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EXTENSIONS OF SOME 2-GROUPS WHICH PRESERVE THE IRREDUCIBILITIES OF INDUCED CHARACTERS

Dedicated to Professor Yukio Tsushima on his 60th birthday

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1. Introduction

Let Q_n and D_n denote the generalized quaternion group and the dihedral group of order $2^{n+1}(n \geq 2)$, respectively. Let SD_n denote the semidihedral group of order $2^{n+1}(n \geq 3)$.

As is stated in [3], these groups have remarkable properties among all 2-groups.

Moreover, Yamada and Iida [4] proved the following interesting result:

Let $\mathbf{Q}$ denote the rational field. Let G be a 2-group and χ a complex irreducible character of G. Then there exist subgroups $H \rhd N$ in G and the complex irreducible character ϕ of H such that $\chi = \phi^G$, $\mathbf{Q}(\chi) = \mathbf{Q}(\phi)$, $N = \mathrm{Ker}\phi$ and

$$H/N \cong Q_n(n \geq 2), \text{ or } D_n(n \geq 3), \text{ or } SD_n(n \geq 3), \text{ or } C_n(n \geq 0),$$

where C_n is the cyclic group of order 2^n, and $\mathbf{Q}(\chi) = \mathbf{Q}(\chi(g), g \in G)$.

In [3], Yamada and Iida considered the case when $N = 1$. Note that ϕ is faithful in this case. They studied the following problem:

Problem. Let ϕ be a faithful irreducible character of H, where $H = Q_n$ or D_n or SD_n. Determine the extension group G of H such that the induced character ϕ^G is also irreducible.

It is well-known that the groups Q_n, D_n and SD_n have faithful irreducible characters. It is also known that they are algebraically conjugate to each other. Hence the irreducibility of ϕ^G, where ϕ is a faithful irreducible character of Q_n or D_n or SD_n, is independent of the choice of ϕ, but depends only on these groups.

In [3], Yamada and Iida solved this problem when $[G : H] = 2$ and 4 for all $H = Q_n$ or D_n or SD_n.

The purpose of this paper is to solve this problem when $[G : H] = 8$ for all $H = Q_n$ or D_n or SD_n.

For other results concerning this problem, see [2].

2. Statements of the results

We use the following notation throught this paper.

- The dihedral group $D_n = \langle a, b\rangle (n \geq 2)$ with
  $a^{2^n} = 1, \quad b^2 = 1, \quad bab^{-1} = a^{-1}.$
- The generalized quaternion group $Q_n = \langle a, b\rangle (n \geq 2)$ with
  $a^{2^n} = 1, \quad b^2 = a^{2^{n-1}}, \quad bab^{-1} = a^{-1}.$
- The semidihedral group $SD_n = \langle a, b\rangle (n \geq 3)$ with
  $a^{2^n} = 1, \quad b^2 = 1, \quad bab^{-1} = a^{-1+2^{n-1}}.$

First, we introduce the following groups defined by Yamada and Iida ([3]):

(i) $G_2^{(1)}(D_n) = \langle a, b, u\rangle$ with
$a^{2^n} = b^2 = u^4 = 1, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}, ub = bu,$

(ii) $G_2^{(2)}(D_n) = \langle a, b, u, w\rangle$ with
$a^{2^n} = b^2 = u^2 = w^2 = 1, bab^{-1} = a^{-1}, waw^{-1} = a^{1+2^{n-1}}, wb = bw,$
$uau^{-1} = aw, ubu^{-1} = bw, uw = wu.$

(iii) $G_2^{(1)}(Q_n) = \langle a, b, u\rangle$ with
$a^{2^n} = 1, b^2 = u^4 = a^{2^{n-1}}, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}, ub = bu,$

(iv) $G_2^{(2)}(Q_n) = \langle a, b, u, w\rangle$ with
$a^{2^n} = 1, b^2 = w^2 = a^{2^{n-1}}, bab^{-1} = a^{-1}, waw^{-1} = a^{1+2^{n-1}}, wb = bw,$
$u^2 = 1, uau^{-1} = a^{1+2^{n-2}}w, ub = bu, uw = wu.$

REMARK. We use the symbols w and u instead of u_1b and u_2 in [3], respectively.

For a finite group G, we denote by $\mathrm{Irr}(G)$ the set of complex irreducible characters of G and by $\mathrm{FIrr}(G)$ ($\subset \mathrm{Irr}(G)$) the set of faithful irreducible characters of G.

Yamada and Iida ([3]) proved the following:

Theorem 0.1. ([3, Theorem 5]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(Q_n)$. Let G be an extension of Q_n such that $[G : Q_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(Q_n)$ or $G_2^{(2)}(Q_n)$.*

Theorem 0.2. ([3, Theorem 5]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(D_n)$. Let G be an extension of D_n such that $[G : D_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(D_n)$ or $G_2^{(2)}(D_n)$.*

Theorem 0.3. ([3, Theorem 6]) *Let $n \geq 4$ and $\phi \in \mathrm{FIrr}(SD_n)$. Let G be an extension of SD_n such that $[G : SD_n] = 2^2$ and $\phi^G \in \mathrm{Irr}(G)$. Then $G \cong G_2^{(1)}(Q_n)$ or $G_2^{(2)}(Q_n)$ or $G_2^{(1)}(D_n)$ or $G_2^{(2)}(D_n)$.*

Further, we also need the following result ([3], [5]).

