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ON SOME ARITHMETICAL PROPERTIES OF ROGERS-RAMANUJAN CONTINUED FRACTION

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1. Introduction

Let $R(z)$ be the Rogers-Ramanujan continued fraction defined by

$$R(z) = 1 + \frac{z}{1} + \frac{z^2}{1} + \frac{z^3}{1} + \cdots \qquad (|z| < 1).$$

For $z = 1/q$ ($q \in \mathbb{N} - \{0, 1\}$), it is easy to transform $R(1/q)$ into the regular continued fraction

$$R(1/q) = 1 + \frac{1}{q} + \frac{1}{q} + \frac{1}{q^2} + \frac{1}{q^2} + \frac{1}{q^3} + \frac{1}{q^3} + \cdots$$

(see e.g. [9; 2.3]). Since this expansion is not ultimately periodic, $R(1/q)$ is not a quadratic number. More generally, as an application of a deep result of Nesterenko on modular functions [12], one can prove that $R(z)$ is transcendental for every algebraic number z ($0 < |z| < 1$) [5]. In this paper, we want to focus on the fact that $R(1/q)$ is not a quadratic number, and generalize this result in two directions.

First, we consider a more general Rogers-Ramanujan continued fraction

$$R(z; x) = 1 + \frac{zx}{1} + \frac{z^2x}{1} + \frac{z^3x}{1} + \cdots \qquad (|z| < 1).$$

Irrationality results on $R(z; x)$ for rational x and z are given in [11], [13], [14]. We will prove the following

Theorem 1. *Let $x = a/b \in \mathbb{Q}^*$ and let $z = 1/q$ with $q \in \mathbb{Z}$, $|q| \geq 2$. Suppose that $a^4 < |q|$. Then $R(1/q; a/b)$ is not a quadratic number.*

It should be noted that Lagrange's theorem on regular continued fractions cannot be applied here, because

$$R(1/q; a/b) = \frac{1}{qb/a} + \frac{1}{q} + \frac{1}{q^2b/a} + \frac{1}{q^2} + \frac{1}{q^3b/a} + \frac{1}{q^3} + \cdots$$

is not a regular continued fraction if $a \neq 1$. Theorem 1 is a direct consequence of the following general result on continued fractions with rational coefficients, which should be compared to Lambert's criterion on irrationality (see e.g. [10; p. 100]).

Theorem 2. *Let $c_1, c_2, c_3, \dots$ be an infinite sequence of rational numbers satisfying the following conditions*

(1) $$|c_n| \geq 2 \qquad \text{for every } n \geq 1$$

(2) $$\sum_{n=1}^{+\infty} |c_n c_{n+1}|^{-1} < \infty$$

(3) *There exists an infinite sequence of rational integers d_n $(n \geq 1)$ such that $d_n c_n \in \mathbb{Z}$ for every $n \geq 1$, and $\liminf_{n\to+\infty} (d_1 d_2 \cdots d_n)^2/c_{n+1} = 0$.*

Then the continued fraction

$$\alpha = 1 + \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n} + \cdots$$

is convergent, and α is not a quadratic number.

Note that, under the hypothesis of Theorem 2, Lambert's criterion implies the irrationality of α.

For the second generalization, we will use Rogers-Ramanujan identities ([6; p. 36], or [8; p. 290], for example), and write

$$R\left(\frac{1}{q}\right) = \frac{\alpha_q^*}{\beta_q^*}$$

with

$$\alpha_q^* = 1 + \sum_{n=1}^{+\infty} (-1)^n q^{-n(5n-1)/2} + \sum_{n=1}^{+\infty} (-1)^n q^{-n(5n+1)/2},$$

$$\beta_q^* = 1 + \sum_{n=1}^{+\infty} (-1)^n q^{-n(5n-3)/2} + \sum_{n=1}^{+\infty} (-1)^n q^{-n(5n+3)/2}.$$

