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ON SYMPLECTIC QUANDLES

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Abstract

We study the structure of *symplectic quandles*, quandles which are also R-modules equipped with an antisymmetric bilinear form. We show that every finite dimensional symplectic quandle over a finite field $\mathbb{F}$ or arbitrary field $\mathbb{F}$ of characteristic other than 2 is a disjoint union of a trivial quandle and a connected quandle. We use the module structure of a symplectic quandle over a finite ring to refine and strengthen the quandle counting invariant.

1. Introduction

A *quandle* is a non-associative algebraic structure whose axioms may be understood as transcriptions of the Reidemeister moves. The term "quandle" was introduced by Joyce [7], though quandles have been studied by other authors under various names such as "distributive groupoids" [9] and (for a certain special case) "Kei" ([14], [13]). Several generalizations of quandles have been defined and studied, including *automorphic sets* (see [3]) and *racks* (see [6]) where the axioms are derived from regular isotopy moves, *virtual quandles* (see [8]) where additional structure is included for modeling virtual Reidemeister moves, and *biquandles* and *Yang-Baxter sets*, which also have axioms derived from the Reidemeister moves but use a different correspondence between algebra elements and portions of link diagrams.

Quandles have found applications in topology as a source of invariants of topological spaces. In particular, finite quandles are useful for defining computable invariants of knotted circles in S^3 and other 3-manifolds as well as generalizations of ordinary knots such as virtual knots, knotted surfaces in S^4, etc.

In [15], an example of a quandle structure defined on a module M over a commutative ring R with a choice of antisymmetric bilinear form $\langle\ ,\ \rangle\colon M \times M \to R$ is given. In this paper we study the structure of this type of quandle, which we call a *symplectic quandle*[1]. Our main result says that every symplectic quandle Q over a field $\mathbb{F}$ (of characteristic other than 2 if $\mathbb{F}$ is not finite) is *almost connected*, that is, Q is a disjoint union in the sense of [3] of a trivial quandle and a connected quandle. Symplectic quandles are not just quandles but also R-modules; we show how to use

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[1]After the completion of this paper, we were reminded that symplectic quandles are also called *quandles of transvections*.

the R-module structure of a finite symplectic quandle to enhance the usual quandle counting invariant.

The paper is organized as follows. In Section 2 we recall the basic definitions and standard examples of quandles. In Section 3 we define symplectic quandles, give some examples and show that symplectic quandles are almost connected. In Section 4 we give an application of symplectic quandles to knot invariants, defining a new family of enhanced quandle counting invariants associated to finite symplectic quandles.

2. Quandle basics

We begin with a definition from [7].

DEFINITION 1. Let Q be a set and $\triangleright\colon Q \times Q \to Q$ a binary operation satisfying
(i) for all $a \in Q$, $a \triangleright a = a$,
(ii) for all $a, b \in Q$, there is a unique $c \in Q$ such that $a = c \triangleright b$, and
(iii) for all $a, b, c \in Q$, $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$.

Axiom (ii) says that the quandle operation $\triangleright$ has a right inverse $\triangleright^{-1}$ such that $(x \triangleright y) \triangleright^{-1} y = x$ and $(x \triangleright^{-1} y) \triangleright y = x$. It is not hard to show that Q is a quandle under $\triangleright^{-1}$ (called the *dual* of $(Q, \triangleright)$) and that the two operations distribute over each other.

Standard examples of quandle structures include:

EXAMPLE 1. Any set Q is a quandle under the operation $x \triangleright y = x$, called a *trivial* quandle. We denote the trivial quandle of order n by T_n.

EXAMPLE 2. The finite abelian group $\mathbb{Z}_n$ is a quandle under $x \triangleright y = 2y - x$. This is sometimes called the *cyclic quandle* of order n.

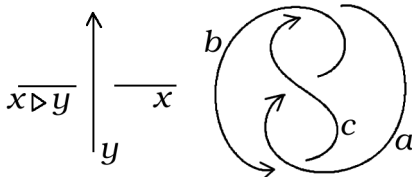
EXAMPLE 3. Any group G is a quandle under the following operations:
- $x \triangleright y = y^{-1}xy$, or
- $x \triangleright y = y^{-n}xy^n$, or
- $x \triangleright y = s(xy^{-1})y$ where $s \in \mathrm{Aut}(G)$.

EXAMPLE 4. Any module over $\mathbb{Z}[t^{\pm 1}]$ is a quandle under

$$x \triangleright y = tx + (1 - t)y.$$

Quandles of this type are called *Alexander quandles*. See [1] and [10] for more.

