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HYPERBOLIC LENGTHS OF SOME FILLING GEODESICS ON RIEMANN SURFACES WITH PUNCTURES

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Abstract

Let $\tilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$ and $n \geq 1$. In this paper, we give a quantitative common lower bound for the hyperbolic lengths of all filling geodesics on $\tilde{S}$ generated by two parabolic elements in the fundamental group $\pi_1(\tilde{S}, a)$.

1. Introduction

Let $\tilde{c}$ be a non-trivial closed curve on a Riemann surface $\tilde{S}$ of type (p, n) with $3p - 3 + n > 0$. The length function

$$l_{\tilde{c}} \colon T(\tilde{S}) \to \mathbb{R}^+$$

on the Teichmüller space $T(\tilde{S})$ is defined by sending each hyperbolic structure $\sigma = \sigma(\tilde{S})$ of $T(\tilde{S})$ to the hyperbolic length $l_{\tilde{c}}(\sigma)$ of the closed geodesic homotopic to $\tilde{c}$ on $\sigma(\tilde{S})$.

It is well known ([11]) that the function $l_{\tilde{c}}$ achieves its positive minimum value when $\tilde{c}$ is a filling curve on $\tilde{S}$ in the sense that every component of $\tilde{S} \setminus \{\tilde{c}\}$ is either a disk or a once punctured disk. The extremal value, of course, depends only on the homotopy class of $\tilde{c}$. In this paper, we give a quantitative common lower bound through $T(\tilde{S})$ for the hyperbolic lengths of certain kind of filling curves on $\tilde{S}$.

Note that if $\tilde{S}$ contains punctures, then some elements $[\alpha]$ in the fundamental group $\pi_1(\tilde{S}, a)$, $a \in \tilde{S}$, are represented by loops α that pass through a and are boundaries of once punctured disks. Let $\mathcal{F}$ denote the set of those elements. $\mathcal{F} \subset \pi_1(\tilde{S}, a)$.

The main result of this paper is the following:

Theorem 1. *Let $\tilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$ and $n \geq 1$. There are infinitely many homotopically independent filling curves $\tilde{c}$ on $\tilde{S}$ that can be expressed as products of two elements in $\mathcal{F}$. For each such $\tilde{c}$ and each hyperbolic structure σ on $\tilde{S}$, we have*:

$$l_{\tilde{c}}(\sigma) \geq 2 \log(\kappa^2 - 5) - 4 \log 2, \tag{1.1}$$

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where $\kappa = 16p + 8n - 21$ *if* $n \geq 3$; $16p + 3$ *if* $n = 2$; *and* $16p + 7$ *if* $n = 1$.

REMARK. From the definition of $\tilde{S}$ we know that $p \geq 0$ if $n \geq 4$; $p \geq 1$ if $n =$ 1, 2, 3.

Let $\mathbb{H} = \{z \in \mathbb{C};\ \operatorname{Im} z > 0\}$ denote the upper half plane with the hyperbolic metric $\rho_{\mathbb{H}}$ given by

$$\rho_{\mathbb{H}}(z)|dz| = \frac{|dz|}{\operatorname{Im} z}. \tag{1.2}$$

Let $\varrho\colon \mathbb{H} \to \tilde{S}$ be the universal covering with a covering group G. G is a torsion free Fichsian group of the fist kind of type (p, n) so that $\mathbb{H}/G \cong \tilde{S}$. The set $\mathcal{F}$ is one-to-one correspondent with the set of parabolic elements of G.

Note that any hyperbolic element g is conjugate in $\mathrm{PSL}_2(\mathbb{R})$ to $z \mapsto \lambda_g\, z$, where $\lambda_g > 1$ is called the multiplier of g. Let A_g be the axis of g. Using (1.2) one calculates the hyperbolic length of $\varrho(A_g)$ is $\log \lambda_g$.

The hyperbolic element g is called essential if every component of $\tilde{S} \setminus \varrho(A_g)$ is either a disk or possibly a once punctured disk. Theorem 1 can be restated as follows:

Theorem 1′. *Let G be a finitely generated Fuchsian group of the first kind of type (p, n) with $3p-3+n > 0$ and $n \geq 1$. There are infinitely many essential hyperbolic elements g of G that are generated by two parabolic elements of G. Furthermore, for each such element g, the multiplier λ_g of g satisfies*:

$$\lambda_g \geq \left\{\frac{1}{4}(\kappa^2 - 5)\right\}^2, \tag{1.3}$$

where κ is given in Theorem 1.

2. Dilatations of pseudo-Anosov maps generated by two positive multi-twists

We first recall the definition and some basic properties of Teichmüller space $T(R)$ of a Riemann surface R of type (p, n), $3p - 3 + n > 0$. For more details see [3, 4, 8].

We define an equivalence class $[\sigma]$ of a conformal structure σ on $\tilde{S}$ as follows. Two conformal structures σ_1 and σ_2 on R are called strongly equivalent if there is an isometry h of $\sigma_1(R)$ onto $\sigma_2(R)$ such that $\sigma_2^{-1} \circ h \circ \sigma_1$, as a self-map of the underlying surface R, is isotopic to the identity. The collection of strong equivalence classes $[\sigma]$ of conformal structures σ form a Teichmüller space $T(R)$. $T(R)$ is naturally equipped with a complex structure.