The numbers α_q^* and β_q^* involve the sequences (u_n) and (v_n) defined by

$$u_0 = 0,\ u_1 = 2,\ u_2 = 3,\ u_3 = 9,\ u_4 = 11, \dots,$$
$$u_{2n-1} = \frac{n(5n-1)}{2},\ u_{2n} = \frac{n(5n+1)}{2}, \dots,$$
$$v_0 = 0,\ v_1 = 1,\ v_2 = 4,\ v_3 = 7,\ v_4 = 13, \dots,$$
$$v_{2n-1} = \frac{n(5n-3)}{2},\ v_{2n} = \frac{n(5n+3)}{2}, \dots.$$

Indeed, one can write $\alpha_q^* = \sum_{n=0}^{+\infty} a(n)q^{-n}$, where $a(n) = \pm 1$ if there exists $k \in \mathbb{N}$ such that $n = u_k$, $a(n) = 0$ otherwise. Similarly, we have $\beta_q^* = \sum_{n=0}^{+\infty} b(n)q^{-n}$, where $b(n) = \pm 1$ if there exists $k \in \mathbb{N}$ such that $n = v_k$, $b(n) = 0$ otherwise. Therefore, we can deduce that $R(1/q)$ is not quadratic for $q \in \mathbb{Z}$ ($|q| \geq 2$) from the following more general result.

Theorem 3. *Let $a(n)$ and $b(n)$ be bounded sequences of rational integers, such that*

$$\begin{cases} a(n) \neq 0 & \text{if there exists } k \in \mathbb{N} \text{ such that } n = u_k, \\ a(n) = 0 & \text{otherwise,} \end{cases}$$

$$\begin{cases} b(n) \neq 0 & \text{if there exists } k \in \mathbb{N} \text{ such that } n = v_k, \\ b(n) = 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{K}$ be any quadratic field. Then, if $q \in \mathbb{Z}$ ($|q| \geq 2$) the three numbers $\alpha_q = \sum_{n=0}^{+\infty} a(n)q^{-n}$, $\beta_q = \sum_{n=0}^{+\infty} b(n)q^{-n}$, and 1, are linearly independent over $\mathbb{K}$.

2. Proof of Theorem 2

We will need the following lemma.

Lemma 1 ([14]). *Let $c_1, c_2, c_3, \ldots$ be an infinite sequence of complex numbers satisfying (1) and (2). Let $P_n = c_n P_{n-1} + P_{n-2}$, $Q_n = c_n Q_{n-1} + Q_{n-2}$ ($n \geq 1$) with $P_0 = Q_{-1} = 0$ and $P_{-1} = Q_0 = 1$. Then $P_n/(c_2c_3\cdots c_n)$ and $Q_n/(c_1c_2\cdots c_n)$ converge to non-zero limits β and γ, and satisfy for every $n \geq 1$*

$$A < \frac{|P_n|}{|c_2c_3\cdots c_n|} < B, \quad A < \frac{|Q_n|}{|c_1c_2\cdots c_n|} < B, \tag{1}$$

where $0 < A = \prod_{n=1}^{+\infty}(1 - 2/|c_nc_{n+1}|) < 1$, $B = \prod_{n=1}^{+\infty}(1 + 2/|c_nc_{n+1}|) > 1$. So the continued fraction $\frac{1}{c_1} \, \frac{1}{+c_2} \, \frac{}{+\cdots} \, \frac{1}{+c_n} \, \frac{}{+\cdots}$ converges to the limit $\alpha = \lim_{n\to+\infty} P_n/Q_n = \beta/(c_1\gamma)$, and

$$\frac{A}{B} < \left|\frac{\beta}{\gamma}\right| < \frac{B}{A}. \tag{2}$$

Proof. Since $|c_n| \geq 2$, we have $|P_n| \geq |2|P_{n-1}| - |P_{n-2}||$. Hence $|P_n| \geq |P_{n-1}|$ for every $n \geq 1$ by induction, and $|P_n| \geq |P_1| = 1$. Therefore $P_n \neq 0$ for every $n \geq 1$, and the same holds for Q_n. We put $u_n = c_n P_{n-1}/P_n$, $v_n = c_n Q_{n-1}/Q_n$ for $n \geq 1$, so that $u_1 = 0$, $v_1 = 1$. Then we have

$$P_n = c_n\left(1 + \frac{u_{n-1}}{c_{n-1}c_n}\right)P_{n-1}, \qquad Q_n = c_n\left(1 + \frac{v_{n-1}}{c_{n-1}c_n}\right)Q_{n-1},$$

and so

$$P_n = c_2c_3\cdots c_n \prod_{k=2}^{n-1}\left(1+\frac{u_k}{c_{k-1}c_k}\right) \quad (n \geq 2),$$

$$Q_n = c_1c_2\cdots c_n \prod_{k=1}^{n-1}\left(1+\frac{v_k}{c_{k-1}c_k}\right) \quad (n \geq 1).$$

