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# A PRESENTATION FOR THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE FROM THE ACTION ON THE COMPLEX OF CURVES

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**Abstract**

We study the action of the mapping class group $\mathcal{M}(F)$ on the complex of curves of a non-orientable surface $F$. Following the outline of [1] we obtain, using the result of [4], a presentation for $\mathcal{M}(F)$ defined in terms of the mapping class groups of the complementary surfaces of collections of curves, provided that $F$ is not sporadic, i.e. the complex of curves of $F$ is simply connected. We also compute a finite presentation for the mapping class group of each sporadic surface.

## 1. Introduction

Presentations for the mapping class group $\mathcal{M}(F_g^n)$ of a compact orientable surface of genus $g$ with $n$ boundary components have been found by various authors. Hatcher and Thurston [10] derived a presentation for $\mathcal{M}(F_g^1)$ from its action on a simply connected 2-dimensional complex, the *cut system complex*. This complex was simplified by Harer [8] and using this simplified complex, Wajnryb [22] obtained a simple presentation for $\mathcal{M}(F_g^1)$ and $\mathcal{M}(F_g^0)$. Starting from Wajnryb's result, Gervais [7] found a simple presentation for $\mathcal{M}(F_g^n)$ for any $n$ and $g \geq 1$. Benvenuti [1] and Hirose [11] showed independently how the Gervais presentation can be recovered using two different modifications of the classical complex of curves introduced by Harvey [9]. Benvenuti used the *ordered complex of curves* and obtained a presentation for $\mathcal{M}(F_g^n)$ in terms of the mapping class groups of the complementary surfaces of collections of curves.

If $F_g^n$ is a non-orientable surface of genus $g$ with $n$ boundary components (i.e. $F_g^n$ is homeomorphic to the connected sum of $g$ projective planes, from which $n$ open discs have been removed), then presentations for $\mathcal{M}(F_g^n)$ are known only for $g \leq 3$ and small $n$. The complex of curves of $F_g^n$ has been studied by various authors. Ivanov [12] determined its homotopy type used it to compute the virtual cohomological dimension of the mapping class group $\mathcal{M}(F_g^n)$.

In this paper we study the action of the mapping class group $\mathcal{M}(F)$ on the complex of curves of a non-orientable surface $F = F_g^n$. Our main result says that $\mathcal{M}(F)$

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can be presented in terms of the isotropy subgroups of the collections of curves, provided that $F$ is not sporadic, i.e. the complex of curves of $F$ is simply connected. On the other hand we show that a presentation for the isotropy subgroup of a collection of curves $A$ can be obtained from a presentation for the mapping class group of the surface obtained by cutting $F$ along $A$. Thus our result recursively produces a presentation for $\mathcal{M}(F)$, provided that we know presentations for the mapping class groups of all sporadic subsurfaces. In this paper we compute an explicit finite presentation for the mapping class group of each sporadic surface.

The paper is organized as follows, In the next two sections we present basic definitions and preliminary results about simple closed curves. In Section 4 we determine the structure of the stabilizer of a simplex of the complex of curves, and in Section 5 we determine $\mathcal{M}(F)$-orbits of simplices. In Section 6 we use the ordered complex of curves to obtain, by a result of Brown [4], a presentation for the mapping class group. Then we show how this presentation can be simplified. Finally, in Section 7 we compute presentations for mapping class groups of sporadic surfaces.

## 2. Basic definitions

Let $F$ denote a smooth, compact, connected surface, orientable or not, possibly with boundary. Define $\mathrm{Diff}(F)$ to be the group of all (orientation preserving if $F$ is orientable) diffeomorphisms $h\colon F \to F$ such that $h$ is the identity on the boundary of $F$. The *mapping class group* $\mathcal{M}(F)$ is the group of isotopy classes in $\mathrm{Diff}(F)$. By abuse of notation we will use the same symbol to denote a diffeomorphism and its isotopy class. If $g$ and $h$ are two diffeomorphisms, then the composition $gh$ means that $h$ is applied first.

By a *simple closed curve* in $F$ we mean an embedding $a\colon S^1 \to F$. Note that $a$ has an orientation; the curve with opposite orientation but same image will be denoted by $a^{-1}$. By abuse of notation, we also use $a$ for the image of $a$. If $a_1$ and $a_2$ are isotopic, we write $a_1 \simeq a_2$.

We say that $a\colon S^1 \to F$ is *non-separating* if $F \setminus a$ is connected and *separating* otherwise. According to whether a regular neighborhood of $a$ is an annulus or a Möbius strip, we call $a$ respectively *two-* or *one-sided*. If $a$ is one-sided, then we denote by $a^2$ its double, i.e. the curve $a^2(z) = a(z^2)$ for $z \in S^1 \subset \mathbb{C}$. Note that although $a^2$ is not simple, it is freely homotopic to a two-sided simple closed curve.

We say that $a$ is *essential* if it neither bounds a disk nor is isotopic to a boundary curve. We say that $a$ is *generic* if it is essential and does not bound a Möbius strip. Note that every one-sided curve is generic.

Define a *generic $r$-family of disjoint curves* to be a $r$-tuple $(a_1, \dots, a_r)$ of generic simple closed curves satisfying:

- $a_i \cap a_j = \emptyset$, for $i \neq j$;
- $a_i$ is neither isotopic to $a_j$ nor to $a_j^{-1}$, for $i \neq j$.

We say that two generic $r$-families of disjoint curves $(a_1, \dots, a_r)$ and $(b_1, \dots, b_r)$ are *equivalent* if there exists a permutation $\sigma \in \Sigma_r$ such that $a_i \simeq b_{\sigma(i)}^{\pm 1}$ for each $1 \leq i \leq r$. We write $[a_1, \dots, a_r]$ for the equivalence class of a generic $r$-family of disjoint curves.

The *complex of curves* of $F$ is the simplicial complex $\mathcal{C}(F)$ whose $r$-simplices are the equivalence classes of generic $(r+1)$-families of disjoint curves in $F$. Vertices of $\mathcal{C}(F)$ are the isotopy classes of unoriented generic curves. The mapping class group $\mathcal{M}(F)$ acts simplicially on $\mathcal{C}(F)$ by $h[a_1, \dots, a_r] = [h \circ a_1, \dots, h \circ a_r]$.

## 3. A few results about simple closed curves

A *bigon* cobounded by two transversal simple closed curves $a$ and $b$ is a region in $F$, whose interior is an open disc and whose boundary is the union of an arc of $a$ and an arc of $b$. Moreover, we assume that except for the endpoints, these arcs are disjoint from $a \cap b$, and that the endpoints do not coincide. If the endpoints coincide (i.e. the arcs are closed curves), then we say that $a$ and $b$ cobound a *degenerate bigon*.

**Lemma 3.1** (Epstein [6]). *Let $a, b$ be two two-sided essential curves in $F$, and suppose $a$ is isotopic to $b$.*
i) *If $a \cap b = \emptyset$, then there exists an annulus in $F$ whose boundary components are $a$ and $b$.*
ii) *If $a \cap b \neq \emptyset$, and they intersect transversely, then $a$ and $b$ cobound a bigon.*

**Lemma 3.2.** *Let $a, b$ be two one-sided simple closed curves and suppose $a$ is isotopic to $b$. Then $a \cap b \neq \emptyset$. If they intersect transversely, then*:
i) *if $|a \cap b| = 1$, then $a$ and $b$ cobound a degenerate bigon,*
ii) *if $|a \cap b| > 1$, then $a$ and $b$ cobound a bigon.*

Proof. We choose a regular neighborhood $N_a$ of $a$, diffeomorphic to the Möbius strip, and denote by $a'$ its boundary curve which is homotopic to $a^2$. Similarly we define $N_b$ and $b'$ homotopic to $b^2$. Now $a'$ and $b'$ are simple closed curves and $a' \simeq b'$, since $a \simeq b$.

If $F$ is the projective plane or the Möbius strip, then the proof is trivial. In the other case $a'$ and $b'$ are essential and we can apply Lemma 3.1.

Assume $a \cap b = \emptyset$. Then we can choose $N_a$ and $N_b$ disjoint. By Lemma 3.1, $a'$ and $b'$ cobound an annulus $A$. But then $F = A \cup N_a \cup N_b$ is diffeomorphic to the Klein bottle and $a$ and $b$ are clearly not isotopic. Thus we have proved that $a$ and $b$ intersect.

Assume that $a$ and $b$ intersect transversely. Then we can choose $N_a$ and $N_b$ in such a way that $a'$ and $b'$ also intersect transversely and $|a' \cap b'| = 4|a \cap b|$. By Lemma 3.1 $a'$ and $b'$ cobound a bigon $D$. If $|a \cap b| = 1$ then $M = N_a \cup N_b \cup D$ is a Möbius strip which contains $a$ and $b$. In this case $a$ and $b$ cobound a degenerate bigon in $M$. Assume that $|a \cap b| \geq 2$. Then there exist an arc $c$ of $a$, an arc $d$ of

$b$ and closed subsets $N_c \subset N_a$ and $N_d \subset N_b$ such that: $|c \cap d| = 2$ and the interior of $N_c \cup N_d \cup D$ is homeomorphic to an open disc. Now $c$ and $d$ cobound a bigon in $N_c \cup N_d \cup D$. □

The next two propositions are proved in [18] (Propositions 3.5 and 3.10) for orientable surfaces. Their proofs are based on Lemma 3.1 and can by applied also in the non-orientable case if the involved curves are two-sided. Therefore, in the proofs we restrain ourselves to the case of one-sided curves, where we use Lemma 3.2 instead of Lemma 3.1.

By a *subsurface* $N$ of $F$ we mean a closed subset which is also a surface. We say furthermore that $N$ is *essential* if no boundary curve of $N$ bounds a disk in $F$.

**Proposition 3.3.** *Let $N$ be an essential subsurface of $F$, and let $a, b\colon S^1 \to N$ be two essential simple closed curves. (In particular $a$ is not isotopic to a boundary curve of $N$.) Then $a$ is isotopic to $b$ in $F$ if and only if $a$ is isotopic to $b$ in $N$.*

Proof. The nontrivial thing to show is that if $a$ and $b$ are isotopic in $F$, then they are also isotopic in $N$. We assume that $a$ and $b$ are one-sided. By Lemma 3.2 they intersect. We may assume that they intersect transversally and argue by induction on $|a \cap b|$.

If $|a \cap b| = 1$, then by Lemma 3.2, $a$ and $b$ cobound a degenerate bigon $D$ in $F$. Since $N$ is essential, $D \cap \partial N = \emptyset$ and hence $D \subset N$. Now we can use $D$ to define an isotopy in $N$ from $a$ to $b^{\pm 1}$. If $a \simeq b^{-1}$ in $N$, then $b \simeq b^{-1}$ in $F$, which can only happen if $F$ is the projective plane (cf. [6], Theorem 1.7). But the projective plane does not contain any essential subsurface. Thus $a \simeq b$ in $N$.

If $|a \cap b| > 1$, then by Lemma 3.2, $a$ and $b$ cobound a bigon $D \subset F$. As above, $D \subset N$ and we can use $D$ to define an isotopy in $N$ from $b$ to a curve $b'$ with $|a \cap b'| = |a \cap b| - 2$. By the inductive hypothesis, $b'$ is isotopic to $a$ in $N$, hence so is $b$. □

**Proposition 3.4.** *Let $(a_1, \dots, a_r)$, $(b_1, \dots, b_r)$ be two generic $r$-families of disjoint curves such that $a_i \simeq b_i$ for all $1 \le i \le r$. Then there exists an isotopy $h_t\colon F \to F$, $t \in [0, 1]$, such that $h_0 =$ identity and $h_1 \circ a_i = b_i$ for all $1 \le i \le r$.*

Proof. We use induction on $r$. The proposition is obvious for $r = 1$ and we assume that it is true for $(r-1)$-families. Replacing each $a_i$ by $h_1 \circ a_i$, we may assume that $a_i = b_i$ for $1 \le i \le r-1$. Then $a_r$ and $b_r$ are disjoint from $a_i = b_i$ for $i < r$ and $a_r \simeq b_r$. Now it suffices to show that there is an isotopy of $F$ which takes $a_r$ to $b_r$ and does not move the curves $a_i = b_i$ for $i < r$. We assume that $a_r$ and $b_r$ are one-sided and intersect transversally. We argue by induction on $|a_r \cap b_r|$.

If $|a_r \cap b_r| = 1$, then by Lemma 3.2, $a_r$ and $b_r$ cobound a degenerate bigon $D$ in $F$. Since the curves $a_i = b_i$ for $i < r$ are generic, they are all disjoint from $D$. Now it is easy to construct an isotopy of $F$, which takes $a_r$ to $b_r$ across $D$ and is equal to the identity outside a neighborhood of $D$, so the other curves do not move.

If $|a_r \cap b_r| > 1$, then by Lemma 3.2, $a_r$ and $b_r$ cobound a bigon $D$ in $F$. As above, the curves $a_i = b_i$ for $i < r$ are disjoint from $D$, and there is an isotopy of $F$, fixed outside a neighborhood of $D$, which takes $a_r$ across $D$ and reduces the number $|a_r \cap b_r|$ without moving the other curves. By the inductive hypothesis there is a final isotopy taking $a_r$ to $b_r$. □

Given a two-sided simple closed curve $a$ we can define a Dehn twist $t_a$ about $a$. Since we are dealing with non-orientable surfaces, it is impossible to distinguish between right and left twists. The direction of a twist $t_a$ has to be specified for each curve $a$. Equivalently we may choose an orientation of a tubular neighborhood of $a$. Then $t_a$ denotes the right Dehn twist with respect to the chosen orientation. Unless we specify which of the two twists we mean, $t_a$ denotes (the isotopy class of) any of the two possible twists.

The next proposition is proved in [18] for orientable surfaces and in [20] for non-orientable surfaces.

**Proposition 3.5.** *Suppose that $F$ is not homeomorphic to the Klein bottle. Consider $r$ two-sided simple closed curves $a_1, \dots, a_r$ satisfying*:

i) $a_i$ *is either generic or isotopic to a boundary curve*;

ii) $a_i \cap a_j = \emptyset$, *for* $i \neq j$;

iii) $a_i$ *is neither isotopic to* $a_j$ *nor to* $a_j^{-1}$, *for* $i \neq j$.

*Then the subgroup of $\mathcal{M}(F)$ generated by Dehn twists $t_{a_1}, \dots, t_{a_r}$ is a free abelian group of rank $r$.*

Note that if $F$ is homeomorphic to the Klein bottle, then up to isotopy there is only one generic two-sided curve $a$, and $t_a$ has order 2.

## 4. The structure of the stabilizer

In this section we follow the outline of Paragraph 6 of [19] to expresses the stabilizer of a simplex of $\mathcal{C}(F)$ by means of the mapping class group of the complementary surface. Our Proposition 4.2 is a generalization to the case of a non-orientable surface of Proposition 6.3 of [19].

Let $A = (a_1, \dots, a_r)$ be a generic $r$-family of disjoint curves. Denote by $F_A$ the compact surface obtained by cutting $F$ along $A$, i.e. the natural compactification of $F \setminus \left(\bigcup_{i=1}^r a_i\right)$. Note that $F_A$ is in general not connected. Denote by $N_1, \dots, N_k$ the connected components of $F_A$. Then we write

$$\mathcal{M}(F_A) = \mathcal{M}(N_1) \times \cdots \times \mathcal{M}(N_k).$$

Denote by $\rho_A\colon F_A \to F$ the continuous map induced by the inclusion of $F \setminus \left(\bigcup_{i=1}^{r} a_i\right)$ in $F$. The map $\rho_A$ induces a homomorphism $\rho_*\colon \mathcal{M}(F_A) \to \mathcal{M}(F)$.

A *pair of pants* is a compact surface homeomorphic to a sphere with 3 holes. We say that the family $A$ determines a *pants decomposition* if each component of $F_A$ is a pair of pants. Such a family exists if and only if the Euler characteristic of $F$ is negative. In such case, a generic family $A$ determines a pants decomposition if and only if $A$ represents a maximal simplex in $\mathcal{C}(F)$. Given a generic family $A = (a_1, \dots, a_r)$ we can always complete it to a pants decomposition, i.e. there exist generic curves $(a_{r+1}, \dots, a_s)$ such that $(a_1, \dots, a_s)$ determines a pants decomposition. Recall that if $N$ is a pair of pants then $\mathcal{M}(N)$ is the free abelian group of rank 3 generated by Dehn twists along the boundary curves.

**Lemma 4.1.** *Assume that $F$ has negative Euler characteristic. Let $A = (a_1, \dots, a_r)$ be a generic family of disjoint curves in $F$ such that $a_1, \dots, a_p$ are two-sided and $a_{p+1}, \dots, a_r$ are one-sided. For each $i \in \{1, \dots, p\}$ let $a'_i$ and $a''_i$ denote the boundary curves of $F_A$ such that $\rho_A \circ a'_i = \rho_A \circ a''_i = a_i$, and choose $t_{a'_i}$ and $t_{a''_i}$ so that $\rho_*(t_{a'_i}) = \rho_*(t_{a''_i})$. For each $j \in \{p+1, \dots, r\}$ let $a'_j$ denote the boundary curve of $F_A$ such that $\rho_A \circ a'_j = a_j^2$. Then $\ker \rho_*$ is generated by $\left\{t_{a'_1} t_{a''_1}^{-1}, \dots, t_{a'_p} t_{a''_p}^{-1}, t_{a'_{p+1}}, \dots, t_{a'_r}\right\}$ and is a free abelian group of rank $r$.*

Proof. Let $G$ denote the subgroup of $\mathcal{M}(F_A)$ generated by

$$\left\{t_{a'_1} t_{a''_1}^{-1}, \dots, t_{a'_p} t_{a''_p}^{-1}, t_{a'_{p+1}}, \dots, t_{a'_r}\right\}.$$

Clearly $G \subseteq \ker \rho_*$ and it follows from Proposition 3.5 that $G$ is a free abelian group of rank $r$. It remains to show that $\ker \rho_* \subseteq G$.

Let $c_1, \dots, c_n$ denote the boundary curves of $F$ and $c'_1, \dots, c'_n$ the corresponding boundary curves of $F_A$ (i.e. $\rho_A \circ c'_i = c_i$). Complete $A$ to a pants decomposition $A' = (a_1, \dots, a_r, a_{r+1}, \dots, a_q, \dots, a_s)$, where $a_{r+1}, \dots, a_q$ are two-sided and $a_{q+1}, \dots, a_s$ one-sided. Let $a'_{r+1}, \dots, a'_s$ denote the generic curves in $F_A$ such that $\rho_A \circ a'_j = a_j$ for $r+1 \le j \le s$.

Let $h$ be an element of $\ker \rho_*$ and $j \in \{r+1, \dots, s\}$. We have $\rho_A \circ h \circ a'_j \simeq \rho_A \circ a'_j$ and it follows by Proposition 3.3 that $h \circ a'_j \simeq a'_j$. Hence, by Proposition 3.4 we may assume that $h \circ a'_j = a'_j$. Now $h$ induces a diffeomorphism of $F_{A'}$, and hence by the structure of the mapping class group of the pair of pants we can write:

$$h = t_{a'_1}^{u_1} t_{a''_1}^{v_1} \cdots t_{a'_p}^{u_p} t_{a''_p}^{v_p} t_{a'_{p+1}}^{u_{p+1}} \cdots t_{a'_q}^{u_q} t_{c'_1}^{w_1} \cdots t_{c'_n}^{w_n},$$

where $u_1, \dots, w_n \in \mathbb{Z}$. The equality

$$1 = \rho_*(h) = t_{a_1}^{u_1+v_1} \cdots t_{a_p}^{u_p+v_p} t_{a_{r+1}}^{u_{r+1}} \cdots t_{a_q}^{u_q} t_{c_1}^{w_1} \cdots t_{c_n}^{w_n}$$

implies by Proposition 3.5:

$$u_1 + v_1 = \cdots = u_p + v_p = u_{r+1} = \cdots = u_q = w_1 = \cdots = w_n = 0,$$

and we have $h = \left(t_{a_1'} t_{a_1''}^{-1}\right)^{u_1} \cdots \left(t_{a_p'} t_{a_p''}^{-1}\right)^{u_p} t_{a_{p+1}'}^{u_{p+1}} \cdots t_{a_r'}^{u_r}$. □

Denote by $[A]$ the simplex in $\mathcal{C}(F)$ represented by the family $A = (a_1, \dots, a_r)$, and by $\mathrm{Stab}([A])$ the stabilizer of $[A]$ in $\mathcal{M}(F)$.

