7. *Let C be an equivalence class in $S_0 + \mathbb{Z}(A \cap \tau_k)$ with respect to $\sim_{S_m,\tau_k}$, and let $\boldsymbol{d} \in C$. If, for any $\boldsymbol{a}, \boldsymbol{a} - \boldsymbol{c} \in C$, $X \in (M_k)_{\boldsymbol{b}_k-\boldsymbol{a}}$, and $P \in D(\check{S}_k)_{\boldsymbol{c}}$, there exists $Q \in D(S_m)_{\boldsymbol{c}}$ such that $X.Q = X.P$, then*

$$\bigoplus_{\boldsymbol{a}'\preceq_{S_m,\tau_k}\boldsymbol{d}} (M_k)_{\boldsymbol{b}_k-\boldsymbol{a}'} \Big/ \bigoplus_{\boldsymbol{a}'\prec_{S_m,\tau_k}\boldsymbol{d}} (M_k)_{\boldsymbol{b}_k-\boldsymbol{a}'}$$

is a Noetherian $\mathbb{Z}^d$-graded right $D(S_m)$-module, where $\prec_{S_m,\tau_k}$ means $\preceq_{S_m,\tau_k}$ and $\not\sim_{S_m,\tau_k}$.

Proposition 5.8. *For each equivalence class C in $S_0 + \mathbb{Z}(A \cap \tau_k)$ with respect to $\sim_{S_m,\tau_k}$, the assumption in* Corollary 5.7 *is satisfied.*

Proof. Similarly to Lemma 5.2, we see

$$\mathbb{I}(\Omega_{S_m}(\boldsymbol{c})) = \langle p_{\boldsymbol{c}} \rangle \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{c} \in S_m + \mathbb{Z}(A\cap\tau_i)} (\boldsymbol{b}_i - \boldsymbol{c} + \tau_i) \right),$$

$$\mathbb{I}(\Omega_{\check{S}_k}(\boldsymbol{c})) = \langle p_{\boldsymbol{c}} \rangle \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{c} \in \check{S}_k + \mathbb{Z}(A\cap\tau_i); i \neq k} (\boldsymbol{b}_i - \boldsymbol{c} + \tau_i) \right).$$

Since $S_m \subseteq \check{S}_k$, $\boldsymbol{b}_i - \boldsymbol{c} \in S_m + \mathbb{Z}(A \cap \tau_i)$ implies $\boldsymbol{b}_i - \boldsymbol{c} \in \check{S}_k + \mathbb{Z}(A \cap \tau_i)$. Hence, if $\boldsymbol{b}_k - \boldsymbol{c} \notin S_m + \mathbb{Z}(A \cap \tau_k)$, then $\mathbb{I}(\Omega_{\check{S}_k}(\boldsymbol{c})) \subseteq \mathbb{I}(\Omega_{S_m}(\boldsymbol{c}))$, i.e., $D(\check{S}_k)_{\boldsymbol{c}} \subseteq D(S_m)_{\boldsymbol{c}}$, and we have nothing to prove.

Suppose that $\boldsymbol{b}_k - \boldsymbol{c} \in S_m + \mathbb{Z}(A \cap \tau_k)$. Let $f \in \mathbb{I}\big(\bigcup_{\boldsymbol{b}_i - \boldsymbol{c} \in \check{S}_k + \mathbb{Z}(A\cap\tau_i); i \neq k} (\boldsymbol{b}_i - \boldsymbol{c} + \tau_i)\big)$. If $f \in \mathbb{I}(\boldsymbol{b}_k - \boldsymbol{c} + \tau_k)$, then $p_{\boldsymbol{c}} f \in \mathbb{I}(\Omega_{S_m}(\boldsymbol{c}))$, and again we have nothing to prove.

Let $f \notin \mathbb{I}(\boldsymbol{b}_k - \boldsymbol{c} + \tau_k)$. Let $X \in (M_k)_{\boldsymbol{b}_k - \boldsymbol{a}}$. Suppose that $\boldsymbol{b}_k - \boldsymbol{a} \notin K\tau_k$. Then there exists a facet $\sigma \succeq \tau_k$ such that $F_\sigma(\boldsymbol{b}_k - \boldsymbol{a}) \neq 0$. Since $F_\sigma(s) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c}) \in \mathbb{I}(\boldsymbol{b}_k - \boldsymbol{c} + \tau_k)$, $p_{\boldsymbol{c}} f (F_\sigma(s) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c})) \in \mathbb{I}(\Omega_{S_m}(\boldsymbol{c}))$. We have

$$\begin{aligned} X.t^{\boldsymbol{c}} p_{\boldsymbol{c}} f (F_\sigma(s) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c})) &= X.(F_\sigma(s - \boldsymbol{c}) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c})) t^{\boldsymbol{c}} p_{\boldsymbol{c}} f \\ &= X(F_\sigma(\boldsymbol{a} - \boldsymbol{c}) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c})) t^{\boldsymbol{c}} p_{\boldsymbol{c}} f \\ &= X F_\sigma(\boldsymbol{b}_k - \boldsymbol{a}) t^{\boldsymbol{c}} p_{\boldsymbol{c}} f. \end{aligned}$$

Here the second equality above holds because

$$X \in (M_k)_{\boldsymbol{b}_k - \boldsymbol{a}} = t^{\boldsymbol{b}_k - \boldsymbol{a}} \mathbb{I}(\Omega_{S_0, S_{k-1}}(\boldsymbol{b}_k - \boldsymbol{a})) / t^{\boldsymbol{b}_k - \boldsymbol{a}} \mathbb{I}(\Omega_{S_0, S_{k-1}}(\boldsymbol{b}_k - \boldsymbol{a})) \cap \mathbb{I}(\boldsymbol{a} + \tau_k).$$

Hence in this case

$$X.t^{\boldsymbol{c}} p_{\boldsymbol{c}} f = X. \frac{1}{F_\sigma(\boldsymbol{b}_k - \boldsymbol{a})} t^{\boldsymbol{c}} p_{\boldsymbol{c}} f (F_\sigma(s) - F_\sigma(\boldsymbol{b}_k - \boldsymbol{c}))$$

as desired.

