CHARACTERIZATION OF PERIODIC POINTS
OF THE NEGATIVE SLOPE ALGORITHM

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Abstract

The notion of negative slope algorithm was introduced by S. Ferenczi, C. Holton, and L. Zamboni in their study of the three letter languages arising from three interval exchange transformations. They characterize eventually periodic points of the negative slope algorithm. In this paper, by using the natural extension of the negative slope algorithm, we give a necessary and sufficient condition for purely periodicity of the negative slope algorithm.

1. Introduction

The negative slope algorithm was introduced by S. Ferenczi, C. Holton, and L. Zamboni [1] to discuss the structure of three-interval exchange transformations. It is a 2-dimensional generalization of the Euclidean algorithm which is arisen from the dynamical system associated with three-interval exchange transformations. This algorithm gives an arithmetic construction generating the symbolic sequence which codes the orbit of a point under a three-interval exchange transformation. It is a kind of multi-dimensional continued fractions algorithm and some arithmetic properties were discussed in [1]. They showed the following theorem.

Theorem 1.1 ([1]). *Suppose iteration by the negative slope algorithm T of $(x, y) \in [0, 1]^2 \setminus \{(x, y) \mid x + y = 1\}$ does not stop. Then the sequence $(T^k(x, y): k \geq 0)$ is ultimately periodic if and only if x and y are in the same quadratic extension of $\mathbb{Q}$.*

In [2], they characterized three symbolic sequences which are the natural codings of three interval exchange transformations, a natural coding means that a sequence $(x_i) = i$, $i = 1, 2, 3$ when the i-th iterate of x_0 lies in the first, second, third interval respectively. In [3], they show the necessary and sufficient condition for three-interval exchange transformations being weak mixing.

In [4], Ishimura and Nakada introduced 4-dimensional map $\bar{T}$ on $[0, 1]^2 \times (-\infty, 0)^2$ which is called the natural extension of the negative slope algorithm.

In this paper, we show the following theorem by using the natural extension of the negative slope algorithm.

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Main Theorem. *Suppose iteration by the negative slope algorithm T of $(x, y) \in [0, 1]^2 \setminus \{(x, y) \mid x + y = 1\}$ does not stop. Then the sequence $(T^k(x, y): k \geq 0)$ is purely periodic if and only if x and y are in the same quadratic extension of $\mathbb{Q}$ and (x^*, y^*) is in $(-\infty, 0)^2$ where x^* denotes the algebraic conjugate of x.*

Then we recover Theorem 1.1 as a corollary of Main Theorem. In other words, we give a second proof of Theorem 1.1 by using the natural extension of the negative slope algorithm. In §2, we define the negative slope algorithm and introduce the representation matrices associated with it. We show some properties of multiplication of their local inverse and some conditions for existence of the negative slope expansion. In §3, we introduce the natural extension of the negative slope algorithm and prove Main Theorem i.e. a characterization for purely periodicity of the negative slope algorithm T by using the natural extension $\bar{T}$ of T. At the last part of §3, we show Theorem 1.1 as a corollary of Main Theorem.

2.   Definitions and basic properties of the negative slope algorithm

2.1.   Definitions of the negative slope algorithm.   First we introduce a map T which is called the negative slope algorithm on the unit square in $[0, 1]$. Let $\mathbb{X} = [0, 1]^2 \setminus \{(x, y) \mid x + y = 1\}$, we define a map T on $\mathbb{X}$ by

$$T(x, y) = \begin{cases} \left(\dfrac{y}{(x+y)-1} - \left[\dfrac{y}{(x+y)-1} \right], \dfrac{x}{(x+y)-1} - \left[\dfrac{x}{(x+y)-1} \right] \right) & \text{if} \quad x+y > 1 \\[3mm] \left(\dfrac{1-y}{1-(x+y)} - \left[\dfrac{1-y}{1-(x+y)} \right], \dfrac{1-x}{1-(x+y)} - \left[\dfrac{1-x}{1-(x+y)} \right] \right) & \text{if} \quad x+y < 1. \end{cases}$$

Using the integer valued functions

$$(n(x, y), m(x, y)) = \begin{cases} \left(\left[\dfrac{x}{(x+y)-1} \right], \left[\dfrac{y}{(x+y)-1} \right] \right) & \text{if} \quad x+y > 1 \\[3mm] \left(\left[\dfrac{1-x}{1-(x+y)} \right], \left[\dfrac{1-y}{1-(x+y)} \right] \right) & \text{if} \quad x+y < 1, \end{cases}$$

and

$$\varepsilon(x, y) = \begin{cases} -1 & \text{if} \quad x+y > 1 \\ +1 & \text{if} \quad x+y < 1, \end{cases}$$

for each $(x, y) \in \mathbb{X}$, we have a sequence

$$((\varepsilon_1(x, y), n_1(x, y), m_1(x, y)), (\varepsilon_2(x, y), n_2(x, y), m_2(x, y)), \dots).$$

We obtain it by

$$\begin{cases} n_k(x, y) = n(T^{k-1}(x, y)) \\ m_k(x, y) = m(T^{k-1}(x, y)) \\ \varepsilon_k(x, y) = \varepsilon(T^{k-1}(x, y)) \end{cases}$$

for $k \geq 1$. Then we note that $n_k, m_k \geq 1$ for $k \geq 1$ and for any sequence $((\varepsilon_i, n_i, m_i),$ $i \geq 1)$, there exists $(x, y) \in \mathbb{X}$ such that $(\varepsilon_i(x, y), n_i(x, y), m_i(x, y)) = (\varepsilon_i, n_i, m_i)$ unless there exists $k \geq 1$ such that $(\varepsilon_i, m_i) = (+1, 1)$ for $i \geq k$ or $(\varepsilon_i, n_i) = (+1, 1)$ for $i \geq k$. We show these properties later as Lemma 2.7. By [1] and [4] we see that if $(x, y) \neq$ $(x', y') \in \mathbb{X}$, then there exists $k \geq 1$ such that

$$(\varepsilon_k(x, y), n_k(x, y), m_k(x, y)) \neq (\varepsilon_k(x', y'), n_k(x', y'), m_k(x', y')).$$

Next we introduce a projective representation of T as follows. We put

$$A_{(+1,n,m)} = \begin{pmatrix} n & n-1 & 1-n \\ m-1 & m & 1-m \\ -1 & -1 & 1 \end{pmatrix}$$

and

$$A_{(-1,n,m)} = \begin{pmatrix} -n & -n+1 & n \\ -m+1 & -m & m \\ 1 & 1 & -1 \end{pmatrix}$$

for $m, n \geq 1$. Then we have

$$A_{(+1,n,m)}^{-1} = \begin{pmatrix} 1 & 0 & n-1 \\ 0 & 1 & m-1 \\ 1 & 1 & n+m-1 \end{pmatrix}$$

and

$$A_{(-1,n,m)}^{-1} = \begin{pmatrix} 0 & 1 & m \\ 1 & 0 & n \\ 1 & 1 & n+m-1 \end{pmatrix}.$$

We identify $(x, y) \in \mathbb{X}$ to $\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha \end{pmatrix}$ for $\alpha \neq 0$. Then we identify $T(x, y)$ to

$$A_{(\varepsilon_1(x,y),n_1(x,y),m_1(x,y))} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

and its local inverse is given by

$$A_{(\varepsilon_1(x,y),n_1(x,y),m_1(x,y))}^{-1}.$$

In this way, we get a representation of $(x, y) \in \mathbb{X}$ by

$$A_{(\varepsilon_1,n_1,m_1)}^{-1} A_{(\varepsilon_2,n_2,m_2)}^{-1} A_{(\varepsilon_3,n_3,m_3)}^{-1} \cdots$$

and T is defined as a multiplication by $A_{(\varepsilon_1, n_1, m_1)}$ from the left and acts as a shift on the set of infinite sequences of matrices

$$\left\{ A^{-1}_{(\varepsilon_1, n_1, m_1)} A^{-1}_{(\varepsilon_2, n_2, m_2)} A^{-1}_{(\varepsilon_3, n_3, m_3)} \cdots \ \middle| \ \varepsilon_k = \pm 1, \ n_k, m_k \geq 1 \text{ for } k \geq 1 \right\}.$$

For a given finite sequence $((\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \ldots, (\varepsilon_k, n_k, m_k))$, we define a cylinder set of length k by

$$\langle (\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \ldots, (\varepsilon_k, n_k, m_k) \rangle$$
$$= \{(x, y) \in \mathbb{X} \mid (\varepsilon_i(x, y), n_i(x, y), m_i(x, y)) = (\varepsilon_i, n_i, m_i), \ 1 \leq i \leq k \}.$$

For simplicity, we write Δ_k for this cylinder set.