Define the *cubic group* $\mathrm{Cub}_r$ to be the group of linear transformations $\phi \in GL(\mathbb{R}^r)$ such that $\phi(e_i) = \pm e_j$ for all $1 \leq i \leq r$, where $\{e_1, \dots, e_r\}$ denotes the canonical basis of $\mathbb{R}^r$. There is a natural homomorphism $\Phi_A\colon \mathrm{Stab}([A]) \to \mathrm{Cub}_r$ defined as follows:

$$\Phi_A(h)(e_i) = \begin{cases} e_j & \text{if } h \circ a_i \simeq a_j, \\ -e_j & \text{if } h \circ a_i \simeq a_j^{-1}. \end{cases}$$

Denote by $\mathrm{Stab}^+([A])$ the kernel of $\Phi_A$. By Proposition 3.4, each element of $\mathrm{Stab}^+([A])$ is represented by a diffeomorphism $h \in \mathrm{Diff}(F)$, such that $h \circ a_i = a_i$ for all $1 \leq i \leq r$. Consider the subgroup $H$ of $\mathrm{Stab}^+([A])$ consisting of the isotopy classes of diffeomorphisms preserving each curve of $A$ with its orientation and preserving orientation of a tubular neighborhood of each two-sided curve of $A$. If $A$ contains $p$ two-sided curves, then there is an obvious homomorphism $\mathrm{Stab}^+([A]) \to (\mathbb{Z}_2)^p$ with kernel $H$. Finally observe that $H$ is equal to $\mathrm{Im}\,\rho_*$.

Now we can summarize the considerations of this section in the following proposition.

**Proposition 4.2.** *Assume that $F$ is a surface of negative Euler characteristic. Let $A$ be a generic $r$-family of disjoint curves containing $p$ two-sided curves* $(0 \leq p \leq r)$. *Then we have the following exact sequences*:

$$1 \to \mathbb{Z}^r \to \mathcal{M}(F_A) \xrightarrow{\rho_*} \mathrm{Stab}^+([A]) \to (\mathbb{Z}_2)^p,$$

$$1 \to \mathrm{Stab}^+([A]) \to \mathrm{Stab}([A]) \xrightarrow{\Phi_A} \mathrm{Cub}_r\ .$$

REMARK 4.3. The homomorphisms $\mathrm{Stab}^+([A]) \to (\mathbb{Z}_2)^p$ and $\Phi_A$ are in general not surjective. By an easy analysis case by case it is possible to describe their images exactly.

## 5. The orbits

For the rest of this paper we assume that $F = F_g^n$ is a *non-orientable* surface of genus $g$ with $n$ boundary components $(n \geq 0)$. Recall that this means that $F$ is diffeomorphic to the connected sum of $g$ projective planes, from which $n$ disjoint open discs have been removed. We also assume that $F$ has negative Euler characteristic, i.e. $g +$

$n > 2$. In this section we determine the $\mathcal{M}(F)$-orbits of simplices of the complex of curves $\mathcal{C}(F)$. We say that two simplices $[A]$ and $[B]$ of $\mathcal{C}(F)$ are *$\mathcal{M}(F)$-equivalent* if they are in the same $\mathcal{M}(F)$-orbit. If $A = (a_1, \ldots, a_r)$, $B = (b_1, \ldots, b_r)$, then $[A]$ and $[B]$ are $\mathcal{M}(F)$-equivalent if and only if there exist $h \in \mathrm{Diff}(F)$ and permutation $\sigma \in \Sigma_r$, such that $h \circ a_i \simeq b^{\pm 1}_{\sigma(i)}$. By Proposition 3.4 that is equivalent to existence of $h \in \mathrm{Diff}(F)$, such that $h \circ a_i = b^{\pm 1}_{\sigma(i)}$.

Let $A = (a_1, \ldots, a_r)$ be a generic family of disjoint curves. Let us fix boundary curves $c_1, \ldots, c_n$ of $F$. By abuse of notation we also denote by $c_i$ the boundary curve $c_i \colon S^1 \to \partial N$ such that $\rho_A \circ c_i = c_i$, where $N$ is a connected component of $F_A$. We say that $c_i$ is an *exterior boundary curve* of $N$.

Let $a_i \colon S^1 \to F$ be a two-sided curve in the family $A$. There exist two connected components $N'$ and $N''$ of $F_A$, and two distinct curves $a'_i \colon S^1 \to \partial N'$ and $a''_i \colon S^1 \to \partial N''$ such that $\rho_A \circ a'_i = \rho_A \circ a''_i = a_i$. We say that $a_i$ is a *separating limit curve* of $N'$ (and $N''$) if $N' \neq N''$, and $a_i$ is a *non-separating two-sided limit curve* of $N'$ if $N' = N''$.

Let $a_i \colon S^1 \to F$ be a one-sided curve in $A$. There exists a component $N$ of $F_A$ and a curve $a'_i \colon S^1 \to \partial N$ such that $\rho_A \circ a'_i = a^2_i$. We say that $a_i$ is a *one-sided limit curve* of $N$.

**Lemma 5.1.** *Suppose that $N$ is a non-orientable connected surface and $c \colon S^1 \to \partial N$ is a boundary curve in $N$. There exists a diffeomorphism $h \colon N \to N$ such that $h$ is the identity on $\partial N \setminus c$ and $h \circ c = c^{-1}$.*

Proof. Let $N'$ be the surface obtained by gluing a disc $D$ to $N$ along $c$. Let $p$ be the center of $D$, and $\alpha \colon (S^1, 1) \to (N' \setminus \partial N', p)$ any one-sided simple loop based at $p$. There exists an isotopy $h_t \colon N' \to N'$, $0 \leq t \leq 1$, such that: $h_0 = \text{identity}$, $h_t(p) = \alpha(e^{2\pi t})$, $h_t$ is the identity on $\partial N'$ for all $t$, and $h_1 \circ c = c^{-1}$. We define $h \colon N \to N$ to be the restriction of $h_1$ to $N$. Such diffeomorphism is called *the boundary slide* (cf. [15]). □

**Proposition 5.2.** *Let $A = (a_1, \ldots, a_r)$ and $B = (b_1, \ldots, b_r)$ be two generic $r$-families of disjoint curves. The simplices $[A]$ and $[B]$ are $\mathcal{M}(F)$-equivalent if and only if there exists a permutation $\sigma \in \Sigma_r$, such that for all subfamilies $A' \subseteq A$ and $B' \subseteq B$, such that $a_i \in A' \Leftrightarrow b_{\sigma(i)} \in B'$, there exists a one to one correspondence between the connected components of $F_{A'}$ and those of $F_{B'}$, such that for every pair $(N, N')$ where $N$ is any component of $F_{A'}$ and $N'$ is the corresponding component of $F_{B'}$, we have*:

- *$N$ and $N'$ are either both orientable or both non-orientable, of the same genus*;
- *if $c_i$ is an exterior boundary curve of $N$, then it is also an exterior boundary curve of $N'$*;
- *if $N$ is orientable and $c_i$ and $c_j$ induce the same orientation of $N$, then they also induce the same orientation of $N'$*;

- *if $a_i$ is a separating limit curve of $N$, then $b_{\sigma(i)}$ is a separating limit curve of $N'$;*
- *if $a_i$ is a non-separating two-sided limit curve of $N$, then $b_{\sigma(i)}$ is a non-separating two-sided limit curve of $N'$;*
- *if $a_i$ is a one-sided limit curve of $N$, then $b_{\sigma(i)}$ is a one-sided limit curve of $N'$.*

Proof. Suppose $[A]$ and $[B]$ are $\mathcal{M}(F)$-equivalent. Then there exist $h \in \mathrm{Diff}(F)$ and $\sigma \in \Sigma_r$, such that $h \circ a_i = b^{\pm 1}_{\sigma(i)}$ for $1 \leq i \leq r$. For each subfamily $A' \subseteq A$, $h$ induces a diffeomorphism $h'\colon F_{A'} \to F_{B'}$, such that $h \circ \rho_{A'} = \rho_{B'} \circ h'$. We define a correspondence between the connected components of $F_{A'}$ and those of $F_{B'}$ as follows. If $N$ is any component of $F_{A'}$ then $N' = h'(N)$ is the corresponding component of $F_{B'}$. Note that we have $h' \circ c_i = c_i$ and hence $c_i$ is an exterior boundary curve of $N$ if and only if it is an exterior boundary curve of $N'$. Furthermore, if $N$ is orientable and $c_i$, $c_j$ induce the same orientation of $N$, then they also induce the same orientation of $N'$. Suppose that $a_i \in A'$ is a two-sided limit curve of $N$. Then $a_i = \rho_{A'} \circ a'_i$ for $a'_i\colon S^1 \to \partial N$ and $b_{\sigma(i)} = h \circ a_i^{\pm 1} = h \circ \rho_{A'} \circ (a'_i)^{\pm 1} = \rho_{B'} \circ h' \circ (a'_i)^{\pm 1}$. Hence $b_{\sigma(i)}$ is a two-sided limit curve of $N'$. Clearly if $a_i$ is separating then so is $b_{\sigma(i)}$. Similarly, if $a_i$ is a one-sided limit curve of $N$ and $a_i^2 = \rho_{A'} \circ a'_i$, then $b^2_{\sigma(i)} = \rho_{B'} \circ h' \circ (a'_i)^{\pm 1}$ and $b_{\sigma(i)}$ is a one-sided limit curve of $N'$.

Assume now that there exists a permutation $\sigma \in \Sigma_r$, such that for each subfamily $A' \subseteq A$ the conditions of the proposition are satisfied. Let us assume, for simplicity, that $\sigma$ is the trivial permutation $\sigma(i) = i$ for $1 \leq i \leq r$. Denote by $N_1, \ldots, N_k$ the connected components of $F_A$, and by $N'_1, \ldots, N'_k$ the corresponding components of $F_B$. By the classification of compact surfaces there exist diffeomorphisms $h_i\colon N_i \to N'_i$, $1 \leq i \leq k$, such that for each exterior boundary curve $c_l\colon S^1 \to \partial N_i$ we have $h_i \circ c_l = c_l^{\pm 1}$, and if $a_j$ is a limit curve of $N_i$, then $\rho_B \circ h_i \circ a'_j = b_j^{\pm 1}$ if $a_j = \rho_A \circ a'_j$, and $\rho_B \circ h_i \circ a'_j = (b_j^2)^{\pm 1}$ if $a_j^2 = \rho_A \circ a'_j$. We will show that we can choose $h_i$ so that for each boundary curve

$$h_i \circ c_l = c_l, \tag{5.1}$$

and for each two-sided limit curve $a_j$ of $N_i$ and $N_m$, if $a_j = \rho_A \circ a'_j = \rho_A \circ a''_j$ then

$$\rho_B \circ h_i \circ a'_j = b_j \iff \rho_B \circ h_m \circ a''_j = b_j. \tag{5.2}$$

Notice that if $h_i$ satisfy (5.1) and (5.2), then they induce $h \in \mathrm{Diff}(F)$ such that $h \circ a_j = b_j^{\pm 1}$ for $1 \leq j \leq r$, which proves Proposition 5.2.

If all $N_i$ are non-orientable, then by Lemma 5.1 we can compose $h_i$ with suitable boundary slides, so that (5.1) and (5.2) are satisfied. Suppose that $N_1, \ldots, N_s$, $1 \leq s \leq k$ are all orientable components of $F_A$. We define $A' \subseteq A$ to be any maximal subfamily consisting of separating limit curves of $N_1, \ldots, N_s$ such that: the surface $M$ obtained by gluing $\coprod_{i=1}^{s} N_i$ along $A'$ is orientable; each $a_i \in A'$ separates $M$, i.e. $M \setminus a_i$ has more connected components than $M$. Notice that $A'$ may be empty. The surface $M$ is in general disconnected and it is the sum of all orientable components of $F_{A\setminus A'}$. Let $M'$ denote the surface obtained by gluing $\coprod_{i=1}^{s} N_i'$ along $B'$, where $b_i \in B' \Leftrightarrow a_i \in A'$. Notice that $M'$ is the sum of all orientable components of $F_{B\setminus B'}$. We claim that we can choose $h_i$ for $i \leq s$, so that (5.2) holds for each $a_j \in A'$. First notice, that after re-numbering the orientable components of $F_A$ if necessary, we may assume that for each $m \leq s$ there is at most one $a_j \in A'$ such that $a_j$ is a separating limit curve of $N_m$ and $N_i$ for $i < m$. Now we define $h_i$ inductively. We choose any $h_1$. Suppose that we have chosen $h_i$ for all $i < m \leq s$. If there is $a_j \in A'$ such that $a_j$ is a separating limit curve of $N_m$ and $N_i$ for $i < m$, then we choose $h_m$ so that (5.2) is satisfied. If there is no such curve, then we choose any $h_m$. Such chosen $h_i$ induce $\tilde{h}\colon M \to M'$, so that $\tilde{h} \circ c_l = c_l^{\pm 1}$ for each exterior boundary curve of $M$. Let $c_i$, $c_j$ be two exterior boundary curves of one component of $M$. Since $A \setminus A'$ and $B \setminus B'$ satisfy the conditions of the proposition, $c_i$ and $c_j$ induce the same orientation of the component of $M$ if and only if they induce the same orientation of the corresponding component of $M'$, hence $\tilde{h} \circ c_i = c_i \Leftrightarrow \tilde{h} \circ c_j = c_j$. Now it is clear that composing if necessary some $h_i$ with orientation reversing diffeomorphism, we can assume $\tilde{h} \circ c_l = c_l$ for each exterior boundary curve of $M$. Thus $h_i$ also satisfy (5.1).

Suppose that $a_j \in A \setminus A'$ is a two-sided limit curve of $N_i$ and $N_m$, $i \leq m \leq s$. Since $A'$ is maximal, $a_j$ is a non-separating limit curve of some component $M_j$ of $M$, i.e. $a_j = \rho_{A\setminus A'} \circ a_j' = \rho_{A\setminus A'} \circ a_j''$ for $a_j', a_j''\colon S^1 \to M_j$. Then $b_j = \rho_{B\setminus B'} \circ b_j' = \rho_{B\setminus B'} \circ b_j''$ for $b_j' = \tilde{h} \circ (a_j')^{\pm 1}$, $b_j'' = \tilde{h} \circ (a_j'')^{\pm 1}$. Note that the surface obtained from $M_j$ by gluing along $a_j$ is orientable if and only if $a_j'$ and $a_j''$ induce opposite orientations of $M_j$. Since $A \setminus (A' \cup \{a_j\})$ and $B \setminus (B' \cup \{b_j\})$ satisfy the conditions of the proposition, the surface obtained from $M$ by gluing along $a_j$ is diffeomorphic to the surface obtained by gluing $M'$ along $b_j$. In particular, one of these surfaces is orientable if and only if the other one is. Hence $a_j'$ and $a_j''$ induce the same orientation of $M_j$ if and only if $b_j'$ and $b_j''$ induce the same orientation of $\tilde{h}(M_j)$. Thus $\tilde{h} \circ a_j' = b_j' \Leftrightarrow \tilde{h} \circ a_j'' = b_j''$ and so (5.2) holds for $a_j$.

Once we have chosen $h_i$ for $i \leq s$, it is easy to construct, using Lemma 5.1, diffeomorphisms $h_i$ for $i > s$ satisfying (5.1) and (5.2) for all curves. $\square$

**Corollary 5.3.** *There are only finitely many $\mathcal{M}(F)$-orbits in $\mathcal{C}(F)$.*

Proof. Let $N$ be a disjoint union of $g+n-2$ pairs of pants. Choose boundary curves of $N$

$$c_1, \ldots, c_n, a_1', \ldots, a_s', a_1'', \ldots, a_r'', \tag{5.3}$$

where $r \leq s$, $n+r+s = 3(g+n-2)$. Consider the surface $M = N/\sim$, where $\sim$ identifies pairs of boundary points as follows: $a_i'(z) = a_i''(z)$ for $i \leq r$, $a_i'(z) = a_i'(z^2)$ for $i > r$. Let $\rho\colon N \to M$ denote the canonical projection. Generic family of disjoint curves $(a_1, \ldots, a_s)$, where $a_i = \rho \circ a_i'$ for $i \leq r$, $a_i^2 = \rho \circ a_i'$ for $i > r$, determines a pants

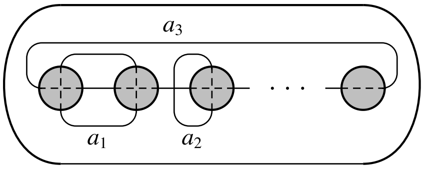


Fig. 1. Non-separating curves.

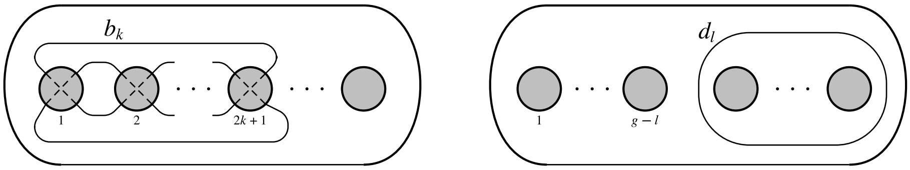


Fig. 2. Separating curves.

decomposition of $M$. Notice that for some choices of curves (5.3) we have $M = F_g^n$, i.e. $M$ is a connected, non-orientable surface of genus $g$. Furthermore, every pants decomposition of $F_g^n$ can be obtained in this way, and thus, by Proposition 5.2, there is at most as many $\mathcal{M}(F)$-orbits of pants decompositions, as the number of different (i.e. not isotopic) choices of curves (5.3). Since that number is finite and every generic family of disjoint curves can be completed to a pants decomposition, there are only finitely many $\mathcal{M}(F)$-orbits in $\mathcal{C}(F)$. $\square$

Let us list all $M(F)$-orbits of the vertices of $\mathcal{C}(F)$. We call a vertex $[a]$ one- or two-sided, and separating or non-separating if $a$ has the appropriate property.

Suppose that $F$ is closed and has genus $g \geq 3$. Consider the three non-separating curves $a_1$, $a_3$, $a_3$ in Fig. 1. In this figure, and also in other figures in this paper, the shaded discs represent crosscaps; this means that their interiors should be removed and then the antipodal points in each boundary component should be identified. We have:

- $a_1$ is two-sided, $F_{a_1}$ is non-orientable;
- $a_2$ is one-sided, $F_{a_2}$ is non-orientable;
- $F_{a_3}$ is orientable, $a_3$ is one-sided if $g$ is odd, and two-sided if $g$ is even.

For each integer $k$, such that $1 \leq k \leq (g/2) - 1$ and for each $l$ such that $2 \leq l \leq g/2$ we define separating generic curves $b_k$ and $d_l$ represented in Fig. 2. We have:

- one component of $F_{b_k}$ is orientable and has genus $k$, the other component is non-orientable and has genus $g - 2k$;
- both components of $F_{d_l}$ are non-orientable and have genera $l$ and $g - l$.

By Proposition 5.2, every vertex of $\mathcal{C}(F)$ is $\mathcal{M}(F)$-equivalent to one of the vertices $[a_1]$, $[a_2]$, $[a_3]$, $[b_k]$, $[d_l]$. Thus we have 3 orbits of non-separating vertices and $2([g/2] - 1)$ orbits of separating vertices, where $[g/2]$ denotes the integer part of $g/2$.

Now suppose that $F$ has boundary, that is $n \geq 1$, and $g$ is arbitrary such that $\chi(F) = 2 - g - n < 0$. For each pair $\{I, I'\}$ of sets such that $I \cup I' = \{1,\dots,n\}$, $I \cap I' = \emptyset$ there is one $\mathcal{M}(F)$-orbit consisting of all non-separating vertices $[a]$ such that

- $F_a$ is orientable, and $c_i$, $c_j$ induce the same orientation of $F_a$ if and only if $\{i, j\} \subseteq I$ or $\{i, j\} \subseteq I'$.

There are $2^{n-1}$ such orbits. The remaining non-separating vertices have form $[a]$, where $F_a$ is non-orientable. If $g = 1$ then there are no such vertices. If $g = 2$ then they are all one-sided and form one $\mathcal{M}(F)$-orbit. If $g \geq 3$ then they form 2 orbits, one contains all one-sided vertices, the other one contains all two-sided vertices.