Finally suppose that $\boldsymbol{b}_k - \boldsymbol{a} \in K\tau_k$. Since $\boldsymbol{a} \sim_{S_m,\tau_k} \boldsymbol{a} - \boldsymbol{c}$, we have

$$\boldsymbol{a} - (\boldsymbol{a} - \boldsymbol{b}_k) \in S_m + \mathbb{Z}(A \cap \tau_k) \Leftrightarrow \boldsymbol{a} - \boldsymbol{c} - (\boldsymbol{a} - \boldsymbol{b}_k) \in S_m + \mathbb{Z}(A \cap \tau_k),$$

or equivalently,

$$\boldsymbol{b}_k \in S_m + \mathbb{Z}(A \cap \tau_k) \Leftrightarrow \boldsymbol{b}_k - \boldsymbol{c} \in S_m + \mathbb{Z}(A \cap \tau_k).$$

But the left hand side is false by the definition of $(\boldsymbol{b}_k, \tau_k)$, and the right hand side is true, which is one of our assumptions. Hence the case when $\boldsymbol{b}_k - \boldsymbol{a} \in K\tau_k$ does not occur, and we have completed the proof of the proposition. □

Corollary 5.9. *M_k is a Noetherian $\mathbb{Z}^d$-graded right $D(S_m)$-module.*

Proof. Since $S_0 + \mathbb{Z}(A \cap \tau_k)$ has only finitely many equivalence classes with respect to $\sim_{S_m,\tau_k}$ by Lemmas 4.6 and 4.10, M_k is a Noetherian $\mathbb{Z}^d$-graded right $D(S_m)$-module by Corollary 5.7 and Proposition 5.8. □

Theorem 5.10. *$D(S_m)$ is right Noetherian.*

Proof. By the sequence (5.3) and Corollary 5.9, $D(S_0)/D(S_0, S_m)$ is a Noetherian $\mathbb{Z}^d$-graded right $D(S_m)$-module, and hence $D(S_m)$ is $\mathbb{Z}^d$-graded right Noetherian by Robson's lemma. Then, by the general theory of $\mathbb{Z}^d$-graded algebras (see [22]), $D(S_m)$ is right Noetherian. □

6. Right modules and left modules

We retain the notation in Sections 3 and 4. Thus S_c is a finitely generated scored $\mathbb{N}A$-set, we have an irredundant expression (3.1):

$$S = S_c \setminus \bigcup_{i=1}^{m} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i))$$

with $\boldsymbol{b}_i \in S_c$, and $F_\sigma(S_c) = F_\sigma(S)$ for all facets σ. In this section, we assume that S is an $\mathbb{N}A$-set. When S satisfies Serre's (S_2) condition, it is not difficult to see that $D(S)$ and $D(\omega(S))$ are anti-isomorphic to each other, where

$$\omega(S) = -1 \times (\text{the weight set of the canonical module of } K[S]).$$

Hence the left Noetherian property of $D(S)$ is derived from the right Noetherian property of $D(\omega(S))$. In this section, we give a sufficient condition for this argument to stay valid.

For $P = \sum_{\boldsymbol{a}} t^{\boldsymbol{a}} f_{\boldsymbol{a}}(s) \in D(\mathbb{Z}^d)$, the operator $P^* = \sum_{\boldsymbol{a}} f_{\boldsymbol{a}}(-s) t^{\boldsymbol{a}}$ is called the formal adjoint operator of P. Then $K[\mathbb{Z}^d] = K[t_1^{\pm 1}, \dots, t_d^{\pm d}]$ is a right $D(S)$-module by taking formal adjoint operators.

Lemma 6.1 (cf. Proposition 4.1.5 in [26]). *Suppose that $\Lambda \subseteq \mathbb{Z}^d$ satisfies that $\boldsymbol{a} \in \Lambda$ and $\boldsymbol{b} \preceq_{S,\{\boldsymbol{0}\}} \boldsymbol{a}$ imply $\boldsymbol{b} \in \Lambda$. Then $K[-\Lambda]$ is a right $D(S)$-submodule of $K[\mathbb{Z}^d]$.*

Proof. Let $f_{\boldsymbol{a}} \in \mathbb{I}(\Omega_S(\boldsymbol{a}))$ and $\boldsymbol{b} \in \Lambda$. Then we have $(t^{\boldsymbol{a}} f_{\boldsymbol{a}}(s))^*.t^{-\boldsymbol{b}} = f_{\boldsymbol{a}}(-s)t^{\boldsymbol{a}}.t^{-\boldsymbol{b}} = f_{\boldsymbol{a}}(-s)t^{\boldsymbol{a}-\boldsymbol{b}} = f_{\boldsymbol{a}}(\boldsymbol{b}-\boldsymbol{a})t^{\boldsymbol{a}-\boldsymbol{b}}$.