For $(x, y) \in \Delta_k$, we denote $T^k(x, y)$ as

$$A_{(\varepsilon_k, n_k, m_k)} \cdots A_{(\varepsilon_1, n_1, m_1)} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

and its local inverse Ψ_{Δ_k} as

$$A^{-1}_{(\varepsilon_1, n_1, m_1)} \cdots A^{-1}_{(\varepsilon_k, n_k, m_k)}.$$

We put

$$\Psi_{\Delta_k} = A^{-1}_{(\varepsilon_1, n_1, m_1)} \cdots A^{-1}_{(\varepsilon_k, n_k, m_k)} = \begin{pmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{pmatrix}$$

for any sequence $((\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \ldots, (\varepsilon_k, n_k, m_k))$, $k \geq 1$. Then it is easy to see that $p_i^{(k)}$ and $r_i^{(k)}$ are non-negative integers and $q_i^{(k)}$ is positive integer for $i = 1, 2, 3$, $k \geq 0$. Since

$$\left\{ \left(\frac{y}{(x + y) - 1}, \frac{x}{(x + y) - 1} \right) : (x, y) \in \mathbb{X}, \ x + y > 1 \right\}$$
$$= \left\{ \left(\frac{1 - y}{1 - (x + y)}, \frac{1 - x}{1 - (x + y)} \right) : (x, y) \in \mathbb{X}, \ x + y < 1 \right\}$$
$$= \{(x', y') : x' \geq 1, \ y' \geq 1\},$$

we see that

$$T^j \{(x, y) \in \mathbb{X} : \varepsilon_k(x, y) = \varepsilon_k, \ n_k(x, y) = n_k, \ m_k(x, y) = m_k, \ 1 \leq k \leq j\} = \mathbb{X}$$

for any $\{(\varepsilon_k, n_k, m_k), \ 1 \leq k \leq j\}$, $\varepsilon_k = \pm 1$, $n_k, m_k \geq 1$ without the boundary of $\mathbb{X}$.

In the following, we show some properties for entries of Ψ_{Δ_k}. After that we give the conditions for $(x, y) \in \mathbb{X}$ when iteration by the negative slope algorithm of (x, y) stops. Finally, we give the condition for existence of an expansion of $(x, y) \in \mathbb{X}$ by the negative slope algorithm.

2.2. Properties of the negative slope algorithm.

Lemma 2.1. *For entries of Ψ_{Δ_k}, we have*

$$\begin{cases} p_1^{(k)} = p_2^{(k)} + \varepsilon_1 \cdots \varepsilon_k \\ r_1^{(k)} = r_2^{(k)} - \varepsilon_1 \cdots \varepsilon_k \\ q_1^{(k)} = q_2^{(k)}. \end{cases}$$

Proof. By simple calculation, we see that

$$\begin{pmatrix} 1 & 0 & n-1 \\ 0 & 1 & m-1 \\ 1 & 1 & n+m-1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (+1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & m \\ 1 & 0 & n \\ 1 & 1 & n+m-1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

So we have

$$A_{(\varepsilon,n,m)}^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \varepsilon \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Then we see that

$$A_{(\varepsilon_1,n_1,m_1)}^{-1} \cdots A_{(\varepsilon_k,n_k,m_k)}^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \varepsilon_1 \cdots \varepsilon_k \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

for $k \geq 1$. Therefore, we obtain

$$\begin{pmatrix} p_1^{(k)} - p_2^{(k)} \\ r_1^{(k)} - r_2^{(k)} \\ q_1^{(k)} - q_2^{(k)} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \cdots \varepsilon_k \\ -\varepsilon_1 \cdots \varepsilon_k \\ 0 \end{pmatrix}. \qquad \square$$

We next give an approximation of $(x + y)$ for $x, y \in \mathbb{X}$.

Lemma 2.2. *For x, $y \in \mathbb{X}$, we have*

$$\left| (x + y) - \left(\frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} \right) \right| < \frac{1}{q_2^{(k)} q_3^{(k)}}.$$

Proof. By taking a determinant of Ψ_{Δ_k}, we have

$$\begin{vmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{vmatrix} = p_1^{(k)} \begin{vmatrix} r_2^{(k)} & r_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} - p_2^{(k)} \begin{vmatrix} r_1^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} + p_3^{(k)} \begin{vmatrix} r_1^{(k)} & r_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix}.$$

From Lemma 2.1, the right hand side is equal to

$$\left(p_2^{(k)} + \delta_k \right) \begin{vmatrix} r_2^{(k)} & r_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} - p_2^{(k)} \begin{vmatrix} r_2^{(k)} - \delta_k & r_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} + p_3^{(k)} \begin{vmatrix} r_2^{(k)} - \delta_k & r_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix}$$

where $\delta_k = \varepsilon_1 \cdots \varepsilon_k$. Since $\det \Psi_{\Delta_k} = 1$, we have

$$(1) \qquad \left(r_2^{(k)} q_3^{(k)} - r_3^{(k)} q_2^{(k)} \right) + \left(p_2^{(k)} q_3^{(k)} - p_3^{(k)} q_2^{(k)} \right) = 1.$$

Substituting $p_1^{(k)} = p_2^{(k)} + \delta_k$, $r_1^{(k)} = r_2^{(k)} - \delta_k$ and $q_1^{(k)} = q_2^{(k)}$ for (1), we see that

$$(2) \qquad \left(r_1^{(k)} q_3^{(k)} - r_3^{(k)} q_1^{(k)} \right) + \left(p_1^{(k)} q_3^{(k)} - p_3^{(k)} q_1^{(k)} \right) = 1.$$

From (1) and (2), we have

$$(3) \qquad \frac{p_1^{(k)} + r_1^{(k)}}{q_1^{(k)}} = \frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} = \frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} + \frac{1}{q_2^{(k)} q_3^{(k)}}.$$

For $(x, y) \in \mathbb{X}$, we put $(x_k, y_k) = T^k(x, y)$, $k \geq 1$. Then we see that

$$\begin{pmatrix} \alpha x \\ \alpha y \\ \alpha \end{pmatrix} = \begin{pmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \\ 1 \end{pmatrix}$$

for $\alpha \neq 0$. Then we obtain

$$(4) \qquad x = \frac{p_1^{(k)} x_k + p_2^{(k)} y_k + p_3^{(k)}}{q_1^{(k)} x_k + q_2^{(k)} y_k + q_3^{(k)}},$$

$$(5) \qquad y = \frac{r_1^{(k)} x_k + r_2^{(k)} y_k + r_3^{(k)}}{q_1^{(k)} x_k + q_2^{(k)} y_k + q_3^{(k)}}.$$

Since $q_3^{(k)} > 0$ for $k \geq 1$, the denominators of the above two equations are not equal to 0. From $p_1^{(k)} = p_2^{(k)} + \delta_k$, $r_1^{(k)} = r_2^{(k)} - \delta_k$ and $q_1^{(k)} = q_2^{(k)}$, we have

$$(6) \qquad x + y = \frac{(p_2^{(k)} + r_2^{(k)})(x_k + y_k) + (p_3^{(k)} + r_3^{(k)})}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}}.$$

Then we see the following.

$$\left| (x + y) - \left(\frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} \right) \right|$$

$$= \left| \frac{(p_2^{(k)} + r_2^{(k)})(x_k + y_k) + (p_3^{(k)} + r_3^{(k)})}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}} - \left(\frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} \right) \right|$$

$$= \left| \left(\frac{p_2^{(k)} + r_2^{(k)}}{q_2^{(k)}} + \frac{-(q_3^{(k)}/q_2^{(k)})(p_2^{(k)} + r_2^{(k)}) + (p_3^{(k)} + r_3^{(k)})}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}} \right) - \left(\frac{p_3^{(k)} + r_3^{(k)}}{q_3^{(k)}} \right) \right|.$$

From (1) and (3), we see that the second line is equal to

$$\left| \frac{1}{q_2^{(k)} q_3^{(k)}} - \frac{1}{q_2^{(k)}} \frac{1}{q_2^{(k)}(x_k + y_k) + q_3^{(k)}} \right| < \left| \frac{1}{q_2^{(k)} q_3^{(k)}} \right|.$$

This is the assertion of this lemma. □

From (3) and from this lemma, we deduce the following approximations.

$$(7) \qquad \left| (x + y) - \frac{p_i^{(k)} + r_i^{(k)}}{q_i^{(k)}} \right| < \frac{2}{q_2^{(k)} q_3^{(k)}}$$

for $i = 1, 2$.