The orbits of separating vertices are of two types, like for closed $F$. For every integer $k$ such that $0 \leq k \leq (g-1)/2$, and pair $\{I, J\}$ of disjoint subsets of $\{1, \dots, n\}$ such that $g + n - 2 \geq 2k + \#(I \cup J) \geq 2$ there is one $\mathcal{M}(F)$-orbit consisting of all separating vertices $[b]$ such that

- $F_b$ has one orientable component $N_o$ of genus $k$ and one non-orientable component $N_n$ of genus $g - 2k$;
- $c_i \subset N_o \Leftrightarrow i \in (I \cup J)$; $c_i$, $c_j$ induce the same orientation of $N_o$ if and only if $\{i, j\} \subseteq I$ or $\{i, j\} \subseteq J$.

For every integer $l$ such that $1 \leq l \leq g/2$ and every $I \subseteq \{1, \dots, n\}$ such that $l + \#I \geq 2$ there is one $\mathcal{M}(F)$-orbit consisting of all separating vertices $[d]$ such that

- $F_d$ has two non-orientable components $N_1$ and $N_2$ of genera $l$ and $g - l$ respectively; $c_i \subset N_1 \Leftrightarrow i \in I$.

## 6. The presentation for $\mathcal{M}(F)$

In [4] Brown describes a method to produce a presentation of a group acting on a simply-connected CW-complex. In [1] Benvenuti uses a special case of Brown's theorem to obtain a presentation for the orientable mapping class group from its action on the ordered complex of curves. In this section we apply the method of [1] to the case of a non-orientable surface.

The following theorem is fundamental for this section.

**Theorem 6.1** (Ivanov [12]). *Let $F = F_g^n$ denote a non-orientable surface of genus $g$ with $n$ boundary components and $\mathcal{C}(F)$ the complex of curves of $F$. Then $\mathcal{C}(F)$ is $(g-3)$-connected if $n \in \{0, 1\}$, and $(g + n - 5)$-connected if $n \geq 2$.*

In particular, except for the surfaces $F_g^n$ where

$$(g, n) \in \{(1, n) \mid n \leq 4\} \cup \{(2, n) \mid n \leq 3\} \cup \{(3, n) \mid n \leq 2\}$$

that we call *sporadic*, the complex of curves of $F_g^n$ is simply connected.

Now we define, following [1], the *ordered complex of curves* of $F$ denoted by $\mathcal{C}^{\text{ord}}(F)$. The $r$-simplices of $\mathcal{C}^{\text{ord}}(F)$ are equivalence classes of *ordered* generic $(r+1)$-families of disjoint curves: $(a_1, \dots, a_r)$ and $(b_1, \dots, b_r)$ represent the same $(r-1)$-simplex in $\mathcal{C}^{\mathrm{ord}}(F)$ if and only if $a_i \simeq b_i^{\pm 1}$ for all $i \in \{1, \ldots, r\}$. We denote by $\langle a_1, \ldots, a_r \rangle$ the simplex of $\mathcal{C}^{\mathrm{ord}}(F)$ represented by the family $(a_1, \ldots, a_r)$. Note that the vertices of $\mathcal{C}^{\mathrm{ord}}(F)$ coincide with those of $\mathcal{C}(F)$ and in general to each $r$-simplex of $\mathcal{C}(F)$ correspond $(r+1)!$ different simplices of $\mathcal{C}^{\mathrm{ord}}(F)$ with the same set of vertices.

The following proposition is proved in [1]. The same proof applies to the case of a non-orientable surface.

**Proposition 6.2.** *If $\mathcal{C}(F)$ is simply connected, then $\mathcal{C}^{\mathrm{ord}}(F)$ is also simply connected.*

The mapping class group $\mathcal{M}(F)$ acts on $\mathcal{C}^{\mathrm{ord}}(F)$ by $h\langle a_1,\ldots,a_r\rangle = \langle h\circ a_1,\ldots,h\circ a_r\rangle$. Two simplices $\langle a_1,\ldots,a_r\rangle$ and $\langle b_1,\ldots,b_r\rangle$ of $\mathcal{C}^{\mathrm{ord}}(F)$ are $\mathcal{M}(F)$-equivalent if and only if the conditions of Proposition 5.2 are satisfied with $\sigma(i) = i$, $i \in \{1, \ldots, r\}$. Observe that to each $\mathcal{M}(F)$-orbit of $r$-simplices of $\mathcal{C}(F)$ correspond $(r+1)!$ orbits in $\mathcal{C}^{\mathrm{ord}}(F)$.

Let $A = (a_1, \ldots, a_r)$ be a generic $r$-family of disjoint curves. Denote by $\mathrm{Stab}(\langle A\rangle)$ the stabilizer in $\mathcal{M}(F)$ of the simplex of $\mathcal{C}^{\mathrm{ord}}(F)$ represented by $A$. The group $\mathrm{Stab}(\langle A\rangle)$ consists of those $h \in \mathcal{M}(F)$ which satisfy $h\circ a_i \simeq a_i^{\pm 1}$ for $i \in \{1,\ldots,r\}$. It is clearly a subgroup of $\mathrm{Stab}([A])$ and by Proposition 4.2, we have the following exact sequence:

$$1 \to \mathrm{Stab}^{+}([A]) \to \mathrm{Stab}(\langle A\rangle) \xrightarrow{\Phi_A} (\mathbb{Z}_2)^r. \tag{6.1}$$

Here $(\mathbb{Z}_2)^r$ is identified with the subgroup of $\mathrm{Cub}_r$ consisting of those $\phi \in GL(\mathbb{R}^r)$ such that $\phi(e_i) = \pm e_i$ for all $1 \le i \le r$.

Denote by $X$ the orbit space $\mathcal{C}^{\mathrm{ord}}(F)/\mathcal{M}(F)$ and by $p\colon \mathcal{C}^{\mathrm{ord}}(F) \to X$ the canonical projection. The space $X$ inherits from $\mathcal{C}^{\mathrm{ord}}(F)$ the structure of a CW-complex; the $r$-cells of $X$ correspond to the $\mathcal{M}(F)$-orbits of $r$-simplices of $\mathcal{C}^{\mathrm{ord}}(F)$.

By Remark 5.3, $X$ is a finite CW-complex. We denote by $X^r$ the $r$-skeleton of $X$. Since the edges of $\mathcal{C}^{\mathrm{ord}}(F)$ are oriented and the action of $\mathcal{M}(F)$ preserves the orientation, the edges of $X$ are also oriented. If $e$ is an edge in either $\mathcal{C}^{\mathrm{ord}}(F)$ or $X$ then we denote by $i(e)$ and $t(e)$ respectively the initial and terminal vertex of $e$. An edge $e \in X^1$ for which $i(e) = t(e) = v$ is called a *loop* based at $v$.

The advantage of the ordered complex of curves over the ordinary complex of curves is that $\mathcal{M}(F)$ acts on $(\mathcal{C}^{\mathrm{ord}}(F))^1$ without inversion, which simplifies the statement of Theorem 6.3 below.

In order to describe a presentation for $\mathcal{M}(F)$ we need to make a number of choices:

(a) We choose a maximal tree $\mathcal{T}$ in $X^1$.

(b) For every $v \in X^0$ we choose a representative $s(v) \in (\mathcal{C}^{\mathrm{ord}}(F))^0$, and for every $e \in X^1$ a representative $s(e) \in (\mathcal{C}^{\mathrm{ord}}(F))^1$ (i.e. $p(s(v)) = v$ and $p(s(e)) = e$), so that $s(i(e)) = i(s(e))$ for every $e \in (\mathcal{C}^{\mathrm{ord}}(F))^1$, and $s(t(e)) = t(s(e))$ for every $e \in \mathcal{T}$. We denote by $S_v$ the stabilizer $\mathrm{Stab}(s(v))$ and by $S_e$ the stabilizer $\mathrm{Stab}(s(e))$.

(c) For every $e \in (\mathcal{C}^{\mathrm{ord}}(F))^1$ we choose $g_e \in \mathcal{M}(F)$ such that

$$g_e(s(t(e))) = t(s(e)).$$

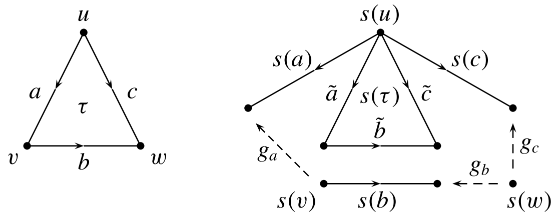


Fig. 3. A triangle in $X$ and its representative in $\mathcal{C}^{\mathrm{ord}}(F)$.

If $e \in \mathcal{T}$ then we take $g_e = 1$. Note, then, that the conjugation map $c_e$ given by $g \mapsto g_e^{-1} g g_e$ maps $\mathrm{Stab}(t(s(e)))$ onto $\mathrm{Stab}(s(t(e)))$; in particular, $c_e(S_e) \subseteq S_{t(e)}$.
(d) For every triangle $\tau \in X^2$, with edges $a$, $b$, $c$ such that $i(c) = i(a) = u$, $t(a) = i(b) = v$, $t(b) = t(c) = w$, we choose a representative $s(\tau)$ in $(\mathcal{C}^{\mathrm{ord}}(F))^2$, such that if $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ are the corresponding edges of $s(\tau)$, then $i(\tilde{c}) = i(\tilde{a}) = s(u)$ (see Fig. 3). We also choose three elements

$$h_{\tau,a} \in S_u, \quad h_{\tau,b} \in S_v, \quad h_{\tau,c} \in S_w,$$

such that $h_{\tau,a}(s(a)) = \tilde{a}$, $h_{\tau,a} g_a h_{\tau,b}(s(b)) = \tilde{b}$, $h_{\tau,a} g_a h_{\tau,b} g_b h_{\tau,c} g_c^{-1}(s(c)) = \tilde{c}$. Let us define $h_\tau = h_{\tau,a} g_a h_{\tau,b} g_b h_{\tau,c} g_c^{-1}$. Observe, that $h_\tau \in S_u$.

The next result is a special case of a general theorem of Brown [4] (cf. Theorem 3 of [1]).

**Theorem 6.3.** *Suppose that $F$ is not sporadic and*:
(i) *for each $v \in X^0$ the group $S_v$ has the presentation $S_v = \langle G_v \mid R_v \rangle$*,
(ii) *for each $e \in X^1$ the stabilizer $S_e$ is generated by $G_e$.*
*Then $\mathcal{M}(F)$ admits the presentation*:

$$\textit{generators} = \bigcup_{v \in X^0} G_v \cup \{g_e \mid e \in X^1\},$$
$$\textit{relations} = \bigcup_{v \in X^0} R_v \cup R^{(1)} \cup R^{(2)} \cup R^{(3)},$$

*where*:

$$R^{(1)} = \{g_e = 1 \mid e \in \mathcal{T}\}.$$
$$R^{(2)} = \{g_e^{-1} i_e(g) g_e = c_e(g) \mid g \in G_e,\, e \in X^1\},$$

*where $i_e$ is the inclusion $S_e \hookrightarrow S_{i(e)}$ and $c_e \colon S_e \to S_{t(e)}$ is as in* (c) *above.*

$$R^{(3)} = \{h_{\tau,a} g_a h_{\tau,b} g_b h_{\tau,c} g_c^{-1} = h_\tau \mid \tau \in X^2\}.$$

In Theorem 6.3, $i_e(g)$, $c_e(g)$, $h_{\tau,a}$, $h_{\tau,b}$, $h_{\tau,c}$ and $h_\tau$ should be expressed as words in the generators $\bigcup_{v \in X^0} G_v$.

Suppose that two of the edges of a triangle $\tau \in (X)_2$ belong to the maximal tree $\mathcal{T}$. Then, using the relations $R^{(1)}$ and $R^{(3)}$ we can express the generating symbol corresponding to the third edge as a product of stabilizers of the representatives for the vertices. The same is true if two of the symbols for the edges were already expressed as products of stabilizers. We say that a symbol $g_e$ is *determinable* (or simply that the corresponding edge $e$ is determinable), if using recursively relations $R^{(1)}$ and $R^{(3)}$, it is possible to express $g_e$ as a product of elements in $\bigcup_{v \in X^0} G_v$. Thus, every edge $e \in \mathcal{T}$ is determinable, and if a triangle in $X^2$ has two determinable edges, then its third edge is also determinable.

**Theorem 6.4.** *Suppose that $F_g^n$ is not sporadic. Then there exists a choice of the maximal tree $\mathcal{T}$ such that all the edges of $X$ are determinable.*

Proof. We fix boundary curves $c_1, \ldots, c_n$. For each generic family of disjoint curves $A$ we identify a generic curve $b$ in $F_A$ with the curve $\rho_A \circ b$ in $F$. For any surface $X$, we denote by $g(X)$ its genus.

**Construction of $\mathcal{T}$ for $g \geq 4$.** Suppose that $g \geq 4$. Let $v_1$ denote the non-separating, two-sided vertex $v_1 = p([a])$, where $F_a$ is non-orientable. For each vertex $v$ different from $v_1$, we define an edge $e_v \in X^1$ with initial vertex $v_1$ and terminal vertex $v$ as follows. We fix a curve $b$, such that $p([b]) = v$ and construct $a$ in $F_b$, such that $p([a]) = v_1$. We consider cases.

CASE 1. $b$ is non-separating and $F_b$ is non-orientable. Since $v \neq v_1$, $b$ must be one-sided and from the comparison of Euler characteristics we know that $g(F_b) \geq 3$. We define $a$ to be any two-sided and non-separating curve in $F_b$, such that $F_{(a,b)}$ is non-orientable.

CASE 2. $b$ is non-separating and $F_b$ is orientable. Now $F_b$ has genus at least 1 and hence it contains a non-separating curve. Let $a$ be any such curve. Note that $F_a$ is non-orientable because we can construct a one-sided curve in $F_a$ by connecting two boundary points of $F_{(a,b)}$ by an arc.

CASE 3. $b$ is separating, $F_b = N \amalg N'$. We consider two sub-cases.

CASE 3a. One of the components, say $N$, is orientable. If $g(N) \geq 1$ then we define $a$ to be any non-separating curve in $N$ (note that $N'$ is non-orientable, and hence so is $F_a$). If $g(N) = 0$, then we define $a$ to be any non-separating, two-sided curve in $N'$, such that $N'_a$ is non-orientable (such curve exists, as $g(N') = g \geq 4$).

CASE 3b. Both components $N$ and $N'$ are non-orientable. Assume $g(N) \geq g(N')$. If $g(N) = g(N')$ and $n \geq 1$, then we assume that $N$ contains the boundary curve $c_1$. If $g(N) \geq 3$ then we define $a$ to be any non-separating, two-sided curve in $N$, such that $N_a$ is non-orientable. If $g(N) = 2$, then we choose for $a$ any non-separating, two-sided curve in $N$, such that all exterior boundary curves of $N$ induce the same orientation of $N_a$. If $F$ is closed and $g(N) = g(N')$, then we can not distinguish between $N$ and $N'$. However, whether we choose $a$ in $N$ or $N'$, we obtain $\mathcal{M}(F)$-equivalent edges $\langle a, b \rangle$.

In each case we have $p([a]) = v_1$ and we define $e_v = p(\langle a, b\rangle)$. By Proposition 5.2 this definitions do not depend on the choices of the curves $a$ and $b$. We define the maximal tree $\mathcal{T} = \{e_v \mid v \neq v_1\}$.

REMARK 6.5. Suppose that $F$ is closed and consider the curves $a_1$, $a_2$, $a_3$, $b_k$, $d_l$ in Figs. 1 and 2. As it was discussed in Section 5, these curves represent all vertices of $X$. Clearly $p([a_1]) = v_1$ and in the construction of the maximal tree described above we can take $b$ to be $a_2$ (Case 1), $a_3$ (Case 2), $b_k$ (Case 3a) or $d_l$ (Case 3b). Then, in each case, we can take $a = a_1$. Thus

$$\mathcal{T} = \left\{ p(\langle a_1, a_2\rangle),\ p(\langle a_1, a_3\rangle),\ p(\langle a_1, b_k\rangle),\ p(\langle a_1, d_l\rangle) \;\middle|\; 2 \leq k+1,\ l \leq \frac{g}{2} \right\}.$$

**Lemma 6.6.** *Suppose that $g \geq 4$ and $\mathcal{T}$ is defined as above. Then the following edges of $X$ are determinable*:
(i) *all the loops based at $v_1$*;
(ii) *all the edges with both ends in non-separating vertices*;
(iii) *all the edges with one end in a non-separating vertex and the other end in a separating vertex*;
(iv) *all the edges with both ends in separating vertices*.

Proof. Let $e = p(\langle a, b\rangle)$ be any edge in $X$ and let $F'$ denote the surface $F_{(a,b)}$.

(i) Suppose $p([a]) = p([b]) = v_1$. The surface $F'$ is either connected or it has two connected components, at least one of which must be non-orientable.

Suppose that $F'$ has a non-orientable connected component of genus at least 2 or it has two non-orientable components. Then there exists a one-sided curve $c$ in $F'$ such that $F_{(a,c)}$ and $F_{(b,c)}$ are non-orientable. By the definition of edges $e_v$ (Case 1), we have that $p(\langle a, c\rangle) = p(\langle b, c\rangle) = e_{p([c])}$, the triangle $p(\langle a, b, c\rangle)$ has two edges in $\mathcal{T}$, and thus $e$ is determinable.

Suppose now that $F'$ is connected and orientable. Let $a'$, $a''$, $b'$, $b''$ denote the boundary curves of $F'$ such that $\rho_{(a,b)} \circ a' = \rho_{(a,b)} \circ a'' = a$, $\rho_{(a,b)} \circ b' = \rho_{(a,b)} \circ b'' = b$. Let $c$ be a separating curve in $F'$ such that $\{a', b'\}$ and $\{a'', b''\}$ are in different components of $F'_c$. Observe that $c$ is non-separating in $F$. Every one-sided curve in $F$ intersects $a \cup b$ odd number of times, thus it intersects $c$. Hence $F_c$ is orientable and $p([c]) \neq v_1$. The triangle $p(\langle a, b, c\rangle)$ has edges $e$, $e_{p([c])}$, $e_{p([c])}$ (Case 2 in the construction of $\mathcal{T}$), thus $e$ is determinable.

Finally suppose that $F'$ has two components $N_1$ and $N_2$, such that $N_1$ is non-orientable of genus 1 and $N_2$ is orientable. Since $g(N_2) \geq 1$, there is a non-separating two-sided curve $c$ in $N_2$. Note that $p([c]) = v_1$ and the loops $p(\langle a, c\rangle)$, $p(\langle b, c\rangle)$ are determinable by previous arguments, because $F_{(a,c)}$ and $F_{(b,c)}$ are connected. Hence $e$ is also determinable, by $p(\langle a, b, c\rangle)$.

(ii) Suppose that both ends of $e$ are non-separating. If both of them are one-sided, then $F'$ is connected and has genus at least 1 if it is orientable, or at least 2 if it is non-orientable. In both cases $F'$ contains a non-separating, two-sided curve $c$. Now $p(\langle c, a\rangle) = e_p([a])$, $p(\langle c, b\rangle) = e_p([b])$ (Case 1 in the construction of $\mathcal{T}$), hence $e$ is determinable by $p(\langle c, a, b\rangle)$.

Suppose that one end of $e$ is one-sided and the other one is two-sided. Then $F'$ is connected and the two-sided end is $v_1$. Thus if $b$ is one-sided, then $e = e_{p([b])}$. If $a$ is one-sided, then we choose any separating curve $c$ in $F'$, such that $F_c$ is connected. Now $p(\langle c, a\rangle) = e_{p([a])}$ and $p(\langle c, b\rangle)$ is a loop at $v_1$, which is determinable by (i). Hence $e$ is determinable by $p(\langle c, a, b\rangle)$.

Suppose that both ends of $e$ are two-sided. We can assume that at least one of the ends is not $v_1$, so $F'$ is orientable. If $F'$ is connected, then we choose a separating generic curve $c$ in $F'$, such that $F_{(a,c)}$ and $F_{(b,c)}$ are connected. Now $p(\langle c, a\rangle)$ is either $e_{p([a])}$ (if $F_a$ is orientable) or a loop at $v_1$ (if $F_a$ is non-orientable) and similarly for $p(\langle c, b\rangle)$. Hence $e$ is determinable by $p(\langle c, a, b\rangle)$. If $F'$ is not connected, then $F_a$ and $F_b$ are orientable. Now $F'$ has a component $N$ with $g(N) \geq 1$ and for any non-separating curve $c$ in $N$ we have $p(\langle c, a\rangle) = e_{p([a])}$ and $p(\langle c, b\rangle) = e_{p([a])}$. Hence $e$ is determinable by $p(\langle c, a, b\rangle)$.