EXAMPLE 5. For any tame link diagram L, there is a quandle $Q(L)$ defined by a Wirtinger-style presentation with one generator for each arc and one relation at each crossing.



$$Q(L) = \langle a, b, c \mid a \triangleright b = c,\ b \triangleright c = a,\ c \triangleright a = b \rangle.$$

This *knot quandle* is in fact a classifying invariant of knots and unsplit links in S^3 and certain other 3-manifolds up to orientation-reversing homeomorphism of the ambient space. Elements of a knot quandle are equivalence classes of quandle words in the arc generators under the equivalence relation generated by the quandle axioms. See [7] and [6] for more.

DEFINITION 2. Let $Q = \{x_1, x_2, \ldots, x_n\}$ be a finite quandle. The matrix M_Q with $M_Q[i, j] = k$ where $x_k = x_i \triangleright x_j$ for all $i, j \in \{1, 2, \ldots, n\}$ is the *quandle matrix* of Q. That is, M_Q is the operation table of Q without the "x"s.

EXAMPLE 6. The quandle $Q = \mathbb{Z}_3 = \{1, 2, 3\}$ (note that we use 3 for the representative of the coset $0 + 3\mathbb{Z}$ so that our row and column numbers start with 1 instead of 0) with $i \triangleright j = 2j - i$ has quandle matrix

$$M_Q = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

3. Symplectic quandles

We begin this section with a definition (see [15]).

DEFINITION 3. Let M be a finite dimensional free module over a commutative ring with identity R and let $\langle\ ,\ \rangle : M \times M \to R$ be an antisymmetric bilinear form such that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in M$. Then M is a quandle with quandle operation

$$\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}.$$

The dual quandle operation is given by

$$\mathbf{x} \triangleright^{-1} \mathbf{y} = \mathbf{x} - \langle \mathbf{x}, \mathbf{y} \rangle \mathbf{y}.$$

If R is a field and the form is non-degenerate, i.e., if $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0}$ for all $\mathbf{y} \in M$ implies $\mathbf{x} = \mathbf{0} \in M$, then M is *symplectic vector space* and $\langle\ ,\ \rangle$ is a *symplectic form*; thus it is

natural to refer to such M as *symplectic quandles*. For simplicity, we will use the term "symplectic quandle over R" to refer to the general case where R is any ring and $\langle\ ,\ \rangle$ is any antisymmetric bilinear form. If $\langle\ ,\ \rangle$ is non-degenerate, we will say $(M, \triangleright)$ is a *non-degenerate symplectic quandle over* R. M and M' are *isometric* if there is an R-module isomorphism $\phi\colon M \to M'$ which preserves the bilinear form $\langle\ ,\ \rangle$.

DEFINITION 4. A quandle is *involutory* if $\triangleright = \triangleright^{-1}$. Note that involutory quandles are also called *kei* (see [7], [13] and [14] for more).

Proposition 1. *If M is a symplectic quandle over a ring R of characteristic 2, then M is involutory.*

Proof. If M is a symplectic quandle over a ring R of characteristic 2, then for any $\mathbf{x}, \mathbf{y} \in M$ we have

$$\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y} = \mathbf{x} - \langle \mathbf{x}, \mathbf{y}\rangle \mathbf{y} = \mathbf{x} \triangleright^{-1} \mathbf{y}. \qquad \square$$

DEFINITION 5. Let Q and Q' be quandles with $Q \cap Q' = \emptyset$. Then we can make $Q \cup Q'$ a quandle by defining $x \triangleright y = x$ when $x \in Q$ and $y \in Q'$ or when $x \in Q'$ and $y \in Q$. This is the *disjoint union* of Q and Q' in the sense of Brieskorn [3]. If Q and Q' are finite then the matrix of $Q \cup Q'$ is the $(n+m) \times (n+m)$ block matrix

$$M_{Q \cup Q'} = \left[\begin{array}{c|c} M_Q & row \\ \hline row & M_{Q'} \end{array}\right]$$

where *row* indicates that all entries are equal to their row number and we denote $Q = \{x_1, \ldots, x_n\}$, and $Q' = \{x_{n+1}, \ldots, x_{n+m}\}$.

Every quandle Q can be decomposed as a disjoint union of a trivial subquandle

$$D = \{x \in Q \mid x \triangleright y = x \text{ and } y \triangleright x = y\ \forall y \in Q\}$$

and a non-trivial subquandle $Q \setminus D$. Both D and $Q \setminus D$ may be empty, and $Q \setminus D$ may contain trivial subquandles. Call D the *maximal trivial component* of Q.