Lemma 2.3. *We put* $(x_k, y_k) = T^k(x, y)$, $k \geq 1$ *for* $(x, y) \in \mathbb{X}$ *and have*

$$x_k + y_k = -\frac{q_3^{(k)}}{q_2^{(k)}} \frac{(x + y) - (p_3^{(k)} + r_3^{(k)})/q_3^{(k)}}{(x + y) - (p_2^{(k)} + r_2^{(k)})/q_2^{(k)}}$$

where $p_i^{(k)}$, $q_i^{(k)}$ *and* $r_i^{(k)}$, $i = 1, 2, 3$, *are entries of* Ψ_{Δ_k}.

Proof. We consider the inverse of Ψ_{Δ_k}. We put $(x_k, y_k) = T^k(x, y)$, $k \geq 1$ for $(x, y) \in \mathbb{X}$. Then we have

$$\begin{pmatrix} \alpha x_k \\ \alpha y_k \\ \alpha \end{pmatrix} = \begin{pmatrix} p_1^{(k)} & p_2^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_2^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_2^{(k)} & q_3^{(k)} \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

for $\alpha \neq 0$. By taking the cofactor matrix of Ψ_{Δ_k}, the inverse is equal to

$$\Psi_{\Delta_k}^{-1} = \begin{pmatrix} \begin{vmatrix} r_2^{(k)} & r_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} & -\begin{vmatrix} p_2^{(k)} & p_3^{(k)} \\ q_2^{(k)} & q_3^{(k)} \end{vmatrix} & \begin{vmatrix} p_2^{(k)} & p_3^{(k)} \\ r_2^{(k)} & r_3^{(k)} \end{vmatrix} \\ -\begin{vmatrix} r_1^{(k)} & r_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} & \begin{vmatrix} p_1^{(k)} & p_3^{(k)} \\ q_1^{(k)} & q_3^{(k)} \end{vmatrix} & -\begin{vmatrix} p_1^{(k)} & p_3^{(k)} \\ r_1^{(k)} & r_3^{(k)} \end{vmatrix} \\ \begin{vmatrix} r_1^{(k)} & r_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix} & -\begin{vmatrix} p_1^{(k)} & p_2^{(k)} \\ q_1^{(k)} & q_2^{(k)} \end{vmatrix} & \begin{vmatrix} p_1^{(k)} & p_2^{(k)} \\ r_1^{(k)} & r_2^{(k)} \end{vmatrix} \end{pmatrix}.$$

Then we have

$$(8) \qquad x_k = \frac{(r_2^{(k)}q_3^{(k)} - r_3^{(k)}q_2^{(k)})x + (-p_2^{(k)}q_3^{(k)} + p_3^{(k)}q_2^{(k)})y + (p_2^{(k)}r_3^{(k)} - p_3^{(k)}r_2^{(k)})}{(r_1^{(k)}q_2^{(k)} - r_2^{(k)}q_1^{(k)})x + (-p_1^{(k)}q_2^{(k)} + p_2^{(k)}q_1^{(k)})y + (p_1^{(k)}r_2^{(k)} - p_2^{(k)}r_1^{(k)})},$$

$$y_k = \frac{(-r_1^{(k)}q_3^{(k)} + r_3^{(k)}q_1^{(k)})x + (p_1^{(k)}q_3^{(k)} - p_3^{(k)}q_1^{(k)})y + (-p_1^{(k)}r_3^{(k)} + p_3^{(k)}r_1^{(k)})}{(r_1^{(k)}q_2^{(k)} - r_2^{(k)}q_1^{(k)})x + (-p_1^{(k)}q_2^{(k)} + p_2^{(k)}q_1^{(k)})y + (p_1^{(k)}r_2^{(k)} - p_2^{(k)}r_1^{(k)})}.$$

By Lemma 2.1, we have the following.

$$x_k + y_k = \frac{(\varepsilon_1 \cdots \varepsilon_k)q_3^{(k)}x + (\varepsilon_1 \cdots \varepsilon_k)q_3^{(k)}y - (\varepsilon_1 \cdots \varepsilon_k)(p_3^{(k)} + r_3^{(k)})}{-(\varepsilon_1 \cdots \varepsilon_k)q_2^{(k)}x - (\varepsilon_1 \cdots \varepsilon_k)q_2^{(k)}y + (\varepsilon_1 \cdots \varepsilon_k)(p_2^{(k)} + r_2^{(k)})}$$

$$= -\frac{q_3^{(k)}}{q_2^{(k)}} \frac{(x + y) - (p_3^{(k)} + r_3^{(k)})/q_3^{(k)}}{(x + y) - (p_2^{(k)} + r_2^{(k)})/q_2^{(k)}}. \qquad \square$$

2.3. The case where the negative slope algorithm stops. Next we define what means that the iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ stops.

DEFINITION 2.4. We denote k-th iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ as $(x_k, y_k) = T^k(x, y)$. Then we say iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ stops if there exists $k_0 \geq 0$ such that $x_{k_0} = 0$ or $y_{k_0} = 0$ or $x_{k_0} + y_{k_0} = 1$.

This implies that iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ stops if there exists $k_0 \geq 0$ s.t. $(x_{k_0}, y_{k_0}) \in \partial \mathbb{X}$. From this definition, we get the following propositions.

Proposition 2.5. *If iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ stops, then (x, y) satisfies one of the following equations.*

$$(p + 1)x + py = q,$$
$$px + (p + 1)y = q,$$
$$px + py = q$$

for some integers $0 \leq q \leq 2p$.

Proof. We put k-th iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ as $(x_k, y_k) = T^k(x, y)$. Suppose $x_k = 0$, then we get the following equation from (8) in Lemma 2.3.

$$0 = \left(r_2^{(k)} q_3^{(k)} - r_3^{(k)} q_2^{(k)} \right) x + \left(-p_2^{(k)} q_3^{(k)} + p_3^{(k)} q_2^{(k)} \right) y + \left(p_2^{(k)} r_3^{(k)} - p_3^{(k)} r_2^{(k)} \right).$$

By (1) in Lemma 2.2, we obtain

$$(p + 1)x + py = q$$

where $p = -p_2^{(k)} q_3^{(k)} + p_3^{(k)} q_2^{(k)}$, $q = -p_2^{(k)} r_3^{(k)} + p_3^{(k)} r_2^{(k)}$. Since

$$x = \frac{q - py}{p + 1} \in [0, 1],$$

we see that the following two cases.
(i) If $0 \leq q - py \leq p + 1$, then

$$0 \leq py \leq q \leq (y + 1)p + 1 \leq 2p + 1.$$

(ii) If $0 \geq q - py \geq p + 1$, then

$$0 \geq py \geq q \geq (y + 1)p + 1 \geq 2p + 1 > 2p.$$

Similarly, we obtain $px + (p + 1)y = 0$ $(0 \leq q \leq 2p)$ for $y_k = 0$.
Suppose $x_k + y_k = 1$, then we get the following by (6) in Lemma 2.2.

$$\left(q_2^{(k)} + q_3^{(k)} \right)(x + y) = \left(p_2^{(k)} + p_3^{(k)} \right) + \left(r_2^{(k)} + r_3^{(k)} \right).$$

Since $0 \leq x + y \leq 2$, we complete the proof. $\qquad\square$

In the following, we show the sufficient condition for the third equation in Proposition 2.5.