Since $u_n = (1+u_{n-1}/c_{n-1}c_n)^{-1}$ and $v_n = (1+v_{n-1}/c_{n-1}c_n)^{-1}$, we see by induction on n that $|u_n| \leq 2$, $|v_n| \leq 2$ for $n \geq 1$, which together with (1) and (2) ensures the convergence of the products $\beta = \prod_{k=2}^{+\infty}(1+u_k/c_kc_{k+1})$ and $\gamma = \prod_{k=1}^{+\infty}(1+v_k/c_kc_{k+1})$, and (1) and (2) follow immediately. □

Lemma 2. *With the notations in* Lemma 1, *there exists* $n_0 \in \mathbb{N}$ *such that* $|\alpha - P_n/Q_n| < 2/|Q_nQ_{n+1}| \leq 1$ *for every* $n \geq n_0$.

Proof. Put $\alpha_n = \frac{1}{c_n} + \frac{1}{c_{n+1}} + \cdots$ $(n \geq 1)$. We have

$$\alpha = \alpha_1 = \frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_{n+1}+\alpha_{n+2}} = \frac{1}{c_1} + \cdots + \frac{1}{c_{n+1}} + \frac{\alpha_{n+2}}{1} = \frac{P_{n+1}+\alpha_{n+2}P_n}{Q_{n+1}+\alpha_{n+2}Q_n},$$

and we get the well-known formula

$$\alpha - \frac{P_n}{Q_n} = \frac{(-1)^n}{Q_nQ_{n+1}(1+\alpha_{n+2}Q_n/Q_{n+1})} \quad (n \geq 1).$$

By (1), we have

$$|Q_n/Q_{n+1}| \leq \frac{B}{A|c_{n+1}|}. \tag{3}$$

By (2) with α_{n+2} in place of $\alpha = \alpha_1$, we get $|\alpha_{n+2}| \leq B/(A|c_{n+2}|)$. Hence $\lim_{n\to+\infty}(1+\alpha_{n+2}Q_n/Q_{n+1}) = 1$ by (2), and Lemma 2 follows. □

Proof of Theorem 2. Suppose that α is a root of $f(x) = ax^2+bx+c$, a, b, $c \in \mathbb{Z}$, $a \neq 0$. It follows from the mean value theorem that $-f(P_n/Q_n) = (\alpha - P_n/Q_n)f'(\theta)$, with $\alpha - 1 \leq \theta \leq \alpha + 1$. By Lemma 2 we get $|f(P_n/Q_n)| \leq 2M/|Q_nQ_{n+1}|$ $(n \geq n_0)$ where $M = \max\{|f'(x)| \mid \alpha - 1 \leq x \leq \alpha + 1\}$. Using (3) yields $|Q_n^2 f(P_n/Q_n)| \leq 2MB/(A|c_{n+1}|)$ $(n \geq n_0)$.

We see by induction on n that $d_1d_2\cdots d_nP_n$ and $d_1d_2\cdots d_nQ_n$ are rational integers; the same holds for $A_n = (d_1d_2\cdots d_n)^2Q_n^2f(P_n/Q_n)$ $(n \geq 1)$. Using (3), we get $\liminf_{n\to+\infty} A_n = 0$, and $A_n = 0$ for infinitely many n, namely $f(P_n/Q_n) = 0$ for infinitely many n. Hence f has infinitely many roots, and $f = 0$. The proof of Theorem 2 is complete. □