Theorem 0.4. *Let $H = Q_n$, D_n or SD_n. Let G be a 2-group which contains H with $[G : H] = 2^r (r \geq 1)$. Let $\phi \in \mathrm{FIrr}(H)$. Suupose that $\phi^G \in \mathrm{Irr}(G)$, then $r \leq n - 2$.*

To state our results, we have to introduce the following groups:

(1) $G_3^{(1)}(D_n) = \langle a, b, u, x\rangle (\rhd G_2^{(1)}(D_n) = \langle a, b, u\rangle)$ with
$a^{2^n} = b^2 = u^4 = 1, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}$,
$ub = bu, xax^{-1} = a^{1+2^{n-3}}, xbx^{-1} = b, xux^{-1} = u$,
$x^2 = u$ (when $n \geq 6$), $x^2 = u^3$ (when $n = 5$).

(2) $G_3^{(2)}(D_n) = \langle a, b, u, x\rangle (\rhd G_2^{(1)}(D_n) = \langle a, b, u\rangle)$ with
$a^{2^n} = b^2 = u^4 = 1, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}$,
$ub = bu, xax^{-1} = a^{1+2^{n-3}}u^2, xbx^{-1} = bu^2, xux^{-1} = u$,
$x^2 = u$ (when $n \geq 6$), $x^2 = u^3$ (when $n = 5$)

(3) $G_3^{(1)}(Q_n) = \langle a, b, u, x\rangle (\rhd G_2^{(1)}(Q_n) = \langle a, b, u\rangle)$ with
$a^{2^n} = 1, b^2 = u^4 = a^{2^{n-1}}, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}$,
$ub = bu, xax^{-1} = a^{1+2^{n-3}}, xbx^{-1} = b, xux^{-1} = u$,
$x^2 = u$ (when $n \geq 6$), $x^2 = u^3$ (when $n = 5$).

(4) $G_3^{(2)}(Q_n) = \langle a, b, u, x\rangle (\rhd G_2^{(1)}(Q_n) = \langle a, b, u\rangle)$ with
$a^{2^n} = 1, b^2 = u^4 = a^{2^{n-1}}, bab^{-1} = a^{-1}, uau^{-1} = a^{1+2^{n-2}}$,
$ub = bu, xax^{-1} = a^{1+2^{n-3}}u^2, xbx^{-1} = b, xux^{-1} = u$,
$x^2 = u^3$ (when $n \geq 6$), $x^2 = u$ (when $n = 5$).

It is easy to see that $[G_3^{(i)}(D_n) : G_2^{(1)}(D_n)] = 2$, and that $[G_3^{(i)}(Q_n) : G_2^{(1)}(Q_n)] = 2$ $(i = 1, 2)$. A direct calculation shows that the number of involutions in $G_3^{(i)}(D_n)$ (resp. $G_3^{(i)}(Q_n)$) are $2^{n-1} + 2^n + 3$ (resp. $2^{n-1} + 3$) for $i = 1, 2$. Therefore $G_3^{(i)}(D_n) \not\cong G_3^{(j)}(Q_n)$ for any i, j, $1 \leq i, j \leq 2$. On the other hand, the number of conjugacy classes of involutions in $G_3^{(1)}(D_n)$ (resp. $G_3^{(2)}(D_n)$) is 5 (resp. 4). Hence $G_3^{(1)}(D_n) \not\cong G_3^{(2)}(D_n)$.

It is easy to see that the center of the groups of $G_3^{(i)}(Q_n)$ are both $\langle a^{2^{n-1}}\rangle$, for $i = 1, 2$. Let $G_3^{(i)}(Q_n) \supset V_i = \{v \in G_3^{(i)}(Q_n) | v^2 = a^{2^{n-1}}\}$. Then a direct calculation shows that the number of conjugacy classes of V_1 in $G_3^{(1)}(Q_n)$ (resp. V_2 in $G_3^{(2)}(Q_n)$) is 5 (resp. 4). Hence $G_3^{(1)}(Q_n) \not\cong G_3^{(2)}(Q_n)$.

Consequently, above four groups $G_3^{(1)}(D_n)$, $G_3^{(2)}(D_n)$, $G_3^{(1)}(Q_n)$ and $G_3^{(2)}(Q_n)$ are not isomorphic to each other.

Our main theorems are the following:

Theorem 1. *Let $\phi \in \mathrm{FIrr}(D_n)$. Suppose that $D_n \subset G$ such that $[G : D_n] = 8$ and $\phi^G \in \mathrm{Irr}(G)$. Then $n \geq 5$ and $G \cong G_3^{(1)}(D_n)$ or $G_3^{(2)}(D_n)$.*

Theorem 2. *Let $\phi \in \mathrm{FIrr}(Q_n)$. Suppose that $Q_n \subset G$ such that $[G : Q_n] = 8$ and $\phi^G \in \mathrm{Irr}(G)$. Then $n \geq 5$ and $G \cong G_3^{(1)}(Q_n)$ or $G_3^{(2)}(Q_n)$.*

Theorem 3. *Let $\phi \in \mathrm{FIrr}(SD_n)$. Suppose that $SD_n \subset G$ such that $[G : SD_n] = 8$ and $\phi^G \in \mathrm{Irr}(G)$. Then $n \geq 5$ and $G \cong G_3^{(1)}(D_n)$ or $G_3^{(2)}(D_n)$ or $G_3^{(1)}(Q_n)$ or $G_3^{(2)}(Q_n)$.*

To prove the theorems, we need some results concerning the criterion of the irreducibility of induced characters.