Proposition 2.6. *If $(x, y) \in \mathbb{X}$ satisfies the following equation,*

$$px + py = q$$

for any integers $0 \leq q \leq 2p$, then there exists $N > 0$ such that the sequence $(T^k(x,y): k \geq 0)$ terminates at $k = N$ for the negative slope algorithm T.

Proof. Suppose $|x + y - 1| = t_1/t_0 < 1$ for $(x, y) \in \mathbb{X}$ where t_1/t_0 is an irreducible fraction. Then by the negative slope algorithm, we see that

$$|x_1 + y_1 - 1| = \left| \frac{t_0}{t_1} - (n_1(x, y) + m_1(x, y)) \right| = \left| \frac{t_2}{t_1} \right| < 1$$

where t_2/t_1 is also an irreducible fraction. Recursively, we get

$$|x_{i+1} + y_{i+1} - 1| = \left| \frac{t_i}{t_{i+1}} - (n_{i+1}(x, y) + m_{i+1}(x, y)) \right| = \left| \frac{t_{i+2}}{t_{i+1}} \right| < 1$$

where t_{i+2}/t_{i+1} is an irreducible fraction. Since $(|t_i|: i \geq 0)$ is a decreasing integer sequence, there exists $N > 0$ s.t. $t_N = 0$. This implies that the sequence $(T^k(x, y): k \geq 0)$ stops at $k = N - 1$. $\qquad\square$

Finally, we give the last lemma of this section. This lemma shows that the condition for existence of an expansion of $(x, y) \in \mathbb{X}$ by the negative slope algorithm.

Lemma 2.7. *For $n_i, m_i \geq 1$, $i \geq 1$ and for any sequence $((\varepsilon_i, n_i, m_i),\ i \geq 1)$, there exists $(x, y) \in \mathbb{X}$ such that $(\varepsilon_i(x, y), n_i(x, y), m_i(x, y)) = (\varepsilon_i, n_i, m_i)$ unless there exists $k \geq 1$ such that $(\varepsilon_i, m_i) = (+1, 1)$ for $i \geq k$ or $(\varepsilon_i, n_i) = (+1, 1)$ for $i \geq k$.*

Proof. Suppose there exists $(x, y) \in \mathbb{X}$ satisfying $(\varepsilon_i(x, y), n_i(x, y)) = (+1, 1)$ for all $i \geq 1$. Then the negative slope expansion of (x, y) is $((+1, 1, m_1), (+1, 1, m_2), \dots)$ for $m_i \geq 1$, $i \geq 1$. We see that

$$A_{(+1,1,m_1)}^{-1} \cdot A_{(+1,1,m_2)}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m_1 - 1 \\ 1 & 1 & m_1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m_2 - 1 \\ 1 & 1 & m_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ m_1 - 1 & m_1 - 1 & m_1(m_2 - 1) \\ m_1 + 1 & m_1 + 1 & 2m_2 - 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_1^{(i+1)} & p_2^{(i+1)} & p_3^{(i+1)} \\ r_1^{(i+1)} & r_2^{(i+1)} & r_3^{(i+1)} \\ q_1^{(i+1)} & q_2^{(i+1)} & q_3^{(i+1)} \end{pmatrix} = \begin{pmatrix} p_1^{(i)} & p_2^{(i)} & p_3^{(i)} \\ r_1^{(i)} & r_2^{(i)} & r_3^{(i)} \\ q_1^{(i)} & q_2^{(i)} & q_3^{(i)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m_{i+1} - 1 \\ 1 & 1 & m_{i+1} \end{pmatrix}$$

$$= \begin{pmatrix} p_1^{(i)} + p_3^{(i)} & p_2^{(i)} + p_3^{(i)} & (m_{i+1} - 1)p_2^{(i)} + (m_{i+1})p_3^{(i)} \\ r_1^{(i)} + r_3^{(i)} & r_2^{(i)} + r_3^{(i)} & (m_{i+1} - 1)r_2^{(i)} + (m_{i+1})r_3^{(i)} \\ q_1^{(i)} + q_3^{(i)} & q_2^{(i)} + q_3^{(i)} & (m_{i+1} - 1)q_2^{(i)} + (m_{i+1})q_3^{(i)} \end{pmatrix}$$

for $i \geq 2$. If we put $p_1^{(i)} = 1$, $p_2^{(i)} = p_3^{(i)} = 0$, then we have $p_1^{(i+1)} = 1$, $p_2^{(i+1)} = p_3^{(i+1)} = 0$. Therefore we obtain $p_1^{(i)} = 1$, $p_2^{(i)} = p_3^{(i)} = 0$ for all $i \geq 1$. From (4) in Lemma 2.2, we see that

$$x = \frac{x_k}{q_1^{(k)} x_k + q_2^{(k)} y_k + q_3^{(k)}}$$

where (x_k, y_k) is k-th iteration by the negative slope algorithm. Since $q_i^{(k)}$, $i = 1, 2, 3$ are increasing integer sequences and $(x_k, y_k) \in \mathbb{X}$, this implies that $x = 0$. Then it is the contradiction to Definition 2.4. $\qquad\square$

3. The natural extension of the negative slope algorithm and characterization of the periodic points

3.1. Necessary part of Main Theorem. In this section, we introduce the 4-dimensional map $\bar{T}$ which is called the natural extension of the negative slope algorithm T. This map has been given in [4]. It was defined as the natural extension of the negative slope algorithm on $\mathbb{R}^4$ as follows. Let $\bar{\mathbb{X}} = \mathbb{X} \times (-\infty, 0)^2$. For $(x, y, z, w) \in \bar{\mathbb{X}}$, we define a map $\bar{T}$ on $\bar{\mathbb{X}}$ by

$$\bar{T}(x, y, z, w)$$
$$= \begin{cases} \left(\dfrac{y}{(x+y)-1} - n(x,y), \dfrac{x}{(x+y)-1} - m(x,y), \dfrac{w}{(z+w)-1} - n(x,y), \dfrac{z}{(z+w)-1} - m(x,y) \right) \\ \hspace{8cm} \text{if} \quad x+y > 1 \\[2mm] \left(\dfrac{1-y}{1-(x+y)} - n(x,y), \dfrac{1-x}{1-(x+y)} - m(x,y), \dfrac{1-w}{1-(z+w)} - n(x,y), \dfrac{1-z}{1-(z+w)} - m(x,y) \right) \\ \hspace{8cm} \text{if} \quad x+y < 1. \end{cases}$$

Then it is easy to see that $\bar{T}$ is bijective on $\bar{\mathbb{X}}$ except for the boundary of $\bar{\mathbb{X}}$. We show our Main Theorem in this section. After the proof of Main Theorem, we show Theorem 1.1 as a corollary of Main Theorem.

Theorem 3.1 (Main Theorem). *Suppose iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ does not stop. Then the sequence $(T^k(x, y): k \geq 0)$ is purely periodic if and only if x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$ where x^* denotes the algebraic conjugate of x.*

We start with proving the necessary condition of Main Theorem. First, we show that if iteration by T of $(x, y) \in \mathbb{X}$ is purely periodic, then there exists $d > 0$ s.t. x, y are in $\mathbb{Q}(\sqrt{d})$. After that, we show that iteration by $\bar{T}$ of $(x, y, z, w) \in \mathbb{X} \times \mathbb{R}^2 \setminus \{(x, y, z, w) \mid z + w = x + y, (z, w) \in \mathbb{X}\}$ goes into $\bar{\mathbb{X}}$.