3. Proof of Theorem 3

To prove Theorem 3, we essentially use the same method as in [2]. We put

$$
\begin{aligned}
\alpha_q^2 &= \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a(k)a(n-k)\right)q^{-n} = \sum_{n=0}^{+\infty} r'(n)q^{-n}, \\
\beta_q^2 &= \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} b(k)b(n-k)\right)q^{-n} = \sum_{n=0}^{+\infty} s'(n)q^{-n}, \qquad (1)\\
\alpha_q\beta_q &= \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{n} a(k)b(n-k)\right)q^{-n} = \sum_{n=0}^{+\infty} t'(n)q^{-n}.
\end{aligned}
$$

As $a(n)$ and $b(n)$ are *bounded* sequences of rational integers, we see that there exists $M > 0$ such that

$$|r'(n)| \le Mr(n), \tag{2}$$

$$|s'(n)| \le Ms(n), \tag{3}$$

$$|t'(n)| \le Mt(n), \tag{4}$$

where $r(n)$, $s(n)$, $t(n)$ are the numbers of solutions $(k, l) \in \mathbb{N}^2$ of the equations $u_k + u_l = n$, $v_k + v_l = n$, $u_k + v_l = n$, respectively.

As in [2], the numbers $r(n)$, $s(n)$ and $t(n)$ can be connected to the number $\rho(n)$ of solutions $(k, l) \in \mathbb{N}^2$ of the equation $k^2 + l^2 = n$. This will be done in the paragraph 3.1, Lemmas 4 and 5. In the paragraph 3.2, we will recall an elementary criterion of irrationality from [1] (Theorem 4) and prove a modified version of [3; Lemma 2] (Theorem 5), concerning the gaps in the sequence $r(n)$. The proof of Theorem 3 will be given in the paragraph 3.3.

3.1. Three technical lemmas

We prove some connections between $r(n)$, $s(n)$, $t(n)$ and $\rho(n)$.

Lemma 3. *Suppose that $n = 2^\alpha \prod p^\beta \prod q^\gamma$, where p and q are primes congruent to 1 and 3 modulo 4, respectively. Then, if n is not a square,*

$$\rho(n) = \prod(\beta + 1)\prod\left(\frac{1 + (-1)^\gamma}{2}\right).$$

Proof. Let $\rho^*(n)$ be the number of decompositions of n as sum of squares of two rational integers. It is well known that the generating function of $\rho^*(n)$ is

$$g^*(x) = \left(\sum_{n=-\infty}^{+\infty} x^{n^2}\right)^2,$$

while the generating function of $\rho(n)$ is

$$g(x) = \left(\sum_{n=0}^{+\infty} x^{n^2}\right)^2.$$

Hence $g^*(x) = \left(2\sum_{n=0}^{+\infty} x^{n^2} - 1\right)^2 = 4g(x) - 4\sum_{n=0}^{+\infty} x^{n^2} + 1$. Thus if n is not a square, $\rho(n) = \rho^*(n)/4$, and Lemma 1 follows directly from [8; (16–9–5) and Theorem 278]. □

Lemma 4. *For every natural integer n, we have*

$$r(n) = \rho(40n + 2), \tag{5}$$

$$s(n) = \rho(40n + 18). \tag{6}$$

Proof. We prove (5). Let (k, l) be a solution of the equation

$$\frac{k(5k+u)}{2} + \frac{l(5l+v)}{2} = n \tag{7}$$

with $u^2 = 1$, $v^2 = 1$. It is easy to verify that this equation is equivalent to

$$(10k + u)^2 + (10l + v)^2 = 40n + 2.$$

Thus every solution (k, l) of (7) yields a solution (k', l') of the equation

$$k'^2 + l'^2 = 40n + 2. \tag{8}$$

Conversely, let (k', l') be a solution of (8). By reduction modulo 5, we obtain $k' = 5k_1 + u$ and $l' = 5l_1 + v$, with $k_1 \in \mathbb{N}$, $l_1 \in \mathbb{N}$, $u^2 = 1$, $v^2 = 1$. But k' and l' must be odd by (8), therefore k_1 and l_1 must be even, and $k' = 10k + u$, $l' = 10l + v$. Thus (k, l) is a solution of (7), and (5) is proved. The proof of (6) is similar. □

