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SOBOLEV'S INEQUALITY FOR RIESZ POTENTIALS OF FUNCTIONS IN NON-DOUBLING MORREY SPACES

YOSHIHIRO MIZUTA, TETSU SHIMOMURA and TAKUYA SOBUKAWA

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Abstract

Our aim in this paper is to give Sobolev's inequality and Trudinger exponential integrability for Riesz potentials of functions in non-doubling Morrey spaces.

1. Introduction

The space introduced by Morrey [13] in 1938 has become a useful tool of the study for the existence and regularity of solutions of partial differential equations. In the present paper, we aim to establish Sobolev's inequality for the Riesz potentials of functions in generalized Morrey spaces in the non-doubling setting, as extensions of Gogatishvili-Koskela [4], Orobitg-Pérez [14] and Sawano-Sobukawa-Tanaka [19].

Let X be a separable metric space with a nonnegative Radon measure μ. For simplicity, write $|x-y|$ for the distance of x and y. We assume that $\mu(\{x\}) = 0$ and $0 < \mu(B(x,r)) < \infty$ for $x \in X$ and $r > 0$, where $B(x,r)$ denotes the open ball centered at x of radius $r > 0$. In this paper, μ may or may not be doubling.

Let G be an open set in X. We define the Riesz potential of order α for a nonnegative measurable function f on G by

$$U_\alpha f(x) = \int_G \frac{|x-y|^\alpha f(y)}{\mu(B(x, 4|x-y|))}\, d\mu(y).$$

Here we introduce the family $L^{p,\nu;k}(G)$ of all measurable functions f on G such that

$$\|f\|^p_{p,\nu,G;k} = \sup_{x\in G, 0<r\le d_G} \frac{r^\nu}{\mu(B(x,kr))} \int_{G\cap B(x,r)} |f(y)|^p\, d\mu(y) < \infty,$$

where $1 < p < \infty$, $\nu > 0$, $k > 1$ and d_G denotes the diameter of G. In case $X = \mathbf{R}^n$ with a nonnegative Radon measure μ, we know that

$$L^{p,\nu;k_1}(G) = L^{p,\nu;k_2}(G)$$

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when $k_2 > k_1 > 1$, but we have an example (see Remark 2.1) in which

$$L^{p,\nu;1}(G) \neq L^{p,\nu;2}(G);$$

see also Sawano-Tanaka [20]. The space $L^{p,\nu;2}(G)$ is referred to as a generalized Morrey space.

To obtain Sobolev type inequalities for Riesz potentials of functions belonging to generalized Morrey spaces, we consider a generalized maximal function defined by

$$M_k f(x) = \sup_{r>0} \frac{1}{\mu(B(x, kr))} \int_{G\cap B(x,r)} |f(y)|\, d\mu(y)$$

for $k \geq 1$ and a locally integrable function f on G. In view of Sawano [18, Corollary 2.1]), M_2 is bounded in $L^p(X)$. Further it is useful to remark that in a certain metric measure space X, M_k fails to be bounded in $L^p(X)$ if and only if $k < 2$ (see Sawano [18, Proposition 1.1]).

By applying the fact that M_2 is a bounded mapping from $L^{p,\nu;2}(X)$ to $L^{p,\nu;4}(X)$ (see Sawano-Tanaka [20, Theorem 2.3]), we first show that $U_\alpha f \in L^{p^\sharp,\nu;4}(X)$ for $f \in L^{p,\nu;2}(X)$, where $1/p^\sharp = 1/p - \alpha/\nu > 0$; in the borderline case $\nu = \alpha p$, we consider the exponential integrability. For this, we also refer the reader to Sawano-Sobukawa-Tanaka [19, Theorem 3.1].

Finally, in our Morrey space setting, we establish an exponential integrability for functions satisfying a Poincaré inequality, as an extension of Gogatishvili-Koskela [4] and Orobitg-Pérez [14].

For related results, see Adams [1], Chiarenza-Frasca [3, Theorem 2], Nakai [15, Theorem 2.2] and the authors [11, 12] in the doubling case.

2. Sobolev's inequality

Throughout this paper, let C denote various constants independent of the variables in question.