(iii) Assume, without loss of generality, that $a$ is separating and $b$ is non-separating. Suppose that both components of $F_a$ have genus $\geq 1$. Let $a_1$ be a generic curve in $F'$ such that $p(\langle a_1, a\rangle) \in \mathcal{T}$, and choose any non-separating curve $c$ in the other component of $F_a$. Notice that $p(\langle c, a_1\rangle)$ is determinable by (ii), and $p(\langle c, a\rangle)$ is determinable by the triangle $p(\langle c, a_1, a\rangle)$. Now if $a_1$ and $b$ belong to different components of $F_a$, then $p(\langle a_1, b\rangle)$ is determinable by (ii), and $e$ is determinable by $p(\langle a_1, a, b\rangle)$. If $a_1$ and $b$ belong to the same component of $F_a$, then $e$ is determinable by $p(\langle c, a, b\rangle)$. If one of the components has genus 0, then $b$ is contained in the other component $N$. Now there exists a two-sided generic curve $a_1$ in $N_b$, such that $N_{a_1}$ is connected and non-orientable. Indeed, if $N_b$ is orientable, then $g(N_b) \geq 1$ and $a_1$ may be any non-separating curve in $N_b$. If $N_b$ is non-orientable, then $g(N_b) \geq 2$ and we may take $a_1$ to be separating in $N_b$. For such $a_1$ we have $p(\langle a_1, a\rangle) \in \mathcal{T}$, and $p(\langle a_1, b\rangle)$ is determinable by (ii). Hence $e$ is determinable by $p(\langle a_1, a, b\rangle)$.

To prove (iv) notice that in this case $F'$ must have a non-orientable component. Choose a one-sided curve $c$ in $F'$ and consider the triangle $p(\langle c, a, b\rangle)$. The assertion follows by (iii). □

This finishes the proof of Theorem 6.4 for $g \geq 4$.

**Construction of $\mathcal{T}$ for $g = 3$.** Suppose that $g = 3$. Since $F$ is not sporadic we have $n \geq 3$. Let $v_1$ denote the non-separating, two-sided vertex $p([a])$, where $F_a$ is non-orientable. Note that this is the only non-separating, two-sided vertex in $X$. As we did for $g \geq 4$, for each $v \neq v_1$ we define an edge $e_v$ form $v_1$ to $v$. We fix $b$ such that $v = p([b])$ and define $a$ in $F_b$ so that $p([a]) = v_1$.

CASE 1. $b$ is one-sided and $F_b$ is non-orientable. Now $F_b$ has genus 2. We define $a$ to be any two-sided and non-separating curve in $F_b$, such that all exterior boundary curves induce the same orientation of $F_b$.

If $F_b$ is connected and orientable (Case 2) or disconnected (Case 3), then we define $a$ in the same way as we did for $g \geq 4$. We only remark that in Case 2, $b$ is one-sided and hence $F_b$ has genus 1; and in Case 3a, if $g(N) = 0$ then $g(N') = 3$, which suffices to choose two-sided and non-separating $a$ with $N'_a$ non-orientable.

As previously we define $e_v = p(\langle a, b\rangle)$ and $\mathcal{T} = \{e_v \mid v \neq v_1\}$.

**Lemma 6.7.** *Suppose that $g = 3$ and $\mathcal{T}$ is defined as above. Then the following edges of $X$ are determinable*:

(i) *all the loops based at $v_1$*;

(ii) *all the edges with one end in $v_1$*;

(iii) *all the edges with at least one edge in one-sided vertex*;

(iv) *all the edges with both ends in separating vertices*.

Proof. First observe that every edge in $X$ satisfies one of the conditions (i)–(iv). Therefore Lemma 6.7 implies Theorem 6.4 for $g = 3$.

Let $e = p(\langle a, b\rangle)$ be any edge in $X$ and $F' = F_{(a,b)}$.

(i) If $p([a]) = p([b]) = v_1$ then $F'$ has two connected components, at least one of which contains two exterior boundary curves. Let $c$ be a curve in $F'$ bounding a pair of pants together with two exterior boundary curves. The edge $e$ is determinable by the triangle $p(\langle a, b, c\rangle)$ having two edges in $\mathcal{T}$.

(ii) Assume $p([a]) = v_1$. If $F_b$ is connected and orientable or it has an orientable component, then $e \in \mathcal{T}$. In the other case $e \in \mathcal{T}$ if and only if all exterior boundary curves induce the same orientation of the orientable component of $F'$. Suppose that $e \notin \mathcal{T}$. Denote by $N$ the connected component of $F_b$ having genus 2 and by $N'$ the orientable component of $F'$ (thus $N' = N_a$). There exists a separating curve $c$ in $N'$, which is non-separating in $N$ and such that any two exterior boundary curves induce opposite orientations of $N'$ if and only if they belong to different components of $N'_c$. The surface $N_c$, which can be obtained from $N'_c$ by gluing along $a$, is the orientable component of $N_{(b,c)}$. Note that all exterior boundary curves induce the same orientation of $N_c$, hence $p(\langle c, b\rangle) = e_{p([b])}$. The loop $p(\langle c, a\rangle)$ is determinable by (i), thus $e$ is determinable by $p(\langle c, a, b\rangle)$.

Now assume $p([b]) = v_1$ and choose any generic curve $d$ in $F'$. The edges $p(\langle b, a\rangle)$ and $p(\langle b, d\rangle)$ have initial vertex $v_1$ and we have already proved that such edges are determinable. Hence $p(\langle a, d\rangle)$ is determinable by $p(\langle b, a, d\rangle)$, and $e$ by $p(\langle a, b, d\rangle)$.

(iii) Suppose that $e$ has both ends in one-sided vertices. Choose any curve $c$ in $F'$ bounding a pair of pants together with two exterior boundary curves. Let $d$ be any two-sided non-separating curve in $F_{(a,c)}$. Then $p([d]) = v_1$, and $p(\langle d, c\rangle)$ and $p(\langle d, a\rangle)$ are determinable by (ii), thus $p(\langle c, a\rangle)$ is determinable by $p(\langle d, c, a\rangle)$. Analogously $p(\langle c, b\rangle)$ is determinable by a different triangle $p(\langle d', c, b\rangle)$. Finally $e$ is determinable by $p(\langle c, a, b\rangle)$.

Suppose that $e$ has one vertex in a one-sided vertex $v$ and the other end in a separating vertex. Assume without loss of generality that $a$ is separating and denote by $N$

the component of $F_a$ which contains $b$, and the other component by $N'$. If $g(N) = 3$ or $g(N) = 1$, then $F'$ contains a non-separating two-sided curve $c$ and $e$ is determinable by $p(\langle c, a, b\rangle)$ and (ii). If $g(N) = 2$, then we choose a one-sided curve $d$ in $N'$ and two-sided, non-separating curve $c$ in $N$. Now $p(\langle a, d\rangle)$ is determinable by $p(\langle c, a, d\rangle)$ and (ii), and $p(\langle b, d\rangle)$ is an edge with two one-sided ends, determinable by previous argument. Finally $e$ is determinable by $p(\langle a, b, d\rangle)$.

(iv) If $e$ has both ends in separating vertices then $F'$ has a non-orientable connected component. Choose a one-sided curve $c$ in $F'$ and consider the triangle $p(\langle a, b, c\rangle)$. The assertion follows by (iii). □

**Construction of $\mathcal{T}$ for $g = 2$.** Suppose that $g = 2$ and $n \geq 4$. Let $v_2$ denote the unique one-sided vertex of $X$. For each separating vertex $v$ we will define an edge $e_v \in X^1$ from $v_2$ to $v$. We fix $b$ such that $p([b]) = v$ and assume $F_b = N \amalg N'$. We define $e_v = p(\langle a, b\rangle)$, where $a$ is a one-sided curve in $F_b$ defined as follows.

CASE 1. One component of $F_b$, say $N$, is orientable. Then we define $a$ to be any one-sided curve in $N'$.

CASE 2. Both components are non-orientable. Assume that $N$ contains the exterior boundary curve $c_1$. We choose $a$ in $N$, so that all exterior boundary curves of $N$ induce the same orientation of $N_a$.

Suppose that $w$ is a two-sided, non-separating vertex of $X$. Let us choose $b$ such that $p([b]) = w$. Now $F_b$ is orientable and has genus 0. We choose a curve $a$ in $F_b$ bounding a pair of pants together with the exterior boundary curves $c_1$ and $c_2$. We define $e_w = p(\langle a, b\rangle)$.

We claim that $\mathcal{T} = \{e_v \mid v \neq v_2\}$ is a maximal tree in $X^1$. First notice that $\mathcal{T}' = \{e_v \mid v \text{ is separating}\}$ is a tree, because every edge $e_v \in \mathcal{T}'$ connects $v$ to $v_2$. Now $\mathcal{T} \setminus \mathcal{T}' = \{e_w \mid w \text{ is two-sided and non-separating}\}$ and every two-sided and non-separating vertex $w$ is connected to exactly one vertex of $\mathcal{T}'$ by $e_w$. It follows that $\mathcal{T}$ indeed is a tree and since it contains all vertices of $X$ it is a maximal tree.

**Lemma 6.8.** *Suppose that $g = 2$ and $\mathcal{T}$ is defined as above. Then the following edges of $X$ are determinable*:

(i) *all the loops based at $v_2$*;

(ii) *all the edges with one end in $v_2$*;

(iii) *all the edges with both ends in two-sided vertices*;

Proof. Let $e = p(\langle a, b\rangle)$ be any edge of $X$ and $F' = F_{(a,b)}$.

(i) Suppose that $p([a]) = p([b]) = v_2$. Choose any separating generic curve $c$ in $F'$ such that one component of $F_c$ is orientable. Then $p(\langle a, c\rangle) = p(\langle b, c\rangle) = e_{p([c])}$ and hence $e$ is determinable by the triangle $p(\langle a, b, c\rangle)$.

(ii) Suppose that $e$ has one end in $v_2$ and the other end in a separating vertex $v$. Assume without loss of generality, that $a$ is separating. If $F_a$ has an orientable component then for each one-sided curve $c$ in $F'$ we have $p(\langle c, a\rangle) \in \mathcal{T}$. Now $p(\langle c, b\rangle)$ is

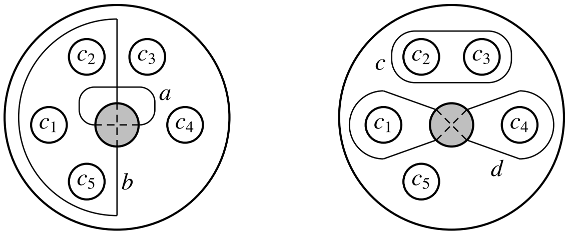


Fig. 4. Representatives of different vertices of the complex $X$ of $F_1^5$: $p([a]) = v_\emptyset$, $p([b]) = v_{\{3,4\}}$, $p([c]) = v_{\emptyset,\{2,3\}}$, $p([d]) = v_{\{1\},\{4\}}$.

determinable by (i), hence $e$ is determinable by $p(\langle c, a, b\rangle)$. Suppose that both components of $F_a$ are non-orientable and let $c$ and $d$ be two one-sided curves in different components of $F_a$, such that $p(\langle c, a\rangle) = e_v$. Since $p(\langle c, d\rangle)$ is determinable by (i), $p(\langle d, a\rangle)$ is determinable by $p(\langle c, d, a\rangle)$. We have $b \cap c = \emptyset$ or $b \cap d = \emptyset$, hence $e$ is determinable by $p(\langle c, a, b\rangle)$ or $p(\langle d, a, b\rangle)$.

(iii) If both ends of $e$ are separating, then there is a one-sided curve $c$ in $F'$ and $e$ is determinable by (ii). Suppose that $e$ has one separating and one non-separating end. Assume without loss of generality, that $a$ is non-separating. Then there is a separating generic curve $c$ in $F'$ such that all boundary curves of $F$ are contained in one connected component of $F_c$. In particular, there is a curve $d$ in $F_{(a,c)}$ bounding a pair of pants together with $c_1$ and $c_2$, that is $p(\langle a, d\rangle) = e_{p([a])}$. The edge $p(\langle c, d\rangle)$ is determinable by the previous argument, hence $p(\langle a, c\rangle)$ is determinable by $p(\langle a, c, d\rangle)$. If $c \simeq b^{\pm 1}$ then we can assume $b \cap d = \emptyset$, and $e$ is determinable by $p(\langle a, b, d\rangle)$. In the other case $e$ is determinable $p(\langle a, b, c\rangle)$. Finally suppose that both ends of $e$ are non-separating. Since $n \geq 4$, $F'$ contains a generic curve $b'$, and $e$ is determinable by $p(\langle a, b, b'\rangle)$. $\square$

**Construction of $\mathcal{T}$ for $g = 1$.** Suppose that $g = 1$ and $n \geq 5$. It follows from Proposition 5.2 that each separating vertex $p([a]) \in X^0$ is uniquely determined by a pair $I, J \subset \{1, \dots, n\}$ such that $I \cap J = \emptyset$, $2 \leq \#I + \#J \leq (n-1)$, and if $N$ is the orientable connected component of $F_a$ then

- $c_i$ is a boundary curve of $N$ if and only if $i \in I \cup J$,
- $c_i$ and $c_j$ induce the same orientation of $N$ if and only if $\{i, j\} \subseteq I$ or $\{i, j\} \subseteq J$.

We denote such vertex by $v_{I,J}$, where we assume $\#I \leq \#J$, and if $\#I = \#J$ then $\min I < \min J$. Each one-sided vertex $p([a])$ is uniquely determined by a subset $I \subset \{1, \dots, n\}$ such that $c_i$ and $c_j$ induce the same orientation of $F_a$ if and only if $\{i, j\} \subseteq I$ or $\{i, j\} \subseteq I'$, where $I' = \{1, \dots, n\} \setminus I$. We denote such vertex by $v_I$, where we assume $\#I \leq n/2$, and if $\#I = n/2$ then $1 \in I$ (see Fig. 4, where we assume that all boundary curves have positive orientations with respect to the standard orientation of the plane of the figure).

If $\#I + \#J \leq \#K + \#L$ then $v_{I,J}$ and $v_{K,L}$ are connected by an edge in $X$ if and only if one of the following conditions is satisfied:

- $I \subseteq K,\ J \subseteq L,\ \#I + \#J < \#K + \#L$;
- $I \subseteq L,\ J \subseteq K,\ \#I + \#J < \#K + \#L$;
- $(I \cup J) \cap (K \cup L) = \emptyset$.

Vertices $v_I$ and $v_{J,K}$ are connected by an edge if and only if either $J \subseteq I,\ K \subseteq I'$ or $K \subseteq I,\ J \subseteq I'$. There are no edges connecting two one-sided vertices because every two one-sided curves in a surface of genus 1 intersect. It follows that $X$ has no loops. Moreover, it follows from Proposition 5.2 that for each pair $v, w \in X^0$ there is at most one edge in $X^1$ with initial vertex $v$ and terminal vertex $w$. If such edge exists, then we denote it by $\langle v; w\rangle$. If every two of three vertices $u,\ v,\ w$ are connected by an edge in $X$, then there are 6 triangles in $X^2$ with vertices $u,\ v,\ w$. We denote by $\langle u; v; w\rangle$ the triangle with edges $\langle u; v\rangle,\ \langle u; w\rangle,\ \langle v; w\rangle$.

We define the maximal tree as

$$\mathcal{T} = \bigcup_{v_{I,J} \in X^0} \{\langle v_I; v_{I,J}\rangle\} \cup \bigcup_{v_I \in X^0 \setminus \{v_\emptyset\}} \{\langle v_I; v_{\emptyset,I'}\rangle\}.$$

**Lemma 6.9.** *Suppose that $g = 1$ and $\mathcal{T}$ is defined as above. Then the following edges of $X$ are determinable*:

(i) *all edges with ends in $v_{I,J}$ and $v_{K,L}$, where $I \subseteq K,\ J \subseteq L$*;

(ii) *all edges with ends in $v_{I,J}$ and $v_{K,L}$, where $(I \cup J) \cap (K \cup L) = \emptyset$*;

(iii) *all edges with ends in $v_{I,J}$ and $v_{K,L}$, where $I \subseteq L,\ J \subseteq K$*;

(iv) *all edges with ends in $v_{I,J}$ and $v_K$.*

Proof. Let $e$ be an edge with ends in $v_{I,J}$ and $v_{K,L}$.

(i) If $I = K$ then $e$ is determinable by a triangle with third vertex $v_I$. Suppose $I \subsetneq K,\ J = L$. The edge $\langle v_{\emptyset,J}; v_{\emptyset,K'}\rangle$ is determinable by the previous argument, hence $\langle v_K; v_{\emptyset,J}\rangle$ is determinable by $\langle v_K; v_{\emptyset,J}; v_{\emptyset,K'}\rangle$. If $I = \emptyset$ then $e$ is determinable by the triangle with edges $e,\ \langle v_K; v_{\emptyset,J}\rangle$ and $\langle v_K; v_{K,J}\rangle$. If $I \neq \emptyset$ then $e$ is determinable by the triangle with edges $e,\ \langle v_{I,J}; v_{\emptyset,J}\rangle,\ \langle v_{K,J}; v_{\emptyset,J}\rangle$, whose last two edges are determinable by the previous argument. Finally, if $I \subsetneq K$ and $J \subsetneq L$ then $e$ is determinable by the triangle with edges $e,\ \langle v_{I,J}; v_{I,L}\rangle,\ \langle v_{K,L}; v_{I,L}\rangle$, because the last two edges are determinable by previous arguments.

(ii) If $\#(I \cup J \cup K \cup L) < n$ then $e$ is determinable by a triangle with third vertex $v_{I\cup K, J\cup L}$, whose remaining two edges are determinable by (i). If $\#(I \cup J \cup K \cup L) = n$ then we assume $\#(I \cup J) \geq 3$. Then there is a vertex $v_{M,N}$ such that $M \subseteq I,\ N \subseteq J$ and $\#(M \cup N) < \#(I \cup J)$. Now $\langle v_{M,N}; v_{K,L}\rangle$ is determinable by the previous argument, and $\langle v_{M,N}; v_{I,J}\rangle$ is determinable by (i). Hence $e$ is also determinable.

(iii) Suppose $J = K,\ I \subsetneq L$. If $\#J \geq 2$ then the edges $\langle v_{\emptyset,L}; v_{J,L}\rangle$ and $\langle v_{\emptyset,J}; v_{\emptyset,L}\rangle$ are determinable by (i) and (ii), hence any edge connecting $v_{\emptyset,J}$ with $v_{J,L}$ is determinable. In particular, $e$ is determinable if $I = \emptyset$, and if $I \neq \emptyset$ then $e$ is determinable by the triangle with edges $e$, $\langle v_{J,L}; v_{\emptyset,J}\rangle$, $\langle v_{I,J}; v_{\emptyset,J}\rangle$, whose last edge is determinable by (i). Suppose $\#J = 1$. Then $\#I = 1$ and $\#L \geq 2$. Now $e$ is determinable by a triangle with third vertex $v_{\emptyset,M}$, where $M = L \setminus I$ if $\#L \geq 3$, and $M = (J \cup L)'$ if $\#L = 2$ ($\#M \geq 2$, since $n \geq 5$). In both cases $e$ is determinable by (i) and (ii). Finally, if $J \subsetneq K$ and $I \subsetneq L$ then $e$ is determinable by the triangle with edges $e$, $\langle v_{I,J}; v_{J,L}\rangle$, $\langle v_{K,L}; v_{J,L}\rangle$, because $\langle v_{I,J}; v_{J,L}\rangle$ is determinable by previous arguments, and $\langle v_{K,L}; v_{J,L}\rangle$ by (i).

(iv) First assume $K = \emptyset$. Then $I = \emptyset$ and if $v_K = i(e)$ then $e \in \mathcal{T}$. Suppose $v_K = t(e)$. Observe that there is a vertex $v_{\emptyset,L}$ such that $L \subsetneq J$ or $J \subsetneq L$. Now $e$ is determinable by $\langle v_{\emptyset,J}; v_\emptyset; v_{\emptyset,L}\rangle$. Now assume $K \neq \emptyset$ and $\#J \geq 2$. Any edge connecting $v_K$ with $v_{\emptyset,J}$ is determinable by a triangle with third vertex $v_{\emptyset,K'}$. In particular, $e$ is determinable if $I = \emptyset$, and if $I \neq \emptyset$ then $e$ is determinable by the triangle with edges $e$, $\langle v_K; v_{\emptyset,J}\rangle$, $\langle v_{I,J}; v_{\emptyset,J}\rangle$, whose last edge is determinable by (i). It remains to consider the case $\#I = \#J = 1$. It is easy to check that then there is a triangle with vertices $v_K$, $v_{I,J}$, $v_{L,M}$, where $I \cup J \subsetneq L \cup M$. The edge connecting $v_K$ with $v_{L,M}$ is determinable by the previous argument, hence $e$ is also determinable.