The condition $\boldsymbol{b} - \boldsymbol{a} \not\preceq_{S,\{\boldsymbol{0}\}} \boldsymbol{b}$ is equivalent to $\mathbb{I}(\Omega_S(\boldsymbol{a})) \subseteq \mathfrak{m}_{\boldsymbol{b}-\boldsymbol{a}}$ by Lemma 4.1 and Theorem 4.11. Hence $(t^{\boldsymbol{a}} f_{\boldsymbol{a}}(s))^*.t^{-\boldsymbol{b}} = 0$ if $\boldsymbol{b} - \boldsymbol{a} \not\preceq_{S,\{\boldsymbol{0}\}} \boldsymbol{b}$. This proves the lemma. □

For the left Noetherian property, we construct an $\mathbb{N}A$-set $\omega(S)$, and show a duality between $D(S)$ and $D(\omega(S))$. To construct $\omega(S)$, we prepare some notation. Let $\tilde{\mathcal{F}}$ denote the union $\mathcal{F} \cup \{\tau_i : i = 1, \dots, m\}$. Set

$$\mathcal{B} := \left\{ (\boldsymbol{b}_\tau)_{\tau \in \tilde{\mathcal{F}}} : \begin{array}{l} \bullet\ \boldsymbol{b}_\tau \in K\tau \cap \mathbb{Z}^d / \mathbb{Z}(A \cap \tau) \text{ for all } \tau \in \tilde{\mathcal{F}}. \\ \bullet\ \text{For all } i \text{ and all } \tau \in \tilde{\mathcal{F}} \text{ with } \tau \succeq \tau_i, \text{ there exists } j \text{ with} \\ \quad \tau_j = \tau \text{ such that } \boldsymbol{b}_i + \boldsymbol{b}_{\tau_i} = \boldsymbol{b}_j + \boldsymbol{b}_{\tau_j} \bmod \mathbb{Z}(A \cap \tau). \end{array} \right\}. \tag{6.1}$$

Throughout this section, we assume

$$\mathcal{B} \neq \emptyset. \tag{6.2}$$

Fix an element $(\boldsymbol{b}_\tau) \in \mathcal{B}$ once for all. We define a subset $\omega(S)$ of $\mathbb{Z}^d$ by

$$\begin{aligned} \omega(S) &:= \{\boldsymbol{a} \in \mathbb{Z}^d : \boldsymbol{b}_\tau \notin E(S)_\tau(-\boldsymbol{a}) \text{ for each } \tau \in \tilde{\mathcal{F}}\} \\ &= \{\boldsymbol{a} \in \mathbb{Z}^d : -\boldsymbol{a} - \boldsymbol{b}_\tau \notin S + \mathbb{Z}(A \cap \tau) \text{ for each } \tau \in \tilde{\mathcal{F}}\}. \end{aligned}$$

By Lemma 6.1, $K[\omega(S)]$ is a right $D(S)$-module.

REMARK 6.2. If S satisfies Serre's (S_2) condition,

$$S = \bigcap_{\sigma \in \mathcal{F}} (S + \mathbb{Z}(A \cap \sigma)), \tag{6.3}$$

then $\tilde{\mathcal{F}} = \mathcal{F}$, and $(\boldsymbol{0}) \in \mathcal{B}$. Hence condition (6.2) is satisfied.

When $\mathbb{N}A$ satisfies Serre's (S_2) condition, $-\omega(\mathbb{N}A)$ for $(\boldsymbol{0}) \in \mathcal{B}$ is the weight set of a right $D(\mathbb{N}A)$-module $H^d_{\mathfrak{m}}(K[\mathbb{N}A])^*$, the Matlis dual of the local cohomology module $H^d_{\mathfrak{m}}(K[\mathbb{N}A])$.

Lemma 6.3. *$\omega(S)$ is an $\mathbb{N}A$-set.*

Proof. Let $\boldsymbol{a} \in \omega(S)$, and $\boldsymbol{b} \in \mathbb{N}A$. Suppose that $\boldsymbol{a} + \boldsymbol{b} \notin \omega(S)$. Then there exists a face $\tau \in \tilde{\mathcal{F}}$ such that $-\boldsymbol{a} - \boldsymbol{b} - \boldsymbol{b}_\tau \in S + \mathbb{Z}(A \cap \tau)$. Then from the $\mathbb{N}A$-stability of S we obtain $-\boldsymbol{a} - \boldsymbol{b}_\tau \in S + \mathbb{Z}(A \cap \tau)$, which contradicts the assumption $\boldsymbol{a} \in \omega(S)$. □

Lemma 6.4. *$D(S)^* \subseteq D(\omega(S))$, where*

$$D(S)^* = \{P^* : P \in D(S)\}.$$

Proof. Since the $\mathbb{N}A$-set S_c is finitely generated, $\omega(S)$ is not empty. We know that $\omega(S)$ is an $\mathbb{N}A$-set by Lemma 6.3. Hence, if $P^* \in D(S)^*$ satisfies $P^*(t^{\boldsymbol{a}}) = 0$ for all $\boldsymbol{a} \in \omega(S)$, then $P^* = 0$. □

Next we show that $\omega(S)$ is of the form considered in Sections 3 and 4. Then we show that $D(S)^* = D(\omega(S))$ under condition (6.2). Thus we deduce the left Noetherian property of $D(S)$ from the right Noetherian property of $D(\omega(S))$ if condition (6.2) is satisfied.

We define the scored closure $S_c(\omega(S))$ of $\omega(S)$ by

$$S_c(\omega(S)) := \bigcap_{\sigma \in \mathcal{F}} \{\boldsymbol{a} \in \mathbb{Z}^d : F_\sigma(\boldsymbol{a}) \in F_\sigma(\omega(S))\}. \tag{6.4}$$

Lemma 6.5. *$S_c(\omega(S))$ is a finitely generated scored $\mathbb{N}A$-set.*

Proof. Clearly $S_c(\omega(S))$ is scored. Since $\omega(S)$ is an $\mathbb{N}A$-set by Lemma 6.3, $S_c(\omega(S))$ is also an $\mathbb{N}A$-set.