For a nonnegative measurable function f on G and $k > 1$, define the maximal function

$$\begin{aligned} M_k f(x) &= \sup_{r>0} \frac{1}{\mu(B(x, kr))} \int_{G\cap B(x,r)} |f(y)|\, d\mu(y) \\ &= \sup_{0<r<d_G} \frac{1}{\mu(B(x, kr))} \int_{G\cap B(x,r)} |f(y)|\, d\mu(y) \end{aligned}$$

for $x \in G$, where d_G denotes the diameter of G. Recall that

$$\| f \|_{p,\nu,G;k}^p = \sup_{x\in G, 0<r\leq d_G} \frac{r^\nu}{\mu(B(x, kr))} \int_{G\cap B(x,r)} |f(y)|^p\, d\mu(y).$$

When $X = \mathbf{R}^n$ with a nonnegative Radon measure μ, we see that if G is an open set of $\mathbf{R}^n$ and $1 < k < 2$, then

$$\| f \|_{p,\nu,G;k} \le C \| f \|_{p,\nu,G;2}$$

for all $f \in L^{p,\nu;2}(G)$, where C is a constant depending only on k and n; for this fact, see e.g. [20, Proposition 1.1].

REMARK 2.1. We set

$$L^{p,\nu;k}(G) = \left\{ f \;\middle|\; \sup_{x \in G, 0 < r \le d_G} \frac{r^\nu}{\mu(B(x,kr))} \int_{G \cap B(x,r)} |f(y)|^p \, d\mu(y) < \infty \right\}.$$

When $G \subset \mathbf{R}^n$, $L^{p,\nu;k}(G) = L^{p,\nu;2}(G)$ for all $k > 1$. We show by an example that

$$L^{p,\nu;1}(G) \ne L^{p,\nu;k}(G)$$

when $k > 1$. For this, consider a measure given by

$$d\mu(y) = e^y \, dy$$

on $\mathbf{R}^1$. For $0 < \beta < 1$, letting $f(y) = y^{-\beta/p}$ for $y > 0$ and $f(y) = 0$ for $y \le 0$, we note the following:

(i) if $0 < x \le r$, then

$$\begin{aligned} \frac{r^\nu}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) &\le \frac{r^\nu}{e^x(e^{2r}-1)} e^{x+r} \int_0^{2r} y^{-\beta} \, dy \\ &\le C \frac{r^{\nu-\beta+1}}{e^r - 1}; \end{aligned}$$

(ii) if $x > r > 0$, then

$$\begin{aligned} \frac{r^\nu}{\mu(B(x,2r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) &\le \frac{r^\nu}{e^x(e^{2r}-1)} \int_{x-r}^{x+r} y^{-\beta} e^y \, dy \\ &\le \frac{r^\nu}{e^x(e^{2r}-1)} \frac{(x+r)^{1-\beta} - (x-r)^{1-\beta}}{1-\beta} e^{x+r} \\ &\le C \frac{r^{\nu-\beta+1}}{e^r - 1}; \end{aligned}$$

(iii) if $x > 0$ and $r > 0$, then

$$\frac{r^\nu}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \ge r^\nu (x+r)^{-\beta}.$$

If $0 < \beta < 1$ and $\beta \le \nu$, then (i) and (ii) imply that $f \in L^{p,\nu;2}(\mathbf{R}^1)$, and if $0 < \beta < 1$ and $\beta < \nu$, then (iii) implies that

$$\limsup_{r\to\infty} \frac{r^\nu}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|^p \, d\mu(y) = \infty$$

for every fixed $x > 0$, so that $f \notin L^{p,\nu;1}(\mathbf{R}^1)$.

In what follows, if f is a function on G, then we assume that $f = 0$ outside G.

First we present the boundedness of maximal functions in the Morrey space $L^{p,\nu;2}(G)$ due to Sawano-Tanaka [20, Theorem 2.3].

Lemma 2.2. *If $\nu > 0$, then*

$$\|M_2 f\|_{p,\nu,G;4} \le C\|f\|_{p,\nu,G;2}$$

for all $f \in L^{p,\nu;2}(G)$.

Proof. Let $\|f\|_{p,\nu,G;2} \le 1$, and fix $x \in G$ and $0 < r \le d_G$. Write $A_0 = B(x,2r)$ and $A_j = B(x, 2^{j+1}r) \setminus B(x, 2^j r)$ for each positive integer j. We set

$$f_j = f\chi_{A_j},$$

where χ_E denotes the characteristic function of E. Note that

$$\begin{aligned}\int_{B(x,r)} M_2 f(z)^p \, d\mu &\le 2^{p-1}\left(\int_{B(x,r)} M_2 f_0(z)^p \, d\mu + \int_{B(x,r)} M_2 g_0(z)^p \, d\mu\right) \\ &\equiv 2^{p-1}(I_1 + I_2),\end{aligned}$$

where $g_0 = \sum_{j=1}^\infty |f_j|$. We have by Sawano [18, Theorem 1.2 and Proposition 1.1]

$$\begin{aligned} I_1 &\le \int M_2 f_0(z)^p \, d\mu \le C \int |f_0(z)|^p \, d\mu \\ &= C \int_{B(x,2r)} |f(z)|^p \, d\mu \le C r^{-\nu} \mu(B(x,4r)). \end{aligned}$$