This completes the proof of Lemma 6.9 and Theorem 6.4 □

We a corollary we obtain the following theorem.

**Theorem 6.10.** *Suppose that $F = F_g^n$ is not sporadic and $\mathcal{T}$ is as in* Lemma 6.4. *Then it is possible to express all the generators $g_e$ appearing in* Theorem 6.3 *as a product of elements in $\bigcup_{v\in X^0} G_v$. Hence, the presentation in* Theorem 6.3 *reduces to*

$$\mathcal{M}(F) = \left\langle \bigcup_{v\in X^0} G_v \;\middle|\; \bigcup_{v\in X^0} R_v \cup \widetilde{R^{(2)}} \cup \widetilde{R^{(3)}} \right\rangle,$$

*where $\widetilde{R^{(i)}}$ are the relations obtained substituting in $R^{(i)}$ the expressions for the generators $g_e$.*

## 7. The sporadic surfaces

Suppose that $F$ is not sporadic. To obtain a finite presentation of the group $\mathcal{M}(F)$ using Theorem 6.10 we need finite presentations for the groups $\mathrm{Stab}(s(v))$ and finite sets of generators of the groups $\mathrm{Stab}(s(e))$. By Proposition 4.2 we can reduce these problems to analogous problems for the groups $\mathcal{M}(N)$, where $N$ is a connected component of $F_{s(v)}$ or $F_{s(e)}$. Note that $N$ has either lower genus than $F$ or equal genus, but less boundary components. If $N$ is orientable then a finite presentation of $\mathcal{M}(N)$ is known (see [7] for the most general case). If $N$ is non-orientable and not sporadic then we can obtain such presentation from Theorem 6.10. Thus applying recursively Theorem 6.10 we obtain a finite presentation for $\mathcal{M}(F)$, provided that we know a finite presentation of the mapping class group of each sporadic subsurface.

The groups $\mathcal{M}(F_1^0)$ and $\mathcal{M}(F_1^1)$ are well known to be trivial (cf. [6]); $\mathcal{M}(F_1^2)$ is generated by Dehn twists along the boundary curves and is isomorphic to $\mathbb{Z}^2$; $\mathcal{M}(F_2^0) = \mathbb{Z}_2 \times \mathbb{Z}_2$ ([16]). Simple presentation for $\mathcal{M}(F_2^1)$ was found in [20], and for $\mathcal{M}(F_3^0)$ in [3]. In this section we determine a finite presentation of $\mathcal{M}(F_g^n)$ for the remaining sporadic surfaces, i.e. for $(g, n) \in \{(1, 3), (1, 4), (2, 2), (2, 3), (3, 1), (3, 2)\}$.

We begin by introducing the pure mapping class group of a punctured surface and Birman's exact sequence, which is our main tool in this section. Let $S$ be an *orientable* surface with $2r$ distinguished points $\Sigma = \{q_1, \dots, q_{2r}\}$ called *punctures*. The *pure mapping class group* $\mathcal{PM}(S, \Sigma)$ is the group of isotopy classes $\operatorname{rel} \Sigma$ of all those diffeomorphisms of $S$ which fix each $q_i$. Up to isomorphism, this group does not depend on the choice of $\Sigma$, only on the number of punctures. We also define $\mathcal{PM}(S, \emptyset)$ to be the ordinary mapping class group $\mathcal{M}(S)$. Forgetting that $q_{2r-1}$ and $q_{2r}$ are distinguished defines a homomorphism $\rho\colon \mathcal{PM}(S, \Sigma) \to \mathcal{PM}(S, \Sigma')$, where $\Sigma' = \Sigma \setminus \{q_{2r-1}, q_{2r}\}$. Let $Q = \{(x_1, x_2) \in (S \setminus \Sigma')^2 \mid x_1 \neq x_2\}$. We define *the pure braid group* $PB_2(S \setminus \Sigma')$ as $\pi_1(Q, (q_{2r-1}, q_{2r}))$. If the Euler characteristic of $S \setminus \Sigma'$ is negative, then there is a short exact sequence due to Birman (see [2]):

$$1 \to PB_2(S \setminus \Sigma') \xrightarrow{j} \mathcal{PM}(S, \Sigma) \xrightarrow{\rho} \mathcal{PM}(S, \Sigma') \to 1,$$

where the homomorphism $j$ is defined as follows. A loop $\beta \in PB_2(S \setminus \Sigma')$ defines an isotopy of 0-dimensional submanifold $(q_{2r-1}, q_{2r}) \subset S \setminus \Sigma'$, which can be extended to an isotopy $h_t \in \operatorname{Diff}(S, \Sigma')$, $0 \le t \le 1$ such that $h_0 = 1$ and $h_1(q_i) = q_i$ for $1 \le i \le 2r$. We define $j(\beta)$ to be the isotopy class in $\operatorname{Diff}(S, \Sigma)$ of $h_1$.

Suppose that $\tau\colon S \to S$ is an orientation reversing involution of $S$, without fixed points, and such that $\tau(q_{2k-1}) = q_{2k}$ for $1 \le k \le r$. Then $S/\tau$ is a non-orientable surface with $r$ distinguished points $\Gamma = \{p_1, \dots, p_r\}$. Consider the subgroup $\mathcal{PM}(S, \Sigma, \tau)$ of $\mathcal{PM}(S, \Sigma)$ consisting of all isotopy classes which admit a representative which commutes with $\tau$. It can be shown that two such representatives are isotopic $\operatorname{rel} \Sigma$ if and only if they are isotopic via an isotopy which commutes with $\tau$ at each time (cf. [3]). Since every diffeomorphism of $S/\tau$ has a unique orientation preserving lift to $S$ which commutes with $\tau$ (the two lifts differ by $\tau$ which is orientation reversing), $\mathcal{PM}(S, \Sigma, \tau)$ can be identified with the group of isotopy classes $\operatorname{rel} \Gamma$ of diffeomorphisms of $S/\tau$ which fix each $p_i$ and preserve the local orientation of $S/\tau$ at each $p_i$.

It follows from the definition of $j$, that $j(\beta) \in \mathcal{PM}(S, \Sigma, \tau)$ if and only if $\beta$ is represented by a loop of the form $t \mapsto (a_t, \tau(a_t))$, where $t \mapsto a_t$ is a loop in $S \setminus \Sigma'$ based at $q_{2r-1}$. Thus the pre-image $j^{-1}(\mathcal{PM}(S, \Sigma, \tau))$ can be identified with $\pi_1(S \setminus \Sigma', q_{2r-1})$ and we obtain the exact sequence:

$$1 \to \pi_1(S \setminus \Sigma') \xrightarrow{j} \mathcal{PM}(S, \Sigma, \tau) \xrightarrow{\rho} \mathcal{PM}(S, \Sigma', \tau) \to 1. \tag{7.1}$$

Suppose now that $F$ is a non-orientable surface of genus $g$ with $r$ punctures $\Gamma = \{p_1, \dots, p_r\}$. Let $\mathcal{PM}(F, \Gamma)$ denote the *pure mapping class group* of $F$. It is defined as

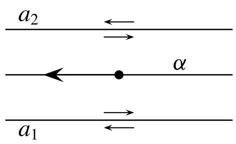


Fig. 5. $j(\alpha) = t_{a_1} t_{a_2}$.

the group of the isotopy classes rel $\Gamma$ of all diffeomorphisms of $F$ which fix each $p_i$. Consider the subgroup $\mathcal{PM}^+(F, \Gamma)$ of $\mathcal{PM}(F, \Gamma)$, consisting of the isotopy classes of those diffeomorphisms which preserve the local orientation of $F$ at each $p_i$. If $S$ is the orientable double cover of $F$ and $F = S/\tau$, then it follows from above considerations that $\mathcal{PM}^+(F, \Gamma)$ can be identified with $\mathcal{PM}(S, \Sigma, \tau)$. Note that $\pi_1(S \setminus \Sigma')$ can be identified with the subgroup $\pi_1^+(F \setminus \Gamma', p_r)$ of $\pi_1(F \setminus \Gamma', p_r)$ consisting of the two-sided loops. With such identifications the sequence (7.1) becomes:

$$1 \to \pi_1^+(F \setminus \Gamma', p_r) \xrightarrow{j} \mathcal{PM}^+(F, \Gamma) \xrightarrow{\rho} \mathcal{PM}^+(F, \Gamma') \to 1, \tag{7.2}$$

where we assume that the Euler characteristic of $F \setminus \Gamma'$ is negative (that is $g + r > 3$).

In this paper we use the same symbol to denote a loop and its homotopy class in the fundamental group. In order for $j$ to be a homomorphism, the product $\alpha\beta$ of two loops should mean first travel along $\beta$ and then along $\alpha$.

If $\alpha$ is a simple loop in $F$ based at $p_r$, then $j(\alpha)$ is the isotopy class of a diffeomorphism obtained by sliding $p_r$ once along $\alpha$.

The next two lemmas are proved in [13], (6.1).

**Lemma 7.1.** *Let $\alpha \in \pi_1^+(F \setminus \Gamma', p_k)$ be a two-sided simple loop and let $a_1$, $a_2$ denote boundary curves of a tubular neighborhood of $\alpha$. Then $j(\alpha) = t_{a_1} t_{a_2}$, where $t_{a_1}$ and $t_{a_2}$ are Dehn twists about $a_1$ and $a_2$ in the directions indicated by arrows in* Fig. 5.

The pure mapping class group $\mathcal{PM}(F, \Gamma)$ acts on $\pi_1^+(F \setminus \Gamma')$ in the obvious way. We denote this action by $h(\alpha)$ for $h \in \mathcal{PM}(F, \Gamma)$ and $\alpha \in \pi_1^+(F \setminus \Gamma')$.

**Lemma 7.2.** *The homomorphism $j$ is $\mathcal{PM}(F, \Gamma)$-equiveriant. That is $j(h(\alpha)) = hj(\alpha)h^{-1}$ for $h \in \mathcal{PM}(F, \Gamma)$ and $\alpha \in \pi_1^+(F \setminus \Gamma')$.*

Suppose that $\tilde{F} = F_g^n$ is a non-orientable surface of negative Euler characteristic (i.e. $g + n > 2$) and let $c_1, \ldots, c_n : S^1 \to \partial\tilde{F}$ denote the boundary curves. Let $F = F_g^0$ be the closed surface with punctures $\Gamma = \{p_1, \ldots, p_n\}$ obtained by gluing a disc with a puncture $p_i$ to $\partial\tilde{F}$ along $c_i$ for $1 \le i \le n$. We identify $\tilde{F}$ with a subsurface of $F$ and denote by $i_* : \mathcal{M}(\tilde{F}) \to \mathcal{PM}^+(F, \Gamma)$ the homomorphism induced by the inclusion $i : \tilde{F} \to F$. It can be proved, using the same methods as in the proof of Proposition 4.1, that $\ker i_*$ is a free abelian group of rank $n$ generated by Dehn twists about the boundary curves $c_i$. Thus we have the exact sequence

$$1 \to \mathbb{Z}^n \to \mathcal{M}(F_g^n) \xrightarrow{i_*} \mathcal{PM}^+(F_g^0, \Gamma) \to 1. \tag{7.3}$$

REMARK 7.3. Note that $\ker i_*$ is a central subgroup of $\mathcal{M}(F_g^n)$. Indeed, for every $i \in \{1, \dots, n\}$ and $h \in \mathcal{M}(F_g^n)$ we have $ht_{c_i}h^{-1} = t_{h(c_i)} = t_{c_i}$.

We record without proof the following easy lemma.

**Lemma 7.4.** *Consider a short exact sequence of groups*

$$1 \to K \xrightarrow{i} G \xrightarrow{p} H \to 1$$

*and suppose that $K$ and $H$ admit presentations*

$$K = \langle G_K \mid R_K \rangle, \quad H = \langle G_H \mid R_H \rangle.$$

*Then $G$ admits the presentation*

$$\langle i(G_K) \cup \widetilde{G_H} \mid i(R_K) \cup \widetilde{R_H} \cup R \rangle, \tag{7.4}$$

*where*:

$$i(G_K) = \{i(k) \mid k \in G_K\}, \quad \widetilde{G_H} = \{\tilde{h} \mid h \in G_H\},$$

*where $\tilde{h}$ is any element in $G$ such that $p(\tilde{h}) = h$,*

$$\begin{aligned} i(R_K) &= \{i(k_1) \cdots i(k_n) \mid k_1 \cdots k_n \in R_K\}, \\ \widetilde{R_H} &= \{\tilde{h}_1 \cdots \tilde{h}_n w(h_1 \cdots h_n) \mid h_1 \cdots h_n \in R_H\}, \\ R &= \{\tilde{h} i(k) \tilde{h}^{-1} w(k, h) \mid h \in G_H, k \in G_k\}, \end{aligned}$$

*where $w(h_1 \cdots h_n)$ and $w(k, h)$ are suitable words in generators $i(G_K)$.*

We can now obtain finite presentations for the mapping class groups $\mathcal{M}(F_g^n)$ of the sporadic surfaces in the following way. Starting from known presentations of the groups $\mathcal{PM}^+(F_1^0, \{p_1, p_2\})$, $\mathcal{PM}^+(F_2^0, \{p_1\})$ and $\mathcal{M}(F_3^0)$, we obtain presentations for all $\mathcal{PM}^+(F_g^0, \Gamma)$, by applying recursively Lemma 7.4 to the sequence (7.2). To do this, we need finite presentations for the groups $\pi_1^+(F_g^0 \setminus \Gamma')$. These can be obtained from standard presentations of fundamental groups $\pi_1(F_g^0 \setminus \Gamma')$ by the Reidemeister-Schreier method (see, for example, [17]). Once we have found the presentations for $\mathcal{PM}^+(F_g^0, \Gamma)$, we obtain presentations for $\mathcal{M}(F_g^n)$, by applying Lemma 7.4 to the sequence (7.3).

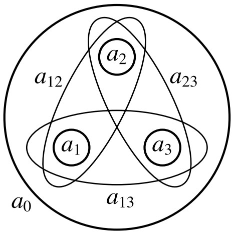


Fig. 6. The curves of the lantern relation.

**7.1. Sporadic surfaces of genus 1.** Until the end of this paper we use the capital letter $A$ to denote a Dehn twist about the curve labelled as $a$. In order for this notation to be unambiguous, we have to specify the direction of the twist $A$ for each curve $a$. Equivalently we may choose an orientation of a tubular neighborhood of $a$. Then $A$ denotes the right Dehn twist with respect to the chosen orientation.

Consider a 2-sphere $S$ with four holes embedded in $F$. Let $a_0$, $a_1$, $a_2$, $a_3$ denote disjoint boundary curves of $S$, and $a_{12}$, $a_{13}$, $a_{23}$ separating generic curves such that $a_{ij}$ separates $a_i$ and $a_j$ from the other two boundary curves of $S$ (Fig. 6). If $A_i$ and $A_{jk}$ are right Dehn twists with respect to the standard orientation of the plane of Fig. 6, then we have the well known lantern relation:

$$A_0A_1A_2A_3 = A_{12}A_{13}A_{23}. \tag{7.5}$$

The lantern relation was discovered by Dehn [5] and rediscovered by Johnson [14]. Note that since $A_{ij}$ commutes with $A_k$, we have:

$$A_{12}A_{13}A_{23} = A_{13}A_{23}A_{12} = A_{23}A_{12}A_{13}. \tag{7.6}$$

Let us fix four points $p_1, \dots, p_4$ in the projective plane $F = F_1^0$ represented in Figs. 7 and 8, where the curve $c_1$ bounds in $F$ a disc containing $p_1$. Let $n \in \{3, 4\}$ and consider the embedding $i\colon \tilde{F} \to F$, where $\tilde{F} = F_1^n$, and the induced homomorphism $i_*\colon \mathcal{M}(\tilde{F}) \to \mathcal{PM}^+(F, \{p_1, \dots, p_n\})$ (if $n = 3$ then we forget that $p_4$ is distinguished). We identify $\tilde{F}$ with $i(\tilde{F})$, and a curve $a$ in $\tilde{F}$ with $i \circ a$ in $F$.

Consider the loops $\alpha_i$, $\alpha_{jk}$, $\beta_{jk}$ represented in Figs. 7 and 8, where we assume, that each of them represents a two-sided simple loop in $\pi_1^+(F \setminus \{p_1, p_2\}, p_3)$ or $\pi_1^+(F \setminus \{p_1, p_2, p_3\}, p_4)$. The boundary of a tubular neighborhood of such loop consist of two two-sided simple closed curves, one of which is trivial (i.e. it either separates a Möbius strip or a disc containing one puncture). We use the symbol $a_i$ or $a_{jk}$ or $b_{jk}$ to denote the non-trivial boundary component of the tubular neighborhood of the corresponding loop (see Fig. 7). Then by Lemma 7.1, we have $j(\alpha_i) = A_i$, $j(\alpha_{jk}) = A_{jk}$, $j(\beta_{jk}) = B_{jk}$. Note that $a_i$, $a_{jk}$, $b_{jk}$ may be chosen to be generic curves in $\tilde{F}$.

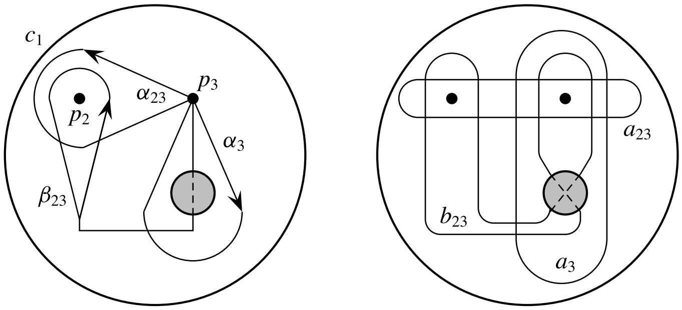


Fig. 7. Generators of $\pi_1^+(F \setminus \{p_1, p_2\}, p_3)$ and generic curves in $F_1^3$.

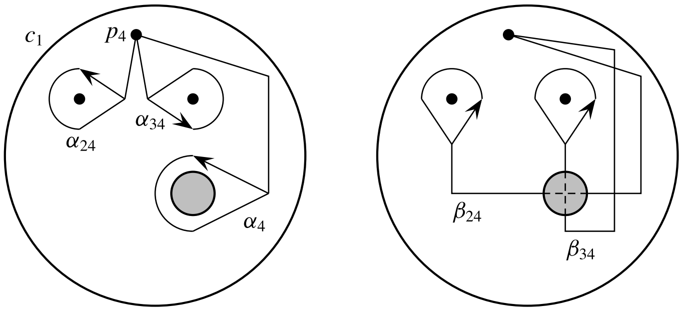


Fig. 8. Generators of $\pi_1^+(F \setminus \{p_1, p_2, p_3\}, p_4)$.

