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ALGEBRAIC REDUCTION OF TWISTOR SPACES OF HOPF SURFACES

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Introduction

Let Z be a compact connected complex manifold of dimension three. Let $a(Z)$ be the algebraic dimension of Z, i.e., the transcendence degree of the meromorphic function field of Z. We have $0 \leq a(Z) \leq 3$. When $a(Z) = 1$, Z admits a meromorphic map $f : Z \to Y$ onto a projective nonsingular curve Y, which induces by the pull-back an isomorphism of the meromorphic function fields. We call such a map f a *meromorphic algebraic reduction* of Z. Some restrictions on the possible structure of general fibers of f are known as in Ueno [10, 12.4, 12.5]. But the question as to which surfaces actually appear in such an algebraic reduction seems still open except in the Kähler case. (See [2] for the general results in the Kähler case.)

Now twistor spaces of compact anti-self-dual surfaces provide an interesting family of non-Kähler compact complex threefolds whose algebraic dimensions can take any values from zero to three. So it should be quite natural to consider the above question for these twistor spaces.

The purpose of this note is to answer the question in one of the simplest cases of the twistor spaces of Hopf surfaces, which, we hope, should serve as useful examples in understanding the situation in the general case. We show for instance that in case $a(Z) = 1$ either 1) a general fiber is a Hopf surface or 2) the normalization of a general fiber, which is in general non-normal, is a nonsingular ruled surface of genus one.

Note that Gauduchon [4] has already determined the algebraic dimensions of the twistor spaces of Hopf surfaces (cf. Proposition 1.1 below), generalizing the previous result of Pontecorvo [8], who has shown that the twistor spaces of special Hopf surfaces of the form $S(r, r)$ (cf. below) are of algebraic dimension two and has determined the structure of their algebraic reductions.

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1. Statement of Results

A *Hopf surface* is a compact analytic surface T whose universal covering is isomorphic to $W := \boldsymbol{C}^2 - \{0\}$. Suppose that T admits an anti-self-dual hermitian metric

k. (Note here that by [3, Theorem 1.4] if T admits more generally an anti-self-dual Riemannian metric whose twistor space has positive algebraic dimension, then up to unramified coverings the metric is necessarily conformally equivalent to a hermitian metric.) Then (T, k) admits a finite unramified covering (S, g) such that S is obtained as a quotient $S = W/\langle h\rangle$, where h is a fixed point free automorphism of W of the form

$$h : (w_1, w_2) \to (\alpha_1 w_1, \alpha_2 w_2), \quad 1 < |\alpha_1| = |\alpha_2|, \ \alpha_i \in \boldsymbol{C}, \quad (w_1, w_2) \in \boldsymbol{C}^2,$$

and the induced anti-self-dual hermitian metric g is a conformally flat hermitian metric induced from the metric

$$\tilde{g} := \frac{dw_1 \cdot d\bar{w}_1 + dw_2 \cdot d\bar{w}_2}{|w|^2}, \quad |w|^2 = |w_1|^2 + |w_2|^2 \tag{1}$$

on W (cf. [8, 3.3]). In general we shall denote the conformally flat primary Hopf surface obtained in this way by

$$S(\alpha_1, \alpha_2).$$

We also write

$$(\alpha_1, \alpha_2) = \left(re^{2\pi\sqrt{-1}\theta_1}, re^{2\pi\sqrt{-1}\theta_2}\right).$$

for some $r > 1$ and $\theta_1, \theta_2 \in \boldsymbol{R}$ with θ_i considered modulo $\boldsymbol{Z}$. $S(\alpha_1, \alpha_2)$ admits an unramified cyclic coverings $S(\alpha_1^k, \alpha_2^k)$ of degree k, one for each $k > 0$.

We first show in our algebro-geometric context the following result of Gauduchon (cf. Corollaire in [4, p. 40]).

Proposition 1.1. *Let $S = S(\alpha_1, \alpha_2)$ with $\alpha_i = re^{2\pi\sqrt{-1}\theta_i}$ be an anti-self-dual hermitian Hopf surface as above and Z the associated twistor space.*

1) *The following conditions are equivalent*:
   i) $a(Z) = 2$.
   ii) *θ_1 and θ_2 are both rational.*
2) *The following conditions are equivalent*:
   i) $a(Z) = 1$.
   ii) *Possibly after passing to a finite unramified covering we may write $\theta_1 = m\theta$ and $\theta_2 = n\theta$ for some non-rational real number θ and a pair of integers m and n with $(m, n) \neq (0, 0)$.*

By [10, Cor. 3.9] Z can never be Moishezon since it contains a non-algebraic surface S as a complex submanifold (cf. (5) below). So the algebraic dimension of Z always satisfies $a(Z) \leq 2$. Therefore we obtain:

Corollary 1.2. *For any* $\alpha_i = re^{2\pi\sqrt{-1}\theta_i}$, $i = 1, 2$, *as above the twistor space* Z *associated to* $S(\alpha_1, \alpha_2)$ *has a positive algebraic dimension if and only if, possibly after passing to a finite unramified covering, for suitable choices of* θ_1 *and* θ_2 *modulo* $\mathbf{Z}$ *their ratio is rational.*

Next we study the structure of algebraic reductions. As follows from 1) of Proposition 1.1, in the case $a(Z) = 2$, S always admits $S(r^k, r^k)$ as a k-fold cyclic unramified covering for some positive integer k. The algebraic reduction of the twistor space Z_k of $S(r^k, r^k)$ is realized by a holomorphic map $f_k : Z_k \to \boldsymbol{P} \times \boldsymbol{P}$ which makes Z a principal elliptic bundle over $\boldsymbol{P} \times \boldsymbol{P}$ (cf. [8]), where $\boldsymbol{P} = \boldsymbol{P}^1$ is the complex projective line. The algebraic reduction for the original Z is then realized by a holomorphic map $f : Z \to Y$ onto a projective orbifold Y which is a cyclic quotient of $\boldsymbol{P} \times \boldsymbol{P}$.

Next we consider the case $a(Z) = 1$. First we introduce some general notations. Let $S = S(\alpha_1, \alpha_2)$ be as above. On the associated twistor space Z there exists a canonical square root $K^{-1/2}$ of the anti-canonical bundle K^{-1}. We denote by $K^{-1/4}$ one of the two square roots of $K^{-1/2}$. (The choice of these two corresponds to the choice of two spin structures of S.)

We may consider h as a generator of the fundamental group $\pi_1(Z)$ ($\cong \pi_1(S) \cong \mathbf{Z}$) of Z. For any $\beta \in \boldsymbol{C}^*$ we denote by $F(\beta)$ the flat line bundle on Z associated to the character $\pi_1(Z) \to \boldsymbol{C}^*$ which sends h to $\beta \in \boldsymbol{C}^*$. We identify $F(\beta)$ with the associated holomorphic line bundle. In what follows the juxstaposition of line bundles denotes their tensor product, e.g., $F(\beta)K^{-1} = F(\beta) \otimes K^{-1}$.

Suppose now that $a(Z) = 1$, so that we are in the case 2) of Proposition 1.1. Let $f : Z \to \boldsymbol{P}$ be the meromorphic algebraic reduction of Z, which is characterized as a unique meromorphic map of Z onto a compact Riemann surface with connected fibers. After passing to a finite unramified covering if necessary, we assume that θ_1, θ_2 satisfy the condition of ii) of Proposition 1.1. In what follows we describe the algebraic reduction of f precisely under this assumption. In the general case the structure of algebraic reduction is easily deduced from the description in this special case (cf. Remark 2).

In order to state our results conveniently, we shall normalize the integers m and n in 2) of the proposition as follows. For two integers m and n we shall denote by $\gcd(m, n)$ the greatest common divisor of $|m|$ and $|n|$. When $m = 0$ (resp. $n = 0$), we understand that $\gcd(m, n) = |n|$ (resp. $|m|$). Note that if we impose the conditions;

$$m + n \geq 0, \quad m > 0 \ \text{ if } \ m + n = 0 \ \text{ and } \ \gcd(m, n) = 1$$

m and n are uniquely determined. Let m_0, and n_0 be the integers taken in this way. Then we define

$$\begin{aligned}(m, n) &= (m_0, n_0) \quad \text{if} \quad m_0 + n_0 \quad \text{is even}\\ &= (2m_0, 2n_0) \quad \text{if} \quad m_0 + n_0 \quad \text{is odd.}\end{aligned}$$

(In the latter case we replace θ by $\theta/2$ accordingly.) In what follows, we use this latter normalization for our m and n. Thus $m+n$ is always divisible by 2 and $\gcd(m,n)=1$ or 2. The structure of f is quite different according to the two cases 1) $m=\pm n$ and 2) $m\neq\pm n$, or under our normalization equivalently, 1) $(m,n)=(1,\pm1)$ and 2) $m<|n|$ or $n<|m|$. The following is our main result.