For the finite generation, by Lemma 3.2, it is enough to prove that each $F_\sigma(S_c(\omega(S))) = F_\sigma(\omega(S))$ is finitely generated as an $F_\sigma(\mathbb{N}A)$-set. For this, it suffices to show that $F_\sigma(\omega(S))$ is bounded below. For each facet σ, we know by Lemma 4.9

$$S + \mathbb{Z}(A \cap \sigma) = [S_c + \mathbb{Z}(A \cap \sigma)] \setminus \bigcup_{\tau_i = \sigma} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \sigma)).$$

Clearly

$$F_\sigma(\omega(S)) \subseteq -(F_\sigma(S)^c \cup \{F_\sigma(\boldsymbol{b}_i) : \tau_i = \sigma\}).$$

This proves the finite generation, since the right hand side is bounded below. □

Corollary 6.6. *$D(\omega(S))$ is right Noetherian.*

Proof. By the Noetherian property of $K[\mathbb{N}A]$, Lemma 6.5, and the similar argument to that in [27, Proposition 3.4], $\omega(S)$ can be written of the form considered in Sections 3 and 4. Hence we can apply Theorem 5.10 to $\omega(S)$. □

Lemma 6.7. *For $\tau \in \tilde{\mathcal{F}}$,*

$$\omega(S) + \mathbb{Z}(A \cap \tau) = \{\boldsymbol{a} \in \mathbb{Z}^d : E(S)_{\tau'}(-\boldsymbol{a}) \not\ni \boldsymbol{b}_{\tau'} \text{ for any } \tau' \in \tilde{\mathcal{F}} \text{ with } \tau' \succeq \tau\}.$$

Proof. The inclusion '$\subseteq$' is clear by definition.

For the inclusion '$\supseteq$', let $\boldsymbol{a} \in \mathbb{Z}^d$ satisfy $E(S)_{\tau'}(-\boldsymbol{a}) \not\ni \boldsymbol{b}_{\tau'}$ for any $\tau' \in \tilde{\mathcal{F}}$ with $\tau' \succeq \tau$. Then we can take $\boldsymbol{b} \in \mathbb{N}(A \cap \tau)$ such that $F_\sigma(-\boldsymbol{a} - \boldsymbol{b}) \notin F_\sigma(S)$ for any facet $\sigma \not\succeq \tau$. Then $E(S)_{\tau'}(-\boldsymbol{a} - \boldsymbol{b}) = \emptyset$ for any face $\tau' \not\succeq \tau$. In particular, $\boldsymbol{a} + \boldsymbol{b} \in \omega(S)$. □

Lemma 6.8. *Let σ be a facet. For $\boldsymbol{\lambda} \in \mathbb{Z}^d \cap K\sigma$,*

$$\boldsymbol{\lambda} \in E(\omega(S))_\sigma(-\boldsymbol{a}) \Leftrightarrow \boldsymbol{b}_\sigma - \boldsymbol{\lambda} \notin E(S)_\sigma(\boldsymbol{a}).$$

Proof. By Lemma 6.7,

$$\omega(S) + \mathbb{Z}(A \cap \sigma) = \{\boldsymbol{a} \in \mathbb{Z}^d : -\boldsymbol{a} \notin \boldsymbol{b}_\sigma + S + \mathbb{Z}(A \cap \sigma)\}.$$

Hence

$$\mathbb{Z}^d = [\omega(S) + \mathbb{Z}(A \cap \sigma)] \amalg -[\boldsymbol{b}_\sigma + S + \mathbb{Z}(A \cap \sigma)].$$

Hence the assertion follows. □

The following proposition may be considered as the duality between S and $\omega(S)$.

Proposition 6.9.

$$\begin{aligned} S &= \{\boldsymbol{a} \in \mathbb{Z}^d : E(S)_\tau(\boldsymbol{a}) \ni \mathbf{0} \textit{ for all } \tau \in \tilde{\mathcal{F}}\} \\ &= \{\boldsymbol{a} \in \mathbb{Z}^d : E(\omega(S))_\tau(-\boldsymbol{a}) \not\ni \boldsymbol{b}_\tau \textit{ for any } \tau \in \tilde{\mathcal{F}}\} \\ &= \omega(\omega(S)). \end{aligned}$$

Proof. For any face τ, we have

$$S + \mathbb{Z}(A \cap \tau) = S_c + \mathbb{Z}(A \cap \tau) \setminus \bigcup_{\tau_i \succeq \tau} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)).$$

Hence

$$S = S_2(S) \cap \bigcap_{\operatorname{codim} \tau_i > 1} (S + \mathbb{Z}(A \cap \tau_i)),$$

where $S_2(S) = \bigcap_{\sigma \in \mathcal{F}} (S + \mathbb{Z}(A \cap \sigma))$ is the S_2-closure of S. This means the first equality of the proposition.