We denote by $\zeta = \zeta_{2^n}$ a primitive 2^n-th root of unity. It is known that, for $H = Q_n$ or D_n, there are $2^{n-1} - 1$ irreducible characters $\phi_\nu (1 \leq \nu < 2^{n-1})$ of H, which are not linear:

$$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{-\nu i}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leq i \leq 2^n).$$

For $H = SD_n$, there are $2^{n-1} - 1$ irreducible characters $\phi_\nu(-2^{n-2} \leq \nu \leq 2^{n-2}$ for odd ν, $1 \leq \nu < 2^{n-1}$ for even ν) of H, which are not linear:

$$\phi_\nu(a^i) = \zeta^{\nu i} + \zeta^{\nu i(-1+2^{n-1})}, \qquad \phi_\nu(a^i b) = 0 \qquad (1 \leq i \leq 2^n).$$

Each irreducible character ϕ_ν of Q_n or D_n or SD_n is induced from a linear character η_ν of the maximal normal cyclic subgroup $\langle a \rangle$: $\eta_\nu(a^i) = \zeta^{\nu i} (1 \leq i \leq 2^n)$. Therefore, for a group $G \supset H = D_n$, or Q_n or SD_n, ϕ_ν^G is irreducible if and only if $\eta_\nu^G = (\eta_\nu^H)^G$ is irreducible. For $H = Q_n$ or D_n or SD_n, an irreducible character ϕ_ν of H is faithful if and only if ν is odd. The faithful irreducible characters ϕ_ν of H are algebraically conjugate to each other.

By the theorem of Shoda (cf. [1, p. 329]), we have the following:

Proposition 1. *Let $\langle a \rangle \subset H \subset G$, where $H = D_n$ or Q_n or SD_n and $\langle a \rangle$ is a maximal normal cyclic subgroup of H. Let ϕ be a faithful irreducible character of H. Then the following conditions are equivalent*

(1) *ϕ^G is irreducible.*

(2) *For each $x \in G - \langle a \rangle$, there exists $y \in \langle a \rangle \cap x\langle a \rangle x^{-1}$ such that $xyx^{-1} \neq y$.*

DEFINITION. When the condition (2) of Proposition 1 holds, we say that G satisfies (EX, H), where $H = D_n$ or Q_n or SD_n.

REMARK. It is easy to see that the group $G_3^{(i)}(D_n)$ (resp. $G_3^{(i)}(Q_n)$) satisfies (EX, D_n) (resp. (EX, Q_n)), for $i = 1, 2$. It is also easy to see that $G_3^{(i)}(D_n)$ and $G_3^{(i)}(Q_n)$ satisfy (EX, SD_n), for $i = 1, 2$.

3. Proof of Theorem 1

By Theorem 0.4, we have $n \geq 5$. Let G be a 2-group, satisfying the conditions of Theorem 1. Then, there exists a subgroup G_2 such that

$$D_n \subset G_2 \subset G$$

and $[G_2 : D_n] = 4$ and $[G : G_2] = 2$, because G is a 2-group. Since G_2 must satisfy the condition (EX, D_n), we have $G_2 \cong G_2^{(1)}(D_n)$ or $G_2^{(2)}(D_n)$. So we can write as

$$G = \langle G_2^{(i)}(D_n), y\rangle$$

where $y \in G - G_2^{(i)}(D_n) = \{g \in G | g \notin G_2^{(i)}(D_n)\}$ and $y^2 \in G_2^{(i)}(D_n)$, for $i = 1$ or 2. Hence we have only to consider the following two cases:

CASE I. $G = \langle G_2^{(1)}(D_n), y\rangle = \langle a, b, u, y\rangle \rhd G_2^{(1)}(D_n) = \langle a, b, u\rangle$

CASE II. $G = \langle G_2^{(2)}(D_n), y\rangle = \langle a, b, w, u, y\rangle \rhd G_2^{(2)}(D_n) = \langle a, b, w, u\rangle$

First, we consider Case I. For the sake of simplicity, we write G_2 instead of $G_2^{(1)}(D_n)$ in this proof. It is well-known that

$$\mathrm{Aut}\langle a\rangle \cong \left(\mathbf{Z}/2^n\mathbf{Z}\right)^* = \langle -1\rangle \times \langle 5\rangle$$

where $(\mathbf{Z}/2^n\mathbf{Z})^*$ is the unit group of the factor ring $(\mathbf{Z}/2^n\mathbf{Z})$ and $\langle -1\rangle$ and $\langle 5\rangle$ are the cyclic subgroups of $(\mathbf{Z}/2^n\mathbf{Z})^*$ generated by -1 and 5 respectively.

First, we consider the element yay^{-1}. Since it is in G_2, it can be represented as $a^i b^k u^j$ for some $i, j, k \in \mathbf{Z}$, $0 \le i \le 2^n - 1$, $0 \le k \le 1$, $0 \le j \le 3$.

Suppose that $yay^{-1} = a^i b u^j$, then

$$ya^8y^{-1} = (a^i b u^j)^8 = 1,$$

by direct calculation. This contradicts the fact that $a^8 \neq 1$.
Thus we must have

$$yay^{-1} = a^i u^j.$$

Since $ya^4y^{-1} = (a^i u^j)^4 = a^{4i(1+2^{n-3}j)}$, i is an odd integer. Furthermore, if $i \in \langle -1\rangle \times \langle 5\rangle - \langle 5\rangle$, then $(by)a(by)^{-1} = a^{-i}u^j$ and $-i \in \langle 5\rangle$. Hence we may assume that $i \in \langle 5\rangle$.