The connection between $t(n)$ and $\rho(n)$ is a bit more difficult to handle, and we only prove:

Lemma 5. *For every integer $n \geq 0$, we have*

$$t(n) = \frac{1}{2}\rho(40n + 10) \quad \text{if} \quad n \not\equiv 1 \pmod 5, \tag{9}$$

$$t(n) \leq \rho(40n + 10), \tag{10}$$

$$t(n) = 2 \quad \text{if} \quad \rho(40n + 10) = 4 \quad \text{and} \quad 8n + 2 \not\equiv 0 \pmod 5. \tag{11}$$

Proof. Let us prove first (16). The equation

$$\frac{k(5k+u)}{2}+\frac{l(5l+v)}{2}=n \tag{12}$$

with $u^2=1$, $v^2=9$ is equivalent to

$$(10k+u)^2+(10l+v)^2=40n+10.$$

Thus every solution (k, l) of (12) yields one solution (k', l') of the equation

$$k'^2+l'^2=40n+10, \tag{13}$$

and (16) is proved.

Next we prove (15). Let (k', l') be a solution of (13). It is easy to verify that only two cases can occur:

Case 1°. $k' \equiv u \pmod 5$ and $l' \equiv v \pmod 5$, with $u, v \in \{1, -1, 3, -3\}$. As k' and l' must be odd by (13), we obtain $k' = 10k+u$ and $l' = 10l+v$.

Case 2°. $k' \equiv 0 \pmod 5$ and $l' \equiv 0 \pmod 5$.

Suppose that $n \not\equiv 1 \pmod 5$. Then Case 2° cannot occur, because $k' = 5k_1$ and $l' = 5l_1$ implies $8n+2 = 5(k_1^2+l_1^2)$ by (13), and reduction modulo 5 yields $n \equiv 1 \pmod 5$. Hence we are in Case 1° and $k' = 10k+u$, $l' = 10l+v$, with $u, v \in \{1, -1, 3, -3\}$. But this gives a solution of (12) only if $u = 1$ or -1 and $v = 3$ or -3. Therefore (15) is proved.

Finally (17) is an immediate consequence of (15). □

3.2. Two theorems The following theorem is proved in [1](see also [4] for a generalization).

Theorem 4. *Let $q \in \mathbb{Z}$ ($|q| \geq 2$). Let $\tau(n)$ be a sequence of rational integers with the following properties* (i), (ii), (iii):

(i) $\tau(n) \neq 0$ *for infinitely many n.*

(ii) *When n is large enough, $|\tau(n)| \leq \omega(n)$ with $\omega(n) > 0$ and* $\limsup_{n\to+\infty} \omega(n+1)/\omega(n) < |q|$.

(iii) *There exists infinitely many $k \in \mathbb{N}$ and integers $n_k \in \mathbb{N}$ such that $\tau(n_k+1) = \tau(n_k+2) = \cdots = \tau(n_k+k) = 0$ and $\lim_{k\to+\infty} \omega(n_k+k+1)/|q|^k = 0$.*

Let $x = \sum_{n=0}^{+\infty} \tau(n)q^{-n}$. Then if $x = \alpha/\beta \in \mathbb{Q}$, we have

$$\alpha q^{n_k} - \beta \sum_{n=0}^{n_k} \tau(n) q^{n_k-n} = 0$$

for all sufficiently large k.

One sees that Theorem 4 is a criterion of irrationality for *gap* series under some conditions. The following result allows to show that $\rho(n)$, $r(n)$, $s(n)$, $t(n)$ are gap series, and to apply Theorem 4 in order to prove Theorem 3.