Next we see that for $z \in B(x,r)$

$$M_2 f_j(z) \le C \sup_{\{t:(2^j-1)r<t<(2^{j+1}+1)r\}} \frac{1}{\mu(B(z,2t))} \int_{B(z,t)} |f(y)| \, d\mu$$

$$
\begin{aligned}
&\le C \sup_{\{t:(2^j-1)r<t<(2^{j+1}+1)r\}} \left(\frac{1}{\mu(B(z,2t))} \int_{B(z,t)} |f(y)|^p \, d\mu \right)^{1/p} \\
&\le C(2^j r)^{-\nu/p},
\end{aligned}
$$

so that

$$M_2 g_0(z) \le \sum_{j=1}^{\infty} M_2 f_j(z) \le C \sum_{j=1}^{\infty} (2^j r)^{-\nu/p} \le C r^{-\nu/p}.$$

Hence it follows that

$$I_2 \le C r^{-\nu} \int_{B(x,r)} d\mu \le C r^{-\nu} \mu(B(x,r)).$$

Thus we obtain

$$\frac{r^\nu}{\mu(B(x,4r))} \int_{B(x,r)} M_2 f(z)^p \, d\mu \le C,$$

which proves the lemma. $\square$

Lemma 2.3. *If f is a nonnegative measurable function on G such that $\|f\|_{p,\nu,G;2} \le 1$, then*

$$\int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,4|x-y|))} \, d\mu(y) \le C \delta^\alpha M_2 f(x)$$

for $x \in G$ and $\delta > 0$.

Proof. We have

$$
\begin{aligned}
\int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,4|x-y|))} \, d\mu(y) &= \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}\delta)\setminus B(x,2^{-j}\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,4|x-y|))} \, d\mu(y) \\
&\le \sum_{j=1}^{\infty} \frac{(2^{-j+1}\delta)^\alpha}{\mu(B(x,2^{-j+2}\delta))} \int_{B(x,2^{-j+1}\delta)} f(y) \, d\mu(y) \\
&\le \delta^\alpha M_2 f(x) \sum_{j=1}^{\infty} 2^{(-j+1)\alpha} \\
&= C \delta^\alpha M_2 f(x),
\end{aligned}
$$

as required. $\square$

Lemma 2.4. *Let $\nu/p \geq \alpha$. Let f be a nonnegative measurable function on G such that $\|f\|_{p,\nu,G;2} \leq 1$. In case $\nu/p > \alpha$,*

$$\int_{G\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,4|x-y|))}\, d\mu(y) \leq C\delta^{\alpha-\nu/p}$$

and in case $\nu/p = \alpha$ and G is bounded,

$$\int_{G\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,4|x-y|))}\, d\mu(y) \leq C \log \frac{1}{\delta}$$

for $x \in G$ and small $\delta > 0$.

Proof. Let j_0 be the smallest integer such that $2^{j_0}\delta \geq d_G$, where d_G is the diameter of G as before. By using Hölder's inequality, we have

$$\begin{aligned}
&\int_{G\setminus B(x,\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,4|x-y|))}\, d\mu(y) \\
&= \sum_{j=0}^{j_0} \int_{B(x,2^{j+1}\delta)\setminus B(x,2^{j}\delta)} \frac{|x-y|^{\alpha} f(y)}{\mu(B(x,4|x-y|))}\, d\mu(y) \\
&\leq \sum_{j=0}^{j_0} (2^{j+1}\delta)^{\alpha} \frac{1}{\mu(B(x,2^{j+2}\delta))} \int_{B(x,2^{j+1}\delta)} f(y)\, d\mu(y) \\
&\leq C\delta^{\alpha} \sum_{j=0}^{j_0} 2^{\alpha j} \left(\frac{1}{\mu(B(x,2^{j+2}\delta))} \int_{B(x,2^{j+1}\delta)} f(y)^p\, d\mu(y) \right)^{1/p} \\
&\leq C\delta^{\alpha} \sum_{j=0}^{j_0} 2^{\alpha j} (2^{j+1}\delta)^{-\nu/p} \\
&= C\delta^{\alpha-\nu/p} \sum_{j=0}^{j_0} 2^{(\alpha-\nu/p)j},
\end{aligned}$$

which proves the required inequality. □

With the aid of Lemmas 2.2, 2.3 and 2.4, we can apply Hedberg's trick (see [6]) to obtain a Sobolev type inequality for Riesz potentials due to Adams [1, Theorem 3.1], Chiarenza and Frasca [3, Theorem 2], Nakai [15, Theorem 2.2] and Sawano-Tanaka [20, Theorem 3.3].