Lemma 3.2. *Suppose iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ does not stop. Then if the sequence $(T^k(x, y): k \geq 0)$ is purely periodic, there exists $d > 0$ such that x and y are in $\mathbb{Q}(\sqrt{d})$.*

Proof. Suppose the sequence $(T^k(x, y): k \geq 0)$ is purely periodic for $(x, y) \in \mathbb{X}$ by the negative slope algorithm T, then there exists $l > 0$ such that $T^l(x, y) = (x, y) \in \mathbb{X}$. From (6) in Lemma 2.2, we see that

$$x + y = \frac{(p_2^{(l)} + r_2^{(l)})(x + y) + (p_3^{(l)} + r_3^{(l)})}{q_2^{(l)}(x + y) + q_3^{(l)}}.$$

Then we have the following quadratic equation with respect to $(x + y)$.

$$q_2^{(l)}(x + y)^2 + (q_3^{(l)} - p_2^{(l)} - r_2^{(l)})(x + y) - (p_3^{(l)} + r_3^{(l)}) = 0.$$

We see that the discriminant d of this equation satisfies

$$d = (q_3^{(l)} - p_2^{(l)} - r_2^{(l)})^2 + 4q_2^{(l)}(p_3^{(l)} + r_3^{(l)}) > 0.$$

Note that the discriminant $d > 0$ is not a square number. In fact, suppose d is a square number, then we see that $x + y \in \mathbb{Q}$. It implies that there exists $N > 0$ such that the sequence $(T^k(x, y): k \geq 0)$ stops at $k = N$ by Proposition 2.6. This contradicts the fact that the sequence $(T^k(x, y): k \geq 0)$ is purely periodic. Therefore, d is not a square number and $x + y \in \mathbb{Q}(\sqrt{d})$. From Lemma 2.1, (4) and (5), we have

$$x = \frac{p_2^{(k)}(x + y) + p_3^{(k)}}{q_2^{(k)}(x + y) + q_3^{(k)} - (\varepsilon_1 \cdots \varepsilon_k)},$$

$$y = \frac{r_2^{(k)}(x + y) + r_3^{(k)}}{q_2^{(k)}(x + y) + q_3^{(k)} + (\varepsilon_1 \cdots \varepsilon_k)}.$$

This is the assertion of this lemma.   □

In the following, we put $(x_k, y_k, z_k, w_k) = \bar{T}^k(x, y, z, w)$, $k \geq 0$ for the natural extension $\bar{T}$ of the negative slope algorithm T. We show that if iteration by T of (x, y) does not stop for $(x, y, z, w) \in \mathbb{X} \times \mathbb{R}^2 \setminus \{(x, y, z, w) \mid z + w = x + y, (z, w) \in \mathbb{X}\}$, then $\exists k_0 > 0$ s.t. $(z_k, w_k) \in (-\infty, 0)^2$ for $k > k_0$. This will yield the necessary condition of Main Theorem.

Lemma 3.3. *Let* $(x, y, z, w) \in \mathbb{X} \times \mathbb{R}^2 \setminus \{(x, y, z, w) \mid z + w = x + y, \ (z, w) \in \mathbb{X}\}$ *and* $(x_k, y_k, z_k, w_k) = \bar{T}^k(x, y, z, w)$, $k \geq 0$ *for the natural extension* $\bar{T}$ *of the negative slope algorithm* T. *Suppose the sequence* $(T^k(x, y) \colon k \geq 0)$ *does not terminate at any finite number* k. *Then there exists* $k_0 > 0$ *such that* $z_k + w_k < 0$ *for* $k > k_0$.

Proof. Suppose the sequence $(T^k(x, y) \colon k \geq 0)$ for $(x, y) \in \mathbb{X}$ does not terminate at any finite number k. Then from Lemma 2.3, we have

$$z_k + w_k = -\frac{q_3^{(k)}}{q_2^{(k)}} \frac{(z + w) - \left(p_3^{(k)} + r_3^{(k)}\right)/q_3^{(k)}}{(z + w) - \left(p_2^{(k)} + r_2^{(k)}\right)/q_2^{(k)}}$$

where $(x_k, y_k, z_k, w_k) = \bar{T}^k(x, y, z, w)$. According to Lemma 2.2 and (7) and $q_i^{(k)} > 0$, $i = 2, 3$, the right hand side converges to $-q_3^{(k)}/q_2^{(k)}$. Then there exists $k_0 > 0$ s.t.

$$z_k + w_k < 0$$

for $k > k_0$. This is the assertion of this lemma. $\qquad\square$

Note that it follows from the definition of $\bar{T}$ that if $z + w = x + y$, then one has that $z_k + w_k = x_k + y_k$ for all $k \geq 0$. From Lemma 2.2 and (7), it is not clear whether the right hand side of the above equation converges. However, we only consider periodic orbits of the sequence $(\bar{T}^k(x, y, z, w) \colon k \geq 0)$. In fact, it is easy to see that the sequence $(\bar{T}^k(x, y, z, w) \colon k \geq 0)$ is not periodic for $z + w = x + y$ even if the sequence $(T^k(x, y) \colon k \geq 0)$ is periodic. Consequently, we can ignore $\{(x, y, z, w) \in \mathbb{X} \times \mathbb{R}^2 \mid z + w = x + y\}$.

The next lemma shows that $\{(z, w) \mid z + w < 0\}$ goes into $(-\infty, 0)^2$ by iterating $\bar{T}$.

Lemma 3.4. *Suppose iteration by the negative slope algorithm* T *of* $(x, y) \in \mathbb{X}$ *does not stop. Then the negative slope expansion of* (x, y) *is an infinite sequence* $((\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \dots)$. *We put*

$$\Lambda_k = \{(z, w) \mid z + w < 0, \ n(z, w) = k \ \text{or} \ m(z, w) = k\}$$

and

$$l_N = \sharp\{l \mid (\varepsilon_l, n_l) \neq (+1, 1) \ \text{or} \ (\varepsilon_l, m_l) \neq (+1, 1), \ l \leq N\}.$$

Then for $(x, y, z, w) \in \mathbb{X} \times \Lambda_k$, *there exists* $N > 0$ *such that for* $l_N > k$,

$$\bar{T}^N(x, y, z, w) \in \bar{\mathbb{X}}.$$

Proof. We know that $\bar{\mathbb{X}}$ is $\bar{T}$-invariant by [4]. Then it is enough to show that the shaded two areas called upper area and lower area at Fig. 1 go into $(-\infty, 0)^2$ by iterating $\bar{T}$.
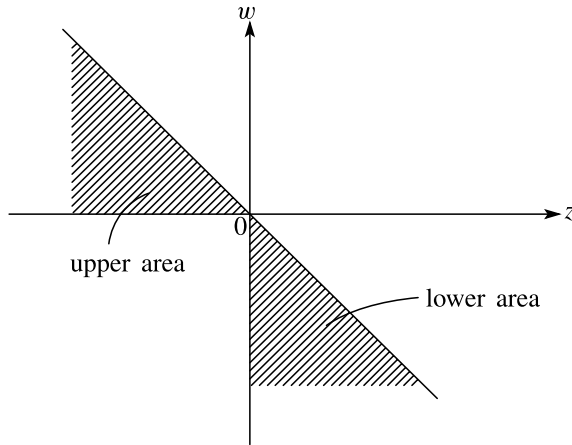
Fig. 1.