Theorem 5. *Let Ω_1 and Ω_2 be two natural integers, with $\Omega_1 \equiv 1 \pmod 4$, Ω_2 odd, $\gcd(\Omega_1, \Omega_2) = \Delta \equiv 1 \pmod 4$, and let $\theta_1 \in \mathbb{N}$, $\theta_2 \in \mathbb{N} - \{0, 1\}$, $\delta \in \mathbb{N} - \{0\}$, $\varepsilon \in]0, 1[$. Denote by $p_1 < p_2 < \cdots < p_n$ a sequence of consecutive rational primes congruent to 3 modulo 4 with the following properties*:

$$(14) \qquad p_n \text{ does not divide } \Omega_2 \text{ for every } n \geq 1,$$

$$\sum_{n=1}^{+\infty} p_n^{-2} \leq \frac{\varepsilon}{2}\,.$$

Then there exists an integer $m_0 = m_0(\Omega_1, \Omega_2, \theta_1, \theta_2, \delta, \varepsilon)$ and a constant $L > 1$ (Linnik's constant [7]) such that, for every $k = p_1 p_2 \cdots p_m$ with $m \geq m_0$, there exists $N_k \in \mathbb{N}$ such that

$$(15) \qquad \rho(N_k - \delta) = \cdots = \rho(N_k - 1) = \rho(N_k + 1) = \cdots = \rho(N_k + k) = 0,$$

(22) $N_k = 2^{\theta_1} \Delta p_k^* h_k^2$, *where p_k^* is a rational prime satisfying $p_k^* \equiv 1 \pmod 4$, and h_k is an integer whose prime divisors are all distinct and congruent to 3 modulo 4,*

$$(23) \qquad N_k \equiv 2^{\theta_1} \Omega_1 \pmod{2^{\theta_1+\theta_2} \Omega_2},$$

$$(24) \qquad N_k \leq (2^{\theta_1+\theta_2} \Omega_2)^L \exp(4L p_{2[\varepsilon k]}).$$

Proof of Theorem 5. We follow the proof of [1; Lemma 2] until the fourth step. We modify the fifth step in the following way. Because of (14), we can choose $\eta \in \{0, 1, \ldots, 2^{\theta_1+\theta_2}\Omega_2 - 1\}$, such that

$$(25) \qquad \eta(p_1 p_2 \cdots p_{m+N+M})^4 + t_m \equiv 2^{\theta_1} \Omega_1 \pmod{2^{\theta_1+\theta_2} \Omega_2}.$$

Then we put for $s \in \mathbb{N}$

$$(26) \qquad \begin{aligned} w_s &= 2^{\theta_1+\theta_2} \Omega_2 (p_1 p_2 \cdots p_{m+N+M})^4 s + \eta (p_1 p_2 \cdots p_{m+N+M})^4 + t_m, \\ D &= \gcd[2^{\theta_1+\theta_2} \Omega_2 (p_1 p_2 \cdots p_{m+N+M})^4,\ \eta(p_1 p_2 \cdots p_{m+N+M})^4 + t_m]. \end{aligned}$$

Using (25) and [1; (30)], we see that

$$(27) \qquad D = 2^{\theta_1} \Delta \prod_{i=1}^{n+N+M} p_i^{\alpha_i}, \quad \text{with } \alpha_i = 0 \text{ or } 2.$$

We write

$$w_s = D(\zeta s + \chi), \text{ with } (\zeta, \chi) \in \mathbb{N} \times \mathbb{N}.$$

Because of [1; (31)], we have $\chi < \eta$ for large m. Then, by Linnik's theorem, there exists a prime number p_k^* and a natural integer σ such that

$$\omega_\sigma = Dp_k^* \le D\left(\frac{2^{\theta_1+\theta_2}\Omega_2(p_1 p_2 \cdots p_{m+N+M})^4}{D}\right)^L. \tag{28}$$

We put $N_k = \omega_\sigma$. By (28) with $L > 1$, we have

$$N_k \le (2^{\theta_1+\theta_2}\Omega_2)^L (p_1 p_2 \cdots p_{m+N+M})^{4L},$$

which leads to (24) by following the sixth step of the proof of [1; Lemma 2].