Theorem 2.5. *Let* $1/p^\sharp = 1/p - \alpha/\nu > 0$. *Then there exists a positive constant* c *such that*

$$\frac{r^\nu}{\mu(B(z,4r))} \int_{B(z,r)} \{U_\alpha f(x)\}^{p^\sharp} d\mu(x) \le c$$

for all $z \in X$ *and* $r > 0$, *whenever* f *is a nonnegative measurable function on* X *satisfying* $\| f \|_{p,\nu,X;2} \le 1$.

Proof. We see from Lemmas 2.3 and 2.4 that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,4|x-y|))} d\mu(y) + \int_{X\setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x,4|x-y|))} d\mu(y) \\ &\le C\delta^\alpha M_2 f(x) + C\delta^{\alpha-\nu/p} \end{aligned}$$

for all $\delta > 0$. Here, letting

$$\delta = \{M_2 f(x)\}^{-p/\nu},$$

we have

$$U_\alpha f(x) \le C\{M_2 f(x)\}^{1-\alpha p/\nu} = C\{M_2 f(x)\}^{p/p^\sharp},$$

which yields

$$\int_{B(z,r)} \{U_\alpha f(x)\}^{p^\sharp} d\mu(x) \le C \int_{B(z,r)} \{M_2 f(x)\}^p d\mu(x)$$

for $z \in X$ and $r > 0$. Hence Lemma 2.2 gives

$$\frac{r^\nu}{\mu(B(z,4r))} \int_{B(z,r)} \{U_\alpha f(x)\}^{p^\sharp} d\mu(x) \le C$$

for such z and r, as required. □

REMARK 2.6. Theorem 2.5 implies that the mapping $f \to U_\alpha f$ is bounded from $L^{p,\nu;2}(X)$ to $L^{p^\sharp,\nu;4}(X)$.

REMARK 2.7. When $X = \mathbf{R}^n$, consider the potential

$$U_{\alpha,k} f(x) = \int \frac{|x-y|^\alpha f(y)}{\mu(B(x,k|x-y|))} d\mu(y)$$

for $k > 1$. Then we can show that the mapping $f \to U_{\alpha,k} f$ is bounded from $L^{p,\nu;2}(\mathbf{R}^n)$ to $L^{p^\sharp,\nu;2}(\mathbf{R}^n)$, when $1/p^\sharp = 1/p - \alpha/\nu > 0$.

REMARK 2.8. We show by an example that the mapping $f \to U_{\alpha,1} f$ fails to be bounded in $L^{p,\nu;2}(\mathbf{R}^n)$.

For this purpose, consider $d\mu(y) = e^y dy$ and

$$f(y) = \begin{cases} y^{-\beta/p} & \text{when} \quad y > 0; \\ 0 & \text{when} \quad y \le 0. \end{cases}$$

In view of Remark 2.1, we see that $f \in L^{p,\nu;2}(\mathbf{R}^1)$ when $0 < \beta < \nu \le 1$. Further we see that

$$U_{\alpha,1} f(x) \ge \int_x^\infty \frac{t^\alpha (x+t)^{-\beta/p}}{e^x(e^t - e^{-t})} e^{x+t} dt = \infty$$

for all $x > 0$ when $\alpha - \beta/p + 1 \ge 0$. This implies that $U_{\alpha,1} f$ does not belong to $L^{p,\nu;2}(\mathbf{R}^1)$ when $0 < \beta < \nu \le 1$.

3. Exponential integrability

Our aim in this section is to discuss the exponential integrability.

Theorem 3.1. *Let G be bounded and $\nu = \alpha p$. Then there exists a positive constant c such that*

$$\frac{r^\nu}{\mu(B(z, 4r))} \int_{G \cap B(z,r)} \{\exp(cU_\alpha f(x)) - 1\} \, d\mu(x) \le 1$$

for all $z \in G$ and $0 < r \le d_G$, whenever f is a nonnegative measurable function on G satisfying $\| f \|_{p,\nu,G;2} \le 1$.