EXAMPLE 7. The quandle Q with matrix M_Q below has maximal trivial component $D = \{x_5\}$ and $Q \setminus D = \{x_1, x_2, x_3, x_4\}$.

$$M_Q = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 2 & 3 & 2 \\ 3 & 3 & 3 & 1 & 3 \\ 4 & 4 & 4 & 4 & 4 \\ 5 & 5 & 5 & 5 & 5 \end{bmatrix}$$

Notice that even though $\{x_1, x_2, x_3\}$ is a trivial subquandle, it is not part of the maximal trivial component because of the way in which it is embedded in the overall quandle.

Proposition 2. *Let Q be a symplectic quandle over R. Then the maximal trivial component of Q is the submodule of R on which $\langle\ ,\ \rangle$ is degenerate.*

Proof. For any $\mathbf{x} \in Q$ we have $\mathbf{x} \triangleright \mathbf{0} = \mathbf{x} + \langle \mathbf{x}, \mathbf{0}\rangle \mathbf{0} = \mathbf{x}$ and $\mathbf{0} \triangleright \mathbf{x} = \mathbf{0} + \langle \mathbf{0}, \mathbf{x}\rangle \mathbf{x} = \mathbf{0}$, so $\mathbf{0}$ is in the maximal trivial component of Q. More generally, let D be the submodule of Q on which $\langle\ ,\ \rangle$ is degenerate, i.e.,

$$D = \{\mathbf{x} \in Q \mid \langle \mathbf{x}, \mathbf{y}\rangle = 0,\ \forall \mathbf{y} \in Q\}.$$

Then for any $\mathbf{d} \in D$ we have $\mathbf{x} \triangleright \mathbf{d} = \mathbf{x} + 0\mathbf{d} = \mathbf{x}$ and $\mathbf{d} \triangleright \mathbf{x} = \mathbf{d} + 0\mathbf{x} = \mathbf{d}$, so D is a trivial subquandle of Q and Q is the disjoint union of D and $Q \setminus D$ in sense of Definition 5. If $\mathbf{x} \notin D$, then there is some $\mathbf{y} \in Q$ with $\langle \mathbf{x}, \mathbf{y}\rangle \neq 0$ so that $\mathbf{x} \triangleright \mathbf{y} \neq \mathbf{x}$; athen $Q \setminus D$ is non-trivial and D is precisely the submodule of Q on which $\langle\ ,\ \rangle$ is degenerate. □

Corollary 3. *If Q is a nondegenerate symplectic quandle, then the maximal trivial component of Q is $D = \{\mathbf{0}\}$.*

We will now restrict our attention to the case where M is a free module over a PID R. It is a standard result (see [2] for example) that such an M equipped with a nondegenerate antisymmetric bilinear form must be even dimensional, with basis $\{\mathbf{b}_i \mid i = 1, \ldots, 2n\}$ such that

$$\langle \mathbf{x}, \mathbf{y}\rangle = \left\langle \sum_{i=1}^{2n} x_i \mathbf{b}_i, \sum_{i=1}^{2n} y_i \mathbf{b}_i \right\rangle = \sum_{i=1}^{2n} \epsilon(i)\alpha_i x_i y_{i+\epsilon(i)} \quad \text{where} \quad \epsilon(i) = \begin{cases} 1 & i \text{ odd} \\ -1 & i \text{ even,} \end{cases}$$

$\alpha_{2i} = \alpha_{2i-1}$, and each α_i is either 1 or a nonunit in R. Such a basis is called a *symplectic basis*. The α_is are called *invariant factors*, and the set with multiplicities of invariant factors determines the symplectic module structure up to isometry (i.e., $\langle\ ,\ \rangle$-preserving isomorphism of R-modules). In particular, if R is a field, then we may choose our basis so that $\alpha_{2i} = \alpha_{2i-1} = 1$ for all $i = 1, \ldots, n$.