Let $[\sigma_1]$ and $[\sigma_2]$ be two points in $T(R)$. Let $h\colon \sigma_1(R) \to \sigma_2(R)$ be a quasi-conformal mapping. Define the complex dilation $\nu(z) = \partial_{\bar{z}}h(z)/\partial_z h(z)$ and denote $\|\nu\| =$

ess.sup$\{|\nu(z)|;\ z \in \sigma_1(R)\}$. By definition, $\|\nu\| < 1$. The maximal dilatation $K(h)$ is defined by

$$K(h) = \frac{1 + \|\nu\|}{1 - \|\nu\|}.$$

The Teichmüller distance between $[\sigma_1]$ and $[\sigma_2]$ is defined as

$$\langle [\sigma_1], [\sigma_2] \rangle = \frac{1}{2} \inf \log K(h),$$

where $h\colon \sigma_1(R) \to \sigma_2(R)$ runs over all maps homotopic to $\sigma_2 \circ \sigma_1^{-1}$. $T(R)$ is a complete metric space with respect to the Teichmüller metric.

The mapping class group (or modular group) Mod_R is the group of isotopy classes f^* of self-maps f of R. f^* acts on $T(R)$ by sending each $[\sigma]$ to $[\sigma \circ f^{-1}]$. Mod_R is a group of isometries with respect to the Teichmüller metric defined above.

An element f^* of Mod_R is called hyperbolic if $\langle [\sigma], f^*[\sigma] \rangle$ assumes a positive minimum value at a point $[\sigma_0]$ in $T(R)$. An isometric image of the real line $\mathbb{R}$ into $T(R)$ is called a Teichmüller geodesic. Similarly, an isometric image of the unit disk $D = \{z \in \mathbb{C};\ |z| < 1\}$ into $T(R)$ is called a Teichmüller disk. According to Bers [4] and Kra [8], f^* is hyperbolic if and only if f^* keeps a Teichmüller geodesic l invariant, which is also equivalent to that f^* keeps a Teichmüller disk D invariant. In this case, for any point $[\sigma] \in l$, we let $[\sigma]$ be represented by R. Then f^* can be realized on R as an absolutely extremal Teichmüller mapping $f_0\colon R \to R$. Let $K(f_0)$ (> 1) denote the maximal dilatation of f_0.

Associated to l (or D), there is a integrable meromorphic quadratic differential ϕ that defines a singular Euclidean metric $|\phi|$ on R. Thus it determines a pair of singular measured foliation $(\mathcal{F}_h, \mathcal{F}_v)$, where $\mathcal{F}_h$ and $\mathcal{F}_v$ are horizontal and vertical foliations respectively.

The absolutely extremal map f_0 takes the two singular foliations into themselves. Away from all singularities, the map stretches the horizontal leaves by the stretching factor $\lambda(f_0) = K(f_0)^{1/2}$, and compress the vertical leaves by the factor $1/\lambda(f_0) = K(f_0)^{-1/2}$. By using the language of Thurston [15], such a map f_0 is also called a pseudo-Anosov diffeomorphism. We use the notations $l_\phi = l$ and $D_\phi = D$ to emphasis that those D and l are determined by ϕ.

In the homotopy class f^* of f, f_0 is a unique pseudo-Anosov diffeomorphism. So the action of f^* on $T(R)$ is analogous to the action of a hyperbolic Möbius transformation on $\mathbb{H}$. In particular, for each hyperbolic element f^*, there is a unique invariant Teichmüller geodesic and a unique invariant Teichmüller disk in $T(R)$.

Let $A = \{\alpha_1, \ldots, \alpha_n\}$ and $B = \{\beta_1, \ldots, \beta_m\}$, $n \geq 1$ and $m \geq 1$ be collections of disjoint simple closed geodesics on R. We assume that A and B intersect minimally, and $A \cup B$ fills R in the sense that every non-trivial loop on R intersects with either A or B or both. Let t_A and t_B denote the positive multi-twists along some elements of A and B, respectively. $A \cup B$ is regarded as a graph on R. According to Proposition 6.4 of [9], when $A \cup B$ is dominant (see [9] for the definition), all elements in $\langle t_A, t_B \rangle$ except conjugates of powers of t_A^n, t_B^m, and possibly of $(t_A \circ t_B)^n$ and $(t_B \circ t_A)^n$, $n, m \in \mathbb{Z}$, are pseudo-Anosov maps. $(t_A \circ t_B)^n$ and $(t_B \circ t_A)^n$ are not pseudo-Anosov if the graph $A \cup B$ is critical (see also [9]).

Let $D_\phi \subset T(R)$ be a Teichmüller disk. We consider the stabilizer $\mathrm{Stab}(D_\phi)$ in Mod_R, which is called a Veech group on a surface in D_ϕ. Since the Teichmüller disk is isometrically the same as $\mathbb{H}$, $\mathrm{Stab}(D_\phi)$ actually determines a subgroup of $\mathrm{PSL}_2(\mathbb{R})$. Thus there defines a map

$$\mathcal{D}\colon\ \mathrm{Stab}(D_\phi) \to \mathrm{PSL}_2(\mathbb{R}).$$

By the main theorem of [10], there exist Teichmüller disks D_ϕ and D_ψ such that $t_A \in \mathrm{Stab}(D_\phi)$ and $t_B \in \mathrm{Stab}(D_\psi)$ and they determine elements $\mathcal{D}(t_A)$ and $\mathcal{D}(t_B)$ in $\mathrm{PSL}_2(\mathbb{R})$. Furthermore, t_A and t_B also act on a common Teichmüller disk D with respect to the flat structure constructed from the dual graph of $A \cup B$.