Proof. Let $\| f \|_{p,\nu,G;2} \le 1$. We see from Lemmas 2.3 and 2.4 that

$$\begin{aligned} U_\alpha f(x) &= \int_{G \cap B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x, 4|x-y|))} \, d\mu(y) + \int_{G \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x, 4|x-y|))} \, d\mu(y) \\ &\le C\delta^\alpha M_2 f(x) + C \log \frac{1}{\delta} \end{aligned}$$

for $x \in G$ and small $\delta > 0$. Here, letting

$$\delta = \{M_2 f(x)\}^{-1/\alpha} \{\log(M_2 f(x))\}^{1/\alpha}$$

when $M_2 f(x)$ is large enough, we have

$$\exp(U_\alpha f(x)) \le C + C\{M_2 f(x)\}^p,$$

so that Lemma 2.2 yields

$$\int_{G\cap B(z,r)} \exp(U_\alpha f(x))\, d\mu(x) \le C\mu(G\cap B(z,r)) + C\int_{G\cap B(z,r)} \{M_2 f(x)\}^p\, d\mu(x)$$
$$\le C\mu(G\cap B(z,r)) + C\frac{\mu(B(z,4r))}{r^\nu}$$

for $z \in G$ and $r > 0$. Hence we find $c > 0$ such that

$$\frac{r^\nu}{\mu(B(z,4r))}\int_{G\cap B(z,r)} \{\exp(cU_\alpha f(x)) - 1\}\, d\mu(x) \le 1$$

for $z \in G$ and $0 < r \le d_G$, whenever $\| f\|_{p,\nu,G;2} \le 1$. Thus the required result is obtained. □

REMARK 3.2. In Theorem 3.1, we can not add an exponent $q > 1$ such that

$$\frac{r^\nu}{\mu(B(z,4r))}\int_{G\cap B(z,r)} \{\exp(cU_\alpha f(x)^q) - 1\}\, d\mu(x) \le 1.$$

For this, consider the potential

$$U(x) = \int_{\mathbf{B}} |x-y|^{\alpha-n}|y|^{-\alpha}\, dy,$$

where $\mathbf{B} = B(0,1) \subset \mathbf{R}^n$. If $\nu = \alpha p < n$ and $f(y) = |y|^{-\alpha}\chi_{\mathbf{B}}(y)$, then

$$r^{\nu-n}\int_{B(x,r)} |f(y)|^p\, dy \le r^{\nu-n}\int_{B(x,r)} |x-y|^{-\alpha p}\, dy$$
$$\le Cr^{\nu-n}r^{n-\alpha p} = C$$

for all $x \in \mathbf{B}$ and $r > 0$, so that $f \in L^{p,\nu;1}(\mathbf{B})$. On the other hand, we see that

$$U(x) \ge \int_{\mathbf{B}\setminus B(x,|x|/2)} |x-y|^{\alpha-n} f(y)\, dy$$
$$\ge 3^{-\alpha}\int_{\mathbf{B}\setminus B(x,|x|/2)} |x-y|^{-n}\, dy$$
$$\ge C\log\frac{2}{|x|}$$

for $x \in \mathbf{B}$, and hence

$$\int_{\mathbf{B}} \exp(cU(x)^q)\, dx = \infty$$

for $c > 0$ and $q > 1$.

Consider the function

$$e_N(t) = e^t - 1 - t - \frac{t^2}{2!} - \cdots - \frac{t^{N-1}}{(N-1)!}.$$

Theorem 3.3. *Let $\nu = \alpha p$. For $\tilde{\nu} > \nu$, take a positive integer N such that*

$$N > \frac{\tilde{\nu} p}{\tilde{\nu} - \alpha p} = \tilde{p}.$$

Then there exists a positive constant c such that

$$\frac{r^\nu}{\mu(B(z, 4r))} \int_{B(z,r)} e_N(cU_\alpha f)\, d\mu(x) \le 1$$

for all $z \in X$ and $r > 0$, whenever f is a nonnegative measurable function on X satisfying $\| f \|_{p,\nu,X;2} + \| f \|_{p,\tilde{\nu},X;2} \le 1$.

Proof. We see from Lemmas 2.3 and 2.4 that

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x, 4|x-y|))}\, d\mu(y) + \int_{X \setminus B(x,\delta)} \frac{|x-y|^\alpha f(y)}{\mu(B(x, 4|x-y|))}\, d\mu(y) \\ &\le C\delta^\alpha M_2 f(x) + C \log \frac{1}{\delta} \end{aligned}$$

for small $\delta > 0$. Here, letting

$$\delta = \{M_2 f(x)\}^{-1/\alpha} \{\log(M_2 f(x))\}^{1/\alpha}$$

when $M_2 f(x)$ is large enough, we have

$$U_\alpha f(x) \le C \log(2 + M_2 f(x)).$$

We write $G_1 = \{x \in X : M_2 f(x) > 2\}$ and $G_2 = \{x \in X : M_2 f(x) \le 2\}$. Then we find $c_1 > 0$ such that