Moreover, as $\Omega_1 \equiv 1 \pmod 4$ and $\theta_2 \ge 2$, we have by using (25) $w_s \equiv 1 \pmod 4$ ($s \in \mathbb{N}$). Thus, by (27) and (28)

$$\Delta \prod_{i=1}^{m+N+M} p_i^{\alpha_i} \cdot p_k^* \equiv 1 \pmod 4.$$

As $\Delta \equiv 1 \pmod 4$ and $\alpha_i = 0$ or 2, we get $p_k^* \equiv 1 \pmod 4$, and p_k^* is a sum of two squares. This proves (22), while (23) results from (26) and the definition of N_k. Finally, (15) is a direct consequence of [1; (35)]. The proof of Theorem 4 is complete. □

3.3. Proof of Theorem 3 For the proof of Theorem 3, it is sufficient to show that the numbers α_q^2, β_q^2, $\alpha_q\beta_q$, α_q, β_q and 1 are linearly independent over $\mathbb{Q}$, because $(a+b\sqrt{d})+(a'+b'\sqrt{d})\alpha_q+(a''+b''\sqrt{d})\beta_q = 0$ implies $(a+a'\alpha_q+a''\beta_q)^2 = d(b+b'\alpha_q+b''\beta_q)^2$. So suppose that, for rational integers A, B, C, D, E, F,

$$A\alpha_q^2 + B\beta_q^2 + C\alpha_q\beta_q + D\alpha_q + E\beta_q + F = 0.$$

Then, if

$$\tau(n) = Ar'(n) + Bs'(n) + Ct'(n) + Da(n) + Eb(n), \tag{29}$$

we have $\sum_{n=0}^{+\infty} \tau(n)q^{-n} = -F$.

First step. Suppose first that $B \ne 0$. Let $\sigma \in \mathbb{N}$ such that q^σ divides none of the numbers $2Bb(u)b(v)$ with $(u, v) \in \mathbb{N}^2$ and $b(u)b(v) \ne 0$; we can choose such a σ because $b(u)$ and $b(v)$ are bounded. In Theorem 5, we put

$$\delta = 40\sigma + 16,$$

and choose ε such that

$$32L - \frac{\log|q|}{320\varepsilon} < 0. \tag{30}$$

Also we put $\Omega_1 = 9$, $\Omega_2 = 5$, $\theta_1 = 1$, $\theta_2 = 2$ and then N_k in (23) satisfies $N_k \equiv 18 \pmod{40}$. We put $n_k = (N_k - 18)/40$.

By using (15) and (22), and Lemma 3, 4, 5, we have

$$s(n_k - \sigma) = \cdots = s(n_k - 1) = s(n_k + 1) = \cdots = s\left(n_k + \left[\frac{k}{40}\right]\right) = 0,$$

$$s(n_k) = 2,$$

$$r(n_k - \sigma) = \cdots = r(n_k - 1) = r(n_k) = \cdots = r\left(n_k + \left[\frac{k}{40}\right]\right) = 0,$$

$$t(n_k - \sigma) = \cdots = t(n_k - 1) = t(n_k) = \cdots = t\left(n_k + \left[\frac{k}{40}\right]\right) = 0,$$

$$a(n_k - \sigma) = \cdots = a(n_k - 1) = a(n_k) = \cdots = a\left(n_k + \left[\frac{k}{40}\right]\right) = 0, \tag{31}$$

$$b(n_k - \sigma) = \cdots = b(n_k - 1) = b(n_k) = \cdots = b\left(n_k + \left[\frac{k}{40}\right]\right) = 0. \tag{32}$$

For the proof of the relations (31) and (32), observe that $t(n) = 0$ implies $a(n) = b(n) = 0$; otherwise, since $u_0 = v_0 = 0$, the equation $u_p + v_m = n$ would have a solution. Thus, by using (2), (3), (4), (29), we get

$$\tau(n_k - \sigma) = \cdots = \tau(n_k - 1) = \tau(n_k + 1) = \cdots = \tau\left(n_k + \left[\frac{k}{40}\right]\right) = 0, \tag{33}$$

$$\tau(n_k) = 2Bb(u)b(v), \text{ with } (u, v) \in \mathbb{N}^2, \quad b(u)b(v) \neq 0. \tag{34}$$

For the proof of the relation (34), by (29) one has $\tau(n_k) = Bs'(n_k) = 2Bb(u)b(v)$ for some $u = v_p$ and $v = v_m$ by (1), where $v_p + v_m = v_m + v_p = n_k$ comes from $s(n_k) = 2$.