**Theorem 7.5.** *The group $\mathcal{PM}^+(F, \{p_1, p_2, p_3\})$ is free, generated by $A_3$, $A_{23}$, $B_{23}$. The group $\mathcal{M}(F_1^3)$ is generated by $A_3$, $A_{23}$, $B_{23}$, $C_1$, $C_2$, $C_3$ and isomorphic to $\mathbb{Z}^3 \times \mathcal{PM}^+(F, \{p_1, p_2, p_3\})$.*

Proof. It can be deduced from Theorem 4.1 of [15] that the group $\mathcal{PM}^+(F, \{p_1, p_2\})$ is trivial. Thus

$$j : \pi_1^+(F \setminus \{p_1, p_2\}, p_3) \to \mathcal{PM}^+(F, \{p_1, p_2, p_3\})$$

is an isomorphism. The fundamental group $\pi_1(F \setminus \{p_1, p_2\}, p_3)$ is free on generators $\alpha_{23}$ and $x$, where $x$ is a one-sided loop, such that $x^2 = \alpha_3^{-1}$, $x\alpha_{23}x^{-1} = \beta_{23}$. Now $\{1, x\}$ is a Schreier system of representatives of right cosets of $\pi_1^+(F \setminus \{p_1, p_2\}, p_3)$ and by the Reidemeister-Schreier method we obtain that the last group is freely generated by the loops $\alpha_3$, $\alpha_{23}$, $\beta_{23}$. Hence the first part of Theorem 7.5. The second part follows from the sequence (7.3). Indeed, the sequence splits as $\pi_1^+(F \setminus \{p_1, p_2\}, p_3)$ is free, and the kernel of $i_*$ is central by Remark 7.3. $\square$

**Theorem 7.6.** *The group $\mathcal{PM}^+(F, \{p_1, p_2, p_3, p_4\})$ admits a presentation with generators $\{A_3, A_4, A_{23}, A_{24}, A_{34}, B_{23}, B_{24}, B_{34}, D\}$ and relations*:
(1) $A_{23}A_4 = A_4A_{23},\ A_{24}A_3 = A_3A_{24}$,
(2) $A_3^{-1}A_4A_{34}B_{34} = B_{34}A_3^{-1}A_4A_{34}$,
(3) $A_4A_{34}A_{24}B_{23} = B_{23}A_4A_{34}A_{24}$,
(4) $A_{34}A_3^{-1}A_{23}B_{24} = B_{24}A_{34}A_3^{-1}A_{23}$,
(5) $A_{34}A_{24}A_{23} = A_{24}A_{23}A_{34} = A_{23}A_{34}A_{24}$,
(6) $B_{34}A_{23}B_{24} = A_{23}B_{24}B_{34} = B_{24}B_{34}A_{23}$,
(7) $A_4A_{34}A_3^{-1} = A_{34}A_3^{-1}A_4 = A_3^{-1}A_4A_{34}$,
(8) $A_{34}^{-1}B_{24}B_{23} = B_{24}B_{23}A_{34}^{-1} = B_{23}A_{34}^{-1}B_{24}$,
(9) $A_{24}B_{23}D^{-1} = B_{23}D^{-1}A_{24} = D^{-1}A_{24}B_{23}$,
(10) $D = A_{34}^{-1}A_4^{-1}B_{34}A_4A_{34}$.
*The group $\mathcal{M}(F_1^4)$ is isomorphic to $\mathbb{Z}^4 \times \mathcal{PM}^+(F, \{p_1, p_2, p_3, p_4\})$.*

Proof. Let us denote, for simplicity,

$$\pi = \pi_1^+(F \setminus \{p_1, p_2, p_3\}, p_4), \quad G = \mathcal{PM}^+(F, \{p_1, p_2, p_3, p_4\}).$$

The fundamental group $\pi_1(F\setminus\{p_1, p_2, p_3\}, p_4)$ is free on generators $\alpha_{24}$, $\alpha_{34}$ and $x$, where $x$ is a one-sided loop, such that $x^2 = \alpha_4$, $x\alpha_{24}x^{-1} = \beta_{24}$, $x\alpha_{34}x^{-1} = \beta_{34}$. Now $\{1, x\}$ is a Schreier system of representatives of right cosets of $\pi$ and by the Reidemeister-Schreier method we obtain that $\pi$ is freely generated by the loops in Fig. 8. By Lemma 7.4 applied to sequence (7.2) and Theorem 7.5, $G$ admits a presentation with generators $A_3$, $A_{23}$, $B_{23}$, $A_4 = j(\alpha_4)$, $A_{k4} = j(\alpha_{k4})$, $B_{k4} = j(\beta_{k4})$, $k = 2, 3$ and relations $hgh^{-1} \in j(\pi)$ for each $h \in \{A_3, A_{23}, B_{23}\}$, $g \in \{A_4, A_{k4}, B_{k4} \mid k = 2, 3\}$. We will show that all these relations are consequences of (1)–(10). We have:

$$(1) \implies A_{23}A_4A_{23}^{-1},\ A_3A_{24}A_3^{-1} \in j(\pi);$$
$$(2) \implies A_3B_{34}A_3^{-1} \in j(\pi);$$
$$(10) \implies D \in j(\pi).$$

From (5) follows

$$A_{23}A_{34}A_{23}^{-1} = A_{24}^{-1}A_{34}A_{24} \in j(\pi),$$
$$A_{23}A_{34}A_{24}A_{23}^{-1} = A_{34}A_{24} \Rightarrow A_{23}A_{24}A_{23}^{-1} \in j(\pi).$$

Analogously we have

$$(6)\text{–}(9) \Rightarrow \{A_{23}B_{24}A_{23}^{-1},\ A_{23}B_{34}A_{23}^{-1},\ A_3A_{34}A_3^{-1},\ A_3A_4A_3^{-1},$$
$$B_{23}A_{34}B_{23}^{-1},\ B_{23}B_{24}B_{23}^{-1},\ B_{23}DB_{23}^{-1},\ B_{23}A_{24}B_{23}^{-1}\} \subset j(\pi).$$

From (3) follows

$$B_{23}A_4A_{34}B_{23}^{-1} \in j(\pi);$$

from this and (8) we have

$$B_{23}A_4B_{23}^{-1} \in j(\pi)$$

and from (10) follows

$$B_{23}B_{34}B_{23}^{-1} \in j(\pi).$$

Finally we have

$$(4) \Rightarrow A_3A_{34}^{-1}B_{24}A_{34}A_3^{-1} = A_{23}B_{24}A_{23}^{-1},$$

and by (6), (7) we have

$$A_3B_{24}A_3^{-1} \in j(\pi).$$

Now we show that relations (1)–(10) are satisfied in $\mathcal{M}(\tilde{F})$, and hence also in $G$. By relation (10), the generator $D$ is a Dehn twist about the curve $A_{34}^{-1}A_4^{-1}(b_{34})$ bounding a pair of pants together with $c_3$ and $c_4$. The relations (1) are obvious. By considering appropriate embeddings of a 2-sphere with four holes in $\tilde{F}$, it is easy to recognize (5)–(9) as relations of type (7.6), i.e. consequences of the lantern relation. In particular, we have lantern relation $A_{12}C_3C_4 = A_4A_{34}A_3^{-1}$, where $A_{12}$ is Dehn twist about a curve bounding a pair of pants together with $c_1$ and $c_2$. Since $B_{34}$ commutes with $A_{12}$, $C_3$ and $C_4$, the relation (2) holds. By Lemma 7.1, we have $j(\alpha_4\alpha_{34}\alpha_{24}) = A_{14} \in G$, where $a_{14}$ bounds a pair of pants in $\tilde{F}$ together with $c_1$ and $c_4$. Thus in $\mathcal{M}(\tilde{F})$ we have $A_4A_{34}A_{24} = A_{14}C$, where $C$ is a product of twists $C_1, \dots, C_4$. Since $B_{23}$ commutes with $A_{14}$ and $C$, (3) holds. Consider a monomorphism $j' \colon \pi_1^+(F \setminus \{p_1, p_2, p_4\}, p_3) \to G$, defined like $j$. There exists exactly one loop $\alpha'_{34}$ such that $j'(\alpha'_{34}) = A_{34} \in G$, and we have $j'(\alpha'_{34}\alpha_3^{-1}\alpha_{23}) = A_{13} \in G$, where $a_{13}$ bounds a pair of pants in $\tilde{F}$ together with $c_1$ and $c_3$. Since $B_{24}$ commutes with $A_{13}$, (4) holds.

We have shown that (1)–(10) are relations in $G$, and all relations from Lemma 7.4 are consequences of (1)–(10). Hence $G$ admits presentation with relations (1)–(10). Since these relations hold also in $\mathcal{M}(\tilde{F})$, the sequence (7.3) splits, and since the kernel of $i_*$ is central, we obtain $\mathcal{M}(\tilde{F}) = \mathbb{Z}^4 \times G$. □

**7.2. Sporadic surfaces of genus 2.** Consider the Klein bottle $K$ with one hole represented in Fig. 9. Let $U$ be a diffeomorphism of $K$ interchanging the shaded discs in Fig. 9 and such that $U^2$ is the Dehn twist about the boundary curve $c$, right with respect to the standard orientation of the plane of the figure. Up to isotopy, $U$ acts on the arc $d$ as it is shown in Fig. 9 (see [21] for precise definition). We fix Dehn twist $A_1$ about the curve $a_1$, in the direction indicated by arrows in Fig. 9. The composition $UA_1$ is the Y-homeomorphism (or cross-cap slide) introduced by Lickorish [16]. The next theorem follows immediately from Theorem A.7 of [20].

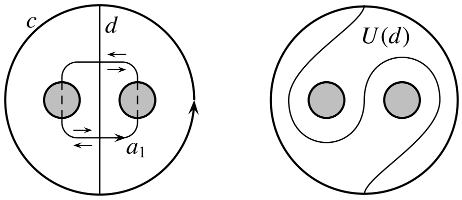


Fig. 9. The diffeomorphism $U$.

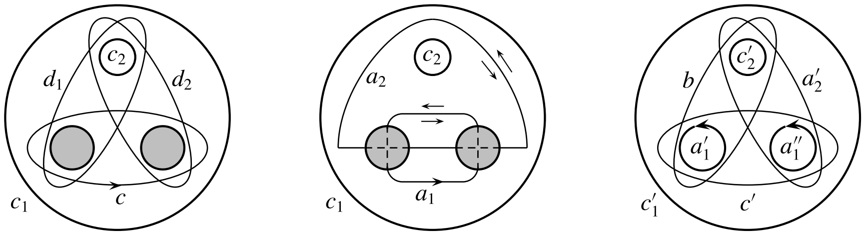


Fig. 10. The surfaces $\tilde{F} = F_2^2$ and $\tilde{F}_{a_1}$.

**Theorem 7.7.** *The mapping class group $\mathcal{M}(K)$ is generated by $A_1$ and $U$ and admits the presentation $\langle A_1, U \mid UA_1U^{-1} = A_1^{-1}\rangle$.*

Let $\tilde{F} = F_2^2$ be the surface obtained by gluing a pair of pants to $K$, and let $c_1$ and $c_2$ denote the boundary curves of $\tilde{F}$ (Fig. 10). We extend $U$ by the identity outside $K$ to a diffeomorphism of $\tilde{F}$. Let $C$, $C_1$, $C_2$ $D_1$, $D_2$ be Dehn twists about the curves represented in Fig. 10, right with respect to the standard orientation of the plane of the figure. We also define Dehn twist $A_1$, $A_2$ in the indicated directions. Note that $U^2 = C$ and $UD_2U^{-1} = D_1$.

The right hand side of Fig. 10 represents the four-holed sphere $\tilde{F}_{a_1}$ obtained by cutting $\tilde{F}$ along $a_1$, where $\rho_{a_1} \circ a_1' = \rho_{a_1} \circ a_1'' = a_1$, $\rho_{a_1}(c_i') = c_i$ for $i = 1, 2$, $\rho_{a_1}(c') = c$, $\rho_{a_1}(a_2') = a_2$, $\rho_{a_1}(b) = U(a_2)$. If $C_i'$, $C'$, $A_1'$, $A_1''$, $A_2'$, $B$ are right Dehn twists with respect to the standard orientation of the plane of Fig. 10, then $\rho_*(C_i') = C_i$, $\rho_*(C') = C$, $\rho_*(A_1'A_1'') = 1$, $\rho_*(A_2') = A_2$, and $\rho_*(B) = UA_2U^{-1}$.

**Lemma 7.8.** *In $\mathcal{M}(\tilde{F})$ we have $(A_2U)^2 = (D_2U)^2 = C_1C_2$.*

Proof. We have the lantern relation $C_1'C_2'A_1'A_1'' = A_2'BC'$. By applying $\rho_*$ to both sides we obtain $C_1C_2 = A_2(UA_2U^{-1})U^2 = (A_2U)^2$. By another lantern relation we have $C_1C_2 = D_2D_1C = D_2(UD_2U^{-1})U^2 = (D_2U)^2$. □

Let $F = F_2^0$ be the Klein bottle obtained by gluing a disc with a puncture $p_i$ to $\partial\tilde{F}$ along $c_i$ for $i = 1, 2$. We identify $U$, $A_1$, $A_2$, $D_2$, with $i_*(U)$, $i_*(A_1)$, $i_*(A_2)$, $i_*(D_2)$

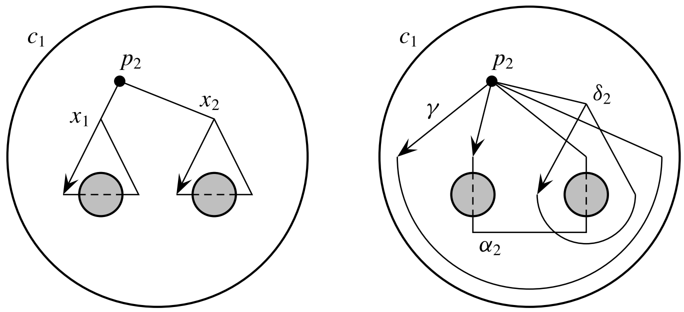


Fig. 11. Generators of $\pi_1(F \setminus \{p_1\}, p_2)$ and $\pi_1^+(F \setminus \{p_1\}, p_2)$.

respectively, where $i_*\colon \mathcal{M}(\tilde{F}) \to \mathcal{PM}^+(F, \{p_1, p_2\})$ is the homomorphism induced by the inclusion of $\tilde{F}$ in $F$.

**Theorem 7.9.** *The group* $\mathcal{PM}^+(F, \{p_1, p_2\})$ *admits a presentation with generators* $\{A_1, A_2, D_2, U\}$ *and relations*: $A_1A_2 = A_2A_1$, $UA_1U^{-1} = A_1^{-1}$, $A_2UD_2 = D_2^{-1}A_2U$, $(A_2U)^2 = (D_2U)^2 = 1$.

Proof. Consider the exact sequence (7.2):

$$1 \to \pi^+(F \setminus \{p_1\}, p_2) \xrightarrow{j} \mathcal{PM}^+(F, \{p_1, p_2\}) \to \mathcal{PM}^+(F, \{p_1\}) \to 1.$$

By Theorem 7.7 and sequence (7.3), $\mathcal{PM}^+(F, \{p_1\})$ has presentation

$$\langle A_1, U \mid UA_1U^{-1} = A_1^{-1},\ U^2 = 1\rangle.$$

The fundamental group $\pi_1(F \setminus \{p_1\}, p_2)$ is free on generators $x_1, x_2$ in Fig. 11. Now $\{1, x_2\}$ is a Schreier system of representatives of cosets of $\pi_1^+(F \setminus \{p_1\}, p_2)$ and by the Reidemeister-Schreier method we obtain that the last group is freely generated by $\delta_2 = x_2^2$, $\alpha_2 = x_2x_1$ and $x_1x_2^{-1}$. It follows that $\pi_1^+(F \setminus \{p_1\}, p_2)$ is free on generators $\delta_2, \alpha_2, \gamma$, where $\gamma = x_2^2(x_1x_2^{-1})(x_2x_1)$. Observe that $j(\gamma) = U^{-2}$, $j(\alpha_2) = A_2A_1^{-1}$, $j(\delta_2) = D_2$. By Lemma 7.4, $\mathcal{PM}^+(F, \{p_1, p_2\})$ admits presentation with generators $U$, $A_1$, $j(\gamma)$, $j(\alpha_2)$, $j(\delta_2)$ and relations $UA_1U^{-1} = A_1^{-1}$, $U^2 = (j(\gamma))^{-1}$, and (by Lemma 7.2):

$$Uj(\gamma)U^{-1} = j(\gamma), \quad Uj(\alpha_2)U^{-1} = j(\alpha_2^{-1}\gamma), \quad Uj(\delta_2)U^{-1} = j(\delta_2^{-1}\gamma),$$
$$A_1j(\gamma)A_1^{-1} = j(\gamma), \quad A_1j(\alpha_2)A_1^{-1} = j(\alpha_2), \quad A_1j(\delta_2)A_1^{-1} = j(\gamma\alpha_2^{-1}\delta_2\alpha_2).$$

Substituting $j(\gamma) = U^{-2}$, $j(\alpha_2) = A_2A_1^{-1}$, $j(\delta_2) = D_2$ we obtain a presentation which can easily be shown to be equivalent to that in Theorem 7.9. □

**Theorem 7.10.** *The group $\mathcal{M}(F_2^2)$ admits a presentation with generators $\{C_1, A_1, A_2, D_2, U\}$ and relations*:

$$C_1 A_i = A_i C_1, \quad \text{for } i = 1, 2,$$
$$C_1 D_2 = D_2 C_1, \quad C_1 U = U C_1,$$
$$A_1 A_2 = A_2 A_1, \quad U A_1 U^{-1} = A_1^{-1}, \quad A_2 U D_2 = D_2^{-1} A_2 U,$$
$$(A_2 U)^2 = (D_2 U)^2.$$

Proof. From sequence (7.3), Theorem 7.9 and Lemma 7.8 we obtain a presentation for $\mathcal{M}(F_2^2)$ with generators $\{C_1, C_2, A_1, A_2, D_2, U\}$ and relations listed in Theorem 7.10 and

$$C_1 C_2 = C_2 C_1, \;\; C_2 D_2 = D_2 C_2, \;\; C_2 U = U C_2, \;\; C_2 A_i = A_i C_2, \tag{7.7}$$

for $i = 1, 2$ and

$$(A_2 U)^2 = C_1 C_2. \tag{7.8}$$

We claim that the relations (7.7) are consequences of the relation (7.8) and relations from Theorem 7.10. Clearly it suffices to check that relations

$$D_2 (A_2 U)^2 = (A_2 U)^2 D_2, \quad U (A_2 U)^2 = (A_2 U)^2 U, \quad A_i (A_2 U)^2 = (A_2 U)^2 A_i,$$

follow from those in Theorem 7.10. Observe that $A_1 (A_2 U)^2 = (A_2 U)^2 A_1$ follows from $A_1 A_2 = A_2 A_1$ and $U A_1 U^{-1} = A_1^{-1}$. From $A_2 U D_2 = D_2^{-1} A_2 U$ we have $D_2^{-1} (A_2 U)^2 D_2 = (A_2 U)^2$ and $U (A_2 U)^2 U^{-1} = U (D_2 U)^2 U^{-1} = D_2^{-1} (D_2 U)^2 D_2 = D_2^{-1} (A_2 U)^2 D_2 = (A_2 U)^2$. Finally we have $A_2^{-1} (A_2 U)^2 A_2 = U (A_2 U)^2 U^{-1} = (A_2 U)^2$. It follows that relations (7.7) are redundant, and hence they can be removed from the presentation. Then the generator $C_2$ can also be removed together with the relation (7.8). □

We fix a point $p_3 \in F \setminus K$, different from $p_2$ and $p_1$, and such that $p_3$ and $p_2$ are in different components of $F \setminus (a_1 \cup a_2)$. We identify $U$, $A_2$, $A_1$ and $D_2$ with elements of $\mathcal{PM}^+(F, \{p_1, p_2, p_3\})$. Let $A_3$ and $D_3$ be such Dehn twists that $j(\alpha_3) = A_3 A_2^{-1}$ and $j(\delta_3) = D_3$, where $\alpha_3$, $\delta_3$ are the loops in Fig. 12, and $j \colon \pi^+(F \setminus \{p_1, p_2\}, p_3) \to \mathcal{PM}^+(F, \{p_1, p_2, p_3\})$ is the monomorphism from sequence (7.2).

**Theorem 7.11.** *The group $\mathcal{PM}^+(F, \{p_1, p_2, p_3\})$ admits a presentation with generators $\{A_1, A_2, A_3, D_2, D_3, U\}$ and relations*:
(1) $A_i A_j = A_j A_i$, *for* $i, j \in \{1, 2, 3\}$;
(2) $U A_1 U^{-1} = A_1^{-1}$;
(3) $A_2 U D_2 = D_2^{-1} A_2 U$;
(4) $(A_2 U)^2 = (D_2 U)^2 = (U D_2)^2$;

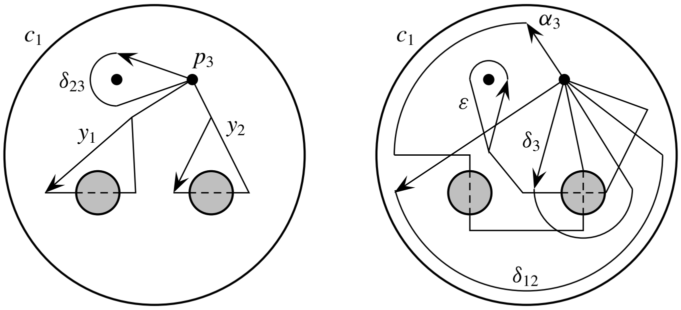


Fig. 12. Generators of $\pi_1(F\setminus\{p_1, p_2\}, p_3)$ and $\pi_1^+(F\setminus\{p_1, p_2\}, p_3)$.