$$\int_{G_1 \cap B(z,r)} e_N(c_1 U_\alpha f(x))\, d\mu(x) \le \int_{B(z,r)} \{M_2 f(x)\}^p\, d\mu(x)$$

and

$$\begin{aligned} \int_{G_2 \cap B(z,r)} e_N(c_1 U_\alpha f(x))\, d\mu(x) &\le \int_{G_2 \cap B(z,r)} \{U_\alpha f(x)\}^{\tilde{p}}\, d\mu(x) \\ &\le \int_{B(z,r)} \{M_2 f(x)\}^p\, d\mu(x) \end{aligned}$$

for $z \in X$ and $r > 0$. Hence Lemma 2.2 gives

$$\frac{r^{\nu}}{\mu(B(z, 4r))} \int_{B(z,r)} e_N(c_2 U_\alpha f(x))\, d\mu(x) \le 1$$

for such z and r, whenever $\| f\|_{p,\nu,X;2} + \| f\|_{p,\tilde{\nu},X;2} \le 1$. This gives the the required result. □

REMARK 3.4. Let $\nu < \alpha p$ and f be a nonnegative measurable function on X belonging to $L^{p,\nu;2}(X)$. Then $U_\alpha f(x)$ is seen to be continuous at $x_0 \in X$ where $\mu(\partial B(x_0, r)) = 0$ for $r > 0$ and

$$\int \frac{|x_0 - y|^{\alpha} f(y)}{\mu(B(x_0, 2|x_0 - y|))}\, d\mu(y) < \infty. \tag{3.1}$$

In fact, for $\delta > 0$, we write

$$\begin{aligned} U_\alpha f(x) &= \int_{B(x_0,3\delta)} \frac{|x - y|^{\alpha} f(y)}{\mu(B(x, 4|x - y|))}\, d\mu(y) + \int_{X\setminus B(x_0,3\delta)} \frac{|x - y|^{\alpha} f(y)}{\mu(B(x, 4|x - y|))}\, d\mu(y) \\ &= U_1(x) + U_2(x). \end{aligned}$$

The proof of Lemma 2.3 implies that

$$U_1(x) \le C\delta^{\alpha-\nu/p}$$

for $x \in B(x_0, \delta)$, when $\alpha - \nu/p > 0$. Note that $\mu(B(x, 4|x - y|)) \to \mu(B(x_0, 4|x_0 - y|))$ as $x \to x_0$ for fixed $y \in X \setminus B(x_0, 3\delta)$ by the assumption that $\mu(\partial B(x_0, r)) = 0$ for $r > 0$. Since $|x - y|^{\alpha}/\mu(B(x, 4|x - y|)) \le C|x_0 - y|^{\alpha}/\mu(B(x_0, 2|x_0 - y|))$ for $x \in B(x_0, \delta)$ and $y \in X \setminus B(x_0, 3\delta)$, we have by Lebesgue's dominated convergence theorem

$$\lim_{x\to x_0} U_2(x) = U_2(x_0),$$

which shows that $U_\alpha f(x)$ is continuous at x_0.

4. Poincaré inequality

Let μ be a nonnegative measure on an open set G. For a measurable function u on G, we define the integral mean over a measurable set $E \subset G$ of positive measure by

$$u_E = \fint_E u(x)\, d\mu = \frac{1}{\mu(E)} \int_E u(x)\, d\mu(x).$$

In this section, we assume that μ satisfies the lower Ahlfors s-regularity condition

$$c_\mu r^s \le \mu(B) < \infty \tag{4.1}$$

for all balls $B = B(x, r) \subset G$, where $s > 0$ and c_μ is a positive constant.

We say that a couple (u, g) satisfies a strong $(1, p_0)$ Poincaré inequality (in G) if

$$\int_B |u(x) - u_B|\, d\mu(x) \le c_P (\mathrm{diam}\, B)^{1+s} \left(\frac{1}{\mu(2B)} \int_B |g(y)|^{p_0}\, d\mu(y) \right)^{1/p_0} \tag{4.2}$$

for each ball $B = B(x, r)$ with $2B = B(x, 2r) \subset G$, where $1 \le p_0 < p$ and c_P is a positive constant.

Set $1/p^\sharp = 1/p - 1/\nu > 0$.

Theorem 4.1. *Let μ be a nonnegative measure on G satisfying* (4.1), *and assume that a couple (u, g) satisfies a strong $(1, p_0)$ Poincaré inequality* (4.2). *If $\|g\|_{p,\nu,G;2} \le 1$, then*

$$\frac{r^\nu}{\mu(2B)} \int_B |u(x) - u_B|^{p^\sharp}\, d\mu(x) \le C$$

for every ball $B = B(x, r)$ with $2B \subset G$.

