Schulze-Pillot, R.
Osaka J. Math.
45 (2008), 965–971

LOCAL THETA CORRESPONDENCE AND THE LIFTING OF DUKE, IMAMOGLU AND IKEDA

RAINER SCHULZE-PILLOT

(Received September 13, 2007)

Abstract

We use results on the local theta correspondence to prove that for large degrees the Duke-Imamoglu-Ikeda lifting of an elliptic modular form is not a linear combination of theta series.

1. Introduction

In the article [7] it is proved as a side remark that a Siegel modular form F of degree $2n$ and weight $n+k$ obtained by the lifting of Duke, Imamoglu and Ikeda [5] (called the DII-lifting in the sequel) from an elliptic modular form of weight $2k$ is not a linear combination of theta series of even unimodular positive definite quadratic forms of rank $m = 2(n+k)$ if n is bigger than k, whereas for $n = k \equiv 0 \bmod 2$ the DII liftings lie in the space generated by theta series subject to a conjecture on L-functions of elliptic cuspidal Hecke eigenforms. The proof uses Böcherer's characterization of the cuspidal Siegel eigenforms that lie in the space of theta series by special values of their standard L-functions.

In this article we use results on the local theta correspondence to give a different proof of the first result. In fact we prove a more general version including theta series attached to arbitrary non degenerate quadratic forms and also theta series with spherical harmonics. The underlying local fact has been noticed by several people (including the author) immediately after the preprint version of [5] became available in 1999 but has apparently never been published.

In the case $n = k$ we show that the local representations attached to a DII-lift are in the image of the local theta correspondence with the orthogonal group of a suitable local quadratic space of dimension $4n = 2(n+k)$ for all places.

I thank Ralf Schmidt for discovering a mistake in an earlier version of this article.

2. The case $n > k$

Let f be an elliptic modular form of weight $2k$ for the full modular group $SL_2(\mathbf{Z})$. It was conjectured by Duke and Imamoglu and proven by Ikeda in [5] that for any

2000 Mathematics Subject Classification. Primary 11F46; Secondary 11F27.

$n \equiv k \bmod 2$ there exists a nonzero Siegel cusp form $F = F_{2n}(f)$ of weight $n+k$ for the group $Sp_{2n}(\mathbf{Z}) \subseteq SL_{4n}(\mathbf{Z})$ whose standard L-function is equal to

$$\zeta(s) \prod_{i=1}^{2n} L(s+k+n-i,\ f),$$

where $L(s,\ f)$ is the usual Hecke L-function of f. We call $F_{2n}(f)$ the DII-lift of degree $2n$ of f.

In fact, it has been shown in [16, Lemma 1.3.1], that the Satake parameters $\alpha_p^{(j)}$ of F at the prime p are given by

$$\text{(2.1)} \qquad \alpha_p^{(0)} = \beta_p^{-n}, \quad \alpha_p^{(j)} = \beta_p p^{-n+j-1/2} \quad \text{for} \quad 1 \leq j \leq 2n,$$

where we write the p-factor of $L(s,\ f)$ as $(1-\beta_p p^{-s+k-1/2})^{-1}(1-\beta_p^{-1} p^{-s+k-1/2})^{-1}$, i.e., we have (by Ramanujan-Petersson) the normalization $|\beta_p| = 1$.

Moreover, Ikeda has recently announced a generalization of this result. In this generalized version f is replaced by an irreducible cuspidal automorphic representation $\tau = \bigotimes \tau_v$ of $GL_2(\mathbf{A}_E)$, where E is a totally real number field and τ_v is assumed to be a principal series representation attached to a character μ_v for all finite places v of E and to be discrete series with minimal weight $\pm\kappa_w$ for the infinite places w of E.