Next, consider the element yuy^{-1}. Write $u_0 = yuy^{-1}$.
Taking the conjugate of both sides of the equality, $ua^4u^{-1} = a^4$, by y, we get

$$u_0 a^{4i(1+2^{n-3}j)} u_0^{-1} = a^{4i(1+2^{n-3}j)}.$$

Since i is odd, we have $u_0 a^4 u_0^{-1} = a^4$. Thus we can write $u_0 = a^{d_0}u^t$ for some $d_0, t \in \mathbf{Z}$. Suppose that t is even, then it is easy to see that $u_0^2 \in \langle a\rangle$. Since $u_0^2 \neq 1$ and $u_0^4 = 1$, we have $u_0^2 = a^{2^{n-1}}$. This contradicts the fact that $\langle a^{2^{n-1}}\rangle$ is the center of G_2 and u^2 is not in the center of G_2. Hence t is odd. Since $1 = u_0^4 = a^{4d_0(1+2^{n-3}t)}$, we have $d_0 \equiv 0 \pmod{2^{n-2}}$. Therefore we may write $d_0 = 2^{n-2}d$ for some $d \in \mathbf{Z}$, and so

$$yuy^{-1} = a^{2^{n-2}d}u^t.$$

Note that

$$ya^{2^{n-2}}y^{-1} = (a^i u^j)^{2^{n-2}} = a^{2^{n-2}i(1+2^{n-3}j)} = a^{2^{n-2}},$$

because $i \in \langle 5 \rangle$. Taking the conjugate of both sides of the equality, $uau^{-1} = a^{1+2^{n-2}}$ by y, we get

$$(a^{2^{n-2}d}u^t)(a^i u^j)(a^{2^{n-2}d}u^t)^{-1} = a^i u^j a^{2^{n-2}}.$$

Hence, we have

$$a^{i(1+2^{n-2})^t}u^j = a^{i+2^{n-2}}u^j.$$

Therefore, $i(1 + t \cdot 2^{n-2}) \equiv i + 2^{n-2} \pmod{2^n}$. But $i \equiv 1 \pmod 4$, so we get $t \equiv 1 \pmod 4$, and hence

$$yuy^{-1} = a^{2^{n-2}d}u.$$

Since

$$y^2ay^{-2} = y(a^i u^j)y^{-1} = (a^i u^j)^i (a^{2^{n-2}d}u)^j,$$

we can write

$$y^2ay^{-2} = a^m u^{ij+j},$$

for some $m \in \mathbf{Z}$. But $y^2 \in G_2$, so $y^2ay^{-2} \in \langle a \rangle$. Hence $ij + j \equiv 0 \pmod 4$. Since $i \equiv 1 \pmod 4$, we have $2j \equiv 0 \pmod 4$. Thus j is even and we can write as $j = 2j_0$ for some integer j_0. Summarizing the results, we can write

$$\begin{aligned} yay^{-1} &= a^i u^{2j_0}, \\ yuy^{-1} &= a^{2^{n-2}d}u. \end{aligned}$$

Hence Case I is divided into the following two cases:

CASE IA. $yay^{-1} = a^i$ ($i \equiv 1 \pmod 4$) and $yuy^{-1} = a^{2^{n-2}d}u$,

CASE IB. $yay^{-1} = a^i u^2$ ($i \equiv 1 \pmod 4$) and $yuy^{-1} = a^{2^{n-2}d}u$.

First we consider Case IA.

We will need the following:

Lemma 1. *Let l be an integer and k be an odd integer. Then there exist an integer c and an odd integer e satisfying the following equalities*

$$\begin{aligned} &c(1 + k \cdot 2^{n-4}) + l \equiv 0 \pmod{2^{n-1}}, \\ &(1 + k \cdot 2^{n-3})^e \equiv 1 + 2^{n-3} \pmod{2^n}. \end{aligned}$$

Since $y^2ay^{-2} = a^{i^2}$ and $y^2 \in G_2$, we have $i^2 \in \langle 1+2^{n-2}\rangle$, where $\langle 1+2^{n-2}\rangle$ is the cyclic subgroup of $(\mathbf{Z}/2^n\mathbf{Z})^*$ generated by $1+2^{n-2}$. Therefore, $i \in \langle 1+2^{n-3}\rangle$. Suppose that $i \in \langle 1+2^{n-2}\rangle$, then there exists $s \in \mathbf{Z}$ such that $(yu^s)a(yu^s)^{-1} = a$. This contradicts the condition (EX, D_n) for G. Hence we can write as

$$yay^{-1} = a^{1+2^{n-3}k},$$

for some odd integer k. Consequently, we have

$$y^2ay^{-2} = a^{1+2^{n-2}k} \text{ (resp. } y^2ay^{-2} = a^{1+(k+2)2^{n-2}}),$$

when $n \geq 6$ (resp. $n = 5$). So, y^2 must be written as $y^2 = a^{l_0}u^m$ for some odd integer m and some integer l_0. Therefore

$$y^2uy^{-2} = y(a^{2^{n-2}d}u)y^{-1} = a^{2^{n-1}d}u = (a^{l_0}u^m)u(a^{l_0}u^m)^{-1} = a^{-l_0 2^{n-2}}u.$$

Hence, we can write $l_0 = 2l$ for some integer l. Thus

$$y^2 = a^{2l}u^m,$$

where m is odd.

Let c, e be the integers satisfying the conditions in Lemma 1, and set $y_1 = (a^c y)^e$. Then we get

$$y_1^2 = (a^c y)^{2e} = (a^{2c(1+k2^{n-4})+2l}u^m)^e = u^{me}$$

So, we have

$$y_1uy_1^{-1} = u,$$

and

$$y_1^2ay_1^{-2} = u^{me}au^{-me} = a^{(1+2^{n-2})^{me}} = a^{1+me2^{n-2}}.$$

On the other hand

$$y_1ay_1^{-1} = (a^c y)^e a(a^c y)^{-e} = a^{(1+2^{n-3}k)^e} = a^{1+2^{n-3}},$$

and

$$y_1^2ay_1^{-2} = a^{(1+2^{n-3})^2} = a^{1+2^{n-2}} \text{ (resp. } a^{1+3\cdot 2^{n-2}}),$$

when $n \geq 6$ (resp. $n = 5$). Hence we get $me \equiv 1 \pmod 4$ when $n \geq 6$, and $me \equiv 3 \pmod 4$ when $n = 5$. Therefore

$$y_1^2 = u \text{ (resp. } u^3),$$

when $n \geq 6$ (resp. when $n = 5$).