Define $N = N_{A,B}$ to be the $n \times m$ matrix whose (i, j)-entry is $i(\alpha_i, \beta_j)$, the intersection number of α_i and β_j, where $\alpha_i \in A$ and $\beta_j \in B$. By assumption, $A \cup B$ fills R. It follows that the graph defined by $A \cup B$ is connected. Hence NN^t is irreducible. Let $\mu(NN^t)$ be the maximum of moduli of the eigenvalues of NN^t (called the Perron-Frobenius eigenvalue in the literature), and set $\mu(A \cup B) = \sqrt{\mu(NN^t)}$. By [9, 16], we have the representations:

$$\mathcal{D}(t_A) = \begin{pmatrix} 1 & \mu(A \cup B) \\ 0 & 1 \end{pmatrix}$$

and

$$\mathcal{D}(t_B) = \begin{pmatrix} 1 & 0 \\ -\mu(A \cup B) & 1 \end{pmatrix}.$$

Let $\mu = \mu(A \cup B)$. The group $\langle \mathcal{D}(t_A), \mathcal{D}(t_B) \rangle$ generated by $\mathcal{D}(t_A)$ and $\mathcal{D}(t_B)$ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{R})$ if $\mu > 2$. In this case, $\langle \mathcal{D}(t_A), \mathcal{D}(t_B) \rangle$ is free of rank 2. Denote $\mathcal{M} = \mathbb{H}/\langle \mathcal{D}(t_A), \mathcal{D}(t_B) \rangle$. By Lemma 6.3 of [9], $\mathcal{M}$ has infinite area and its convex core is a twice punctured disk. These results will lead to the following Lemma 1.

Let ϵ_μ be the larger root of the quadratic equation

$$x^2 + (2 - \mu^2)x + 1 = 0.$$

That is,

$$\epsilon_\mu = \frac{1}{2}(\mu^2 - 2 + \mu\sqrt{\mu^2 - 4}).$$

It is easy to check that ϵ_μ is an increasing function with respect to μ. Notice that different metric scales were used in different papers, we need to review those arguments presented in [9] and [8] for the sake of consistency.

Let f be a pseudo-Anosov element in $\langle t_A, t_B \rangle$. f determines a Teichmüller disk D isometric to $\mathbb{H}$ with respect to the Teichmüller metric on D and the hyperbolic metric on $\mathbb{H}$. Thus f induces a hyperbolic Möbius transformation $\mathcal{D}(f)$ on $\mathbb{H}$. Denote $F = \mathcal{D}(f)$. Let τ denote the translation length of F:

$$\tau = \inf_z \rho_{\mathbb{H}}(z, F(z)).$$

From Section 7.34 of Beardon [2], we know that

$$\frac{1}{2}|\mathrm{trace}(F)| = \cosh\frac{\tau}{2} = \frac{\exp(\tau/2) + \exp(-\tau/2)}{2}. \tag{2.1}$$

By isometry we obtain

$$\frac{1}{2}\log K(f) = \tau. \tag{2.2}$$

Since $\lambda(f) = K(f)^{1/2}$, from (2.2), we get

$$\log\lambda(f) = \tau. \tag{2.3}$$

Let $\xi = \exp(\tau/2)$. A simple calculation shows that ξ satisfies

$$\xi^2 - |\mathrm{trace}(F)|\xi + 1 = 0.$$

By Lemma 6.3 of [9], $\xi \geq \epsilon_\mu$, i.e., $\exp(\tau/2) \geq \epsilon_\mu$, or $\tau/2 \geq \log\epsilon_\mu$. It follows that

$$\tau \geq 2\log\epsilon_\mu.$$

Together with (2.3), we obtain

$$\log\lambda(f) \geq 2\log\epsilon_\mu.$$

Hence, we have $\lambda(f) \geq \epsilon_\mu^2$. We summarize the result in the following lemma.

Lemma 1 (Leininger [9]). *Assume that $\mu > 2$. For any pseudo-Anosov element f of $\langle t_A, t_B \rangle$, we have that*

$$\lambda(f) \geq \epsilon_\mu^2.$$

REMARK. Due to different metric scales the original result stated in [9] takes the inequality $\lambda(f) \geq \epsilon_\mu$.

Since $\lambda(f) = K(f)^{1/2}$, from Lemma 1, we obtain:

$$K(f) \geq \left\{ \frac{1}{2}(\mu^2 - 2 + \mu\sqrt{\mu^2 - 4}) \right\}^4. \tag{2.4}$$

3. Translation lengths of essential hyperbolic elements

Let $\tilde{S}$ be as in the introduction. Let $a \in \tilde{S}$ and let $S = \tilde{S} \setminus \{a\}$. S is of type $(p, n+1)$. Associated to $T(\tilde{S})$ there is a fiber space $F(\tilde{S})$ defined as follows. For each $[\nu] \in T(\tilde{S})$, by Ahlfors-Bers [1], there is normalized quasiconformal automorphism w^ν of the complex plane $\mathbb{C}$ such that the restriction $w^\nu|_{\mathbb{H}^*}$ to the lower half plane $\mathbb{H}^*$ is conformal, and its Beltrami coefficient $\partial_{\bar{z}} w^\nu / \partial_z w^\nu$ on $\mathbb{H}$ projects to a conformal structure that determines $[\nu]$. We form the Bers fiber space

$$F(\tilde{S}) = \{([\nu], z);\ [\nu] \in T(\tilde{S}),\ z \in w^\nu(\mathbb{H})\}.$$

Note that in this setting $\mathbb{H}$ is considered the central fiber and the group G acts on $F(\tilde{S})$ in a natural manner.