Orobitg-Pérez [14] gave a version of Sobolev's inequalities in the L^p space setting.

Proof of Theorem 4.1. Let $2B = B(x_0, 2r) \subset G$. For $x \in B$, set

$$B_i(x) = B(x, 2^{-i} r).$$

By the Lebesgue differentiation theorem we have

$$\lim_{i \to \infty} u_{B_i(x)} = u(x)$$

for μ-a.e. x and hence we may assume that our fixed point x has this property. Let $N = N(x)$ be a positive integer whose value will be determined later. Letting $B_i = B_i(x)$ for $i = 1, 2, \ldots,$ we have by (4.1)

$$\begin{aligned}
|u(x) - u_B| &\le |u_{B_1} - u_B| + \sum_{i=1}^\infty |u_{B_i} - u_{B_{i+1}}| \\
&\le \frac{1}{\mu(B \cap B_1)} \int_B |u - u_B|\, d\mu + \frac{1}{\mu(B \cap B_1)} \int_{B_1} |u - u_{B_1}|\, d\mu \\
&\quad + \sum_{i=1}^\infty \frac{1}{\mu(B_{i+1})} \int_{B_i} |u - u_{B_i}|\, d\mu
\end{aligned}$$

$$\begin{aligned}
&\le Cr^{-s} \int_B |u - u_B| \, d\mu + C \sum_{i=1}^{N} (2^{-i} r)^{-s} \int_{B_i} |u - u_{B_i}| \, d\mu \\
&\quad + C \sum_{i=N+1}^{\infty} (2^{-i} r)^{-s} \int_{B_i} |u - u_{B_i}| \, d\mu \\
&= I_0 + I_1 + I_2.
\end{aligned}$$

By using Hölder's inequality and a strong $(1, p_0)$ Poincaré inequality, we find

$$\begin{aligned}
I_0 &\le C(\operatorname{diam} B) \left(\frac{1}{\mu(2B)} \int_B |g|^{p_0} \, d\mu \right)^{1/p_0} \\
&\le C(\operatorname{diam} B) \left(\frac{1}{\mu(2B)} \int_B |g|^{p} \, d\mu \right)^{1/p} \\
&\le Cr^{1-\nu/p}
\end{aligned}$$

and

$$\begin{aligned}
I_1 &\le C \sum_{i=0}^{N} 2^{-i} r \left(\frac{1}{\mu(2B_i)} \int_{B_i} |g|^{p_0} \, d\mu \right)^{1/p_0} \\
&\le C \sum_{i=0}^{N} 2^{-i} r \left(\frac{1}{\mu(2B_i)} \int_{B_i} |g|^{p} \, d\mu \right)^{1/p} \\
&\le C \sum_{i=0}^{N} (2^{-i} r)^{1-\nu/p} \\
&\le C(2^{-N} r)^{1-\nu/p}.
\end{aligned}$$

According to the estimation of I_1, we obtain

$$\begin{aligned}
I_2 &\le C \sum_{i=N+1}^{\infty} 2^{-i} r \left(\frac{1}{\mu(2B_i)} \int_{B_i} |g|^{p_0} \, d\mu \right)^{1/p_0} \\
&\le C \sum_{i=N+1}^{\infty} 2^{-i} r \{M_2 g_0(x)\}^{1/p_0} \\
&\le C 2^{-N} r \{M_2 g_0(x)\}^{1/p_0},
\end{aligned}$$

where $g_0(y) = |g(y)|^{p_0} \chi_B(y)$ with χ_B denoting the characteristic function of B. Now, considering N to be the integer part of $(2^{-N} r)^{-\nu/p} = \{M_2 g_0(x)\}^{1/p_0}$, we establish

$$|u(x) - u_B| \le C[r^{-\nu/p^\sharp} + \{M_2 g_0(x)\}^{p/(p^\sharp p_0)}]. \tag{4.3}$$

Therefore, it follows from Lemma 2.2 that

$$\begin{aligned}\int_B |u(x)-u_B|^{p^\sharp}\,d\mu(x) &\le C\int_B [r^{-\nu}+\{M_2 g_0(x)\}^{p/p_0}]\,d\mu(x)\\ &\le C\left[r^{-\nu}\mu(B)+\int g_0(y)^{p/p_0}\,d\mu(y)\right]\\ &= C\left[r^{-\nu}\mu(B)+\int_B |g(y)|^p\,d\mu(y)\right]\\ &\le Cr^{-\nu}\mu(2B),\end{aligned}$$

as required. □

Corollary 4.2. *Let μ be a nonnegative measure on G satisfying* (4.1), *and assume that a couple* (u, g) *satisfies a strong* $(1, p_0)$ *Poincaré inequality. Then*

$$\left(⨍_B |u(x)-u_B|^{p^*}\,d\mu(x)\right)^{1/p^*} \le Cr^{-s/p^*}\mu(B)^{1/p}\left(⨍_B |g(y)|^p\,d\mu(y)\right)^{1/p}$$

for every $B = B(x, r)$ *with* $2B \subset G$, *where* $1/p^* = 1/p - 1/s > 0$.