We start with lower area at Fig. 1. Suppose iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ does not stop, then (x, y) has an infinite expansion by the negative slope algorithm

$$((\varepsilon_1, n_1, m_1), (\varepsilon_2, n_2, m_2), \ldots, (\varepsilon_l, n_l, m_l), \ldots).$$

In the following, we consider the two partitions $\left\{ \Lambda_k^{(+)} \mid k = 1, 2, \ldots \right\}$ and $\left\{ \Lambda_k^{(-)} \mid k = 1, 2, \ldots \right\}$ in lower area. First, we define the partition $\Lambda_k^{(\pm)}$ and $f_\pm(z, w)$ as follows.
(i) For $\varepsilon_1 = +1$, we define

$$\Lambda_k^{(+)} := \left\{ (z, w) \,\middle|\, \left[\frac{1 - w}{1 - (z + w)} \right] = k \right\},$$

$$f_+(z, w) := \left(\frac{1 - w}{1 - (z + w)}, \frac{1 - z}{1 - (z + w)} \right).$$

(ii) For $\varepsilon_1 = -1$, we define

$$\Lambda_k^{(-)} := \left\{ (z, w) \,\middle|\, \left[\frac{w}{(z + w) - 1} \right] = k \right\},$$

$$f_-(z, w) := \left(\frac{w}{(z + w) - 1}, \frac{z}{(z + w) - 1} \right).$$

In the case of $\varepsilon_1 = +1$, we see that

$$\mathbb{T}\left(x, y, \Lambda_k^{(+)}\right) = \left(x_1, y_1, f_+\left(\Lambda_k^{(+)}\right) - (n_1, m_1)\right)$$

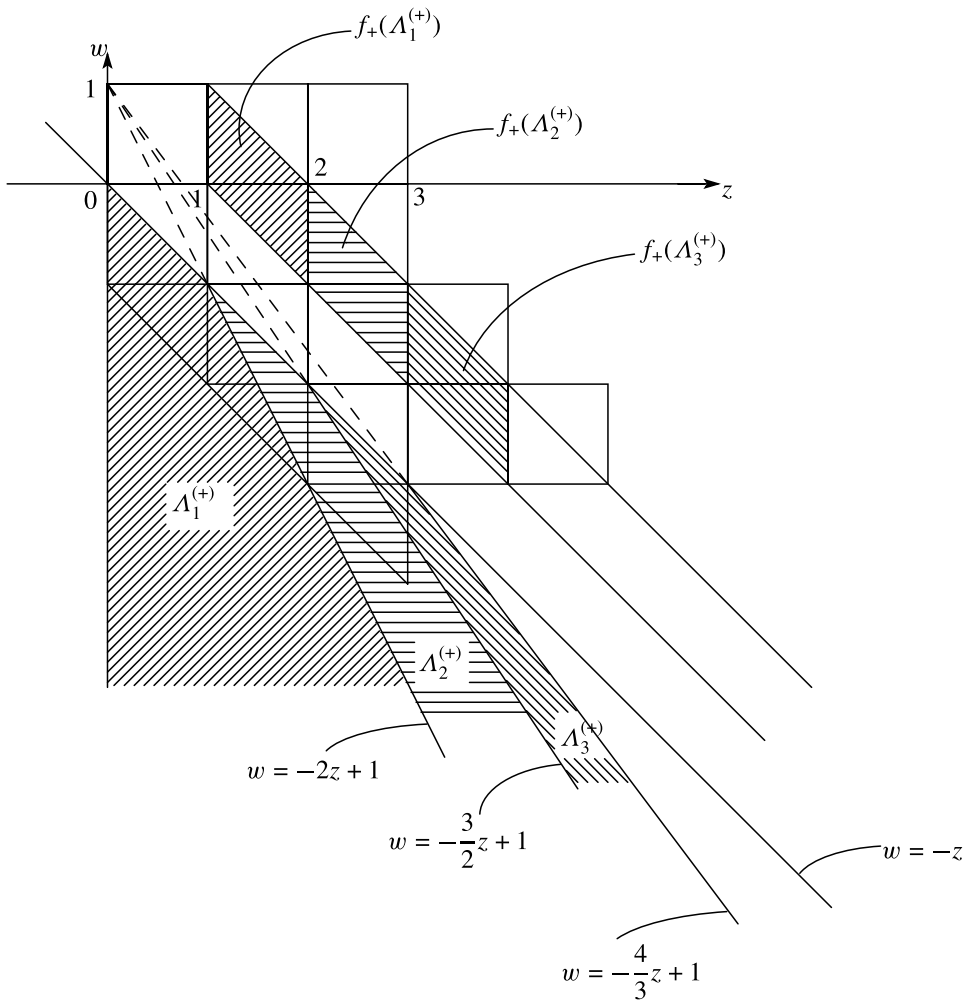(see Fig. 2). We have the following three cases for images by $\bar{T}$ of $\Lambda_k^{(+)}$.

Fig. 2.

(i-a) For $n_1 = 1$, we see that

$$f_+\big(\Lambda_k^{(+)}\big) - (1, m_1) \subset \Lambda_1^{(+)} \cup \Lambda_2^{(+)} \cup \cdots \cup \Lambda_k^{(+)} \quad \text{or} \quad \Lambda_1^{(-)} \cup \Lambda_2^{(-)} \cup \cdots \cup \Lambda_k^{(-)}.$$

(i-b) For $1 < n_1 < k + 1$, we see that

$$f_+\big(\Lambda_k^{(+)}\big) - (n_1, m_1) \subset \Lambda_1^{(+)} \cup \Lambda_2^{(+)} \cup \cdots \cup \Lambda_{k-n_1+1}^{(+)} \quad \text{or} \quad \Lambda_1^{(-)} \cup \Lambda_2^{(-)} \cup \cdots \cup \Lambda_{k-n_1+1}^{(-)}.$$

(i-c) For $n_1 \geq k + 1$, we see that

$$f_+\big(\Lambda_k^{(+)}\big) - (n_1, m_1) \subset (-\infty, 0)^2.$$

In the case of $\varepsilon_1 = -1$, we see that

$$\mathbb{T}\big(x, y, \Lambda_k^{(-)}\big) = \big(x_1, y_1, f_-\big(\Lambda_k^{(-)}\big) - (n_1, m_1)\big)$$

(see Fig. 3). We have the following two cases for images by $\bar{T}$ of $\Lambda_k^{(-)}$.
(ii-a) For $n_1 < k$, we see that

$$f_-\big(\Lambda_k^{(-)}\big) - (n_1, m_1) \subset \Lambda_1^{(+)} \cup \Lambda_2^{(+)} \cup \cdots \cup \Lambda_{k-n_1}^{(+)} \quad \text{or} \quad \Lambda_1^{(-)} \cup \Lambda_2^{(-)} \cup \cdots \cup \Lambda_{k-n_1}^{(-)}.$$

(ii-b) For $n_1 \geq k$, we see that

$$f_-\big(\Lambda_k^{(-)}\big) - (n_1, m_1) \subset (-\infty, 0)^2.$$

From (i) and (ii), we obtain

$$\big\{f_+\big(\Lambda_k^{(+)}\big) - (n_1, m_1)\big\} \subset \Lambda_1^{\pm} \cup \cdots \cup \Lambda_{k-1}^{(\pm)}$$

except for $(n_1, m_1) = (1, m_1)$ and

$$\big\{f_-\big(\Lambda_k^{(-)}\big) - (n_1, m_1)\big\} \subset \Lambda_1^{\pm} \cup \cdots \cup \Lambda_{k-1}^{(\pm)}$$

for any (n_1, m_1). From Lemma 2.7, there does not exist $l_0 > 0$ s.t. $(\varepsilon_l, n_l) = (+1, 1)$ or $(\varepsilon_l, m_l) = (+1, 1)$ for all $l > l_0$. Therefore, there exists $k_0 > 0$ such that $\Lambda_k^{(+)}$ and k-th iteration by $\bar{T}$ of $\Lambda_k^{(+)}$ are disjoint for $k > k_0$. So we put

$$l_N = \sharp\{l : (\varepsilon_l, n_l) \neq (+1, 1) \text{ or } (\varepsilon_l, m_l) \neq (+1, 1), l \leq N\}.$$