Theorem 1.3. *Let the notations and assumptions be as above. In particular, we assume that $a(Z)=1$.*
1) *Suppose that $(m,n)=(1,1)$ (resp. $=(1,-1)$). Then f is a holomorphic fiber bundle over $\boldsymbol{P}$ with typical fiber the Hopf surface $S=S(\alpha,\alpha)$ (resp. $S(\alpha,\bar{\alpha})$), $\alpha=re^{2\pi\sqrt{-1}\theta}$, and is given by the linear system $|F(r^{1/2})K^{-1/4}|$ (resp. $|F(r^{-1/2})K^{-1/4}|$).*
2) *Suppose that $m<|n|$ (resp. $n<|m|$). Then f is given by the linear system $|F(r^m)K^{-n/4}|$ (resp. $|F(r^n)K^{-m/4}|$) and never is holomorphic. There exist precisely two reducible fibers whose underlying reduced subspaces are union of nonsingular Hopf surfaces intersecting transversally. The other fibers are all irreducible and mutually isomorphic. If R denotes a general fiber, and if $\nu:\hat{R}\to R$ is the normalization, $\hat{R}$ is isomorphic to a $\boldsymbol{P}$-bundle over a nonsingular elliptic curve C of the form $\boldsymbol{P}(1\oplus L)$, where C is isomorphic to $\boldsymbol{C}^*/\langle\alpha_2\rangle$ (resp. $\boldsymbol{C}^*/\langle\alpha_1\rangle$), and L is a holomorphic line bundle of degree zero on C of infinite order.*

All the above results can be read off from an explicit description of the geometry of Z, which in fact gives us more precise informations on the structure of f. We shall state some of them in the following:

REMARK 1. 1) In 1) of the above theorem the relevant surfaces $S(\alpha,\bar{\alpha})$ and $S(\alpha,\alpha)$ are precisely those primary Hopf surfaces which admit a coordinate quaternionic structure, or equivalently, those which admit a hyperhermitian structure (cf. Kato, Ma. [5, Prop. 8], and Boyer [1]). The map f coincides with the fibering associated to the hyperhermitian structure.
2) Suppose that we are in the situation of 2) of the above theorem.
a) Only in the cases $S(r,re^{2\pi\sqrt{-1}\theta})$ or $S(re^{2\pi\sqrt{-1}\theta},r)$, f is given by the linear system of the form $|K^{-n/4}|$ for some integer n; and in this case by $|K^{-1/2}|$.
b) The two reducible fibers are mutually conjugate by the real structure σ of Z and are the unions $S_0\cup S_3$ and $S_1\cup S_2$ (resp. $S_0\cup S_2$ and $S_1\cup S_3$) in the notation of (5) below. The multiplicities of the irreducible components are always $|n-m|/2$ and $(n+m)/2$ so that f admits no 'multiple' fibers.
c) The base locus of f is a disjoint union of mutually σ-conjugate two nonsingular elliptic curves C_{02} and C_{13} (resp. C_{03} and C_{12}) in the notation of (6) below. These are isomorphic to C above.
d) The general fiber R is nonsingular (so that ν is isomorphic) if and only if either $m+n=2$ or $|n-m|=2$.

e) The twistor space Z is in fact a G-equivariant compactification of a commutative complex Lie group $G \cong \boldsymbol{C}^{*3}/\boldsymbol{Z}$ and the G-action on Z induces a natural G-action on $\boldsymbol{P}$ making f equivariant. The two reducible fiber corresponds to the fixed points of this action on $\boldsymbol{P}$.

REMARK 2. If T is a general anti-self-dual Hopf surface, it admits a finite unramified covering S which is a primary Hopf surface of the form of the theorem. Then the twistor space Z_T of T is a quotient of that of S. When $a(Z) = 1$, the algebraic reduction $f_T : Z_T \to \boldsymbol{P}$ of Z_T is thus a quotient of the algebraic reduction $f : Z \to \boldsymbol{P}$ for S by the induced action of the covering transformation group of $S \to T$. The general fiber of f_T admits a general fiber of f as a finite unramified covering.

2. Twistor Spaces

The twistor space $\tilde{Z}$ of the anti-self-dual manifold $(W, \tilde{g})$ is naturally identified with the Zariski open subset $U := \boldsymbol{P}^3 - l_0 \cup l_\infty$ of $\boldsymbol{P}^3(z_0 : z_1 : z_2 : z_3)$, where $l_0 = \{z_0 = z_1 = 0\}$ and $l_\infty = \{z_2 = z_3 = 0\}$. W is realized as a subset of U defined by $z_0 = 0$. Moreover, the induced action of h on $\tilde{Z}$ takes the form

$$\tilde{h} : (z_0 : z_1 : z_2 : z_3) \to (dz_0 : \bar{d}z_1 : az_2 : \bar{a}z_3), \tag{2}$$

where a, d are complex numbers with $|d| = 1$, determined (up to simultaneous inversion of the signs) by the condition

$$(ad, \bar{a}d) = (\alpha_1, \alpha_2)$$

(cf. [9]). If we set

$$a = re^{2\pi\sqrt{-1}s}, \quad d = e^{2\pi\sqrt{-1}t}, \tag{3}$$

then we have the following transformation rules modulo $\boldsymbol{Z}$;

$$\theta_1 \equiv s + t, \quad \theta_2 \equiv -s + t \quad \text{or} \quad 2s \equiv \theta_1 - \theta_2, \quad 2t \equiv \theta_1 + \theta_2. \tag{4}$$