Set $b_0 = y_1 b y_1^{-1}$. Taking the conjugate of both sides of the equality, $bab^{-1} = a^{-1}$, by y_1, we get

$$b_0 a^{1+2^{n-3}} b_0^{-1} = a^{-1-2^{n-3}}.$$

So, we have $b_0 a b_0^{-1} = a^{-1}$, hence we can write as $b_0 = a^t b$ for some $t \in \mathbf{Z}$. On the other hand, since $y_1^2 = u$ (or u^3), we have

$$b = y_1^2 b y_1^{-2} = y_1(a^t b)y_1^{-1} = a^{t(1+2^{n-3})} a^t b = a^{2t(1+2^{n-4})} b.$$

So $t \equiv 0 \pmod{2^{n-1}}$ and we can write $y_1 b y_1^{-1} = a^{2^{n-1} t_0} b$ where $t_0 = 0$ or 1. Summarizing the results, we get

$$\begin{aligned} &y_1 a y_1^{-1} = a^{1+2^{n-3}},\\ &y_1 u y_1^{-1} = u,\\ &y_1 b y_1^{-1} = a^{2^{n-1} t_0} b,\\ &y_1^2 = u \text{ (resp. } y_1^2 = u^3) \text{ when } n \geq 6 \text{ (resp. } n = 5). \end{aligned}$$

When $t_0 = 0$, these relations are the same as that of $G_3^{(1)}(D_n)$. So, the group $G = \langle a, b, u, y_1 \rangle$ is clearly isomorphic to $G_3^{(1)}(D_n)$.

When $t_0 = 1$, we set $u_1 = a^{2^{n-1}} u$ and $y_2 = a^{2^{n-2}} y_1$. Then we have $u_1^4 = 1$ and $u_1 b = b u_1$ and $u_1 a u_1^{-1} = a^{1+2^{n-3}}$. So, $\langle a, b, u_1 \rangle = \langle a, b, u \rangle = G_2^{(1)}(D_n)$.

Further, we have

$$\begin{aligned} &y_2 a y_2^{-1} = a^{1+2^{n-3}},\\ &y_2 u_1 y_2^{-1} = u_1,\\ &y_2 b y_2^{-1} = b,\\ &y_2^2 = u_1 \text{ (resp. } y_2^2 = u_1^3) \text{ when } n \geq 6 \text{ (resp. } n = 5). \end{aligned}$$

Thus, in this case also, the group $G = \langle a, b, u_1, y_2 \rangle$ is isomorphic to $G_3^{(1)}(D_n)$.

Next, we consider Case IB.

Let $yay^{-1} = a^i u^2$. We have

$$ya^2 y^{-1} = a^{2i(1+2^{n-2})} = a^{2i+2^{n-1}}.$$

By the condition (EX, D_n), we must have

$$i \notin \langle 1 + 2^{n-2} \rangle.$$

Since $y^2 a^2 y^{-2} = a^{2i^2(1+2^{n-2})^2} = a^{2i^2}$ and $y^2 \in G_2$, we get

$$i^2 \in \langle 1 + 2^{n-2} \rangle.$$

Hence we can write $i = 1 + 2^{n-3}k$, so

$$yay^{-1} = a^{1+2^{n-3}k}u^2,$$

where k is an odd integer. Therefore

$$ya^4y^{-1} = (a^{1+2^{n-3}k}u^2)^4 = a^{4(1+2^{n-3}k)} = a^{4+2^{n-1}},$$

and

$$ya^{2^t}y^{-1} = a^{2^t},$$

for $t \geq 3$. Thus we have

$$y^2ay^{-2} = y(a^{1+2^{n-3}k}u^2)y^{-1} = y(aa^{2^{n-3}k}u^2)y^{-1} = a^{1+2^{n-2}k_0},$$

where $k_0 = k + 2d$ (resp. $k_0 = k + 2 + 2d$) when $n \geq 6$ (resp. $n = 5$). In any cases, k_0 is an odd integer. Since $y^2 \in G_2$, we can write $y^2 = a^{l_0}u^m$ for some $l_0, m \in \mathbf{Z}$, and m is odd. By the same way as in Case IA, we can write $l_0 = 2l$ for some $l \in \mathbf{Z}$. So,

$$y^2 = a^{2l}u^m.$$

We can show easily

Lemma 2. (1) *Let l be an even integer. Then there exists an integer t_0 satisfying the following equality*

$$4t_0(1 + 2^{n-3} + k \cdot 2^{n-4}) + 2l \equiv 0 \pmod{2^n}.$$

(2) *Let l be an odd integer. Then there exists an integer t_1 satisfying the following equality*

$$2(2t_1 + 1)(1 + 2^{n-3} + k \cdot 2^{n-4}) + 2l - 2^{n-2} \equiv 0 \pmod{2^n}.$$

Let t_0 and t_1 be the integers satisfying the conditions in Lemma 2.
When l is even, we set $y_1 = a^{2t_0}y$. Then we have

$$y_1^2 = (a^{2t_0}y)^2 = a^{4t_0(1+2^{n-3}+k2^{n-4})+2l}u^m = u^m.$$

When l is odd, we set $y_1 = a^{2t_1+1}y$. Then we have

$$y_1^2 = (a^{2t_1+1}y)^2 = a^{2(2t_1+1)(1+2^{n-3}+k2^{n-4})+2l-2^{n-2}}u^{2+m} = u^{2+m}.$$