For the second equality, first note that by Lemma 6.8

$$E(S)_\sigma(\boldsymbol{a}) \ni \mathbf{0} \Leftrightarrow E(\omega(S))_\sigma(-\boldsymbol{a}) \not\ni \boldsymbol{b}_\sigma \tag{6.5}$$

for any facet σ. We have

$$\begin{aligned} S_2(S) &= \{\boldsymbol{a} \in \mathbb{Z}^d : E(S)_\sigma(\boldsymbol{a}) \ni \mathbf{0} \text{ for all facets } \sigma\} \\ &= \{\boldsymbol{a} \in \mathbb{Z}^d : E(\omega(S))_\sigma(-\boldsymbol{a}) \not\ni \boldsymbol{b}_\sigma \text{ for any facet } \sigma\}. \end{aligned}$$

Suppose that $\boldsymbol{a} \in S_2(S)$ and $\boldsymbol{b}_{\tau_i} \in E(\omega(S))_{\tau_i}(-\boldsymbol{a})$ for some τ_i with $\operatorname{codim} \tau_i > 1$. Then $-\boldsymbol{a} - \boldsymbol{b}_{\tau_i} \in \omega(S) + \mathbb{Z}(A \cap \tau_i)$. By Lemma 6.7, $E(S)_{\tau_i}(\boldsymbol{a} + \boldsymbol{b}_{\tau_i}) \not\ni \boldsymbol{b}_{\tau_i}$, or equivalently, $E(S)_{\tau_i}(\boldsymbol{a}) \not\ni \mathbf{0}$. We have thus proved the inclusion '$\subseteq$' of the second equation.

Since $S = S_2(S) \setminus \bigcup_{\operatorname{codim} \tau_i > 1}(\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i))$, and since the right hand side of the second equality is included in $S_2(S)$, to prove the inclusion '$\supseteq$', it suffices to show that $\boldsymbol{b} \in \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)$ with $\operatorname{codim} \tau_i > 1$ does not belong to the right hand side. Since $(\boldsymbol{b}_\tau)$ belongs to $\mathcal{B}$, for any $\tau \in \tilde{\mathcal{F}}$ with $\tau \succeq \tau_i$ there exists j with $\tau_j = \tau$ such that

$$\boldsymbol{b} + \boldsymbol{b}_{\tau_i} - \boldsymbol{b}_\tau \in \boldsymbol{b}_j + \mathbb{Z}(A \cap \tau).$$

In particular,

$$\boldsymbol{b} + \boldsymbol{b}_{\tau_i} - \boldsymbol{b}_\tau \notin S + \mathbb{Z}(A \cap \tau).$$

Hence

$$\boldsymbol{b}_\tau \notin E(S)_\tau(\boldsymbol{b} + \boldsymbol{b}_{\tau_i}) \quad \text{for any} \quad \tau \in \tilde{\mathcal{F}} \quad \text{with} \quad \tau \succeq \tau_i.$$

This means by Lemma 6.7

$$-\boldsymbol{b} - \boldsymbol{b}_{\tau_i} \in \omega(S) + \mathbb{Z}(A \cap \tau_i).$$

This is equivalent to

$$\boldsymbol{b}_{\tau_i} \in E(\omega(S))_{\tau_i}(-\boldsymbol{b}).$$

Hence $\boldsymbol{b}$ does not belong to the right hand side of the second equality of the proposition. □

Theorem 6.10. *Under condition* (6.2),

$$D(\omega(S)) = D(S)^*.$$

Proof. By Lemma 6.4 and Proposition 6.9, $D(\omega(S))^* \subseteq D(S)$. Hence

$$D(\omega(S)) = D(\omega(S))^{**} \subseteq D(S)^* \subseteq D(\omega(S)).$$

Hence $D(\omega(S)) = D(S)^*$ and $D(S) = D(\omega(S))^*$. □

Theorem 6.11. *Assume that S satisfies condition* (6.2). *Then there exist one-to-one correspondences between left modules, left ideals, right modules, right ideals of $D(S)$ and right modules, right ideals, left modules, left ideals of $D(\omega(S))$, respectively.*

Proof. This is immediate from Theorem 6.10. □

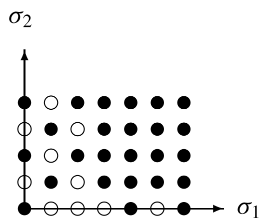

Fig. 1. The semigroup $\mathbb{N}A$ in Example 6.15

Theorem 6.12. *If S satisfies condition* (6.2), *then $D(S)$ is left Noetherian. In particular, $D(S)$ is left Noetherian, if S satisfies Serre's condition* (S_2).

Proof. This is immediate from Corollary 6.6 and Theorem 6.11. □

Theorem 6.13. *Assume that S satisfies condition* (6.2) *and that there exists $\boldsymbol{a} \in \mathbb{Z}^d$ such that $\omega(S) = \boldsymbol{a} + S$. Then there exist one-to-one correspondences between left modules, left ideals of $D(S)$ and its right modules, right ideals, respectively.*

Proof. We have

$$D(\omega(S)) = D(\boldsymbol{a} + S) = t^{\boldsymbol{a}} D(S) t^{-\boldsymbol{a}} \simeq D(S)$$

as K-algebras. Hence the theorem follows from Theorem 6.11. □

Corollary 6.14. *Assume that $K[\mathbb{N}A]$ is Gorenstein. Then there exist one-to-one correspondences between left modules, left ideals of $D(\mathbb{N}A)$ and its right modules, right ideals, respectively.*

Proof. In this case, $\mathbb{N}A$ satisfies the assumption of Theorem 6.13. □

EXAMPLE 6.15. Let

$$A = \begin{bmatrix} 0 & 1 & 3 & 3 & 4 & 4 & 6 & 6 \\ 2 & 1 & 1 & 2 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The semigroup $\mathbb{N}A$ is illustrated in Fig. 1. The scored extention $S_c(\mathbb{N}A)$ equals $\mathbb{N}^2$. We have

$$\mathbb{Z}(A \cap \sigma_1) = \mathbb{Z}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbb{Z}(A \cap \sigma_2) = \mathbb{Z}\begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

The standard expression of $\mathbb{N}A$ is

$$\mathbb{N}A = \mathbb{N}^2 \setminus \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) \setminus \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \setminus \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$$
$$\setminus \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \right) \setminus \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The S_2-closure $S_2(\mathbb{N}A)$ equals $\mathbb{N}A \cup \left\{ \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$, and thus $\mathbb{N}A$ does not satisfy (S_2) condition.