Ikeda then proves the existence of an irreducible cuspidal automorphic representation $\pi = \pi(m, \tau)$ of the metaplectic group $\widetilde{Sp}_m(\mathbf{A}_E)$ with local components π_v described in terms of τ_v as follows: For a real place w the representation π_w is the lowest weight representation of lowest K-type $\det^{\kappa_w+m/2}$, for a finite place v the representation π_v is a degenerate principal series representation which is induced from a character $\mu_v^{(m)}$ on the maximal parabolic $\tilde{P}_m$ derived from μ_v

If $m = 2n$ is even one can obtain from π a representation of $Sp_{2n}(\mathbf{A}_E)$, also denoted by $\pi = \pi(2n, \tau)$; this representation is induced from the $2n$-tuple of characters $\mu_v|\ \ |^{-n+j-1/2}$ for $1 \leq j \leq 2n$. If (for $E = \mathbf{Q}$) in addition f is as above and τ is the automorphic representation of $GL_2(\mathbf{A}_\mathbf{Q})$ associated to f this gives the representation of $Sp_{2n}(\mathbf{A}_\mathbf{Q})$ associated to $F_{2n}(f)$.

We are interested in the question whether the Siegel modular form $F_{2n}(f)$ can be obtained as a linear combination of theta series, respectively in the more general situation whether the representation $\pi(2n, \tau)$ is in the image of the theta correspondence between Sp_{2n} and a suitable orthogonal group.

We let V be a vector space over E of even dimension $2r$ with a nondegenerate quadratic form q and associated symmetric bilinear form $B(x,y) = q(x+y)-q(x)-q(y)$ on it. If $E = \mathbf{Q}$ and q is positive definite we consider a homogenous pluriharmonic form $P \in \mathbf{C}[\{X_{ij} \mid 1 \leq i \leq 2r,\ 1 \leq j \leq m\}]$ in $2rm$ variables of weight ν (see [3, III, 3.5]) as a function on $(V \otimes \mathbf{R})^m$ by identifying the latter space with $\mathbf{R}^{2rm}$ using a basis of V which is orthonormal with respect to the symmetric bilinear form B.

The theta series of degree m of L with respect to P is defined for any $\mathbf{Z}$-lattice L of full rank $2r$ on V by

$$\vartheta^{(m)}(Z, L, P) = \sum_{\mathbf{x}\in L^m} P(\mathbf{x}) \exp(2\pi i \operatorname{tr}(q(\mathbf{x})Z)),$$

where $Z \in \mathfrak{H}_m$ is in the Siegel upper half space $\mathfrak{H}_m \subseteq M_m^{\mathrm{Sym}}(\mathbf{C})$ and where $q(\mathbf{x}) = (1/2)(B(x_i, x_j))$ is half the Gram matrix of the m-tuple $\mathbf{x} = (x_1, \ldots, x_m)$. It is a Siegel modular form of weight $r + \nu$ for some congruence subgroup of $Sp_m(\mathbf{Z})$. One has a similar definition for arbitrary totally real E and totally positive definite q.

If L is even unimodular the theta series is a Siegel modular form for the full modular group $Sp_m(\mathbf{Z})$.

More generally one considers for arbitrary E and (non degenerate) q the theta correspondence associating to a subset of the set of irreducible automorphic representations of the adelic orthogonal group $O_{(V,q)}(\mathbf{A}_E)$ a subset of the set of irreducible automorphic representations of the adelic group $Sp_m(\mathbf{A}_E)$. There is by now a vast literature (starting with Weil's [18] and Howe's [4]) but no textbook reference on this correspondence, for definitions and properties of it see [14, 11, 12, 8, 10].

Theorem 2.1. *Let E, V, q, τ be as above.*
(1) *If $2n > r - 1$ and there is a finite place v of E for which the completion V_v of the quadratic space (V, q) is not split (i.e. is not an orthogonal sum of hyperbolic planes) the representation $\pi(2n, \tau)$ is not in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A}_E)$.*
(2) *If $2n > r$ the representation $\pi(2n, \tau)$ is not in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A}_E)$.*

Corollary 2.2. *Let f be an elliptic modular form of weight $2k$ as above, $\nu \in \mathbf{N}_0$. Then for $n > k - \nu$ the DII-lift $F_{2n}(f)$ is not a linear combination of theta series of positive definite quadratic forms with pluriharmonic forms of degrees $\nu' \geq \nu$.*

In particular for $n > k$ the DII-lift $F_{2n}(f)$ is not a linear combination of theta series attached to positive definite quadratic forms (with or without pluriharmonic forms).