To show this, first suppose $\int_B |g(y)|^p\,d\mu(y) \le 1$. Then the decay condition (4.1) implies that $\|g\|_{p,s,G;1}$ is bounded. Now we see from the inequalities after (4.3) that

$$\int_B |u(x)-u_B|^{p^*}\,d\mu(x) \le Cr^{-s}\mu(B).$$

Hence we obtain

$$\left(\int_B |u(x)-u_B|^{p^*}\,d\mu(x)\right)^{1/p^*} \le C(r^{-s}\mu(B))^{1/p^*}\left(\int_B |g(y)|^p\,d\mu(y)\right)^{1/p}$$

for a general g, which gives the required result.

REMARK 4.3. Let G be an open set in $\mathbf{R}^n$. We assume that a couple (u, g) satisfies a $(1, p_0)$ Poincaré inequality in G, that is,

$$⨍_B |u(x)-u_B|\,d\mu(x) \le c'_P\mu(B)^{1/s}\left(⨍_B |g(y)|^{p_0}\,d\mu(y)\right)^{1/p_0} \tag{4.4}$$

for all balls $B \subset G$, where $1 < p_0 < p$ and c'_P is a positive constant independent of (u, g). We further assume that $\mu(\partial B) = 0$ and

$$\mu(B)^{\nu/s}⨍_B |g(y)|^p\,d\mu(y) \le 1$$

for each ball $B \subset G$, where $1 < p < \nu$. Then

$$\sup_{B \subset G} \mu(B)^{\nu/s} \fint_B |u(x) - u_B|^{p^\sharp} \, d\mu(x) \le C.$$

For this, we also refer to Hajłasz-Koskela [5].

For a proof of this fact, let $x \in G$ be a Lebesgue point of u. As in the proof of Theorem 1.1 by Gogatishvili-Koskela [4], we take a sequence of balls $\{B_j\}$ such that $x \in B_{j+1} \subset B_j \subset B$ and $\mu(B_j) = 2^{-j}\mu(B)$. Then, as in (4.3), we can prove

$$|u(x) - u_B|^{p^\sharp} \le C[\mu(B)^{-\nu/s} + \{M_1 g_0(x)\}^{p/p_0}],$$

which gives the required inequality by the boundedness of the maximal operator M_1.

Finally we discuss the exponential integrability in the same manner as in Theorem 4.1.

Theorem 4.4. *Let G be bounded and $\nu = p$. Let μ be a nonnegative measure on G satisfying* (4.1), *and assume that a couple (u, g) satisfies a strong* $(1, p_0)$ *Poincaré inequality. Then there exists a positive constant c such that*

$$\frac{r^\nu}{\mu(2B)} \int_B \{\exp(c|u(x) - u_B|) - 1\} \, d\mu(x) \le 1$$

for every ball $B = B(z, r)$, whenever $2B \subset G$ and $\|g\|_{p,\nu,G;2} \le 1$.

REMARK 4.5. Let μ be a nonnegative measure on $\mathbf{R}^n$ satisfying (4.1), and assume that a couple (u, g) satisfies a strong $(1, p_0)$ Poincaré inequality. If $g \in L^{p,\nu;2}(\mathbf{R}^n)$ and $p > \nu$, then u can be corrected almost everywhere to be continuous on $\mathbf{R}^n$; for this, see [10].

In fact, the first part of the proof of Theorem 4.1 implies that

$$|u(x) - u_B| \le Cr^{(p-\nu)/p}$$

for almost every $x \in B$, which proves

$$|u(x) - u(y)| \le C|x - y|^{(p-\nu)/p}$$

for almost every $x, y \in B$.

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Yoshihiro Mizuta
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima 739–8521
Japan
e-mail: yomizuta@hiroshima-u.ac.jp

Tetsu Shimomura
Department of Mathematics
Graduate School of Education
Hiroshima University
Higashi-Hiroshima 739–8524
Japan
e-mail: tshimo@hiroshima-u.ac.jp

Takuya Sobukawa
Department of Mathematics Education
Faculty of Education
Okayama University
Tsushima-naka 700–8530
Japan
e-mail: sobu@cc.okayama-u.ac.jp