In [3], Bers established an isomorphism $\varphi\colon F(\tilde{S}) \to T(S)$ that is unique up to a modular transformation on $T(S)$.

The isomorphism φ determines an embedding φ^* of G into the mapping class group Mod_S such that each element in the image $\varphi^*(G)$ projects to the trivial mapping class on $\mathrm{Mod}_{\tilde{S}}$ defined by adding the puncture a back into S. Conversely, Bers [3] also showed that if a mapping class θ can be projected to the trivial one in $\mathrm{Mod}_{\tilde{S}}$, then θ lies in the image $\varphi^*(G)$.

It was shown in [8, 12] that $g \in G$ is parabolic if and only if $g^* = \varphi^*(g)$ is induced by a Dehn twist along a boundary curve $\partial\Delta$, where Δ is a twice punctured disk on S enclosing a and another puncture b, where b is regarded as a puncture of $\tilde{S}$ that is determined by the conjugacy class of g. In [8] Kra also proved that g is essential (that is, the complement of the projection of its axis A_g consists of disks and possibly once punctured disks) if and only if g^* is a pseudo-Anosov mapping class in Mod_S. By abuse of language, in the sequel we denote by $K(g^*)$ the dilatation of the corresponding absolutely extremal map on a surface S that realizes the mapping class g^*.

Let $g \in G$ be an essential hyperbolic element with axis A_g. For simplicity we denote $\tilde{c} = \varrho(A_g) \subset \tilde{S}$. Then $\tilde{c}$ is a filling geodesic on $\tilde{S}$ and by a theorem of [11], the length function $l_{\tilde{c}}\colon T(\tilde{S}) \to \mathbb{R}^+$ attains a minimum value at $[\mu_0] \in T(\tilde{S})$. We may assume that $[\mu_0] = [0]$. We need to see how the number $K(g^*)$ is dominated by $l_{\tilde{c}}([0])$. We review:

Lemma 2 (Kra [8]). *With the conditions above, we let λ_g denote the multiplier of g, i.e., g is conjugate to $z \mapsto \lambda_g z$. Then*

$$K(g^*) \leq \lambda_g^2. \tag{3.1}$$

Outline of proof. For any point $x \in A_g \subset \mathbb{H}$, the translation length $\rho_{\mathbb{H}}(x, g(x)) = l_{\tilde{c}}([0])$. By Royden's theorem [13] (see also Earle-Kra [6]), the Teichmüller metric on $T(S)$ coincides with the Kobayashi metric on $T(S)$. Since the restriction $\varphi|_{\mathbb{H}} \colon \mathbb{H} \to T(S)$ is holomorphic, it cannot increase the distance. Therefore,

$$\langle \varphi(x), g^* \circ \varphi(x) \rangle = \langle \varphi(x), \varphi(g(x)) \rangle \leq \rho_{\mathbb{H}}(x, g(x)). \tag{3.2}$$

By definition, the translation length of g^* is no larger than the distance $\langle \varphi(x), g^* \circ \varphi(x) \rangle$. It follows from (3.2) that

$$\begin{aligned} \frac{1}{2} \log K(g^*) &\leq \langle \varphi(x), g^* \circ \varphi(x) \rangle \leq \rho_{\mathbb{H}}(x, g(x)) \\ &= l_{\tilde{c}}([0]) = \int_1^{\lambda_g} \frac{1}{y}\, dy = \log \lambda_g. \end{aligned}$$

It follows that

$$K(g^*) \leq \lambda_g^2, \tag{3.3}$$

as asserted. □

Let $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_m\}$, $n \geq 1$ and $m \geq 1$, be defined in Section 2. First we consider the case that A and B are restricted to contain only one element (the case of $n = m = 1$). In this case, it is well known (see [7]) that there are pseudo-Anosov maps f on S that are not represented by elements in the group $\langle t_A, t_B \rangle$ generated by t_A and t_B. In general case, it is not completely clear whether every essential element $g \in G$, g^* is in the group $\langle t_A, t_B \rangle$. However, there exist infinitely many pairs $\{\alpha_1, \beta_1\}$ so that the group $\langle t_{\alpha_1}, t_{\beta_1} \rangle$ contains pseudo-Anosov mapping classes of forms g^*, where $g \in G$ is essential. In particular, there are infinitely many pairs $\{\alpha_1, \beta_1\}$, where α_1 and β_1 are peripheral on $\tilde{S}$, so that $\langle t_{\alpha_1}, t_{\beta_1} \rangle$ contains pseudo-Anosov mapping classes of form g^* for $g \in G$ being essential hyperbolic.