Proof of the theorem. If $\pi(2n, \tau)$ is in the image of the global theta correspondence with $O_{(V,q)}(\mathbf{A}_E)$ its local components $\pi_v(2n, \tau)$ are in the image of the local theta correspondence with $O_{(V,q)}(E_v)$ for all places v of E; we denote by π_v' the corresponding representation of this orthogonal group.

The local theta correspondence has been described explicitly in terms of the Bernstein-Zelevinsky data of the representations in [14, 8]. By construction the Bernstein-Zelevinsky data of $\pi_v(2n, \tau)$ are $\mu_v|\ |^{-n+j-1/2}$ for $1 \leq j \leq 2n$. The characters $\mu_v|\ |^{-n+j-1/2}$ are never of the type $|\ |^i$ for some integral i; this is clear if μ_v is ramified, and follows from $|\mu_v(\omega_v)| < q_v^{5/34}$ (see [6]) if μ_v is unramified (where ω_v is a prime element at the place v and q_v is the norm of ω_v). Since by the results of [14, 8] only characters of the type $|\ |^i$ for some integral i can be missing in the Bernstein-Zelevinsky data of the representation π' on the orthogonal side we see that all the $\mu_v|\ |^{-n+j-1/2}$ for $1 \le j \le 2n$ have to appear in these Bernstein-Zelevinsky data. Obviously the rank of the orthogonal group $O_{(V,q)}(E_v)$ has to be at least $2n$ for this to be possible. If V_v is split this requires $r \ge 2n$, if V_v is not split it requires $r - 1 \ge 2n$ (and even $r - 2 \ge 2n$ if the anisotropic kernel of V_v is of dimension 4). □

Proof of the corollary. It is well known that one can associate to a cuspidal Siegel modular form F for $Sp_{2n}(\mathbf{Z})$ which is an eigenfunction of all Hecke operators an irreducible cuspidal automorphic representation $\pi(F)$ of $Sp_{2n}(\mathbf{A_Q})$ with the same Satake parameters, see e.g. [2]. Moreover, if F has weight $\tilde{k}$ and can be written as a linear combination of theta series of positive definite quadratic forms with pluriharmonic forms of weights $\nu' \ge \nu$, the fact that the space generated by theta series with pluriharmonic forms of weight ν' for lattices on a fixed quadratic space is Hecke invariant guarantees that there is a positive definite quadratic space (V, q) of dimension $2r = 2(\tilde{k} - \nu')$ for some $\nu' \ge \nu$ such that the irreducible representation $\pi(F)$ is in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A_Q})$.

Considering this for the DII lift $F_{2n}(f)$ and ν, ν', k as in the assertion we have $\tilde{k} = k + n$ and therefore $r = k + n - \nu' \le k + n - \nu$ and $k < n + \nu$, hence $r < 2n$.

By the theorem $\pi(F_{2n}(f))$ is not in the image of the theta correspondence with $O_{(V,q)}(\mathbf{A_Q})$, so $F_{2n}(f)$ can not be a linear combination of theta series as described above. □

REMARK. (1) If we omit the restriction that the quadratic form is positive definite we still get the same result as long as we restrict attention to the holomorphic theta series of weight r associated to indefinite quadratic forms of rank $2r$ that have been constructed by Siegel and Maaß [17, 9]. Although there is no method known to use suitable test functions in the oscillator representation in order to construct holomorphic theta series of weight $k + n - \nu'$ associated (by the theta correspondence) to indefinite quadratic forms of rank larger than $2(k + n - \nu')$, the results of Rallis [15] seem to imply that such a construction is indeed possible. Such a construction would then yield $F_{2n}(f)$ without contradicting the theorem.