In any cases, we can write as $y_1^2 = u^{m_0}$ where m_0 is an odd integer. Hence we have

$$y_1uy_1^{-1} = u.$$

We also have

$$y_1 a y_1^{-1} = a^{1+k_1 2^{n-3}} u^2,$$

where, $k_1 = k - 4$ (resp. $k_1 = k$) when l is odd (resp. l is even).
When $n \geq 6$, a direct calculation shows that

$$y_1^s a y_1^{-s} = a^{1+sk_1 2^{n-3}} u^{2s},$$

for any integer s $(1 \leq s \leq 7)$.
When $n = 5$, we have

$$y_1^s a y_1^{-s} = a^{1+sk_1 2^{n-3}+2^{n-1}e} u^{2s},$$

for any integer s $(1 \leq s \leq 7)$, where $e = 0$ for $s = 1$ or 5, and $e = 1$ for $s = 3$ or 7.
In any cases, we can take s_0 satisfying the following equality

$$y_1^{s_0} a y_1^{-s_0} = a^{1+2^{n-3}} u^2.$$

Set $y_2 = y_1^{s_0}$, then we have

$$y_2 a y_2^{-1} = a^{1+2^{n-3}} u^2,$$
$$y_2 u y_2^{-1} = u,$$
$$y_2^2 = u^{m_0 s_0}.$$

Since $y_2^2 a y_2^{-2} = a^{1+2^{n-2}}$ (resp. $a^{1+3\cdot 2^{n-2}}$) for $n \geq 6$ (resp. $n = 5$), we have

$$y_2^2 = u \text{ (resp. } u^3\text{)},$$

for $n \geq 6$ (resp. $n = 5$).

Finally, we consider $y_2 b y_2^{-1}$. Write $b_0 = y_2 b y_2^{-1}$. Taking the conjugate of $a^{-2} = ba^2 b^{-1}$ by y_2, we have

$$a^{-2(1+2^{n-3})(1+2^{n-2})} = b_0 a^{2(1+2^{n-3})(1+2^{n-2})} b_0^{-1}.$$

Since $(1 + 2^{n-3})(1 + 2^{n-2})$ is odd, we have,

$$a^{-2} = b_0 a^2 b_0^{-1}.$$

So, we can write as $y_2 b y_2^{-1} = a^t b u^r$ for some $t, r \in \mathbf{Z}$. We also take the conjugate of $a^{-1} = bab^{-1}$ by y_2, then

$$a^{-(1+2^{n-3})(1+2^{n-1})} u^2 = (a^t b u^r)(a^{1+2^{n-3}} u^2)(a^t b u^r)^{-1} = a^{-(1+2^{n-3})(1+2^{n-2}r)-2^{n-1}t} u^2.$$

Hence we get

$$-(1+2^{n-3})(1+2^{n-1}) \equiv -(1+2^{n-3})(1+r\cdot 2^{n-2}) - t\cdot 2^{n-1} \pmod{2^n},$$

so

$$2^{n-2}(1+2^{n-3})(2-r) - t\cdot 2^{n-1} \equiv 0 \pmod{2^n}.$$

Therefore $r \equiv 0 \pmod 2$. If we write $r = 2r_1$, where $r_1 \in \mathbf{Z}$, we have $r_1 + t \equiv 1 \pmod 2$. Since $y_2^2 = u$ (or u^3), we get

$$b = y_2^2 b y_2^{-2} = y_2(a^t b u^{2r_1})y_2^{-1} = (a^{1+2^{n-3}}u^2)^t(a^t b u^{2r_1})u^{2r_1}.$$

Hence we have

$$(a^{1+2^{n-3}}u^2)^t a^t = 1.$$

Since

$$(a^{1+2^{n-3}}u^2)^t = a^{-t} \in \langle a\rangle,$$

we have $t \equiv 0 \pmod 2$. Therefore r_1 is odd. Denote by $t = 2t_1$ where $t_1 \in \mathbf{Z}$. We have

$$1 = (a^{1+2^{n-3}}u^2)^{2t_1}a^{2t_1} = a^{4t_1(1+2^{n-3}+2^{n-4})},$$

so $t \equiv 0 \pmod{2^{n-1}}$. If we write $t = 2^{n-1}t_2$, we have

$$y_2 b y_2^{-1} = a^{2^{n-1}t_2} b u^{2r_1} = a^{2^{n-1}t_2} b u^2.$$

Summarizing the results, we get

$$\begin{aligned} &y_2 a y_2^{-1} = a^{1+2^{n-3}}u^2,\\ &y_2 u y_2^{-1} = u,\\ &y_2 b y_2^{-1} = a^{2^{n-1}t_2} b u^2,\\ &y_2^2 = u \text{ (resp. } y_2^2 = u^3) \text{ when } n \geq 6 \text{ (resp. } n = 5). \end{aligned}$$

When $t_2 = 0$, these relations are the same as that of $G_3^{(2)}(D_n)$. So, the group $G = \langle a, b, u, y_2\rangle$ is clearly isomorphic to $G_3^{(2)}(D_n)$.

When $t_2 = 1$, we set $u_1 = a^{2^{n-1}}u$ and $x_2 = a^{2^{n-2}}y_2$. Then we have $u_1^4 = 1$, $u_1 b = b u_1$ and $u_1 a u_1^{-1} = a^{1+2^{n-2}}$.

So,

$$\langle a, b, u_1\rangle \cong \langle a, b, u\rangle = G_2^{(1)}(D_n).$$

Further, we have

$$
\begin{aligned}
&x_2ax_2^{-1} = a^{1+2^{n-3}}u^2 = a^{1+2^{n-3}}u_1^2,\\
&x_2u_1x_2^{-1} = u_1,\\
&x_2bx_2^{-1} = bu^2 = bu_1^2,\\
&x_2^2 = u_1 \text{ (resp. } x_2^2 = u_1^3) \text{ when } n \geq 6 \text{ (resp. } n = 5).
\end{aligned}
$$

Thus, in the case also, the group $G = \langle a, b, u_1, x_2\rangle$ is isomorphic to $G_3^{(2)}(D_n)$, as desired.