Let $S = S(\alpha_1, \alpha_2)$ as in Proposition 1.1. The twistor space Z of S is then the quotient of $\tilde{Z}$ by the cyclic group $\langle \tilde{h} \rangle$. There exist four distinct smooth divisors $S_j, 0 \leq j \leq 3$, in Z; they are the natural images of the divisors $\{z_j = 0\}$ in $\tilde{Z}$. In view of (3) and (4) each S_j is a primary Hopf surface such that

$$S_0 \cong S(\alpha_1, \alpha_2), \quad S_1 \cong S(\bar{\alpha}_1, \bar{\alpha}_2), \quad S_2 \cong S(\bar{\alpha}_1, \alpha_2), \quad S_3 \cong S(\alpha_1, \bar{\alpha}_2). \tag{5}$$

S_0 and S_1 (resp. S_2 and S_3) are conjugate to each other via the real structure σ of Z which in turn is induced by that on $\tilde{Z}$; $(z_0 : z_1 : z_2 : z_3) \to (\bar{z}_1 : -\bar{z}_0 : \bar{z}_2 : -\bar{z}_3)$.

Moreover, $S_0 \bigcap S_1 = S_2 \bigcap S_3 = \emptyset$ and any of the other intersections

$$C_{ij} := S_i \bigcap S_j \tag{6}$$

is isomorphic to a nonsingular elliptic curve.

All the S_j are elementary divisors on Z. Hence, Z is actually considered as the twistor space of any of S_i up to complex conjugations. For instance, the twistor spaces of $S_0 \cong S(\alpha_1, \alpha_2)$ and $S_2 \cong S(\alpha_1, \bar{\alpha}_2)$ are complex conjugate to each other. (This duality corresponds to the double statements in the above theorem and remarks.) Indeed, the orientation reversing diffeomorphism $(z_1, z_2) \to (z_1, \bar{z}_2)$ of W induces an isometry $S(\alpha_1, \alpha_2) \to S(\alpha_1, \bar{\alpha}_2)$. Especially, the group of isometries of both space are the same. But, note for instance that on $S(\alpha, \alpha)$ the isometry group $U(2)$ is realized by biholomorphic automorphisms, while on $S(\alpha, \bar{\alpha})$ this is not the case.

Set $\Delta = \bigcup_j S_j$. Then $\tilde{Z}$ is an equivariant partial compactification of $\boldsymbol{C}^{*3} = \boldsymbol{P}^3 - \{z_0 z_1 z_2 z_3 = 0\}$. We may think of $\tilde{h}$ as generating a discrete infinite cyclic group of $\boldsymbol{C}^{*3}$ and then $Z = \tilde{Z}/\langle\tilde{h}\rangle$ is considered an equivariant compactification of the commutative complex Lie group $G := \boldsymbol{C}^{*3}/\langle\tilde{h}\rangle$ with boundary an anti-canonical divisor Δ (cf. Lemma 2.1 below).

We fix a lift of the action of $\tilde{h}$ to $\boldsymbol{C}^4$ as

$$\tilde{h} : (z_0, z_1, z_2, z_3) \to (dz_0, \bar{d}z_1, az_2, \bar{a}z_3). \tag{7}$$

The proof of 1) of Proposition 1.1 is very simple.

Proof of 1) of Proposition 1.1. Suppose that ii) is true. Since $a = re^{\pi\sqrt{-1}(\theta_1 - \theta_2)}$ and $d = e^{\pi\sqrt{-1}(\theta_1+\theta_2)}$ and since $\theta_1 \pm \theta_2$ are rational, there exists a positive integer k such that $a^k = r$ and $d^k = 1$. Then in view of (7) we see that $(z_1/z_0)^k$ and $(z_3/z_2)^k$ give two algebraic independent meromorphic functions on Z; thus i) holds true.

Conversely, we assume that $a(Z) = 2$. Then by [10, Th. 3.8, 2)] the hypersurfaces $S_0 \cong S(\alpha_1, \alpha_2)$ and $S_2 \cong S(\bar{\alpha}_1, \alpha_2)$ in Z must both be of algebraic dimension one. By a theorem of Kodaira [6, Th. 31] this happens only when $\alpha_1^m = \alpha_2^m$ and $\alpha_1^m = \bar{\alpha}_2^m$ for some positive integers m, or equivalently, θ_1 and θ_2 are both rational. Thus ii) holds. □

Let L_j, $0 \le j \le 3$, be the line bundles on Z corresponding to the divisors S_j. We shall identify these line bundles.