In matrix notation with $\mathbf{x}$, $\mathbf{y}$ row vectors, we have $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}A\mathbf{y}^T$ where A is a block diagonal matrix of the form

$$A = \left[\begin{array}{cc|cc|c|cc} 0 & \alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & 0 & 0 & \alpha_4 & \cdots & 0 & 0 \\ 0 & 0 & -\alpha_4 & 0 & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{2n} \\ 0 & 0 & 0 & 0 & \cdots & -\alpha_{2n} & 0 \end{array}\right].$$

It is clear that isometric R-modules are isomorphic as quandles. Conversely, a symplectic quandle structure on R^n determines the antisymmetric bilinear form $\langle\ ,\ \rangle$ uniquely up to choice of basis: for a basis $\{\mathbf{b}_i \mid i = 1, \ldots, 2n\}$ of R^{2n} we have

$$\mathbf{b}_i \triangleright \mathbf{b}_j - \mathbf{b}_i = \langle \mathbf{b}_i, \mathbf{b}_j \rangle \mathbf{b}_j = \alpha_{ij} \mathbf{b}_j$$

and since $\{\mathbf{b}_i\}$ is a basis, the α_{ij} thus determined is unique. Changing bases to get a symplectic basis, we then obtain the invariant factors. Thus the quandle structure together with the R-module structure of M determine the invariant factors and hence determine $\langle\ ,\ \rangle$, and we have:

Theorem 4. *Let Q and Q' be non-degenerate $2n$-dimensional symplectic quandles over a PID R. Then Q and Q' are isomorphic as quandles iff they are isometric.*

Our search through examples of finite symplectic quandles of small cardinality over $\mathbb{Z}_n$ for n non-prime has failed to yield any examples of symplectic quandles which are isomorphic as quandles but not isometric as R-modules. Thus, we have

Conjecture 1. *Two symplectic quandles of the same dimension over $\mathbb{Z}_n$ are isomorphic as quandles if and only if they are isometric.*

The following example shows that cardinality alone does not determine R or the rank of Q.

EXAMPLE 8. Let $R = \mathbb{Z}_2$ and $F = \mathbb{Z}_2[t]/(t^2 + t + 1)$. Both R and F are fields of characteristic 2, and the symplectic vector spaces

$$V = R^4, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{y}^T$$

and

$$V' = F^2, \quad \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y}^T$$

are both symplectic quandles of order 16. From their quandle matrices

$$M_V = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 4 & 3 & 2 & 2 & 8 & 7 & 2 & 2 & 12 & 11 & 2 & 2 & 16 & 15 \\
3 & 4 & 3 & 2 & 3 & 8 & 3 & 6 & 3 & 12 & 3 & 10 & 3 & 16 & 3 & 14 \\
4 & 3 & 2 & 4 & 4 & 7 & 6 & 4 & 4 & 11 & 10 & 4 & 4 & 15 & 14 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 13 & 14 & 15 & 16 & 9 & 10 & 11 & 12 \\
6 & 6 & 8 & 7 & 6 & 6 & 4 & 3 & 14 & 13 & 6 & 6 & 10 & 9 & 6 & 6 \\
7 & 8 & 7 & 6 & 7 & 4 & 7 & 2 & 15 & 7 & 13 & 7 & 11 & 7 & 9 & 7 \\
8 & 7 & 6 & 8 & 8 & 3 & 2 & 8 & 16 & 8 & 8 & 13 & 12 & 8 & 8 & 9 \\
9 & 9 & 9 & 9 & 13 & 14 & 15 & 16 & 9 & 9 & 9 & 9 & 5 & 6 & 7 & 8 \\
10 & 10 & 12 & 11 & 14 & 13 & 10 & 10 & 10 & 10 & 4 & 3 & 6 & 5 & 10 & 10 \\
11 & 12 & 11 & 10 & 15 & 11 & 13 & 11 & 11 & 4 & 11 & 2 & 7 & 11 & 5 & 11 \\
12 & 11 & 10 & 12 & 16 & 12 & 12 & 13 & 12 & 3 & 2 & 12 & 8 & 12 & 12 & 5 \\
13 & 13 & 13 & 13 & 9 & 10 & 11 & 12 & 5 & 6 & 7 & 8 & 13 & 13 & 13 & 13 \\
14 & 14 & 16 & 15 & 10 & 9 & 14 & 14 & 6 & 5 & 14 & 14 & 14 & 14 & 4 & 3 \\
15 & 16 & 15 & 14 & 11 & 15 & 9 & 15 & 7 & 15 & 5 & 15 & 15 & 4 & 15 & 2 \\
16 & 15 & 14 & 16 & 12 & 16 & 16 & 9 & 8 & 16 & 16 & 5 & 16 & 3 & 2 & 16
\end{bmatrix}$$