(2) The corollary could in principle be proved without using representation theoretic tools by computing the possible Hecke eigenvalues of a linear combination of theta series with the help of results of Andrianov [1] and Yoshida [19] and comparing with the eigenvalues of a DII-lift; we expect this to be a rather tedious and unpleasant computation.

3. The case $n = k$

With the notations of the previous section we assume now $E = \mathbf{Q}$ and $n = k$, so that the weight $k + n$ of a DII lift is equal to the rank $2n$ of the symplectic group considered.

Theorem 3.1. *For $n = k$ the local components π_p of the representation $\pi(F_{2n}(f))$ of the Duke-Imamoglu-Ikeda lift $F_{2n}(f)$ are in the image of the local theta correspondence with the split quadratic space over $\mathbf{Q}_p$ of dimension $4n = 2(k+n)$ (which is the orthogonal sum of $2n$ hyperbolic planes) for all (finite) primes p.*

The component π_∞ at the real place is in the image of the theta correspondence with the orthogonal group of the positive definite quadratic space over $\mathbf{R}$ of dimension $4n$ and also in the image of the theta correspondence with the orthogonal group of the quadratic space of dimension $4n$ and signature $(4n-1, 1)$ over $\mathbf{R}$.

Proof. The assertion for the finite primes is again an immediate consequence of the results of [14, 8]. The assertion at the real place follows from Theorem 15 of [13] since π_∞ is a limit of discrete series. □

REMARK. (1) At the real place it follows from Theorem 15 of [13] that π_∞ occurs also in the images of the theta correspondence with the orthogonal groups of the quadratic spaces of dimension $4n+2$ and signatures $(4n+1, 1)$ and $(4n, 2)$ over $\mathbf{R}$, but not for any other quadratic space of dimension $4n$ or $4n+2$.
(2) Since the split quadratic space of dimension $4n$ has square discriminant, there is no quadratic space over $\mathbf{Q}$ which is split at all finite primes and of signature $(4n-1, 1)$. It is moreover well known that there is a positive definite quadratic space over $\mathbf{Q}$ of even dimension $2r$ which is split at all finite primes if and only if r is divisible by 4 and that the same condition is necessary and sufficient for the existence of a quadratic space over $\mathbf{Q}$ of even dimension $2r+2$ and signature $(2r+1, 1)$ which is split at all finite primes

We see that $\pi(F_{2n}(f))$ can (in the case $n = k$) be in the image of the global theta correspondence with some quadratic space of dimension $4n = 2(n+k)$ only if n is even and the quadratic space is the unique positive definite space of that dimension which is split at all finite places.

In a similar way for even n there is also a unique quadratic space (V, q) over $\mathbf{Q}$ of dimension $4n+2$ and signature $(4n+1, 1)$ for which π_v is in the image of the local theta correspondence with the orthogonal group of the completion V_v for all (finite or infinite) places v of $\mathbf{Q}$; this space is again split at all finite primes.

There are also spaces of signature $(4n, 2)$ for which π_v is in the image of the local theta correspondence with the orthogonal group of the completion V_v for all (finite or infinite) places v of $\mathbf{Q}$; such a space can be obtained as the orthogonal sum of an imaginary quadratic field equipped with the norm form scaled by some negative number and a positive definite space of dimension $4n$ which is split at all finite places. In all cases our purely local methods allow no statement about occurrence in the global theta correspondence for the respective space.

4. Iterated theta liftings

One might ask whether it is possible to construct the Duke-Imamoglu-Ikeda lift $F_{2n}(f)$ or the associated automorphic representation by a sequence of theta liftings between groups G_i, where for each i the pair G_i, G_{i+1} consists (in either order) of a symplectic or metaplectic group and an orthogonal group, starting with the representation associated to f on SL_2 or the representation on the metaplectic group $\widetilde{SL}_2$ associated to the form g which corresponds to f under the Shimura correspondence. We have more generally:

Proposition 4.1. *Let the notations be as in* Sections 1 *and* 2 *and* $n > 1$. *The generalized Duke-Imamoglu-Ikeda lift* $\pi(2n, \tau)$ *can not be constructed by a series of theta liftings as described above.*

Proof. If one starts out with the metaplectic group $\widetilde{SL}_2$ all subsequent groups G_i will be orthogonal groups of quadratic spaces in odd dimension or metaplectic groups $\widetilde{Sp}_m$ (with a genuine representation of $\widetilde{Sp}_m$ on it) so that we will never arrive at Sp_{2n}, with the exception that the initial step may lead to $SO(3, 2)$ identified with $PGSp_2$ or to $SO(2, 1)$ identified with PGL_2 (in which case we obtain a Saito-Kurokawa lift respectively a Shimura lift). After this initial step there have to appear correspondences which raise the rank of the group, and again by [14, 8] all local representations occurring will (at each finite place v of E) have at most two terms $\mu_v|\ |^j$ with $j \in (1/2)\mathbf{Z} \setminus \mathbf{Z}$ among their Bernstein-Zelevinsky data, so $\pi_v(2n, \tau)$ can never occur for $n > 1$.

If one starts with SL_2 the representation $\pi_v(2n, \tau)$ can not occur if $n > 1$ for the same reason. □

References

[1] A.N. Andrianov: Quadratic Forms and Hecke Operators, Grundlehren der Mathematischen Wissenschaften **286**, Springer, Berlin, 1987.

[2] M. Asgari and R. Schmidt: *Siegel modular forms and representations*, Manuscripta Math. **104** (2001), 173–200.

[3] E. Freitag: Siegelsche Modulfunktionen, Grundlehren der Mathematischen Wissenschaften **254**, Springer, Berlin, 1983.

[4] R. Howe: *θ-series and invariant theory*; in Automorphic Forms, Representations and L-Functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, 275–285.

[5] T. Ikeda: *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. (2) **154** (2001), 641–681.
[6] H.H. Kim and F. Shahidi: *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and functorial symmetric cube for* GL_2, C.R. Acad. Sci. Paris Sér. I Math. **331** (2000), 599–604.
[7] W. Kohnen and R. Salvati Manni: *Linear relations between theta series*, Osaka J. Math. **41** (2004), 353–356.
[8] S.S. Kudla: *On the local theta-correspondence*, Invent. Math. **83** (1986), 229–255.
[9] H. Maaß: *Modulformen zu indefiniten quadratischen Formen*, Math. Scand. **17** (1965), 41–55.
[10] C. Mœglin, M.-F. Vignéras and J.-L. Waldspurger: Correspondances de Howe sur un Corps p-Adique, Lecture Notes in Math. **1291**, Springer, Berlin, 1987.
[11] C. Mœglin: *Non nullité de certains relévements par séries théta*, J. Lie Theory **7** (1997), 201–229.
[12] C. Mœglin: *Quelques propriétés de base des séries théta*, J. Lie Theory **7** (1997), 231–238.
[13] A. Paul: *On the Howe correspondence for symplectic-orthogonal dual pairs*, J. Funct. Anal. **228** (2005), 270–310.
[14] S. Rallis: *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), 469–515.
[15] S. Rallis: *On the Howe duality conjecture*, Compositio Math. **51** (1984), 333–399.
[16] R. Schmidt: *On the spin* L*-function of Ikeda's lifts*, Comment. Math. Univ. St. Pauli **52** (2003), 1–46.
[17] C.L. Siegel: *Indefinite quadratische Formen und Modulfunktionen*; in Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, Interscience Publisher, Inc., New York, 1948, 395–406.
[18] A. Weil: *Sur la formule de Siegel dans la théorie des groupes classiques*, Acta Math. **113** (1965), 1–87.
[19] H. Yoshida: *The action of Hecke operators on theta series*; in Algebraic and Topological Theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, 197–238.

Fachrichtung 6.1 Mathematik
Universität des Saarlandes (Geb. E2.4)
Postfach 151150, 66041 Saarbrücken
Germany
e-mail: schulzep@math.uni-sb.de