Lemma 2.1. *The product $L_0 L_1 L_2 L_3$ of L_i is isomorphic to the anti-canonical bundle K^{-1} of Z.*

Proof. Let $V_j = \{z_j \neq 0\}$ for $j = 0, 1, 2, 3$. Consider the meromorphic differential form ψ on $\boldsymbol{P}^3$ which is written as

$$\psi = d\log\frac{z_3}{z_0} \wedge d\log\frac{z_1}{z_0} \wedge d\log\frac{z_2}{z_0}$$

on V_0 and is written in a smilar way on the other $V_i, i = 1, 2, 3$. The form ψ has no zeroes and its polar locus is precisely the union $\bigcup_j \{z_j = 0\}$; moreover, it is clearly $\tilde{h}$-invariant and descends to a meromorphic form whose associated divisor is precisely the union $\bigcup_j S_j$. This implies that $\sum_j S_j$ is an anti-canonical divisor. □

Lemma 2.2. *We have*

$$L_j \cong F(c_j r^{-1/2})K^{-1/4},$$

where $c_j = d, \bar{d}, a, \bar{a}$ *according as* $j = 0, 1, 2, 3$ *respectively. In particular*

$$L_0L_1 \cong F(r^{-1})K^{-1/2} \quad \textit{and} \quad L_2L_3 \cong F(r)K^{-1/2}.$$

Proof. Clearly we have $L_1 = F(\bar{d}/d)L_0$, $L_2 = F(a/d)L_0$, $L_3 = F(\bar{a}/d)L_0$. On the other hand, by Lemma 2.1 we have $L_0L_1L_2L_3 \cong K^{-1}$. Thus we get $L_0^4 = F(d^4r^{-2})K^{-1}$, and hence, by a suitable choice of $K^{-1/4}$ we may write $L_i = F(c_i r^{-1/2})K^{-1/4}$. □

We consider the induced $\tilde{h}$-action on the space of monomials in z_i. We are interested in pairs of coprime monomials P and Q of the same degree such that both are eigenvectors of $\tilde{h}$-action with the same eigenvalue and that the hypersurface $P = cQ$ is irreducible for a general constant c.

Let p (resp. q) be the number of (effective) variables contained in P (resp. Q); e.g., $p = 2$ if $P = z_0^2 z_1^3$. We assume that $p \geq q$.

Lemma 2.3. *Under the above notations and assumptions the possible types of the pairs* $\{P, Q\}$ *as above are as follows*:

a) $(p, q) = (1, 1)$ *and*

$$\{P, Q\} = \{z_0, z_1\} \quad (\textit{resp.} \quad \{z_2, z_3\}).$$

which occurs precisely when $2t \in \boldsymbol{Z}$ (*resp.* $2s \in \boldsymbol{Z}$).

b) $(p, q) = (2, 2)$ *and the possibilities are*:

$$\{P, Q\} = \left\{z_0^i z_2^j, z_1^i z_3^j\right\} \quad \left(\textit{resp.} \quad \left\{z_0^i z_3^j, z_1^i z_2^j\right\}\right),$$

for some positive integers $i, j > 0$ *with* $\gcd(i, j) = 1$. *This occurs precesely when* $ti = -sj$ (*resp.* $ti = sj$) *for suitable representatives* s *and* t *modulo* $\boldsymbol{Z}/2$.

Proof. By the coprime property we see that P and Q contain no common variables. Thus the possibilities for the pairs (p, q) are: $(3, 1), (2, 2), (2, 1), (1, 1)$. Moreover if we take into account the effect of the weight r we have that z_2 and z_3 are never contained in the same monomial, and that if one of them is contained in P, then the other must be contained in Q with the same power. This excludes the possibility $(3, 1), (2, 1)$, leaving only the cases $(2, 2), (1, 1)$. In the $(1, 1)$ case clearly only possibilities are: $\{P, Q\} = \{z_0, z_1\}$ or $\{z_2, z_3\}$. In the $(2, 2)$ case possibilities are similarly reduced to: $\{P, Q\} = \{z_0^i z_2^j, z_1^i z_3^j\}$ or $\{z_0^i z_3^j, z_1^i z_2^j\}$, for some positive integers $i, j > 0$ with $(i, j) = 1$, where the last condition of coprimeness of i and j follows from the irreducibility of $P = cQ$ for general c.