and

$$M_{V'} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 6 & 5 & 8 & 7 & 14 & 16 & 15 & 13 & 10 & 11 & 9 & 12 \\
3 & 3 & 3 & 3 & 11 & 9 & 10 & 12 & 7 & 6 & 8 & 5 & 15 & 16 & 13 & 14 \\
4 & 4 & 4 & 4 & 16 & 13 & 15 & 14 & 12 & 11 & 10 & 9 & 8 & 6 & 5 & 7 \\
5 & 6 & 8 & 7 & 5 & 2 & 16 & 11 & 5 & 14 & 12 & 3 & 5 & 10 & 4 & 15 \\
6 & 5 & 7 & 8 & 2 & 6 & 9 & 13 & 10 & 3 & 6 & 15 & 14 & 4 & 12 & 6 \\
7 & 8 & 6 & 5 & 15 & 10 & 7 & 2 & 3 & 9 & 13 & 7 & 11 & 7 & 16 & 4 \\
8 & 7 & 5 & 6 & 12 & 14 & 2 & 8 & 16 & 8 & 3 & 11 & 4 & 13 & 8 & 9 \\
9 & 11 & 10 & 12 & 9 & 3 & 6 & 16 & 9 & 7 & 14 & 4 & 9 & 15 & 2 & 8 \\
10 & 12 & 9 & 11 & 14 & 7 & 3 & 10 & 6 & 10 & 4 & 16 & 2 & 5 & 10 & 13 \\
11 & 9 & 12 & 10 & 3 & 11 & 13 & 5 & 15 & 4 & 11 & 8 & 7 & 2 & 14 & 11 \\
12 & 10 & 11 & 9 & 8 & 15 & 12 & 3 & 4 & 13 & 5 & 12 & 16 & 12 & 6 & 2 \\
13 & 16 & 15 & 14 & 13 & 4 & 11 & 6 & 13 & 12 & 7 & 2 & 13 & 8 & 3 & 10 \\
14 & 15 & 16 & 13 & 10 & 8 & 14 & 4 & 2 & 5 & 9 & 14 & 6 & 14 & 11 & 3 \\
15 & 14 & 13 & 16 & 7 & 12 & 4 & 15 & 11 & 15 & 2 & 6 & 3 & 9 & 15 & 5 \\
16 & 13 & 14 & 15 & 4 & 16 & 5 & 9 & 8 & 2 & 16 & 10 & 12 & 3 & 7 & 16
\end{bmatrix}$$

we can easily see that V and V' are not isomorphic as quandles by checking that the quandle polynomials $qp_V(s, t) = s^{16}t^{16} + 15s^8t^8$ and $qp_{V'}(s, t) = s^{16}t^{16} + 15s^4t^4$ are not equal (see [11] for more).

DEFINITION 6. A quandle Q is *connected* if it has a single orbit, i.e., if every element $z \in Q$ can be obtained from every other element $x \in Q$ by a sequence of quandle operations $\triangleright$ and dual quandle operations $\triangleright^{-1}$. A quandle is *almost connected* if it is a disjoint union in the sense of Definition 5 of its maximal trivial component and a single connected subquandle.

Our main result says that symplectic quandles Q over a finite field or infinite field $\mathbb{F}$ of characteristic other than 2 are almost connected; in particular, if $\langle\ ,\ \rangle$ is non-degenerate then the subquandle $Q \setminus \{\mathbf{0}\}$ is a connected quandle. Connected quandles are of particular interest for defining knot invariants since knot quandles for knots (i.e., single-component links) are always connected. In particular, the image of a quandle homomorphism $f\colon Q(L) \to T$ from a knot quandle to T always lies within a single orbit of the codomain quandle T, though of course f need not be surjective.

For the remainder of this section, let Q be a symplectic quandle over a field $\mathbb{F}$ and choose a symplectic basis $\{\mathbf{b}_i\}$ with invariant factors $\alpha_{2i} = 1$ for $i = 1, \ldots, n$.