(5) $(UD_3)^2 = (D_3U)^2$;
(6) $D_3UD_2U^{-1} = UD_2U^{-1}D_3$;
(7) $A_3UD_2D_3 = UD_2D_3A_3^{-1}$;
(8) $(UA_3)^2 = (UD_2D_3)^{-2}$;
(9) $A_2(A_3UD_2)^2 = (A_3UD_2)^2A_2$;
(10) $A_2A_1^{-1}D_3A_1A_2^{-1} = A_3UD_2D_3^{-1}(A_3UD_2)^{-1}$;
(11) $A_1(A_3UD_2)^2A_1^{-1} = (UD_2)^{-1}(A_3UD_2)^2UD_2$.

Proof. Let us denote, for simplicity,

$$\pi = \pi_1^+(F \setminus \{p_1, p_2\}, p_3), \quad G = \mathcal{PM}^+(F, \{p_1, p_2, p_3\}).$$

The fundamental group $\pi_1(F \setminus \{p_1, p_2\}, p_2)$ is free on generators $\delta_{23}, y_1, y_2$ in Fig. 12. Now $\{1, y_2\}$ is a Schreier system of representatives of cosets of $\pi$ and by the Reidemeister-Schreier method we obtain that the last group is freely generated by $\delta_{23}$, $\delta_3 = y_2^2$, $\varepsilon = y_2\delta_{23}y_2^{-1}$, $y_2y_1$ and $y_1y_2^{-1}$. It follows that $\pi$ is free on generators $\delta_{23}, \delta_3, \varepsilon, \alpha_3, \delta_{12}$, where $\delta_{12} = \delta_3(y_1y_2^{-1})(y_2y_1)$, $\alpha_3 = y_2y_1\delta_{23}$. By Lemmas 7.1 and 7.8 we have

$$j(\delta_{23}) = (UA_3)^2, \quad j(\delta_{12}\delta_{23}) = (UD_2)^{-2}. \tag{7.9}$$

First we show that relations (1)–(11) are satisfied in $G$: (1) and (6) are obvious; (4) and (5) follow from Lemma 7.8; (2), (3), (7) are relations of type $ht_ah^{-1} = t_{h(a)}^{\pm 1}$ and hence they can be checked by looking at the effect of $h$ on the curve $a$; (10) follows from $A_2A_1^{-1}(\delta_3) = A_3UD_2(\delta_3^{-1})$; (8) is equivalent to $UD_2D_3D_2^{-1}U^{-1} = (UA_3)^{-2}D_3^{-1}(UD_2)^{-2}$, which follows from $UD_2(\delta_3) = \delta_{23}^{-1}\delta_3^{-1}\delta_{12}\delta_{23}$. It can be checked that $\varepsilon\delta_3 = A_3((\delta_{12}\delta_{23})^{-1}\delta_3)$ and hence $j(\varepsilon) = A_3(UD_2)^2D_3A_3^{-1}D_3^{-1}$; from this and (7) we obtain

$$j(\varepsilon) = (A_3UD_2)^2. \tag{7.10}$$

Now (9) and (11) follow from (7.10) and the equalities $A_2(\varepsilon) = \varepsilon$ and $A_1(\varepsilon) = (UD_2)^{-1}(\varepsilon)$.

By Theorem 7.9 and sequence (7.2), $G$ admits presentation with generators $\{A_1, A_2, D_2, U, j(\alpha_3), j(\delta_3), j(\delta_{12}), j(\delta_{23}), j(\varepsilon)\}$ and relations (2), (3), $A_1A_2 = A_2A_1$, $(A_2U)^2 = (D_2U)^2 = j(\delta_{23}^{-1}\delta_{12}^{-1})$ and:

(i) $Uj(\alpha_3)U^{-1} = j(\delta_{23}\alpha_3^{-1}\delta_{12}\delta_{23})$;

(ii) $Uj(\delta_3)U^{-1} = j(\delta_3^{-1}\delta_{12})$;

(iii) $Uj(\delta_{23})U^{-1} = j(\delta_{23})$;

(iv) $Uj(\delta_{12})U^{-1} = j(\delta_{12})$;

(v) $Uj(\varepsilon)U^{-1} = j(\delta_3^{-1}\delta_{12}\delta_{23}\alpha_3^{-1}\varepsilon\alpha_3\delta_{23}^{-1}\delta_{12}^{-1}\delta_3)$;

(vi) $D_2j(\alpha_3)D_2^{-1} = j(\delta_{23}^{-1}\delta_3^{-1}\varepsilon\delta_3\alpha_3)$;

(vii) $D_2j(\delta_3)D_2^{-1} = j(\delta_{23}^{-1}\delta_3\delta_{23})$;

(viii) $D_2j(\delta_{23})D_2^{-1} = j(D_2(\delta_3^{-1})\delta_3\delta_{23})$;

(ix) $D_2j(\delta_{12})D_2^{-1} = j(\delta_{12}\delta_{23}D_2(\delta_{23}^{-1}))$;

(x) $D_2j(\varepsilon)D_2^{-1} = j(D_2(\alpha_3)\alpha_3^{-1}\delta_{23})$;

(xi) $A_2j(\alpha_3)A_2^{-1} = j(\alpha_3)$;

(xii) $A_2j(\delta_3)A_2^{-1} = j(\delta_{12}\delta_{23}\alpha_3^{-1}\varepsilon\delta_3\alpha_3)$;

(xiii) $A_2j(\delta_{23})A_2^{-1} = j(\alpha_3^{-1}\delta_{23}\alpha_3)$;

(xiv) $A_2j(\delta_{12})A_2^{-1} = j(\delta_{12}\delta_{23})A_2j(\delta_{23}^{-1})A_2^{-1}$;

(xv) $A_2j(\varepsilon)A_2^{-1} = j(\varepsilon)$;

(xvi) $A_1j(\alpha_3)A_1^{-1} = j(\alpha_3)$;

(xvii) $A_1j(\delta_3)A_1^{-1} = j(\delta_{12}\delta_{23}\alpha_3^{-1}\delta_3\alpha_3\delta_{23}^{-1})$;

(xviii) $A_1j(\delta_{23})A_1^{-1} = j(\delta_{23})$;

(xix) $A_1j(\delta_{12})A_1^{-1} = j(\delta_{12})$;

(xx) $A_1j(\varepsilon)A_1^{-1} = j((UD_2)^{-1}(\varepsilon))$.

It remains to check, that the relations (i)–(xx) above are consequences of (1)–(11) in Theorem 7.11 and (7.9), (7.10), $j(\delta_3) = D_3$. We have:

$$\text{(i)} \Longleftrightarrow UA_3A_2^{-1}U^{-1} = (UA_3)^2A_3^{-1}A_2(A_2U)^{-2} \Longleftrightarrow (A_2U)^2 = (UA_2)^2 \Longleftarrow (4);$$

$$\begin{aligned}\text{(ii)} \Longleftrightarrow UD_3U^{-1} &= D_3^{-1}(UD_2)^{-2}(UA_3)^{-2} \overset{(8)}{=} D_3^{-1}D_2^{-1}U^{-1}D_3UD_2D_3\\ &\overset{(6)}{=} D_3^{-1}U^{-1}D_3UD_3\\ \Longleftrightarrow (5);\end{aligned}$$

$$\text{(iii)} \Longleftrightarrow (UA_3)^2 = (A_3U)^2 \Longleftarrow (7),\ (8);$$

$$\text{(iv)} \Longleftrightarrow U(UA_3)^2(UD_2)^2U^{-1} = (UA_3)^2(UD_2)^2 \Longleftarrow (4),\ (7),\ (8);$$

$$\begin{aligned}\text{(v)} \overset{(9)}{\Longleftrightarrow} U(A_3UD_2)^2U^{-1} &= D_3^{-1}(UD_2)^{-1}(A_3UD_2)^2(UD_2)D_3\\ &\overset{(7)}{=} A_3^{-1}D_3^{-1}A_3(UD_2)^2D_3\\ \overset{(4),(6),(7)}{\Longleftrightarrow} D_3(A_3U)^2D_2 &= (A_3U)^2D_2D_3\\ \overset{(8)}{\Longleftrightarrow} D_3(UD_2D_3)^{-2}D_2 &= (UD_2D_3)^{-2}D_2D_3\end{aligned}$$

$$\begin{aligned}
&\Longleftarrow (4),\ (5),\ (6);\\
\text{(vi)} &\Longleftrightarrow D_2A_3A_2^{-1}D_2^{-1} = (A_3U)^{-2}D_3^{-1}(A_3UD_2)^2D_3A_3A_2^{-1}\\
&\overset{(3),\,(7)}{\Longleftrightarrow} D_2 = (A_3U)^{-2}D_3^{-1}A_3(UD_2)^2D_3UD_2^{-1}U^{-1}A_3^{-1}\\
&\qquad \overset{(6),\,(4)}{=} (A_3U)^{-2}D_3^{-1}(A_3U)^2D_2D_3 \overset{(8)}{=} (UD_2D_3)^2D_3^{-1}(UD_2D_3)^{-2}D_2D_3\\
&\Longleftarrow (4),\ (5),\ (6);\\
\text{(vii)} &\Longleftrightarrow D_2D_3D_2^{-1} = (UD_2D_3)^2D_3(UD_2D_3)^{-2} \Longleftarrow (4),\ (5),\ (6);\\
\text{(viii)} &\Longleftrightarrow D_2(UD_2D_3)^{-2}D_2^{-1} = D_2D_3^{-1}D_2^{-1}D_3(UD_2D_3)^{-2} \Longleftarrow (4),\ (6);\\
\text{(ix)} &\Longleftrightarrow D_2(\delta_{12}\delta_{23}) = \delta_{12}\delta_{23} \Longleftrightarrow (UD_2)^2 = (D_2U)^2 \Longleftarrow (4);\\
\text{(x)} &\Longleftrightarrow D_2(A_3UD_2)^2D_2^{-1} = D_2A_3A_2^{-1}D_2^{-1}A_2A_3^{-1}(A_3U)^2 \Longleftarrow (3);\\
(1) &\Longrightarrow \text{(xi)};\\
\text{(xii)} &\Longleftrightarrow A_2D_3A_2^{-1} = (UD_2)^{-2}A_2A_3^{-1}(A_3UD_2)^2D_3A_3A_2^{-1}\\
&\overset{(7)}{\Longleftrightarrow} A_2(UD_2)^2 = (UD_2)^2A_2\\
&\Longleftarrow (4);\\
\text{(xiii)} &\Longleftrightarrow A_2(UA_3)^2A_2^{-1} = A_2A_3^{-1}(UA_3)^2A_3A_2^{-1}\\
&\overset{(1)}{\Longleftrightarrow} (UA_3)^2 = (A_3U)^2 \Longleftarrow (7),\ (8);\\
\text{(xiv)} &\Longleftrightarrow A_2(UD_2)^2 = (UD_2)^2A_2 \Longleftarrow (4);\\
(9) &\Longrightarrow \text{(xv)};\\
\text{(xvii)} &\Longleftrightarrow A_1D_3A_1^{-1} = (UD_2)^{-2}A_2A_3^{-1}D_3A_3A_2^{-1}(UA_3)^{-2}\\
&\overset{(1),\,(2),\,(4)}{\Longleftrightarrow} A_2A_1^{-1}D_3A_1A_2^{-1} = A_3(UD_2)^2D_3(UA_3)^2A_3^{-1} \overset{(8)}{\Longleftrightarrow} (10);\\
(1),\ (2),\ (4) &\Longrightarrow \text{(xvi), (xviii), (xix)};\\
\text{(xx)} &\Longleftrightarrow (11).
\end{aligned}$$

□

Let $\tilde{F} = F_2^3$ be a subsurface of $F$ such that boundary curve $c_i : S^1 \to \partial\tilde{F}$ bounds in $F$ a disc with puncture $p_i$ for $i = 1, 2, 3$. We identify $\{A_1, A_2, A_3, D_2, D_3, U\}$ with elements of $\mathcal{M}(\tilde{F})$.

**Theorem 7.12.** *The group* $\mathcal{M}(F_2^3)$ *admits a presentation with generators* $\{A_1, A_2, A_3, D_2, D_3, U, C_1, C_2, C_3\}$ *and relations* (1)–(7), (9)–(11) *from* Theorem 7.11 *and* (8′) $(UA_3)^2(UD_2D_3)^2 = (C_1C_2C_3)^2$, $C_iC_j = C_jC_i$, $C_iA_j = A_jC_i$, $C_iD_k = C_iD_k$, $C_iU = UC_i$, *for* $i, j \in \{1, 2, 3\}$, $k \in \{2, 3\}$.

Proof. Let $H$ denote the subgroup of $\mathcal{M}(\tilde{F})$ generated by the twists $\{C_1, C_2, C_3\}$. It is easy to see that relations (1)–(7) and (10) are satisfied in $\mathcal{M}(\tilde{F})$. In the proof

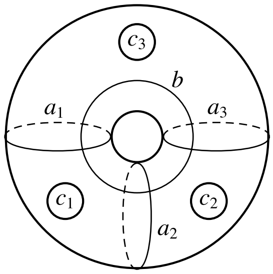


Fig. 13. The torus $T_3$.

of Theorem 7.11 we showed that $j(\varepsilon) = (A_3 U D_2)^2$ in $\mathcal{PM}^+(F, \{p_1, p_2, p_3\})$. On the other hand, by Lemma 7.1, $j(\varepsilon)$ is equal to a Dehn twist $E$ about a generic curve $e$. Thus in $\mathcal{M}(\tilde{F})$ we have $E(A_3 U D_2)^{-2} \in H$. It can be checked that in $\mathcal{M}(\tilde{F})$ we have $A_2 E A_2^{-1} = E$ and $A_1 E A_1^{-1} = (U D_2)^{-1} E (U D_2)$, and hence (9) and (11) hold, since $H$ is central.

Let $d_{23}$ and $l$ denote boundary curves of tubular neighborhoods of the loops $\delta_{23}$ and $\delta_3 \delta_{23}$, such that in $\mathcal{PM}^+(F, \{p_1, p_2, p_3\})$ we have $D_{23} = j(\delta_{23})$, $L D_2^{-1} = j(\delta_3 \delta_{23})$. The curves $d_{23}$ and $c_1$ bound in $\tilde{F}$ a Klein bottle with two holes, while $l$, $c_2$, $c_3$ bound a 4-holed sphere, together with a curve bounding a Möbius strip. Thus we have lantern relation $L C_2 C_3 = D_{23} D_2 D_3$ and relation $(U A_3)^2 = (U L)^2 = C_1 D_{23}$ from Lemma 7.8. Now

$$\begin{aligned}(U A_3)^2 &= (U L)^2 = (U D_{23} D_2 D_3 (C_2 C_3)^{-1})^2 = D_{23}^2 (C_2 C_3)^{-2} (U D_2 D_3)^2 \\ &= (U A_3)^4 (C_1 C_2 C_3)^{-2} (U D_2 D_3)^2 \\ &\iff (8').\end{aligned}$$

Theorem 7.12 follows from Theorem 7.11 and sequence (7.3). □

**7.3. Sporadic surfaces of genus 3.** Consider a torus with three holes $T_3$ represented in Fig. 13, and let $T_2$ be the torus with two holes obtained by gluing a disc to the boundary of $T_3$, along the curve $c_2$. We fix in $T_3$ and $T_2$ the orientation induced by the standard orientation of the plane of Fig. 13, and let $C_i$, $A_i$, $B$, $i = 1, 2, 3$ denote Dehn twists along the curves in the figure, right with respect to that orientation. The next theorem follows from the main result of [7].

**Theorem 7.13.** *The group $\mathcal{M}(T_3)$ admits presentation with generators $\{C_i, A_i, B \mid i = 1, 2, 3\}$ and relations*:

(7.11) $$C_i C_j = C_j C_i, \quad C_i A_j = A_j C_i, \quad C_i B = B C_i,$$

(7.12) $$A_i A_j = A_j A_i, \quad A_i B A_i = B A_i B,$$

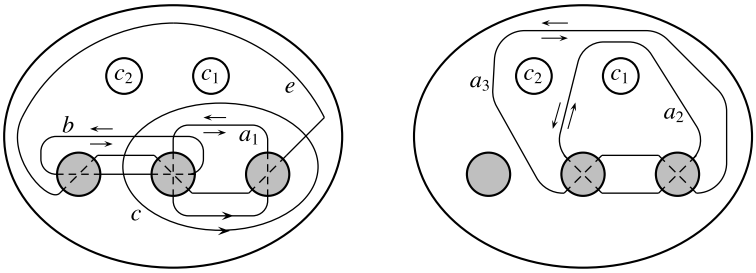


Fig. 14. The surface $\tilde{F} = F_3^2$.

*for* $i, j = 1, 2, 3,$ *and*

$$(A_1A_2A_3B)^3 = C_1C_2C_3. \tag{7.13}$$

*A presentation for* $\mathcal{M}(T_2)$ *may be obtained by adding to the above presentation relations* $C_2 = 1$ *and* $A_2 = A_3$.

REMARK 7.14. The relation (7.13)) is called "star" in [7]. In $\mathcal{M}(T_2)$ it takes form $(A_1A_2^2B)^3 = C_1C_3$, and it follows from relations (7.12) that $(A_1A_2^2B)^3 = (A_1^2A_2B)^3$.

Let $\tilde{F} = F_3^2$ be the surface obtained by gluing a Möbius strip $M$ to the boundary of $T_3$ along $c_3$. We identify $\tilde{F}$ with the surface represented in Fig. 14, where $M$ is a regular neighborhood of the one-sided curve $e$. Consider an embedding $\phi\colon K \to \tilde{F}$, where $K$ is the holed Klein bottle in Fig. 9, such that $\phi \circ c = c$ and $\phi \circ a_1 = a_1$. We define $U = \phi_*(U)$, where $U\colon K \to K$ is defined in Subsection 7.2. We identify $A_1$, $A_2$, $A_3$, and $B$ with elements of $\mathcal{M}(\tilde{F})$ (the directions of these twists are indicated by arrows in Fig. 14).

Let $F = F_3^0$ be the closed surface obtained by gluing two discs to $\partial\tilde{F}$. We fix a point $p_1 \in F$ inside the disc bounded by $c_1$, and $p_2 \in F$ inside the disc bounded by $c_2$.

**Theorem 7.15.** *The group* $\mathcal{PM}^+(F, \{p_1\})$ *admits a presentation with generators* $\{A_1, A_2, B, U\}$ *and relations*:
(1) $A_1A_2 = A_2A_1$;
(2) $A_1BA_1 = BA_1B$, $A_2BA_2 = BA_2B$;
(3) $UA_1U^{-1} = A_1^{-1}$;
(4) $UBU^{-1} = A_2^{-1}B^{-1}A_2$;
(5) $(UA_2)^2 = 1$;
(6) $(A_1A_2^2B)^3 = 1$.

Proof. Let us denote $G = \mathcal{PM}^+(F, \{p_1\})$. Notice that relations (1)–(6) are satisfied in $G$: (1) is obvious; (2), (3), (4) are relations of type $ht_ah^{-1} = t_{h(a)}^{\pm 1}$; (5) follows from Lemma 7.8; (6) is a star relation (cf. Remark 7.14).

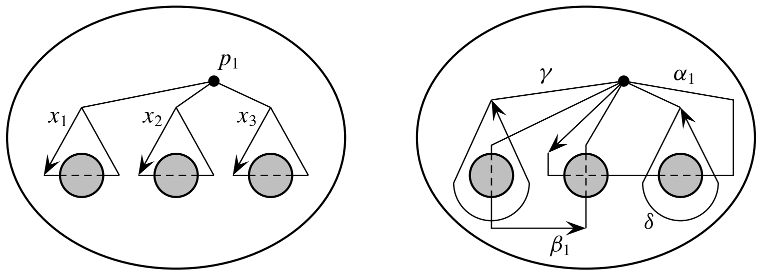


Fig. 15. Generators of $\pi_1(F, p_1)$ and $\pi_1^+(F, p_1)$.