Conversely, we show that the above pairs of monomials are in fact contained in a common eigenspace of $\tilde{h}$-action. First of all, in the $(1, 1)$-case, z_0 and z_1 are in the same eigenspace if and only if $d = \bar{d}$, or equivalently, $2t \in \mathbf{Z}$. The case of z_2 and z_3 can be similarly treated.

In the $(2, 2)$-case $z_0^i z_2^j$ and $z_1^i z_3^j$ are in the same eigenspace if and only if $2(ti + sj) \in \mathbf{Z}$, or equivalently, $ti+sj = 0$ if we replace s and t suitably modulo $\mathbf{Z}/2$. Indeed, since $(i, j) = 1$, there exist integers m and n such that $mi+nj = 1$. Then, if $ti+sj = l \in \mathbf{Z}/2$, subtracting $mli + njl = l$ from the left hand side, we get $i(t - ml) + j(s - nl) = 0$; thus if we replace s and t by $s - nl$ and $t - ml$ respectively, we get that $sj = -ti$. The case of $z_0^i z_3^j$ and $z_1^i z_2^j$ is similarly treated. □

In terms of the corresponding Hopf surface $S(\alpha_1, \alpha_2)$ with $\alpha_i = e^{2\pi\sqrt{-1}\theta_i}$, the above conditions are stated as follows.

CASE a). The pair $\{z_0, z_1\}$ (resp. $\{z_2, z_3\}$) occurs if and only if

$$\theta_1 \equiv -\theta_2 \bmod \mathbf{Z} \quad (\text{resp.} \quad \theta_1 \equiv \theta_2 \bmod \mathbf{Z});$$

in other words $S = S(\alpha, \bar{\alpha})$ (resp. $S(\alpha, \alpha)$) for some $\alpha = re^{2\pi\sqrt{-1}\theta}$, $r > 1$.

CASE b). The pair $\{z_0^i z_2^j, z_1^i z_3^j\}$ (resp. $\{z_0^i z_3^j, z_1^i z_2^j\}$)) occurs if and only if

$$(i + j)\eta_1 = (j - i)\eta_2 \quad (\text{resp. } (j - i)\eta_1 = (i + j)\eta_2),$$

where $\eta_1 = s+t$ and $\eta_2 = -s+t$ with s and t replaced modulo $\mathbf{Z}/2$ so that the relation $ti + sj = 0$ already holds. Thus if we set

$$m = j - i, \quad n = i + j \quad (\text{resp. } m = i + j, \quad n = j - i), \tag{8}$$

and then define

$$\mu = n\eta_1 = m\eta_2 \quad \text{and} \quad \theta = \frac{\mu}{mn},$$

we have $\eta_1 = m\theta$ and $\eta_2 = n\theta$, where $m < |n|$ (resp. $n < |m|$). (Here, when $mn = 0$, or equivalently, when $i = j = 1$, we understand that $\theta = \eta_2/2$ (resp. $\eta_1/2$).)

Now in view of (4) $\eta_i \equiv \theta_i$ modulo $\boldsymbol{Z}/2$. Thus we conclude that exactly for the case $S = S(\varepsilon_1 r e^{2\pi\sqrt{-1}m\theta}, \varepsilon_2 r e^{2\pi\sqrt{-1}n\theta})$ for some $\varepsilon_i \in \{1, -1\}$ with m, n and θ as above, the pair of monomials in question actually occurs. For instance in the case $m = 0$ we have $S = S(\varepsilon_1 r, \varepsilon_2 r e^{4\pi\sqrt{-1}\theta})$.

We may, and we shall, consider CASE a) as the case $m = \pm n = \pm 1$ in what follows.

Proof of 2) of Proposition 1.1. Note that the two conditions i) and ii) are both invariant under passing to finite unramified coverings (as far as we treat the surfaces of type $S(\alpha_1, \alpha_2)$ for ii)). Suppose first that ii) is true. By the above remark it suffices to show that the twistor space associated to the Hopf surface $S(re^{2\pi\sqrt{-1}m\theta}, re^{2\pi\sqrt{-1}n\theta})$ with m, n, and θ as in the proposition are of algebraic dimension one. Since $a(Z)$ is at most one by 1) as $\theta \notin \boldsymbol{Q}$, we have only to find a non-constant meromorphic function on Z. When $m < |n|$, we define $i = (n+m)/2$, $j = (n-m)/2$ and consider the pair of monomials $\{z_0^i z_2^j, z_1^i z_3^j\}$ on $\tilde{Z}$. Then by what we have seen above these monomials are in the common eigenspace of $\tilde{h}$-action so that their ratio $z_0^i z_2^j / z_1^i z_3^j$ descends to a non-constant meromorphic function on Z as desired. In case $m = n$ we consider the pair $\{z_2, z_3\}$ and the same argument shows that their ratio again gives rise to a nonconstant meromorphic function on Z. The cases $n < |m|$ and $m = -n$ are similarly treated.