Lemma 5. *If any component x_i of $\mathbf{x} = \sum_{i=1}^{2n} x_i\mathbf{b}_i \in Q$ is nonzero then for any $j \in \{1, \ldots, 2n\}$ there is a $\mathbf{z} = \mathbf{x} \triangleright \mathbf{y} \in Q$ with $z_j \neq 0$ for some $\mathbf{y} \in Q$. That is, we can change a zero component to a nonzero component using a quandle operation, provided at least one other component of $\mathbf{x}$ is nonzero.*

Proof. Suppose $x_i \neq 0$ and $x_j = 0$. Then choose $\beta \in \mathbb{F}$ such that $\beta \neq \epsilon(i)x_i/(\epsilon(j)x_{j+\epsilon(j)})$ or, if $x_{j+\epsilon(j)} = 0$, $\beta \neq -\epsilon(i)x_i$. and define $\mathbf{y} = \mathbf{b}_{i+\epsilon(i)} + \beta\mathbf{b}_j$. Then we have

$$\mathbf{x} \triangleright \mathbf{y} = \mathbf{x} + (\epsilon(i)x_i - \epsilon(j)x_{j+\epsilon(j)}\beta)\mathbf{y}$$

and the j-th component of $\mathbf{z} = \mathbf{x} \triangleright \mathbf{y}$ is

$$z_j = 0 + (\epsilon(i)x_i - \beta\epsilon(j)x_{j+\epsilon(j)})\beta$$

which is nonzero by our choice of β. □

Lemma 6. *For any $\mathbf{x} = \sum_{i=1}^{2n} x_i\mathbf{b}_i \in Q$ and for any $\beta \in \mathbb{F}$, we can add (or subtract) $\beta^2 x_i$ to (or from) $x_{i+\epsilon(i)}$ with quandle operations and dual quandle operations.*

Proof.

$$\mathbf{x} \triangleright \beta\mathbf{b}_{i+\epsilon(i)} = \mathbf{x} + (\epsilon(i)x_i\beta)\beta\mathbf{b}_{i+\epsilon(i)} = \mathbf{x} + \epsilon(i)\beta^2 x_i\mathbf{b}_{i+\epsilon(i)}$$

and similarly

$$\mathbf{x} \triangleright^{-1} \beta\mathbf{b}_{i+\epsilon(i)} = \mathbf{x} - \epsilon(i)\beta^2 x_i\mathbf{b}_{i+\epsilon(i)}.$$ □

Lemma 7. *If the characteristic of $\mathbb{F}$ is not 2, then for any $\mathbf{x} \neq 0$, we can change any component x_i of $\mathbf{x}$ to any value $z \in \mathbb{F}$ with quandle operations and dual quandle operations.*

Proof. Write $x_i = z + w$. By Lemma 5 we may assume that $x_{i+\epsilon(i)} \neq 0$. Then

$$\mathbf{x} \triangleright w\mathbf{b}_i = \mathbf{x} + (\epsilon(i + \epsilon(i))x_{i+\epsilon(i)}w)w\mathbf{b}_i$$

and the new quandle element has i-th component equal to

$$\begin{aligned} x_i + \epsilon(i + \epsilon(i))x_{i+\epsilon(i)}w^2 &= z + w + \epsilon(i + \epsilon(i))x_{i+\epsilon(i)}w^2 \\ &= z + \epsilon(i + \epsilon(i))x_{i+\epsilon(i)}\left(\frac{w}{\epsilon(i + \epsilon(i))x_{i+\epsilon(i)}} + w^2\right). \end{aligned}$$

Let us denote $j = i + \epsilon(i)$. If the characteristic of $\mathbb{F}$ is not 2, then we can complete the square to obtain

$$\begin{aligned} x_i + \epsilon(j)x_j w^2 &= z + \epsilon(j)x_j\left(\frac{1}{4x_j^2} + \frac{w}{\epsilon(j)x_j} + w^2\right) - \epsilon(j)x_j\frac{1}{4x_j^2} \\ &= z + \epsilon(j)x_j\left(\frac{1}{2x_j} + \epsilon(j)w\right)^2 - \epsilon(j)x_j\left(\frac{1}{2x_j}\right)^2. \end{aligned}$$

Then by Lemma 6 we can remove both terms via quandle operations and dual quandle operations to obtain z in the i-th component, as required. □

Lemma 8. *In a finite field $\mathbb{F}$ of characteristic 2, every element of $\mathbb{F}$ is a square.*

Proof. The map $f\colon \mathbb{F} \to \mathbb{F}$ given by $f(x) = x^2$ is a homomorphism of fields since

$$f(x + y) = (x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 = f(x) + f(y)$$

and

$$f(xy) = (xy)^2 = x^2y^2.$$

Then $\ker(f) = \{0\}$ since $\mathbb{F}$ has no zero divisors; thus f is injective and, since $\mathbb{F}$ is finite, surjective. In particular, every $\alpha \in \mathbb{F}$ satisfies $\alpha = \beta^2$ for some $\beta \in \mathbb{F}$. □

Taken together, Lemmas 5, 6, 7 and 8 imply:

Theorem 9. *Let $\mathbb{F}$ be a field of characteristic other than 2, or a finite field of characteristic 2. Then every symplectic quandle over $\mathbb{F}$ is almost connected.*

If R is not a field, then symplectic quandles over R need not be almost connected, as the next example shows.