Now, we consider Case II. For the sake of simplicity, we write $G_2^{(2)}$ instead of $G_2^{(2)}(D_n)$ in this proof.

First, we consider the element yay^{-1}.
Write $a_0 = yay^{-1}$. Since it is in $G_2^{(2)}$, it can be represented as $a^ib^jw^lu^m$ for some $i, j, l, m \in \mathbf{Z}$, $0 \leq i \leq 2^n - 1$, $0 \leq j, l, m \leq 1$.
By a direct calculation, we have

$$(a^ibw^lu^m)^4 = 1,$$

for any $i, l, m \in \mathbf{Z}$. So we must have $yay^{-1} = a^iw^lu^m$.

Suppose that $yay^{-1} = a^iw^l$.
When $l = 1$ we have $yay^{-1} = a^iw$. Then

$$ya^2y^{-1} = (a^iw)^2 = a^{2i(1+2^{n-2})},$$

so i must be an odd integer. If we write $i = 2i_0 + 1, i_0 \in \mathbf{Z}$, we have

$$(uy)a(uy)^{-1} = u(a^{2i_0+1}w)u^{-1} = (aw)^{2i_0+1}w = a^{2i_0(1+2^{n-2})+1}.$$

When $l = 0$ we have $yay^{-1} = a^i$. Consequently, when $yay^{-1} = a^iw^l$, there exists an element $g \in G - G_2^{(2)}$ such that $gag^{-1} \in \langle a\rangle$. Write

$$gag^{-1} = a^s,$$

where $s \in \mathbf{Z}$.
Then, by the same way as in the proof of Case I, we must have

$$gag^{-1} = a^{1+2^{n-2}k},$$

for some odd integer k.
Then we have

$$(ug)\langle a\rangle(ug)^{-1} \not\subseteq \langle a\rangle,$$

since $u\langle a\rangle u^{-1} \not\subseteq \langle a\rangle$. Further

$$(ug)a^2(ug)^{-1} = u(a^{2(1+2^{n-2}k)})u^{-1} = (aw)^{2(1+2^{n-2}k)} = a^{2(1+2^{n-2})(1+2^{n-2}k)} = a^2$$

since k is odd. Therefore we have

$$\langle a\rangle \cap (ug)\langle a\rangle(ug)^{-1} = \langle a^2\rangle.$$

This contradicts the hypothesis that G satisfies the condition (EX, D_n). Thus we must have

$$a_0 = yay^{-1} = a^i w^l u.$$

In this case, we have $ya^4y^{-1} = (a^i w^l u)^4 = a^{4i}$, so i must be odd. As usual, we may assume that $i \in \langle 5\rangle$. If we write $i = 2i_0 + 1, i_0 \in \mathbf{Z}$, then

$$ya^2y^{-1} = (a^i w^l u)^2 = a^{2i+(i_0+l)2^{n-1}} w \notin \langle a\rangle$$

and $ya^4y^{-1} = a^{4i}$. Hence

$$\langle a\rangle \cap y\langle a\rangle y^{-1} = \langle a^4\rangle.$$

Therefore, by the condition (EX, D_n) for G, we must have $a^{4i} \neq a^4$. Thus

$$i \notin \langle 1 + 2^{n-2}\rangle.$$

On the other hand, $y^2a^4y^{-2} = a^{4i^2}$ and $y^2 \in G_2^{(2)}$, we get

$$i^2 \in \langle 1 + 2^{n-2}\rangle.$$

Hence we can write $i = 1 + 2^{n-3}k$, for some odd integer k. So we must have

$$a_0 = yay^{-1} = a^{1+2^{n-3}k} w^l u.$$

We denote by $C_{G_2^{(2)}}(\langle a\rangle)$, the centralizer of $\langle a\rangle$ in $G_2^{(2)}$.
It is clear that $C_{G_2^{(2)}}(\langle a\rangle) = \langle a\rangle$.
So

$$C_{G_2^{(2)}}(\langle a_0\rangle) = \langle a_0\rangle.$$

By a direct calculation, we have

$$a_0^8 = (a^{1+k2^{n-3}} w^l u)^8 = a^8.$$

Write $w_0 = ywy^{-1}$. Then, by taking the conjugate of $waw^{-1} = a^{1+2^{n-1}}$ by y, we get

$$w_0 a_0 w_0^{-1} = a_0^{1+2^{n-1}} = a_0 \cdot a_0^{2^{n-1}} = a_0 \cdot a^{2^{n-1}} = (a^{1+k2^{n-3}} w^l u)a^{2^{n-1}}.$$

Therefore we have

$$ww_0a_0w_0^{-1}w^{-1} = w(a^{1+k2^{n-3}}w^lu)a^{2^{n-1}}w^{-1} = a_0.$$

Thus $ww_0 \in C_{G_2^{(2)}}(\langle a_0\rangle) = \langle a_0\rangle$. So we can write

$$w_0 = ywy^{-1} = wa_0^{j_0} = w(a^{1+2^{n-3}k}w^lu)^{j_0},$$

for some integer j_0. Since $w_0^2 = 1$, we have

$$\begin{aligned}1 &= (wa_0^{j_0})^2 = \{w(a^{1+k2^{n-3}}w^lu)^{j_0}\}^2\\ &= w(a^{1+k2^{n-3}}w^lu)^{j_0}w(a^{1+k2^{n-3}}w^lu)^{j_0}\\ &= (a^{1+k2^{n-3}}w^lua^{2^{n-1}})^{j_0}(a^{1+k2^{n-3}}w^lu)^{j_0}\\ &= (a^{1+k2^{n-3}}w^lu)^{2j_0}a^{2^{n-1}j_0}.\end{aligned}$$