We have $\tilde{\mathcal{F}} = \{\sigma_1, \sigma_2, \{\mathbf{0}\}\}$.

The set $\mathcal{B}$ consists of only one element $(\boldsymbol{b}_\tau)$:

$$\mathbf{0} = \boldsymbol{b}_{\{\mathbf{0}\}} \quad \begin{cases} \boldsymbol{b}_{\sigma_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \boldsymbol{b}_{\sigma_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{cases}$$

We can check that, for this $(\boldsymbol{b}_\tau)$, $\omega(\mathbb{N}A) = \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \mathbb{N}A$. Hence, by Theorems 6.11 and 6.12, the category of left $D(\mathbb{N}A)$-modules and that of right $D(\mathbb{N}A)$-modules are equivalent, and $D(\mathbb{N}A)$ is Noetherian.

7. Left Noetherian property

In this section, we consider the left Noetherian property; we give a sufficient condition in Subsection 7.1 and a necessary condition in Subsection 7.2.

Let S be a semigroup $\mathbb{N}A$, and S_2 its S_2-closure.

Let

$$S = S_c \setminus \bigcup_{i=1}^{m} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)),$$
$$(7.1) \qquad S_2 = S_c \setminus \bigcup_{i=1}^{l} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i))$$

be the standard expressions for S and S_2. Hence τ_i $(i \leq l)$ are facets, while τ_i $(l < i \leq m)$ are not.

Lemma 7.1.

$$D(S) \subseteq D(S_2).$$

Proof. Recall that

$$D(S_2)_{\boldsymbol{a}} = D(S_c)_{\boldsymbol{a}} \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{a} \in S_2 + \mathbb{Z}(A\cap\tau_i), i \le l} \boldsymbol{b}_i - \boldsymbol{a} + K\tau_i \right),$$

$$D(S)_{\boldsymbol{a}} = D(S_c)_{\boldsymbol{a}} \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{a} \in S + \mathbb{Z}(A\cap\tau_i), i \le m} \boldsymbol{b}_i - \boldsymbol{a} + K\tau_i \right).$$

Note that for $i \le l$

$$S_2 + \mathbb{Z}(A \cap \tau_i) = S + \mathbb{Z}(A \cap \tau_i)$$

since τ_i is a facet. Hence $D(S) \subseteq D(S_2)$. □

7.1. A sufficient condition. Since $D(S_2)$ is left Noetherian by Theorem 6.12, the following lemma is proved similarly to Lemma 5.1.

Lemma 7.2. *If $D(S_2)/D(S_2, S)$ is a Noetherian left $D(S)$-module, then $D(S)$ is left Noetherian.*

Theorem 7.3. *Assume that $S_2 \setminus S$ is a finite set. If, for all $i > l$, the intersection*

$$\bigcap_{\boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j, j \le l} \tau_j$$

equals the origin, then $D(S)$ is left Noetherian.

Proof. We show that $D(S_2)/D(S_2, S)$ is finite-dimensional. Then the theorem follows from Lemma 7.2.

Note that all τ_i with $i > l$ are the origin $\{\mathbf{0}\}$, since $S_2 \setminus S$ is finite.

First we show that $D(S_2)_{\boldsymbol{a}} = D(S_2, S)_{\boldsymbol{a}}$ for all but finite $\boldsymbol{a} \in \mathbb{Z}^d$. Recall that

$$D(S_2)_{\boldsymbol{a}} = D(S_c)_{\boldsymbol{a}} \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{a} \in S_2 + \mathbb{Z}(A\cap\tau_i), i \le l} \boldsymbol{b}_i - \boldsymbol{a} + K\tau_i \right),$$

$$D(S_2, S)_{\boldsymbol{a}} = D(S_c)_{\boldsymbol{a}} \cdot \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i - \boldsymbol{a} \in S_2 + \mathbb{Z}(A\cap\tau_i), i \le m} \boldsymbol{b}_i - \boldsymbol{a} + K\tau_i \right).$$

Hence $D(S_2)_{\boldsymbol{a}} \neq D(S_2, S)_{\boldsymbol{a}}$ if and only if there exists $i > l$ such that

(7.2)
- $\boldsymbol{b}_i - \boldsymbol{a} \in S_2$,
- $\boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j \ (j \le l) \ \Rightarrow \ \boldsymbol{b}_j - \boldsymbol{a} \notin S_2 + \mathbb{Z}(A \cap \tau_j)$.

It suffices to show that for a fixed $i > l$ there exists only finitely many $\boldsymbol{a} \in \mathbb{Z}^d$ with (7.2). Take M as in (3.2), i.e.,

$$M > \max F_\sigma(S)^c \cup \{F_\sigma(\boldsymbol{b}_i) : i\} \quad \text{for all facets} \quad \sigma.$$

(Note that $\min F_\sigma(S) \cup \{F_\sigma(\boldsymbol{b}_i) : i\} = 0$ in our case.)