By combining Lemma 1 and Lemma 2, we can readily obtain the following lemma:

Lemma 3. *Assume that a hyperbolic element $g \in G$ is essential and $g^* \in \langle t_A, t_B \rangle$ for certain $A = \{\alpha_1, \dots, \alpha_n\}$ and $B = \{\beta_1, \dots, \beta_n\}$, $n \geq 1$ and $m \geq 1$. Then $A \cup B$ fills S and*

$$\mu(A \cup B) \leq \max\left\{2, \frac{1}{2}\left(1 + \sqrt{5 + 4\sqrt{\lambda_g}}\right)\right\}.$$

Proof. Denote $\mu = \mu(A \cup B)$. By Lemma 1 and (3.3), we have

$$\lambda_g^2 \geq K(g^*) = \lambda(g^*)^2 \geq \left\{\frac{1}{2}(\mu^2 - 2 + \mu\sqrt{\mu^2 - 4})\right\}^4.$$

Note that $\lambda_g = \exp(l_{\tilde{c}}[0])$. We assume that $\mu > 2$. Then clearly $\mu - 2 \le \sqrt{\mu^2 - 4}$. It follows that

$$\mu^2 - \mu - (1 + \sqrt{\lambda_g}) \le 0.$$

The lemma then follows immediately. □

REMARK. The estimation in the above lemma can be sharpened by applying a theorem of [5] that states that if $\mu > 2$ then in fact $\mu > 2.0065936$. This implies that there is an integer N such that

$$\mu - \left(2 - \frac{1}{N}\right) \le \sqrt{\mu^2 - 4}.$$

A simple calculation shows that $N \ge 7$.

4. Peripheral simple curves on punctured Riemann surfaces

In this section we prove that there are infinitely many essential elements $g \in G$ generated by two parabolic elements.

Let $\tilde{S}$ be of type (p, n) with $3p - 3 + n > 0$ and $n \ge 1$. Let $a \in \tilde{S}$ and $S = \tilde{S} \setminus \{a\}$. Then S is of type $(p, n + 1)$. Let $x_1 = a, x_2, \dots, x_{n+1}$, $n \ge 1$, denote the punctures of S. Let $\mathcal{P}(S, a)$ denote the set of equivalence classes of paths α on S connecting a and another puncture, where two paths α_1 and α_2 are considered equivalent if they are homotopic to each other by a homotopy fixing the end punctures. Let $\mathcal{E}(S, a)$ denote the set of equivalence classes of twice punctured disks on S that enclose a and another puncture, where two such disks are equivalent if their boundary curves are homotopic to each other without interfering with any other punctures.

Given a path representative $\alpha \in \mathcal{P}(S, a)$, we can always fatten α, giving rise to an element in $\mathcal{E}(S, a)$. Conversely, for every element $\Delta \in \mathcal{E}(S, a)$, there is a path α connecting the two end punctures and lying entirely in Δ. α is unique up to a homotopy, i.e., any two such paths are homotopic within Δ and fix the end punctures. We thus obtain a bijection:

$$j : \mathcal{P}(S, a) \to \mathcal{E}(S, a). \tag{4.1}$$

two elements $\alpha, \beta \in \mathcal{P}(S, a)$ are called to fill S if every component of $S \backslash \{\alpha, \beta\}$ is either a disk or a once punctured disk. We need the following lemmas.

Lemma 4. *Let $\alpha, \beta \in \mathcal{P}(S, a)$ and assume that $\{\alpha, \beta\}$ fills S, then $\{\partial j(\alpha), \partial j(\beta)\}$ must also fill S in a regular sense.*

Proof. Denote $\Delta_\alpha = j(\alpha)$ and $\Delta_\beta = j(\beta)$. It is easy to see that $\Delta_\alpha \cap \Delta_\beta$ consists of quadrilateral (that are homeomorphic to disks) and two or one punctured disk components (depending on whether or not α and β share both end punctures). The rest of

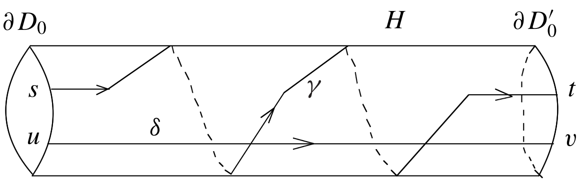

Fig. 1.

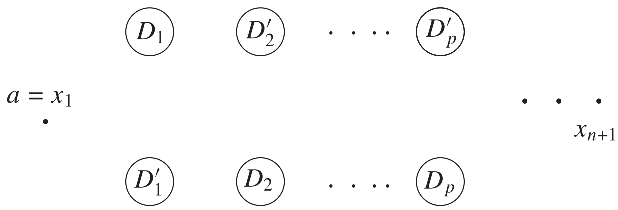

Fig. 2.

components in $\Delta_\alpha \cup \Delta_\beta$ include components of $\Delta_\alpha \setminus \Delta_\beta$ and components of $\Delta_\beta \setminus \Delta_\alpha$, all of which are homeomorphic to disks. The remaining components of $S \setminus \{\Delta_\alpha \cup \Delta_\beta\}$ are essentially the same as the components in $\overline{S} \setminus \{\alpha \cup \beta\}$ which are either disks or punctured disks. This proves Lemma 4. □

The following lemma comes from referee's comments:

Lemma 5. *Let S be of type $(p, n+1)$, $3p+n > 3$, $n \geq 1$. There are infinitely many pairs (α, β) of paths in $\mathcal{P}(S, a)$ so that $\{\alpha, \beta\}$ fills S.*

Proof. Observe that S can be thought of as a Riemann sphere with p handles and n punctures. Let H be a handle with $(\partial D_0, \partial D'_0)$ the two boundary components. Let γ, δ be two curves on H that are not to be homotopic and $\{\gamma, \delta\}$ fills H. Note that γ can be winded around δ as many time as possible. The end points of γ are denoted by s, t, and the end points of δ are denoted by u, v. See Fig. 1.