EXAMPLE 9. The symplectic quandle $V'' = (\mathbb{Z}_4)^2$ with bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \mathbf{y}^T$$

has quandle matrix

$$M_{V''} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 10 & 12 & 10 & 12 & 2 & 2 & 2 & 2 & 10 & 12 & 10 & 12 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 12 & 10 & 12 & 10 & 4 & 4 & 4 & 4 & 12 & 10 & 12 & 12 \\
5 & 7 & 5 & 7 & 5 & 15 & 5 & 15 & 5 & 7 & 5 & 7 & 5 & 15 & 5 & 15 \\
6 & 8 & 6 & 8 & 14 & 6 & 14 & 6 & 6 & 8 & 6 & 8 & 14 & 6 & 14 & 6 \\
7 & 5 & 7 & 5 & 7 & 13 & 7 & 13 & 7 & 5 & 7 & 5 & 7 & 13 & 7 & 13 \\
8 & 6 & 8 & 6 & 16 & 8 & 16 & 8 & 8 & 6 & 8 & 6 & 16 & 8 & 16 & 8 \\
9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 & 9 \\
10 & 10 & 10 & 10 & 2 & 4 & 2 & 4 & 10 & 10 & 10 & 10 & 2 & 4 & 2 & 4 \\
11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 & 11 \\
12 & 12 & 12 & 12 & 4 & 2 & 4 & 2 & 12 & 12 & 12 & 12 & 4 & 2 & 4 & 2 \\
13 & 15 & 13 & 15 & 13 & 7 & 13 & 7 & 13 & 15 & 13 & 15 & 13 & 7 & 13 & 7 \\
14 & 16 & 14 & 16 & 6 & 14 & 6 & 14 & 14 & 16 & 14 & 16 & 6 & 14 & 6 & 14 \\
15 & 13 & 15 & 13 & 15 & 5 & 15 & 5 & 15 & 13 & 15 & 13 & 15 & 5 & 15 & 5 \\
16 & 14 & 16 & 14 & 8 & 16 & 8 & 16 & 16 & 14 & 16 & 14 & 8 & 16 & 8 & 16
\end{bmatrix}.$$

V'' has maximal trivial component $D = \{x_1, x_3, x_9, x_{11}\}$, but the nontrivial component $V'' \setminus D$ has disjoint orbit subquandles $\{x_2, x_4, x_{10}, x_{12}\}$, $\{x_5, x_7, x_{13}, x_{15}\}$ and $\{x_6, x_8, x_{14}, x_{16}\}$ and hence is not connected. For comparison with the order 16 symplectic quandles in Example 8, the quandle polynomial for V'' is $qp_{V''}(s, t) = 4s^{16}t^{16} + 12s^8t^8$.

4. Symplectic quandles and knot invariants

The primary application for finite quandles has so far been in the construction of link invariants. Given a finite quandle T we have the quandle counting invariant $|\mathrm{Hom}(Q(L), T)|$, the quandle 2-cocycle invariants $\Phi_\chi(L, T)$ and the specialized subquandle polynomial invariants $\Phi_{\mathrm{qp}}(L)$ described in [4] and [11] respectively. The connected component of a symplectic quandle over a finite field is a finite connected quandle which generally has a number of nontrivial subquandles, making this type of quandle well suited for the specialized subquandle polynomial invariant. In this section we describe two additional ways of getting extra information about the knot or link type from the set of homomorphisms from a link quandle into a finite symplectic quandle.

One easy way to get more information out of the set $|\mathrm{Hom}(Q(L), T)|$ is to count the cardinalities of the image subquandles for each $f \in \mathrm{Hom}(Q(L), T)$; even if T is connected, the smallest subquandle of T containing the images of generators of $Q(L)$ need not be the entire quandle T. If instead of counting 1 for each homomorphism f, we count the cardinality of the image of f, we obtain a set with multiplicities of integers, which we can convert into a polynomial for easy comparison with other invariant values by converting the elements of the set to exponents of a variable q and converting the multiplicities to coefficients. Thus we have

DEFINITION 7. The *enhanced quandle counting invariant* of a link L with respect to a finite target quandle T is given by

$$\Phi_E(L, T) = \sum_{f \in \mathrm{Hom}(Q(L), T)} q^{|\mathrm{Im}(f)|}.$$

This enhanced quandle counting invariant can be understood as a decomposition of the usual quandle counting invariant into a sum of counting invariants over all subquandles of our target quandle with the restriction that we only count *surjective* homomorphisms onto each subquandle.