Suppose that $\boldsymbol{a} \in \mathbb{Z}^d$ satisfies (7.2). Then

(7.3)
- $F_\sigma(\boldsymbol{b}_i - \boldsymbol{a}) \geq 0 \quad$ for all facets $\quad \sigma$,
- $F_{\tau_j}(\boldsymbol{b}_i - \boldsymbol{a}) = F_{\tau_j}(\boldsymbol{b}_j - \boldsymbol{a}) \leq M \quad$ for all $\quad j \leq l \quad$ with $\quad \boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j$.

There exists only finitely many such $\boldsymbol{a} \in \mathbb{Z}^d$, since the intersection $\bigcap_{\boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j, j \leq l} \tau_j$ equals the origin.

Now it is left to show that each $D(S_2)_{\boldsymbol{a}}/D(S_2, S)_{\boldsymbol{a}}$ is finite-dimensional. Let $I := \mathbb{I}(\{\boldsymbol{b}_i - \boldsymbol{a} : \boldsymbol{b}_i - \boldsymbol{a} \in S_2, l < i \leq m\})$. Then $D(S_2)_{\boldsymbol{a}} I \subseteq D(S_2, S)_{\boldsymbol{a}}$. There exist surjective $K[s]$-module homomorphisms

$$D(S_2)_{\boldsymbol{a}}/D(S_2, S)_{\boldsymbol{a}} \leftarrow D(S_2)_{\boldsymbol{a}}/D(S_2)_{\boldsymbol{a}} I \leftarrow K[s]/I.$$

The latter is an isomorphism, since $D(S_2)_{\boldsymbol{a}}$ is a singly generated $K[s]$-module by [25, Proposition 7.7]. Hence $D(S_2)_{\boldsymbol{a}}/D(S_2, S)_{\boldsymbol{a}}$ is finite-dimensional, and we have completed the proof. □

7.2. A necessary condition. Let S be a semigroup $\mathbb{N}A$, and let

$$S = S_c \setminus \bigcup_{i=1}^{m} (\boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i))$$

be the standard expression, where S_c is the scored extention of S. In this subsection, we assume that $\operatorname{codim} \tau_m > 1$, and that

$$\left(\bigcap_{\tau_i \succ \tau_m; \boldsymbol{b}_i - \boldsymbol{b}_m \in K\tau_i} \tau_i \right) \neq \tau_m, \tag{7.4}$$

and we show that $D(S)$ is not left Noetherian. We construct a strictly increasing sequence of left ideals of $D(S)$.

Let ρ be a ray of $\mathbb{R}_{\geq 0} A$ contained in $\left(\bigcap_{\tau_i \succ \tau_m; \boldsymbol{b}_i - \boldsymbol{b}_m \in K\tau_i} \tau_i\right) \setminus (\tau_m \setminus \{\boldsymbol{0}\})$. Fix a vector $\boldsymbol{d}_\rho \in \mathbb{N}(A \cap \rho)$. Similarly to Lemma 5.2, we have the following lemma.

Lemma 7.4.

$$D(S)_{-k\boldsymbol{d}_\rho} = \mathbb{I}\left(\bigcup_{\boldsymbol{b}_i + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A \cap \tau_i)} (\boldsymbol{b}_i + K\tau_i) \right) \cdot P_{-k\boldsymbol{d}_\rho}, \tag{7.5}$$

where $D(S_c)_{-k\boldsymbol{d}_\rho} = K[s]P_{-k\boldsymbol{d}_\rho}$.

Lemma 7.5. *For* $k \gg 0$,

$$\boldsymbol{b}_m + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A \cap \tau_m). \tag{7.6}$$

Proof. Since $\boldsymbol{d}_\rho \in S$, $\boldsymbol{b}_m + k\boldsymbol{d}_\rho \in S_c$. Suppose that $\boldsymbol{b}_m + k\boldsymbol{d}_\rho \notin S + \mathbb{Z}(A \cap \tau_m)$. Then there exists i with $\tau_i \succeq \tau_m$ such that $\boldsymbol{b}_m + k\boldsymbol{d}_\rho \in \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)$ for $k \gg 0$. Hence $\boldsymbol{d}_\rho \in \mathbb{Z}(A \cap \tau_i)$. Thus we have $\boldsymbol{b}_m \in \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)$, and then $\boldsymbol{b}_m + \mathbb{Z}(A \cap \tau_m) \subseteq \boldsymbol{b}_i + \mathbb{Z}(A \cap \tau_i)$. By the irredundancy of the standard expression, we have $i = m$. But, since $\rho \not\preceq \tau_m$, we have $\boldsymbol{b}_m + k\boldsymbol{d}_\rho \notin \boldsymbol{b}_m + \mathbb{Z}(A \cap \tau_m)$. □

Lemma 7.6. *Suppose that* $\tau_i \succ \tau_m$ *and* $\boldsymbol{b}_i - \boldsymbol{b}_m \in K\tau_i$. *Then*

$$\boldsymbol{b}_i + k\boldsymbol{d}_\rho \notin S + \mathbb{Z}(A \cap \tau_i). \tag{7.7}$$

Proof. By the definition of ρ, $\rho \preceq \tau_i$ for such τ_i. Hence $\boldsymbol{d}_\rho \in \mathbb{Z}(A \cap \tau_i)$. Then we see $\boldsymbol{b}_i + k\boldsymbol{d}_\rho \notin S + \mathbb{Z}(A \cap \tau_i)$. □

For each i with $\boldsymbol{b}_i + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A \cap \tau_i)$ and $\boldsymbol{b}_i + K\tau_i \neq \boldsymbol{b}_m + K\tau_m$, we take a facet $\sigma_i \succeq \tau_i$ as follows:

(1) If $\tau_i \not\succeq \tau_m$, then take a facet $\sigma_i \succeq \tau_i$ such that $\sigma_i \not\succeq \tau_m$.