We remove p pairs (D_i, D'_i) of small disks and $n+1$ points $x_1 = a, x_2, \ldots x_{n+1}$, $n \geq 1$, from the Riemann sphere $\mathbb{S}^2$, obtaining S_0. S_0 is drawn in Fig. 2 in the case that p is even (if p is odd, the positions of D_p and D_p in Fig. 2 are switched).

For $i = 1, \ldots, p$, let (u_i, s_i) and (v_i, t_i) be pairs of marked points on ∂D_i and $\partial D'_i$ respectively. Paste p copies of H to S_0 in such a way that $(\partial D_0, \partial D_i)$ and $(\partial D'_0, \partial D'_i)$ are glued together with $u_i = u$, $v_i = v$, $s_i = s$, and $t_i = t$. Then we can define $\alpha \in \mathcal{P}(S, a)$ as follows. Connect $a = x_1$ and s_1, followed with γ, then connect t_1 and s_2, and then followed with γ again, and so forth. After p steps, we connect t_p and x_{n+1} by a path away from all punctures other than the end punctures. Similarly, we can define $\beta \in \mathcal{P}(S, a)$ to be a path that goes from $a = x_1$ to v_1, followed with the inverse

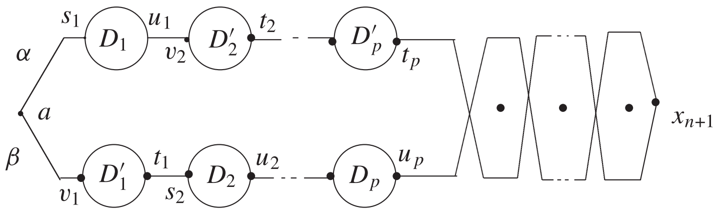

Fig. 3.

δ^{-1} of δ, then go to u_1, then connect u_1 and v_2, and so forth. After p steps, we draw a path connecting u_p to x_{n+1} in such a way that the component of $S\backslash\{\alpha, \beta\}$ that includes x_i, $i = 2, \ldots x_n$, is a once punctured disk (this could occur only when $n \geq 3$). Fig. 3 below shows the case that p is even and the two paths α and β are in $\mathcal{E}(S, a)$. One can easily check that any component of $S \setminus \{\alpha, \beta\}$ is either a disk or a once punctured disk, which says $\{\alpha, \beta\}$ fills S. □

From Lemma 4 and Lemma 5, one obtains:

Lemma 6. *There are infinitely many essential elements $g \in G$ that are generated by two parabolic elements.*

Proof. From the construction there are infinitely many pairs (α, β) that fills S. According to Lemma 4, there are infinitely many pairs $(j(\alpha), j(\beta))$ that fill S. By Lemma 3 of [17], there are parabolic elements T_i, $i = 1, 2$, so that $\varphi^*(T_1) = \partial j(\alpha)$ and $\varphi^*(T_2) = \partial j(\beta)$. Since $\partial j(\alpha)$ and $\partial j(\beta)$ are homotopic to trivial loops as a is filled in, we see that any finite product

$$\prod_i \left(t^{n_i}_{\partial j(\alpha)} \circ t^{m_i}_{\partial j(\beta)}\right), \quad n_i, m_i \in \mathbb{Z}, \tag{4.2}$$

projects to a trivial mapping class. It follows from Bers [4] that (4.2) is of form $\varphi^*(g)$ for an essential element $g \in G$. Clearly, g is generated by T_1 and T_2. This proves the lemma. □

5. Minimal intersections of two peripheral curves

In the previous section we constructed two curves α and β that are boundaries of twice punctured disks enclosing a. In this section we give an estimate of lower bound of intersections of α and β. We first prove:

Lemma 7. *Let $\alpha, \beta \in \mathcal{P}(S, a)$. Suppose that $\{\alpha, \beta\}$ fills S. Then in addition to a and another end puncture, α intersects with β at least $2p - 3 + n$ points. In particular, if $n = 1, 2$, then α intersects with β at least $2p$ points.*

Proof. Note that if punctures on S are considered distinguished points on the compactification $\overline{S}$ of S, $\alpha \cup \beta$ defines a graph on S with a number E of edges, a number F of vertices, and a number V of vertices. We know that the Euler characteristic $\chi(\overline{S}) = 2 - 2p$.

Assume that in addition to a and possible another end point, α intersects with β k times. Let n' be the number of once punctured disk components of $S \setminus \{\alpha, \beta\}$. If $k = 0$, then $\alpha \cup \beta$ is a binary tree or a circle. In former case, $E = 2$, $V = 3$. Since $F + V - E = 2 - 2p$, $F = 1 - 2p$, which implies that $F = 1$. In order for α, β to fill S, we must have $n' \leq 1$ and S is of type $(0, n+1)$ for $n \leq 3$, contradicting to our hypothesis.

In later case, $E = 2$ and $V = 2$. Since $F + V - E = 2 - 2p$, $F = 2 - 2p$. This implies that $F = 2$. So we must have $n' \leq 2$, and S is of type $(0, n+1)$ for $n \leq 3$. Again, this is a contradiction.

Now we assume that $k > 0$ and that all the intersections are distinct. There are two cases to consider.

CASE 1. α and β share only one endpoint a. In this case, we have $\underline{V} = k + 3$, $E = 2(k+1)$. Since $\alpha \cup \beta$ fills S, $F \geq n'$, where $n' + 3 = n + 1$. Now from $\chi(\overline{S}) = 2 - 2p$ we obtain

$$2 - 2p = V + F - E \geq (k+3) + (n-2) - 2(k+1).$$

It follows that $k \geq 2p + n - 3$.