Conversely, we assume that $a(Z) = 1$ and show that S is of the desired form. Take an algebraic reduction $(g : \hat{Z} \to \boldsymbol{P}, \mu : \hat{Z} \to Z)$, where $\hat{Z}$ is a complex manifold, μ is a bimeromorphic morphism, and g is a surjective holomorphic map with connected fibers. The associated meromorphic map $f = g\mu^{-1} : Z \to \boldsymbol{P}$ is nothing but the meromorphic algebraic reduction of Z. Let F be a general fiber of g. Then after passing to a finite unramified covering of S if necessary, we may assume that the natural map $\pi_1(F) \to \pi_1(Z)$ is surjective (cf. [3]). It then follows easily that the general fiber of the induced meromorphic map $\tilde{f} : \tilde{Z} \to \boldsymbol{P}$ is irreducible as well as for f itself.

Now $\tilde{f}$ extends to a meromorphic map $\hat{f} : \boldsymbol{P}^3 \to \boldsymbol{P}$ since $\boldsymbol{P}^3 - \tilde{Z} = l_0 \cup l_\infty$, which is of codimension two in $\boldsymbol{P}^3$. Let H_1 be the line bundle of degree one on $\boldsymbol{P}$. Then its pull-back $\hat{f}^* H_1$ can be defined on $\boldsymbol{P}^3$ as a holomorphic line bundle. Hence we have $\hat{f}^* H_1 \cong H^\kappa$ for some positive integer κ, where H is the hyperplane bundle on $\boldsymbol{P}^3$. Take two linearly independent sections on $\boldsymbol{P}$ and pull them back by $\hat{f}$ to obtain two holomorphic sections, say P and Q, of H^κ. (Again note that $\boldsymbol{P}^3 - \tilde{Z}$ is of codimension two in $\boldsymbol{P}^3$ so that P and Q extend to holomorphic sections of H^κ on the whole $\boldsymbol{P}^3$.) We may consider them as homogeneous polynomials of degree κ in the variables z_0, z_1, z_2, z_3.

With respect to the $\tilde{h}$-action the space P_κ of homogeneous polynomials of degree κ is decomposed into a direct sum of its eigenspaces, each of which is spanned by monomials. Since P/Q descends to a well-defined meromorphic function which defines f, P and Q should be in one and the same eigenspace, say E, of $\tilde{h}$ and $\dim E \geq 2$. We get a natural meromorphic map $\tilde{Z} \to \boldsymbol{P}(E^*)$, which descends to a meromorphic map defined on Z, where E^* denotes the dual of E. But this latter map must coincide with f itself as it dominates f. This implies that actually we have $\dim E = 2$. We may choose P and Q so that they are monomials. Since $\tilde{f}$ is induced by P/Q, $P = cQ$, $c \in \boldsymbol{C}$, are defining equations of fibers of $\tilde{f}$. By what we have noted above the general fiber of $\tilde{f}$ is irreducible. Hence, the hypersurfaces $P = cQ$ in $\boldsymbol{P}^3$ are irreducible for general c; in particular P and Q are coprime. Therefore, these P and Q satisfy all the assumptions in Lemma 2.3. Thus by that lemma together with the argument following it, we see immediately that in this case S is necessarily of the form $S(\varepsilon_1 re^{2\pi\sqrt{-1}m\theta}, \varepsilon_2 re^{2\pi\sqrt{-1}n\theta})$, $\varepsilon_i = \pm 1$, the case $m = \pm n = \pm 1$ precisely corresponding to CASE a) of Lemma 2.3. Here, we have $\theta \notin \boldsymbol{Q}$ from 1) of the proposition. Then for the double covering $S(re^{2\pi\sqrt{-1}2m\theta}, re^{2\pi\sqrt{-1}2n\theta})$, the condition of ii) is realized. □

Proof of Theorem 1.3. The argument in the first part of the proof of 2) of Proposition 1.1 shows that the meromorphic algebraic reduction f is induced by an $\tilde{h}$-invariant meromorphic function P/Q on $\tilde{Z}$ with $\{P, Q\}$ one of the pairs obtained in Lemma 2.3.

We consider CASE a) and b) separately corresponding respectively to the cases 1) and 2) of the theorem.

CASE a). $S = S(\alpha, \bar{\alpha})$, or $= S(\alpha, \alpha)$.