Consider the exact sequence (7.2):

$$1 \to \pi^+(F, p_1) \xrightarrow{j} G \to \mathcal{M}(F) \to 1.$$

The fundamental group $\pi_1(F, p_1)$ is generated by the loops $x_1, x_2, x_3$ in Fig. 15 satisfying one defining relation $x_3^2x_2^2x_1^2 = 1$. Now $\{1, x_3\}$ is a Schreier system of representatives of cosets of $\pi_1^+(F, p_1)$ and by the Reidemeister-Schreier method we obtain that the last group is generated by $u_1 = x_1x_3^{-1}$, $u_2 = x_2x_3^{-1}$, $u_3 = x_3x_1$, $u_4 = x_3x_2$ and $u_5 = x_3^2$ satisfying two defining relations: $u_5u_2u_4u_1u_3 = 1$, $u_5u_4u_2u_3u_1 = 1$. After Tietze transformations (cf. [17]) we obtain

$$\pi_1^+(F, p_1) = \langle \alpha_1, \beta_1, \delta, \gamma \mid \beta_1^{-1}\delta^{-1}\gamma^{-1}\alpha^{-1}\delta\alpha_1\beta_1\gamma = 1\rangle,$$

where $\alpha_1 = u_4$, $\delta = u_5$, $\beta_1 = u_2u_3$, $\gamma = u_1u_3$ are the loops in Fig. 15. It follows from Theorem 2 of [3] that $\mathcal{M}(F)$ admits a presentation with generators $\{A_1, B, U\}$ and relations $A_1BA_1 = BA_1B$, $UA_1U^{-1} = A_1^{-1}$, $UBU^{-1} = A_1^{-1}B^{-1}A_1$, $U^2 = 1$, $(A_1^3B)^3 = 1$. The last relation is a special form of the star relation (7.13) and it can be checked that in $G$ we have $(A_1^3B)^3 = j(\beta_1^{-1}\alpha_1\beta_1\alpha_1^{-1})$. We also have $UBU^{-1}A_1^{-1}BA_1 = j(\beta_1^{-1}\alpha_1^{-1})$. By Lemma 7.4, $G$ admits presentation with generators $\{A_1, B, U, j(\alpha_1), j(\beta_1), j(\gamma), j(\delta)\}$ and relations:

(i) $A_1BA_1 = BA_1B$;
(ii) $UA_1U^{-1} = A_1^{-1}$;
(iii) $UBU^{-1}A_1^{-1}BA_1 = j(\beta_1^{-1}\alpha_1^{-1})$;
(iv) $U^2 = j(\gamma)$;
(v) $(A_1^3B)^3 = j(\beta_1^{-1}\alpha_1\beta_1\alpha_1^{-1})$;
(vi) $j(\beta_1^{-1}\delta^{-1}\gamma^{-1}\alpha^{-1}\delta\alpha_1\beta_1\gamma) = 1$;
(vii) $A_1j(\alpha_1)A_1^{-1} = j(\alpha_1)$;
(viii) $A_1j(\beta_1)A_1^{-1} = j(\alpha_1^{-1}\beta_1)$;
(ix) $A_1j(\gamma)A_1^{-1} = j(\gamma)$;
(x) $A_1j(\delta)A_1^{-1} = j(\gamma^{-1}\alpha_1^{-1}\delta\alpha_1)$;
(xi) $Bj(\alpha_1)B^{-1} = j(\alpha_1\beta_1)$;

(xii) $Bj(\beta_1)B^{-1} = j(\beta_1)$;
(xiii) $Bj(\gamma)B^{-1} = j(\beta_1^{-1}\gamma\delta\beta_1)$;
(xiv) $Bj(\delta)B^{-1} = j(\delta)$;
(xv) $Uj(\alpha_1)U^{-1} = j(\alpha_1^{-1}\gamma^{-1})$;
(xvi) $Uj(\beta_1)U^{-1} = j(\gamma\delta\alpha_1\beta_1)$;
(xvii) $Uj(\gamma)U^{-1} = j(\gamma)$;
(xviii) $Uj(\delta)U^{-1} = j(\delta^{-1}\gamma^{-1})$.

We have:

$$j(\gamma) = U^2, \quad j(\alpha_1) = A_2A_1^{-1}, \quad j(\beta_1) = A_1A_2^{-1}BA_2A_1^{-1}B^{-1}. \tag{7.14}$$

It can be checked that $U^{-1}B(\alpha_1) = \delta\beta_1$, and hence

$$j(\delta) = U^{-1}BA_2A_1^{-1}B^{-1}UBA_1A_2^{-1}B^{-1}A_2A_1^{-1}. \tag{7.15}$$

Let $H$ denote the subgroup of $G$ generated by $\{A_1, A_2, B\}$. Consider the homomorphism $i_*\colon \mathcal{M}(T_2) \to G$ induced by the inclusion of $T_2$ in $F$. It can be proved, using the same methods as in the proof of Lemma 4.1, that $\ker i_*$ is generated by $\{C_1, C_3\}$. Now it follows from Theorem 7.13 that $i_*(\mathcal{M}(T_2)) = H$ and every relation in $H$ is a consequence of (1), (2), (6).

We will show that relations (i)–(xviii) after replacing $j(\alpha_1)$, $j(\beta_1)$, $j(\gamma)$ and $j(\delta)$ by expressions (7.14), (7.15), are consequences of (1)–(6). Relations (i), (ii) are the same as (2), (3); (iv), (xi), (xvii) are trivial; (v), (vii), (viii), (xii) are relations in $H$, hence they follow from (1), (2), (6). We have

$$UBU^{-1}A_1^{-1}BA_1 \overset{(4)}{=} A_2^{-1}B^{-1}A_2A_1^{-1}BA_1 \overset{(2)}{=} BA_2^{-1}A_1B^{-1} \iff \text{(iii)};$$

$$(3) \implies \text{(ix)};$$

$$(1),\ (3),\ (5) \implies \text{(xv)};$$

(x), (xiii), (xiv) can easily be reduced to relations in $H$, by using (1)–(4).

Let $X = UBA_2^{-1}A_1B^{-1}A_1^{-1}A_2BA_1^{-1}A_2B^{-1}U$, and note that to prove (1)–(6) $\Rightarrow$ (xvi), (xviii), it suffices to show (1)–(6) $\Rightarrow X \in H$. By (2), (3), (4) we have

$$UA_1U^{-1} \in H, \quad UBU^{-1} \in H, \quad BA_2B^{-1} = UB^{-1}U^{-1},$$

thus

$$X \in H \iff UA_2^{-1}B^{-1}A_1^{-1}A_2BA_1^{-1}A_2B^{-1}U \in H \iff UA_2^{-1}B^{-1}A_1^{-2}B^{-1}U \in H.$$

It can be checked that from (1), (2), (6) follows $A_2^{-1}B^{-1}A_1^{-2}B^{-1}A_2^{-1} = A_1BA_1^2BA_1$, hence $X \in H \Leftrightarrow UA_2U \in H \Leftarrow (5)$. Finally, we have

$$\begin{aligned} j(\beta_1^{-1}\delta^{-1}\gamma^{-1}\alpha_1^{-1}\delta\alpha_1\beta_1\gamma) &\overset{\text{(xvi)}}{=} j(\beta_1^{-1}\delta^{-1})U^{-2}j(\alpha_1^{-1})U^{-1}j(\beta_1)U \\ &= U^{-1}j(\beta_1^{-1}\alpha_1^{-1})U^{-1}j(\alpha_1^{-1})U^{-1}j(\beta_1)U, \end{aligned}$$

thus

$$(\mathrm{vi}) \Longleftrightarrow (Uj(\alpha_1))^2 = 1 \Longleftarrow (1),\ (3),\ (5). \qquad \square$$

**Theorem 7.16.** *The group* $\mathcal{M}(F_3^1)$ *admits a presentation with generators* $\{A_1, A_2, B, U\}$, *relations* (1)–(4) *from* Theorem 7.15 *and* $(A_2U)^2 = (UA_2)^2 = (A_1^2A_2B)^3$.

Proof. Consider the surface $F_3^1$ obtained by gluing a disc to the boundary of $\tilde{F}$ along $c_2$. Observe that relations (1)–(4) from Theorem 7.15 are satisfied in $\mathcal{M}(F_3^1)$, and we have $(A_1^2A_2B)^3 = C_1$ (star) and $(A_2U)^2 = C_1$ (Lemma 7.8). After replacing the generator $C_1$ in the presentation of $\mathcal{M}(F_3^1)$ resulting from applying Lemma 7.4 to sequence (7.3), we obtain Theorem 7.16. $\square$

**Theorem 7.17.** *The group* $\mathcal{PM}^+(F, \{p_1, p_2\})$ *admits a presentation with generators* $\{A_1, A_2, A_3, B, D_1, D_2, D_3, U\}$ *and relations*:
(1) $A_iA_j = A_jA_i,\ i, j = 1, 2, 3;$
(2) $A_iBA_i = BA_iB,\ i = 1, 2, 3;$
(3) $UA_1U^{-1} = A_1^{-1};$
(4) $UBU^{-1} = A_3^{-1}B^{-1}A_3;$
(5) $UD_1 = D_1U;$
(6) $UD_3 = D_3U;$
(7) $BD_2 = D_2B;$
(8) $(UA_2)^2 = D_1;$
(9) $(A_1^2A_3B)^3 = (UA_3)^2 = D_3;$
(10) $A_2^{-1}UD_2U^{-1}A_2 = UB^{-1}D_1^{-1}BU^{-1};$
(11) $(UD_2)^2D_1D_3 = U^2;$
(12) $(A_1A_2A_3B)^3 = 1.$

Proof. Let us denote $G = \mathcal{PM}^+(F, \{p_1, p_2\})$. The fundamental group $\pi_1(F \setminus \{p_1\}, p_2)$ is free on generators $y_1, y_2, y_3$ in Fig. 16. Now $\{1, y_3\}$ is a Schreier system of representatives of cosets of $\pi_1^+(F \setminus \{p_1\}, p_2)$ and by the Reidemeister-Schreier method we obtain that the last group is freely generated by $v_1 = y_1y_3^{-1}$, $v_2 = y_2y_3^{-1}$, $v_3 = y_3y_1$, $v_4 = y_3y_2$, $v_5 = y_3^2$. It follows that $\pi_1^+(F \setminus \{p_1\}, p_2)$ is free on generators $\delta_2 = v_5$, $\delta_1 = v_1v_3$, $\beta_2 = v_2v_3$, $\delta_3 = \delta_2v_2v_4\delta_1$, $\alpha_2 = \delta_3v_4$ (see Fig. 16). We introduce Dehn twists $D_i = j(\delta_i)$, $i = 1, 2, 3$. We also have

$$j(\alpha_2) = A_3A_2^{-1}, \quad j(\beta_2) = A_3^{-1}A_2BA_2^{-1}A_3B^{-1}.$$

Let us check that relations (1)–(12) are satisfied in $G$: (1), (2), (12) follow from Theorem 7.13; (3), (4), (10) are relations of type $ht_ah^{-1} = t_{h(a)}^{\pm 1}$; (5), (6), (7) are obvious; (8), (9) follow from Lemma 7.8 and star relation; (11) follows from the equality $U(\delta_2) = \delta_2^{-1}\delta_3^{-1}\delta_1^{-1}$ and relations (5), (6).

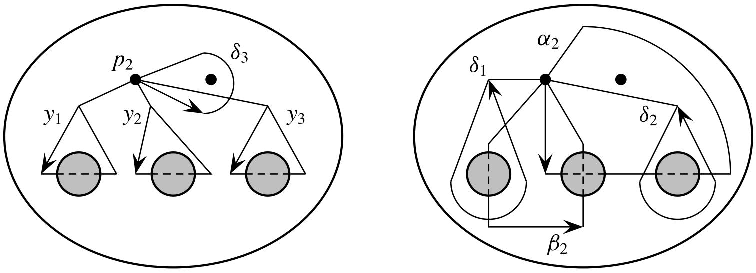


Fig. 16. Generators of $\pi_1(F \setminus \{p_1\}, p_2)$ and $\pi_1^+(F \setminus \{p_1\}, p_2)$.

By Theorem 7.15 and Lemma 7.4 for sequence (7.2), $G$ admits a presentation with generators $\{A_1, A_2, B, U, j(\alpha_2), j(\beta_2), j(\delta_i) \mid i = 1, 2, 3\}$ and relations (1), (2), (3) and:
(i) $UBU^{-1}A_2^{-1}BA_2 = j(\beta_2^{-1}\alpha_2^{-1})$;
(ii) $(UA_2)^2 = j(\delta_1)$;
(iii) $(A_1A_2^2B)^3 = j(\beta_2^{-1}\delta_3^{-1}\alpha_2\beta_2\alpha_2^{-1})$;
(iv) $A_1j(\alpha_2)A_1^{-1} = A_2j(\alpha_2)A_2^{-1} = j(\alpha_2)$;
(v) $A_1j(\beta_2)A_1^{-1} = j(\alpha_2^{-1}\delta_3\beta_2)$;
(vi) $A_1j(\delta_1)A_1^{-1} = A_2j(\delta_1)A_2^{-1} = Uj(\delta_1)U^{-1} = j(\delta_1)$;
(vii) $A_1j(\delta_3)A_1^{-1} = Bj(\delta_3)B^{-1} = Uj(\delta_3)U^{-1} = j(\delta_3)$;
(viii) $A_1j(\delta_2)A_1^{-1} = j(\delta_3^{-1}\delta_1^{-1}\alpha_2^{-1}\delta_3\delta_2\delta_3^{-1}\alpha_2)$;
(ix) $A_2j(\beta_2)A_2^{-1} = j(\alpha_2^{-1}\beta_2)$;
(x) $A_2j(\delta_3)A_2^{-1} = j(\alpha_2^{-1}\delta_3\alpha_2)$;
(xi) $A_2j(\delta_2)A_2^{-1} = j(\alpha_2^{-1}\delta_3^{-1}\alpha_2\delta_3\delta_2\beta_2\delta_1^{-1}\beta_2^{-1}\alpha_2^{-1}\delta_3\alpha_2)$;
(xii) $Bj(\alpha_2)B^{-1} = j(\alpha_2\beta_2)$;
(xiii) $Bj(\beta_2)B^{-1} = j(\beta_2)$;
(xiv) $Bj(\delta_1)B^{-1} = j(\beta_2^{-1}\delta_1\delta_3\delta_2\beta_2)$;
(xv) $Bj(\delta_2)B^{-1} = j(\delta_2)$;
(xvi) $Uj(\alpha_2)U^{-1} = j(\delta_3\alpha_2^{-1}\delta_1^{-1})$;
(xvii) $Uj(\beta_2)U^{-1} = j(\delta_1\delta_3\delta_2\delta_3^{-1}\alpha_2\beta_2)$;
(xviii) $Uj(\delta_2)U^{-1} = j(\delta_2^{-1}\delta_3^{-1}\delta_1^{-1})$.

We will show that relations (i)–(xviii) after substituting $j(\alpha_2) = A_3A_2^{-1}$, $j(\beta_2) = A_3^{-1}A_2BA_2^{-1}A_3B^{-1}$, $j(\delta_i) = D_i$, are consequences of (1)–(12).

Let $H$ denote the subgroup of $G$ generated by $\{A_1, A_2, A_3, B\}$. As in the proof of Theorem 7.15, we have $H = i_*(\mathcal{M}(T_3))$, where $i_*$ is the homomorphism induced by the inclusion of $T_3$ in $F$, and every relation in $H$ is a consequence of (1), (2), (12), by Theorem 7.13. Note that by the star relation (9), $D_3 \in H$.

Relations (i)–(vii), (ix), (x), (xii), (xiii), (xv) follow easily from (1)–(12) or are relations in $H$;

$$(8),\ (9) \implies \text{(xvi)};$$

$$(5),\ (6),\ (11)\ \Longrightarrow \text{(xviii)};$$

by (5), (8) we have $A_2D_1 = D_1A_2$ and

$$\begin{aligned}
\text{(xiv)} &\overset{\text{(xviii)}}{\Longleftrightarrow} j(\beta_2)BD_1B^{-1}j(\beta_2^{-1}) = Uj(\delta_2^{-1})U^{-1}\\
&\Longleftrightarrow A_3^{-1}BA_3D_1A_3^{-1}B^{-1}A_3 = A_2^{-1}UD_2^{-1}U^{-1}A_2\\
&\Longleftarrow (4),\ (5),\ (10);\\
\text{(xvii)} &\overset{\text{(xviii)}}{\Longleftrightarrow} Uj(\beta_2)U^{-1} = UD_2^{-1}U^{-1}D_3^{-1}j(\alpha_2\beta_2)\\
&\overset{(5),(6),(11)}{\Longleftrightarrow} j(\beta_2) = UD_2U^{-1}D_1U^{-1}j(\alpha_2\beta_2)U\\
&\overset{\text{(xiv), (xviii)}}{\Longleftrightarrow} BD_1B^{-1} = j(\beta_2^{-1})D_1U^{-1}j(\alpha_2\beta_2)U\\
&\Longleftrightarrow D_1^{-1}A_2A_3^{-1}BA_3A_2^{-1}D_1 = U^{-1}j(\alpha_2\beta_2)UB\\
&\overset{(4),(8)}{\Longleftrightarrow} A_2^{-1}B^{-1}A_2 = j(\alpha_2\beta_2)UBU^{-1}\\
&\Longleftarrow UBU^{-1} \in H \Longleftarrow (4);\\
\text{(viii)} &\overset{\text{(vii)}}{\Longleftrightarrow} A_1(\delta_3\delta_2\delta_3^{-1}) = \delta_1^{-1}\alpha_2^{-1}\delta_3\delta_2\delta_3^{-1}\alpha_2\delta_3^{-1}\\
&\overset{\text{(xvii)}}{\Longleftrightarrow} A_1(\delta_1^{-1}U(\beta_2)\beta_2^{-1}\alpha_2^{-1}) = \delta_1^{-1}\alpha_2^{-1}\delta_1^{-1}U(\beta_2)\beta_2^{-1}\delta_3^{-1}\\
&\overset{\text{(iv), (v), (vi)}}{\Longleftrightarrow} A_1U(\beta_2) = \alpha_2^{-1}\delta_1^{-1}U(\beta_2)\\
&\overset{(8)}{\Longleftrightarrow} A_1Uj(\beta_2)U^{-1}A_1^{-1} = A_3^{-1}U^{-1}A_2^{-1}j(\beta_2)U^{-1}\\
&\overset{(3)}{\Longleftrightarrow} A_2UA_3UA_1^{-1}j(\beta_2)A_1 = j(\beta_2)\\
&\Longleftarrow UA_3U \in H \Longleftarrow (9);\\
\text{(xi)} &\overset{\text{(xiv), (xviii)}}{\Longleftrightarrow} A_3D_2A_3^{-1} = D_3^{-1}A_3A_2^{-1}D_3D_2B^{-1}UD_2U^{-1}BA_2A_3^{-1}D_3\\
&\overset{\text{(vii), (7), (11)}}{\Longleftrightarrow} D_3D_2D_3^{-1} = A_2^{-1}B^{-1}D_1^{-1}BA_2\\
&\overset{\text{(xvii)}}{\Longleftrightarrow} D_1^{-1}Uj(\beta_2)U^{-1}j(\beta_2^{-1}\alpha_2^{-1}) = A_2^{-1}B^{-1}D_1^{-1}BA_2\\
&\overset{(8)}{\Longleftrightarrow} U^{-1}A_3^{-1}BA_3A_2^{-1}B^{-1}U^{-1}BA_3^{-1}A_2B^{-1}A_2^{-1} = B^{-1}D_1^{-1}B\\
&\Longleftarrow (2),\ (4),\ (8).
\end{aligned}$$

□

**Theorem 7.18.** *The group $\mathcal{M}(F_3^2)$ admits a presentation with generators $\{A_1, A_2, A_3, B, D_1, D_2, D_3, U, C_1, C_2\}$ and relations* (1)–(7), (9), (10) *from* Theorem 7.17 *and*
(8′) $(UA_2)^2 = D_1C_1$,
(11′) $(UD_2)^2D_1D_3 = U^2C_1C_2^2$,
(12′) $(A_1A_2A_3B)^3 = C_1C_2 = C_2C_1$,
$C_iA_j = A_jC_i,\ C_iD_k = D_kC_i,\ C_iB = BC_i,\ C_iU = UC_i,$ *for* $i = 1, 2,\ j, k = 1, 2, 3.$

Proof. The relations (1)–(7), (9), (10) from Theorem 7.17 are satisfied in $\mathcal{M}(\tilde{F}) = \mathcal{M}(F_3^2)$; (8′) follows from Lemma 7.8; (12′) is the star relation; (11′) follows from Lemma 7.8 and lantern relation $C_1C_2U^2 = ((UD_2)^2C_2^{-1})D_1D_3$. Now Theorem 7.18 follows from Theorem 7.17 and Lemma 7.4 for sequence (7.2). □

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