CASE 2. α and β share both end punctures. In this case, $\alpha \cup \beta$ is closed when the two endpoints are added. We must have $V = k + 2$, $E = 2(k+1)$ and $F \geq n'$, where $n' + 2 = n + 1$. Hence

$$2 - 2p = V + F - E \geq (k+2) + (n-1) - 2(k+1).$$

It follows that $k \geq 2p + n - 3$.

In the case of $n = 1, 2$, the author was informed by the referee that $k \geq 2p$. In fact, we first assume that $n = 1$. Then α and β share the same end punctures. So $\alpha \cup \beta$ is a closed when the two endpoints are added. Since each component of $S \setminus \{\alpha, \beta\}$ is a disk, $S \setminus \{\alpha, \beta\}$ is not connected. Hence $F \geq 2$. Recall that $V = k + 2$, and $E = 2(k+1)$. We have

$$2 - 2p = V + F - E \geq (k+2) + 2 - 2(k+1).$$

It follows that $k \geq 2p$. In the case of $n = 2$, when α, β share the same end punctures, by the same argument as above, we have $k \geq 2p$. Otherwise, we assume that α terminates x_2 and β terminates x_3 with $x_2 \neq x_3$. Then $V = k + 3$ and $E = 2(k+1)$. Since $F \geq 1$, we have

$$2 - 2p = V + F - E \geq (k+3) + 1 - 2(k+1).$$

Thus $k \geq 2p$, as asserted. □

Let $\#\{c_1, c_2\}$ denote the set of the minimal intersection points of arbitrary two curves c_1, c_2 on S, and $i(c_1, c_2)$ the intersection number of c_1 and c_2. We have

Lemma 8. *Let α and β be defined as in* Lemma 4. *Then any point in $\#\{\alpha, \beta\}$ other than end punctures of α and β contributes at least* 4 *intersection points to $\#\{c_1, c_2\}$ for $c_1 = \partial j(\alpha)$, and $c_2 = \partial j(\beta)$.*

Proof. We only handle the case that α and β share both end punctures, as drawn in Fig. 3. Let y_i be such an intersection in $\#\{\alpha \cap \beta\}$. By hypothesis, $b = x_{n+1}$ is the other endpoint of α and β, respectively. Let $c_1' \sim c_1$, $c_2' \sim c_2$ be representatives of $\partial j(\alpha)$ and $\partial j(\beta)$, respectively. Assume that c_1' and c_2' are very close to α and β respectively. Observe that y_i contributes 4 intersections to $\#\{c_1', c_2'\}$. In fact, the intersection near y_i is a quadrilateral. Then the lemma follows from the fact that a homotopy does not decrease the intersection number. □

Together with Lemma 6, Lemma 7, and Lemma 8, we are able to prove the following:

Lemma 9. *Let $\tilde{S}$ be of type (p, n) with $3p-3+n > 0$ and $n \geq 1$. Then there are infinitely many pairs (c_1, c_2) of simple closed curves on S with the following properties*:
(1) *$c_1 = \partial\Delta_1$ and $c_2 = \partial\Delta_2$ for $\Delta_1, \Delta_2 \in \mathcal{E}(S, a)$,*
(2) *$\{c_1, c_2\}$ fills S in the regular sense, and*
(3) *the intersection number $i(c_1, c_2) \geq 8p+4n-10$ if $n \geq 3$; $i(c_1, c_2) \geq 8p+4$ if $n = 1$; and $i(c_1, c_2) \geq 8p+2$ if $n = 2$.*

Proof. First we consider the case of $n \geq 3$. If α, β share only one end puncture a, by Lemma 7, there are at least $2p-3+n$ distinct intersection points y_i in $\#\{\alpha, \beta\}$. By Lemma 8, each y_i, $1 \leq i \leq 2p-3+n$, contributes at least 4 intersections to $\#\{c_1, c_2\}$. The puncture a contributes at least 2 intersections in $\#\{c_1, c_2\}$. Therefore,

$$i(c_1, c_2) \geq 4(2p + n - 3) + 2 = 8p + 4n - 10.$$

If α and β share both end punctures (a and b), by Lemma 7 again, α and β cross at least $2p - 3 + n$ times. Let y_i, $1 \leq i \leq 2p - 3 + n$ denote these intersections. By Lemma 8, each y_i contributes at least 4 intersections to $\#\{c_1, c_2\}$, The punctures a and b each contributes at least 2 intersections to $\#\{c_1, c_2\}$. We conclude that

$$i(c_1, c_2) \geq 4(2p + n - 3) + 2 + 2 = 8p + 4n - 8.$$

It follows that $i(c_1, c_2) \geq 8p + 4n - 10$ if $k \geq 3$.

If $n = 1$, then S has only two punctures and $\alpha \cup \beta$ has to be closed as the two end punctures are filled in. By Lemma 7 and Lemma 8, we have $i(c_1, c_2) \geq 4(2p) + 4 = 8p + 4$. If $n = 2$, then S has three punctures. By Lemma 7 and Lemma 8 again, we have $i(c_1, c_2) \geq 4(2p) + 2$ if other than a α and β have different terminal punctures; and $i(c_1, c_2) \geq 4(2p) + 4$ if α and β have the same end punctures ($\alpha \cup \beta$ is closed as the end punctures are filled in). Overall we have $i(c_1, c_2) \geq 4(2p) + 2$ if $n = 2$. This proves the lemma. □

6. Proof of Theorem 1′

The fact that there are infinitely many essential elements g of G that are generated by two parabolic elements was proved in Section 4. Let $g \in G$ be an essential element generated by two parabolic elements T_1 and T_2. Let $t_1 = \varphi^*(T_1)$ and $t_2 = \varphi^*(T_2)$. By Theorem 2 of [8, 12], t_1 and t_2 are Dehn twists along c_1 and c_2 for $c_1 = \partial\Delta_1$ and $c_1 = \partial\Delta_2$, where $\Delta_1, \Delta_2 \in \mathcal{E}(S, a)$.