Since the order of the element $a^{1+2^{n-3}k}w^lu$ is 2^n, we have $j_0 \equiv 0 \pmod{2^{n-1}}$. If we write $j_0 = 2^{n-1}j$ where $j \in \mathbf{Z}$, then we have

$$w_0 = ywy^{-1} = wa_0^{2^{n-1}j} = wa^{2^{n-1}j}.$$

Next, we write $u_0 = yuy^{-1}$. Then, by taking the conjugate of $uau^{-1} = aw$ by y, we get

$$u_0(a^{1+k2^{n-3}}w^lu)u_0^{-1} = (a^{1+k2^{n-3}}w^lu)(a^{2^{n-1}j}w).$$

Hence

$$uw^ju_0(a^{1+k2^{n-3}}w^lu)u_0^{-1}w^{-j}u^{-1} = a^{1+k2^{n-3}}w^lu.$$

So, we have

$$uw^ju_0 \in C_{G_2^{(2)}}(\langle a_0\rangle) = \langle a_0\rangle.$$

Therefore we can write

$$u_0 = yuy^{-1} = w^ju(a_0)^m = w^ju(a^{1+k2^{n-3}}w^lu)^m,$$

for some integer m. On the other hand, by taking the conjugate of $u = wuw^{-1}$ by y, we get

$$u_0 = w_0u_0w_0^{-1} = (a^{2^{n-1}j}w)\{w^ju(a^{1+2^{n-3}k}w^lu)^m\}(a^{2^{n-1}j}w)^{-1} = u_0a^{2^{n-1}m}.$$

Hence, m is even, so we can write as $m = 2m_0$ for some integer m_0. And

$$u_0 = w^ju(a^{1+2^{n-3}k}w^lu)^{2m_0} = w^ju(a^{2(1+2^{n-3}k)+2^{n-1}l}w)^{m_0} = ua^{2m_0(1+2^{n-3}k)+m_0\cdot l\cdot 2^{n-1}}w^{j+m_0}.$$

Since $u_0^2 = 1$ we have

$$1 = u_0^2 = a^{4m_0+m_0 2^{n-1}+m_0 k 2^{n-1}}.$$

Therefore $4m_0 \equiv 0 \pmod{2^n}$, so we can write $m = 2m_0 = 2^{n-1}m_1$ for some integer m_1. Thus

$$u_0 = yuy^{-1} = w^j u(a^{1+k2^{n-3}} w^l u)^{2^{n-1}m_1} = w^j u a^{2^{n-1}m_1}.$$

Summarizing the results, we must have

$$\begin{aligned} yay^{-1} &= a^{1+k2^{n-3}} w^l u, \\ yuy^{-1} &= w^j u a^{2^{n-1}m_1}, \\ ywy^{-1} &= wa^{2^{n-1}j}. \end{aligned}$$

Using these relations, we have

$$\begin{aligned} y^2ay^{-2} &= y(a^{1+k2^{n-3}} w^l u)y^{-1} = a^{1+k2^{n-2}+(jl+m_1)2^{n-1}} w^j \\ \text{(resp. } &= a^{1+k2^{n-2}+(1+jl+m_1)2^{n-1}} w^j), \end{aligned}$$

when $n \geq 6$ (resp. $n = 5$). Set $k_1 = k+2(jl+m_1)$ when $n \geq 6$ and set $k_1 = k+2(1+jl+m_1)$ when $n = 5$. Then k_1 is odd and

$$y^2ay^{-2} = a^{1+k_1 2^{n-2}} w^j.$$

Suppose that j is even, then $y^2ay^{-2} = a^{1+k_1 2^{n-2}}$. This contradicts the fact that $y^2 \in G_2^{(2)}$. Suppose that j is odd, then we have

$$uy^2a(uy^2)^{-1} = u(a^{1+k_1 2^{n-2}} w)u^{-1} = a^{1+k_1 2^{n-2}}$$

and $uy^2 \in G_2^{(2)}$, contradiction. Consequently, Case II does not occur. Thus the proof of Theorem 1 is completed.

4. Proof of Theorems 2 and 3

Proof of Theorem 2 is similar to that of Theorem 1, so we omit some of the details. By Theorem 0.4, we must have $n \geq 5$. Let G be a 2-group, satisfying the conditions in Theorem 2. Then, by the same way as in the proof of Theorem 1, we have only to consider the following two cases:

CASE I. $G = \langle G_2^{(1)}(Q_n), y\rangle = \langle a, b, u, y\rangle \rhd G_2^{(1)}(Q_n) = \langle a, b, u\rangle$

CASE II. $G = \langle G_2^{(2)}(Q_n), y\rangle = \langle a, b, w, u, y\rangle \rhd G_2^{(2)}(Q_n) = \langle a, b, w, u\rangle$,

where, $[G; G_2^{(i)}(Q_n)] = 2$, $y \notin G_2^{(i)}(Q_n)$ and $y^2 \in G_2^{(i)}(Q_n)$, for $i = 1$ or 2.

Furthermore, Case I can be devided into the following two cases:

CASE IA. $yay^{-1} = a^i (i \equiv 1 \pmod 4)$ and $yuy^{-1} = a^{2^{n-2}d}u$

CASE IB. $yay^{-1} = a^i u^2 (i \equiv 1 \pmod 4)$ and $yuy^{-1} = a^{2^{n-2}d}u$

In Case IA, we can show that $G \cong G_3^{(1)}(Q_n)$ and, in Case IB, we can show that $G \cong G_3^{(2)}(Q_n)$ by the same way as in the proof of Theorem 1.

On the other hand, we can show that Case II does not occur, by the same argument as in the proof of Theorem 1.

So, the proof of Theorem 2 is completed.

Theorem 3 follows from Theorem 0.3, Theorem 1 and Theorem 2.

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