For any subquandle $S \subseteq T$ of a finite quandle T, let $\mathrm{SH}(Q(L), S)$ be the set of surjective quandle homomorphisms from a link quandle $Q(L)$ onto S and let $SQ(T)$ be the set of all subquandles of T. Then

$$\Phi_E(L, T) = \sum_{S \in SQ(T)} |\mathrm{SH}(Q(L), S)| q^{|S|}.$$

Because symplectic quandles are not just quandles but also R-modules, we can take advantage of the R-module structure of a finite symplectic quandle T to further enhance the counting invariant.

DEFINITION 8. Let T be a finite symplectic quandle over a (necessarily finite) ring R and let L be a link. Then for each $f \in \mathrm{Hom}(Q(L), T)$, let $\rho(f)$ be the cardinality of the R-submodule spanned by $\mathrm{Im}(f) \subseteq T$ (note that $\mathrm{Im}(f)$ itself need not be a submodule). Then the *symplectic quandle polynomial* of L with respect to T is

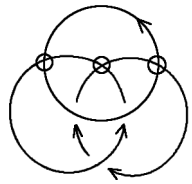
$$\Phi_{\mathrm{sqp}}(L, T) = \sum_{f \in \mathrm{Hom}(Q(L), T)} q^{|\mathrm{Im}(f)|} z^{\rho(f)}.$$

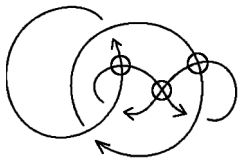
Note that in Definition 8 the finite target quandle T has a fixed R-module structure; in the case of a counterexample to Conjecture 1, i.e., if two symplectic quandles exist which are isomorphic as quandles but not as modules, then we would expect two such symplectic quandles to define distinct symplectic quandle polynomial invariants.

In particular, if R is not a field then we must be careful to specify the R-module structure of T and our choice of bilinear form.

The following example demonstrates that Φ_{sqp} contains more information than the quandle counting invariant alone.

EXAMPLE 10. The two pictured virtual links have the same value for the quandle counting invariant with respect to the symplectic quandle $T = (\mathbb{Z}_3)^2$ but different values for $\Phi_{\text{sqp}}(L, T)$.

L_1: 

L_2: 

$|\text{Hom}(Q(L_1), T)| = 105$
$\Phi_{\text{sqp}}(L_1, T) = 9qz + 72q^2z^3 + 24qz^3$

$|\text{Hom}(Q(L_2), T)| = 105$
$\Phi_{\text{sqp}}(L_2, T) = qz + 72q^2z^3 + 24q^3z^3 + 8qz^3$

Proposition 10. *If $T = K^{2n}$ is the nondegenerate symplectic quandle of dimension $2n$ over the Galois field $K = GF(p^m)$ for a prime p, then $\Phi_{\text{sqp}}(\text{Unknot}, T) = qz + (p^{2nm} - 1)qz^{p^m}$.*

Proof. Every element of $\text{Hom}(Q(\text{Unknot}), T)$ is a constant map into a single element of T. The zero map contributes $q^1z^1 = qz$ to the sum, while each of the nonzero constant maps has image subquandle consisting of a single element of T which spans a dimension 1 subspace; hence each of these $p^{2nm} - 1$ maps contributes qz^{p^m} to the sum. □

Specializing $z = 1$ and $q = 1$ in $\Phi_{\text{sqp}}(L, T)$ yields the quandle counting invariant $|\text{Hom}(Q(L), T)|$. Specializing $z = 1$ yields the enhanced quandle counting invariant $\Phi_E(L, T)$.

Our initial computations suggest that these symplectic quandle polynomial invariants are quite non-trivial for virtual links, though the fact that finite symplectic quandles tend to have rather large cardinality ($|R|^{2n}$) means that more efficient computing algorithms may be required to explore these invariants in greater detail. Our `Maple` software is able to compute $\Phi_{\text{sqp}}(L, T)$ for links with smallish numbers of crossings for symplectic quandles of order ≤ 81 in a relatively short amount of time, but the time requirement increases rapidly as $|R|$ and n increase.

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