We remark that in our situation the fixed point z_i of T_i, $i = 1, 2$, cannot be vertices of a common fundamental region of G. For otherwise, let $\omega \colon G \to \pi_1(\tilde{S}, a)$ denote a canonical isomorphism. Then we have that $\omega(T_1^{\pm 1} \circ T_2^{\pm 1})$ is either a simple loop bounding 2 punctures of $\tilde{S}$, or a "figure 8" loop on $\tilde{S}$ that is not a filling loop unless $\tilde{S}$ is of type (0, 3), which has been excluded by our assumption.

From Lemma 9, the intersection number $i(c_1, c_2) \geq \kappa_0$, where $\kappa_0 = 8p + 4n - 10$ if $n \geq 3$; $\kappa_0 = 8p + 4$ if $n = 1$; and $\kappa_0 = 8p + 2$ if $n = 2$. Since $A = \{c_1\}$ and $B = \{c_2\}$ consist single element, we have $n = m = 1$ in the discussion of Section 2. Hence by definition, $\mu(NN^t) = i(c_1, c_2)^2$. Thus

$$\mu(A \cup B) = \sqrt{\mu(NN^t)} = i(c_1, c_2) \geq \kappa_0. \tag{6.1}$$

In particular, since $p \geq 0$, and $p \geq 1$ if $n \leq 3$, we see that $\kappa_0 > 2$. It follows that $\mu(A \cup B) > 2$. Since $\lambda_g > 1$,

$$\frac{1}{2}\left(1 + \sqrt{5 + 4\sqrt{\lambda_g}}\right) > 2.$$

Now from Lemma 3 along with (6.1), we have that

$$\kappa_0 \leq \mu(A \cup B) \leq \frac{1}{2}\left(1 + \sqrt{5 + 4\sqrt{\lambda_g}}\right),$$

where $\lambda_g = \exp\{l_{\tilde{c}}([0])\}$. A simple calculation shows that

$$\lambda_g \geq \left\{\frac{1}{4}(\kappa^2 - 5)\right\}^2,$$

where $\kappa = 16p + 8n - 21$ if $n \geq 3$; $\kappa = 16p + 3$ if $n = 2$; and $\kappa = 16p + 7$ if $n = 1$. This proves Theorem 1′. Since $l_{\tilde{c}}([0]) = \log \lambda_g$, we obtain

$$l_{\tilde{c}}(\sigma) \geq 2 \log(\kappa^2 - 5) - 4 \log 2.$$

This proves Theorem 1. □

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References

[1] L. Ahlfors and L. Bers: *Riemann's mapping theorem for variable metrics*, Ann. of Math. (2) **72** (1960), 385–404.
[2] A.F. Beardon: The Geometry of Discrete Groups, Springer, New York, 1983.
[3] L. Bers: *Fiber spaces over Teichmüller spaces*, Acta Math. **130** (1973), 89–126.
[4] L. Bers: *An extremal problem for quasiconformal mappings and a theorem by Thurston*, Acta Math. **141** (1978), 73–98.
[5] A.E. Brouwer and A. Neumaier: *The graphs with spectral radius between* 2 *and* $\sqrt{2+\sqrt{5}}$, Linear Algebra Appl. **114**/**115** (1989), 273–276.
[6] C.J. Earle and I. Kra: *On holomorphic mappings between Teichmüller spaces*; in Contributions to Analysis (a collection of papers dedicated to Lipman Bers), Academic Press, New York, 1974, 107–124.
[7] P. Hubert and E. Lanneau: *Veech groups without parabolic elements*, preprint (2005).
[8] I. Kra: *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces*, Acta Math. **146** (1981), 231–270.
[9] C.J. Leininger: *On groups generated by two positive multi-twists*: *Teichmüller curves and Lehmer's number*, Geom. Topol. **8** (2004), 1301–1359.
[10] A. Marden and H. Masur: *A foliation of Teichmüller space by twist invariant disks*, Math. Scand. **36** (1975), 211–228, and Addendum, **39** (1976), 232–238.
[11] B. Maskit and J.P. Matelski: *Minimum geodesics on surfaces*, manuscript.
[12] S. Nag: *Nongeodesic discs embedded in Teichmüller spaces*, Amer. J. Math. **104** (1982), 399–408.
[13] H.L. Royden: *Automorphisms and isometries of Teichmüller space*; in Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies **66**, Princeton Univ. Press, Princeton, N.J., 1971, 369–383
[14] R.C. Penner: *Probing mapping class group using arcs*, manuscript, 2005.
[15] W.P. Thurston: *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), 417–431.
[16] W.A. Veech: *Teichmüller curves in moduli space*, *Eisenstein series and an application to triangular billiards*, Invent. Math. **97** (1989), 553–583.
[17] C. Zhang: *Hyperbolic mapping classes and their lifts on Bers fiber space*, to appear, 2005.

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