(2) If $\tau_i \succeq \tau_m$ and $\boldsymbol{b}_i - \boldsymbol{b}_m \notin K\tau_i$, then take a facet $\sigma_i \succeq \tau_i$ such that $F_{\sigma_i}(\boldsymbol{b}_i) \neq F_{\sigma_i}(\boldsymbol{b}_m)$.

(3) We do not need to consider the case where $\tau_i \succ \tau_m$ and $\boldsymbol{b}_i - \boldsymbol{b}_m \in K\tau_i$ by Lemma 7.6.

Finally take a facet σ_m containing ρ and τ_m. Then define $E(k)$ by

$$E(k) := (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m)) \prod_{\substack{\boldsymbol{b}_i + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A\cap\tau_i),\\ \boldsymbol{b}_i + K\tau_i \neq \boldsymbol{b}_m + K\tau_m}} (F_{\sigma_i} - F_{\sigma_i}(\boldsymbol{b}_i)) \cdot P_{-k\boldsymbol{d}_\rho}. \tag{7.8}$$

Then, by Lemma 7.4, $E(k) \in D(S)_{-k\boldsymbol{d}_\rho}$.

Lemma 7.7. $E(k) \notin (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m))D(S)_{-k\boldsymbol{d}_\rho}$ *for* $k \gg 0$.

Proof. By the definitions of σ_i, $\prod_{\substack{\boldsymbol{b}_i + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A\cap\tau_i),\\ \boldsymbol{b}_i + K\tau_i \neq \boldsymbol{b}_m + K\tau_m}} (F_{\sigma_i} - F_{\sigma_i}(\boldsymbol{b}_i)) \notin \mathbb{I}(\boldsymbol{b}_m + K\tau_m)$. Hence the assertion follows from Lemmas 7.4 and 7.5. □

Theorem 7.8. $D(S)$ *is not left Noetherian.*

Proof. We construct a strictly increasing sequence of left ideals.

First we claim that for $k, l \gg 0$

$$D(S)_{-l\boldsymbol{d}_\rho} \cdot E(k) \subseteq (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m))D(S)_{-(l+k)\boldsymbol{d}_\rho}. \tag{7.9}$$

Let $f(s)P_{-l\boldsymbol{d}_\rho} \in D(S)_{-l\boldsymbol{d}_\rho}$. Note that, by Lemma 7.4, for $k, l \gg 0$ we have $f(s)P_{-k\boldsymbol{d}_\rho} \in D(S)_{-k\boldsymbol{d}_\rho}$ if and only if $f(s)P_{-l\boldsymbol{d}_\rho} \in D(S)_{-l\boldsymbol{d}_\rho}$, and recall from [27, Lemma 5.10] that $P_{-(l+k)\boldsymbol{d}_\rho} = P_{-l\boldsymbol{d}_\rho}P_{-k\boldsymbol{d}_\rho}$ for $k, l \gg 0$. Thus for $k, l \gg 0$

$$\begin{aligned} f(s)P_{-l\boldsymbol{d}_\rho}E(k) &= f(s)P_{-l\boldsymbol{d}_\rho}(F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m)) \prod_{\substack{\boldsymbol{b}_i + k\boldsymbol{d}_\rho \in S + \mathbb{Z}(A\cap\tau_i),\\ \boldsymbol{b}_i + K\tau_i \neq \boldsymbol{b}_m + K\tau_m}} (F_{\sigma_i} - F_{\sigma_i}(\boldsymbol{b}_i))P_{-k\boldsymbol{d}_\rho} \\ &\in (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m))K[s]f(s)P_{-l\boldsymbol{d}_\rho}P_{-k\boldsymbol{d}_\rho} \\ &= (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m))K[s]f(s)P_{-(l+k)\boldsymbol{d}_\rho} \\ &\subseteq (F_{\sigma_m} - F_{\sigma_m}(\boldsymbol{b}_m))D(S)_{-(l+k)\boldsymbol{d}_\rho}. \end{aligned}$$

Thus we have proved claim (7.9).

Take $k_0 \in \mathbb{N}$ large enough. By Lemma 7.7 and claim (7.9), for $l > h$,

$$E(lk_0) \notin \left(\sum_{k=1}^{h} D(S)\cdot E(kk_0)\right)_{-lk_0\boldsymbol{d}_\rho}.$$

Therefore $\left\{\sum_{k=1}^{h} D(S)\cdot E(kk_0)\colon h = 1, 2, \ldots\right\}$ is a strictly increasing sequence of left ideals of $D(S)$. □

Finally we make a conjecture of the condition for $D(S)$ to be left Noetherian.

Conjecture 7.9. *Let* (7.1) *be the standard expressions of S and S_2. Then the following are equivalent.*
(1) *$D(S)$ is left Noetherian.*
(2) *$D(S_2)/D(S_2, S)$ is a Noetherian left $D(S)$-module.*
(3) *$D(S_2)/D(S)$ is a Noetherian left $D(S)$-module.*
(4) *For all $i > l$,*

$$\left(\bigcap_{\tau_j \succ \tau_i,\, \boldsymbol{b}_i - \boldsymbol{b}_j \in K\tau_j,\, j \le l} \tau_j\right) = \tau_i.$$

REMARK 7.10. Lemma 7.2 says that (2) implies (1). Clearly (2) and (3) are equivalent under (1). The implication (1) $\Rightarrow$ (4) is Theorem 7.8. Note also that (4) is satisfied when the set $\mathcal{B}$ is not empty (cf. Theorem 6.12).

By Theorems 7.3 and 7.8, the conjecture is true for $d \le 2$.

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