We know that both have conjugate-isomorphic twistor spaces; so it suffices to consider the case of $S = S(\alpha, \alpha)$. In this case the unitary group $U(2)$ acts on S as a group of isometries, and hence, we have the induced action of $GL(2, \boldsymbol{C})$ on the twistor space Z. The algebraic reduction is induced by z_0/z_1 and is holomorphic. Indeed the induced action of $GL(2, \boldsymbol{C})$ on $\boldsymbol{P}$ is transitive and $f : Z \to \boldsymbol{P}$ is a homogeneous fiber bundle whose typical fiber may be identified with $S_3 = S(\alpha, \bar{\alpha})$. Furthermore, f is induced by the linear system $|L_2L_3|$ and the second assertion follows from Lemma 2.2. Thus 1) is proved.

CASE b). Consider the case $\{P, Q\} = \{z_0^i z_3^j, z_1^i z_2^j\}$ corresponding to the case $m < |n|$. (The other case can be treated similarly; or one may refer to the duality as in CASE a).) Now the algebraic reduction $f : Z \to \boldsymbol{P} = \boldsymbol{C}(w) \cup \{\infty\}$ is induced by the $\tilde{h}$-invariant meromorphic function $z_0^i z_3^j / z_1^i z_2^j$, where $\{i, j\}$ and $\{m, n\}$ are related by the formula (8). So it is given by the linear system $|L_0^i L_3^j| = |L_1^i L_2^j|$ and by Lemma 2.2 we deduce

$$L_0^i L_3^j \cong L_1^i L_2^j \cong F(r^{(j-i)/2})K^{-(i+j)/4} = F(r^{m/2})K^{-n/4},$$

which is never isomorphic to $K^{-n/4}$ except for the case $S \cong S(r, re^{2\pi\sqrt{-1}\theta})$, and $i = j = 1$; in the latter case both line bundles are isomorphic to $K^{-1/2}$.

Now we consider the $\boldsymbol{C}^*$-action on $\tilde{Z}$ defined by

$$(z_0 : z_1 : z_2 : z_3) \to (tz_0 : z_1 : z_2 : tz_3), \quad t \in \boldsymbol{C}^*.$$

This induces the action on $\boldsymbol{P}$ via $w \to t^{i+j}w$, making f equivariant. This shows that fibers over $\{w \neq 0, \infty\}$ are all isomorphic; consider the fiber R over $\{w = 1\}$. Its inverse image $\tilde{R}$ is the hypersurface $\tilde{R} : z_0^i z_3^j = z_1^i z_2^j$ in $\tilde{Z}$. Define a holomorphic map $g : \boldsymbol{P} \times \boldsymbol{C}^* \to \tilde{R}$ defined by

$$(u, v) \to (u^j : 1 : vu^i : v) \in \tilde{R}, \quad (u, v) \in (\boldsymbol{P}, \boldsymbol{C}^*),$$

which is easily checked to be bijective. In particular, g is the normalization of $\tilde{R}$. (It is isomorphic if and only if either $i = 1$ or $j = 1$.) The $\tilde{h}$-action on $\tilde{R}$ lifts to $\boldsymbol{P} \times \boldsymbol{C}^*$

$$\tilde{h} : (u, v) \to \left(e^{2\pi\sqrt{-1}\beta}u, re^{2\pi\sqrt{-1}(t-s)}v\right).$$

where $\beta := 2t/j = 2s/i$. Hence, the normalization of $R = \tilde{R}/\boldsymbol{Z}$ is isomorphic to $(\boldsymbol{P} \times \boldsymbol{C}^*)/\boldsymbol{Z}$, where $\boldsymbol{Z}$ is the infinite cyclic group generated by $\tilde{h}$. The latter is isomorphic to a $\boldsymbol{P}$-bundle $\boldsymbol{P}(1 \oplus L)$ over the elliptic curve $C := \boldsymbol{C}^*/\langle\alpha_2\rangle$, where L is the flat line bundle $\boldsymbol{C}^* \times_{\boldsymbol{Z}} \boldsymbol{C} \to C = \boldsymbol{C}^*/\boldsymbol{Z}$ associated to the representation $\alpha_2 \to e^{2\pi\sqrt{-1}(2\theta)}$, and thus of infinite order.

The induced real structure on $\boldsymbol{P}$ interchanges 0 and ∞; the reducible fibers, the fibers over 0 and ∞, are mutually conjugate to each other, its underlying reduced subspaces are the unions $S_0 \cup S_3$ or $S_1 \cup S_2$ respectively.

The base locus of the map f consists of two mutually conjugate nonsingular elliptic curves C_{02} and C_{13}. In particular, f never is holomorphic. This shows all the statements of Theorem 1.3. $\square$

Most of the assertions in Remark 1 also have been shown in the course of the above proof. The rest is also immediate to see.

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