Then for $(z, w) \in \Lambda_k^{(\pm)}$, there exists $N > 0$ s.t. for $l_N > k$,

$$\mathbb{T}^N(x, y, z, w) \in \bar{\bar{\mathbb{X}}}.$$

It is the same as upper area in Fig. 1, which completes the proof. $\qquad\square$
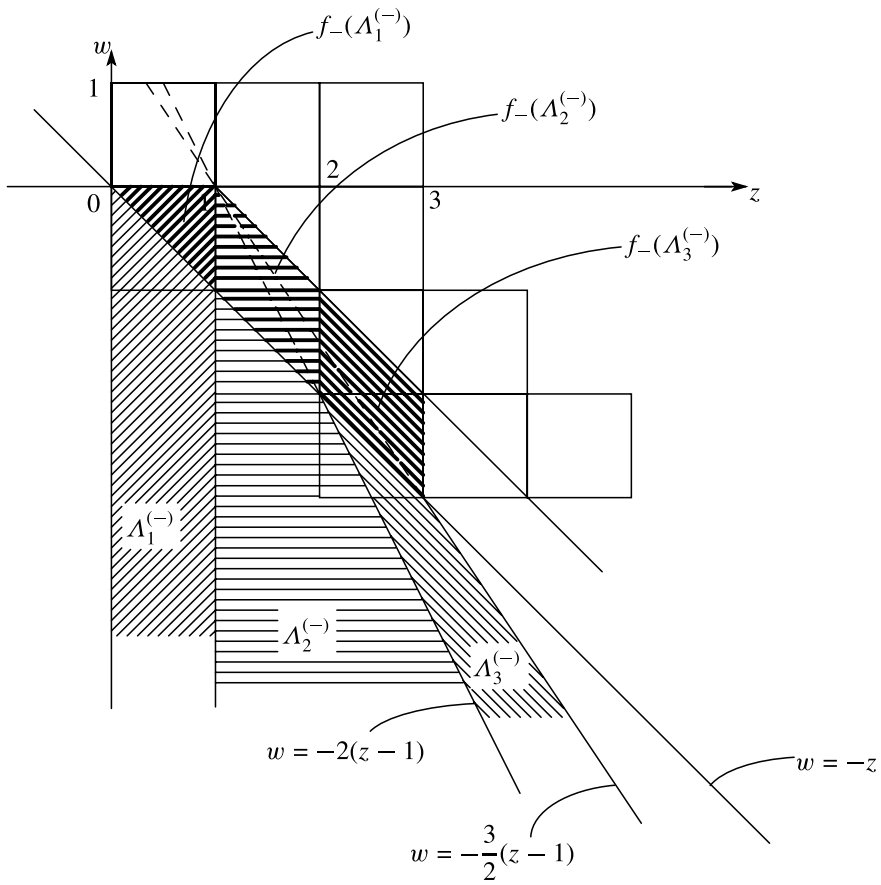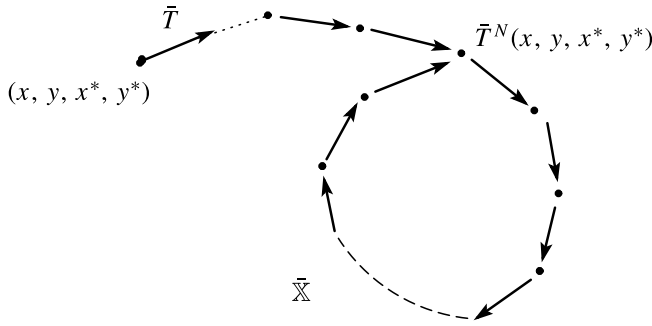
Fig. 3.

Fig. 4.

REMARK 3.5.   In the following proof, we use the fact that $\bar{T}^k(x, y, x^*, y^*) = (x_k, y_k, (x_k)^*, (y_k)^*)$. This is easy to show from the definition of the map $\bar{T}$.

$$\bar{T}(x,y,x^*,y^*)$$
$$= \begin{cases} \left(\dfrac{y}{(x+y)-1} - n(x,y), \dfrac{x}{(x+y)-1} - m(x,y), \dfrac{y^*}{(x^*+y^*)-1} - n(x,y), \dfrac{x^*}{(x^*+y^*)-1} - m(x,y) \right) \\ \hfill \text{if} \quad x+y > 1 \\[2mm] \left(\dfrac{1-y}{1-(x+y)} - n(x,y), \dfrac{1-x}{1-(x+y)} - m(x,y), \dfrac{1-y^*}{1-(x^*+y^*)} - n(x,y), \dfrac{1-x^*}{1-(x^*+y^*)} - m(x,y) \right) \\ \hfill \text{if} \quad x+y < 1. \end{cases}$$

Since $n(x, y)$ and $m(x, y)$ are positive integers, we see that for $x + y > 1$,

$$\frac{y^*}{(x^* + y^*) - 1} - n = \left(\frac{y}{(x + y) - 1} - n \right)^*,$$
$$\frac{x^*}{(x^* + y^*) - 1} - m = \left(\frac{x}{(x + y) - 1} - m \right)^*.$$

These are the same as $x + y < 1$. Then we obtain

$$\bar{T}(x, y, x^*, y^*) = (x_1, y_1, (x_1)^*, (y_1)^*).$$

Proof (necessary part of Main Theorem).   Suppose the sequence $(T^k(x, y): k \geq 0)$ is purely periodic for $(x, y) \in \mathbb{X}$ by the negative slope algorithm T. Then from Lemma 3.2, x and y are in the same quadratic extension of $\mathbb{Q}$. It is easy to see that $(\bar{T}^k(x, y, x^*, y^*): k \geq 0)$ is purely periodic if $(T^k(x, y): k \geq 0)$ is purely periodic, where x^* is the algebraic conjugate of x. We show this fact later as Remark 3.5. From Lemma 3.3 and Lemma 3.4, we see that there exists $N > 0$ such that $\bar{T}^N(x, y, x^*, y^*) \in \mathbb{X}$. Since $\bar{\mathbb{X}}$ is $\bar{T}$-invariant, we obtain that $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$ (see Fig. 4). $\square$

3.2. Sufficient part of Main Theorem. Next, we show the sufficient part of Main Theorem. Suppose x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$. Then we show that the number of $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$ is finite and the orbit of (x, y, x^*, y^*) by $\bar{T}$ is purely periodic. We start with some definitions for quadratic irrational numbers.

DEFINITION 3.6.   If α is a quadratic irrational number, then it satisfies the following quadratice equation:

$$ax^2 + bx + c = 0$$

where $a, b, c \in \mathbb{Z}$ and the great common measure (GCM) of a, b and c is equal to 1. Then we say

$$D = a^2(\alpha - \alpha^*)^2 = b^2 - 4ac$$

is the discriminant of α, where α^* is the algebraic conjugate of α. We also say that D is the discriminant of $ax^2 + bx + c$.

We denote the discriminant α and $f(x)$ as D_α and D_f, respectively.

DEFINITION 3.7.   The numbers α and α' are equivalent with respect to modular transformations if they satisfy

$$\alpha = \frac{s\alpha' + t}{u\alpha' + v}$$

where $s, t, u, v \in \mathbb{Z}$ and $sv - tu = \pm 1$.

From Definition 3.7, we deduce the following lemmas.

Lemma 3.8. *If α' is equivalent to a quadratic irrational number α with respect to modular transformations, then the discriminant of α' and α are equal.*

Proof.   Assume that α is a quadratic irrational number with a discriminant D and α' is equivalent to α w.r.t. modular transformations. Then we have the following:

(9) $$a\alpha^2 + b\alpha + c = 0, \quad GCM(a, b, c) = 1, \quad D = b^2 - 4ac.$$

(10) $$\alpha = \frac{s\alpha' + t}{u\alpha' + v}, \quad sv - ts = \pm 1.$$

From (9), (10) and simple calculation, we see that α' is the root of the following equation.

(11) $$(as^2 + bsv + cu^2)\alpha'^2 + (2ast + b(sv + tu) + 2cuv)\alpha' + (at^2 + btv + cv^2) = 0.$$

Then we obtain the discriminant D' of the above equation as follows.

$$D' = (2ast + b(sv + tu) + 2cuv)^2 - 4(as^2 + bsv + cu^2)(at^2 + btv + cv^2)$$
$$= b^2 - 4ac.$$

From assumption, $GCM(as^2 + bsv + cu^2, 2ast + b(sv + tu) + 2cuv, at^2 + btv + cv^2)$ is equal to 1. This is the assertion of this lemma. $\square$

Lemma 3.9. *The cardinality of quadratic equations $ax^2 + bx + c = 0$ with fixed discriminant where $a, b, c \in \mathbb{Z}$, $GCM(a, b, c) = 1$, $ac < 0$ is finite.*

Proof. Let D be the fixed discriminant of $ax^2 + bx + c = 0$ where $a, b, c \in \mathbb{Z}$, $GCM(a, b, c) = 1$, $ac < 0$. Then we see that

$$D = b^2 - 4ac = b^2 + 4|ac| > b^2.$$

This implies that b is bounded by D and the cardinality of a pair (a, c) is finite for each b. This is the assertion of this lemma. $\square$

Note that if α is the root of quadratic equations of Lemma 3.9, then the cardinality of such α is also finite.

Lemma 3.10. *Assume that α and β are in the same quadratic extension of $\mathbb{Q}$ and $(\alpha, \beta, \alpha^*, \beta^*) \in \bar{\mathbb{X}}$, then $D_{\alpha+\beta}$ is greater than D_α and D_β.*

Proof. From assumption, we see that

$$(\alpha, \alpha^*) = \left(\frac{-b + c\sqrt{\theta}}{a}, \frac{-b - c\sqrt{\theta}}{a} \right), \quad a, c > 0, \quad GCM(a, b, c) = 1,$$

$$(\beta, \beta^*) = \left(\frac{-q + r\sqrt{\theta}}{p}, \frac{-q - r\sqrt{\theta}}{p} \right), \quad p, r > 0, \quad GCM(p, q, r) = 1,$$

where θ does not contain square numbers as factors. Then α and β satisfy the following quadratic equations.

$$a^2 x^2 + 2abx + b^2 - c^2\theta = 0,$$
$$p^2 x^2 + 2pqx + q^2 - r^2\theta = 0.$$

On the other hand, $x + y$ satisfies

$$x + y = \frac{(-bp - aq) + (pc + ar)\sqrt{\theta}}{ap}.$$

Then the quadratic equation of $x + y$ is

$$a^2 p^2 (x+y)^2 + 2ap(bp+aq)(x+y) + (bp+aq)^2 - (pc+ar)^2\theta = 0.$$

It is enough to show the following four cases.
(1) If $GCM(a, b, p, q, \theta) = 1$, then we see that

$$D_{\alpha+\beta} = 4a^2 p^2 (pc+ar)^2, \quad D_\alpha = 4a^2 c^2 \theta, \quad D_\beta = 4p^2 r^2 \theta.$$

This implies that $D_{\alpha+\beta} > D_\alpha, D_\beta$.
(2) If $GCM(a, p, \theta) = i > 1$, $GCM(a, b, \theta) = 1$ and $GCM(p, q, \theta) = 1$, then we see that

$$D_{\alpha+\beta} = 4i^2 (a')^2 (p')^2 (p'c+a'r)^2 \theta, \quad D_\alpha = 4a^2 c^2 \theta, \quad D_\beta = 4p^2 r^2 \theta$$

where $a = ia'$ and $p = ip'$. This implies that $D_{\alpha+\beta} > D_\alpha, D_\beta$.
(3) If $GCM(a, b, p, \theta) = j > 1$ and $GCM(p, q, \theta) = 1$, then we see that

$$D_{\alpha+\beta} = 4j^2 (a')^2 (p')^2 (p'c+a'r)^2 \theta, \quad D_\alpha = 4(a')^2 c^2 \theta, \quad D_\beta = 4p^2 r^2 \theta$$

where $a = ja'$ and $p = jp'$. This implies that $D_{\alpha+\beta} > D_\alpha, D_\beta$.
(4) If $GCM(a, b, p, q, \theta) = l > 1$, then we see that

$$D_{\alpha+\beta} = 4(a')^2 (p')^2 (p'c+a'r)^2 \theta, \quad D_\alpha = 4(a')^2 c^2 \theta, \quad D_\beta = 4(p')^2 r^2 \theta$$

where $a = la'$ and $p = lp'$. This implies that $D_{\alpha+\beta} > D_\alpha, D_\beta$.
It is clear that $D_\alpha < 4a^2 c^2 \theta$, $D_\beta < 4p^2 r^2 \theta$ if $GCM(a, b, \theta) > 1$, $GCM(p, q, \theta) > 1$. This completes the proof. □

We give the last lemma to complete Main Theorem. We show that if $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$, then the sequence $(\bar{T}^k(x, y, x^*, y^*): k \geq 0)$ is purely periodic.

Lemma 3.11. *Suppose iteration by T of $(x, y) \in \mathbb{X}$ does not stop. Then the sequence $(\bar{T}^k(x, y, x^*, y^*): k \geq 0)$ is purely periodic if x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$, where x^* denotes the algebraic conjugate of x.*

Proof. If x and y are in the same quadratic extension of $\mathbb{Q}$ and $(x, y, x^*, y^*) \in \bar{\mathbb{X}}$, then we see that $x + y$ is equivalent to $x_k + y_k$, $k \geq 1$ w.r.t. the negative slope algorithm from (1) and (6). It implies that D_{x+y} is equal to $D_{x_k+y_k}$ by Lemma 3.8. From Lemma 3.10, D_{x_k} and D_{y_k} are bounded by D_{x+y} for each $k \geq 1$. This implies that the cardinality of (x_k, y_k) for $k \geq 0$ is finite from Lemma 3.9. Since $\bar{T}$ is bijective on $\bar{\mathbb{X}}$, there exists $l \geq 1$ s.t. for any $k > l$,

$$\mathbb{T}^k(x, y, x^*, y^*) = \mathbb{T}^{k+l}(x, y, x^*, y^*).$$

Note that (x, y, x^*, y^*) doesn't converge to the boundary of $\bar{\bar{\mathbb{X}}}$. Indeed, if $(x^*, y^*) = (z, 0)$ for $z < 0$, then we see that

$$
\bar{T}(z, 0) = \begin{cases} \left(\dfrac{1}{1 - z} - n, \dfrac{1 - z}{1 - z} - m \right) & \text{if} \quad \varepsilon = +1 \\[3mm] \left(\dfrac{0}{z - 1} - n, \dfrac{z}{z - 1} - m \right) & \text{if} \quad \varepsilon = -1. \end{cases}
$$

From Lemma 2.7, there does not exist $k_0 > 0$ such that $\bar{T}^k(z,0) \in \partial(-\infty, 0)^2$ for $k \geq k_0$.

Since $\mathbb{T}$ is bijective on $\bar{\bar{\mathbb{X}}}$, we see that

$$
\mathbb{T}^{k-1}(x, y, x^*, y^*) = \mathbb{T}^{k+l-1}(x, y, x^*, y^*).
$$

By induction, we obtain

$$
(x, y, x^*, y^*) = \mathbb{T}^l(x, y, x^*, y^*).
$$

This completes the proof of Main Theorem. □

Then we have the following corollary of Main Theorem.

Corollary 3.12 ([1]). *Suppose iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ does not stop. Then x and y are in the same quadratic extension of $\mathbb{Q}$ if and only if the sequence $(T^k(x, y): k \geq 0)$ is eventually periodic.*

Proof. Suppose iteration by the negative slope algorithm T of $(x, y) \in \mathbb{X}$ does not stop and x and y are in the same quadratic extension of $\mathbb{Q}$. Then from Lemma 3.4, there exists $N' > 0$ such that $\bar{T}^{N'}(x, y, x^*, y^*) = (x_{N'}, y_{N'}, (x_{N'})^*, (y_{N'})^*) \in \bar{\bar{\mathbb{X}}}$. Therefore we see that the sequence $(\bar{T}^k(x, y, x^*, y^*): k \geq N')$ is purely periodic by Main Theorem. It implies that the sequence $(T^k(x, y): k \geq 0)$ is eventually periodic. Conversely, if the sequence $(T^k(x, y): k \geq 0)$ is eventually periodic, then there exists $N > 0$ such that the sequence $(T^j(x_N, y_N): j \geq 0)$ is purely periodic. By Main Theorem, we see that x_N and y_N are in the same quadratic extension of $\mathbb{Q}$ and $(x_N, y_N, (x_N)^*, (y_N)^*)$ is in $\bar{\bar{\mathbb{X}}}$. This is the assertion of this corollary. □

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