But we know that

$$\rho(n) \le d(n) \le \exp\left(\frac{\log n}{\log\log n}\right)$$

for large n ([8; p. 262, Th. 317, and §18-7, p. 270]). Using (29), (2), (3), (4), and Lemmas 4 and 5, we get for large n

$$\tau(n) \le \exp\left(\frac{2\log n}{\log\log n}\right) = \omega(n). \tag{35}$$

Moreover, Theorem 5 and (24) yield

$$n_k + \left[\frac{k}{40}\right] + 1 \le 40^{L-1}\exp(4Lp_{2[\varepsilon k]}) + \left[\frac{k}{40}\right] + 1. \tag{36}$$

Using the prime number theorem in arithmetic progressions, we have for large k

$$\frac{1}{4}\frac{p_{2[\varepsilon k]}}{\log p_{2[\varepsilon k]}} \le 2\varepsilon k \le \frac{p_{2[\varepsilon k]}}{\log p_{2[\varepsilon k]}}, \tag{37}$$

so that by (36)

$$n_k + \left[\frac{k}{40}\right] + 1 \le \exp(8Lp_{2[\varepsilon k]}).$$

Hence we get by (35)

$$\log \omega\Big(n_k + \left[\frac{k}{40}\right] + 1\Big) \le \frac{16Lp_{2[\varepsilon k]}}{\log 8L + \log p_{2[\varepsilon k]}}$$

and so for large k

$$\omega\Big(n_k + \left[\frac{k}{40}\right] + 1\Big) \le \frac{32Lp_{2[\varepsilon k]}}{\log p_{2[\varepsilon k]}}. \tag{38}$$

We put $h = p_{2[\varepsilon k]}/(\log p_{2[\varepsilon k]})$. Then h tends to infinity as k does. By using (38) and (37), we get

$$\frac{\omega(n_k + [k/40] + 1)}{|q|^{[k/40]}} \le \frac{\exp 32L\ h}{|q|^{(h/320\varepsilon)-1}}.$$

Therefore, by the choice of ε in (30), we have

$$\lim_{k\to+\infty} \frac{\omega(n_k + [k/40] + 1)}{|q|^{[k/40]}} = 0.$$

Noting that $\lim_{k\to+\infty} \omega(n+1)/\omega(n) = 1$, and recalling (29) and (33), we can apply Theorem 4 and obtain

$$Fq^{n_k} + \sum_{n=0}^{n_k} \tau(n) q^{n_k - n} = 0.$$

By using (33) and (34), we now have for some $(u, v) \in \mathbb{N}^2$

$$Fq^{n_k} + 2Bb(u)b(v) + \sum_{n=0}^{n_k-\sigma-1} \tau(n) q^{n_k - n} = 0.$$

Thus $q^{\sigma+1}$ divides $2Bb(u)b(v)$, and this contradiction proves that $B = 0$.

Second step. We now suppose that $C \neq 0$, and we choose σ such that q^σ does not divide any of the numbers $2Ca(u)b(v)$ for $(u, v) \in \mathbb{N}^2$ with $a(u)b(v) \neq 0$. In Theorem 5, we put

$$\delta = 40\sigma + 16,$$

and choose ε as in (30), $\Omega_1 = 5$, $\Omega_2 = 5$, $\theta_1 = 1$, $\theta_2 = 2$. Then N_k in (23) satisfies $N_k \equiv 10 \pmod{40}$ and we put $n_k = (N_k - 10)/40$. By using (15), (22), and Lemmas 3, 4 and 5 (observe that $N_k = 40n_k + 10$, so that (17) in Lemma 5 applies), we get

$$\begin{aligned}
&t(n_k - \sigma) = \cdots = t(n_k - 1) = t(n_k + 1) = \cdots = t\Big(n_k + \Big[\frac{k}{40}\Big]\Big) = 0,\\
&t(n_k) = 2,\\
&r(n_k - \sigma) = \cdots = r(n_k - 1) = r(n_k) = \cdots = r\Big(n_k + \Big[\frac{k}{40}\Big]\Big) = 0,\\
&s(n_k - \sigma) = \cdots = s(n_k - 1) = s(n_k) = \cdots = s\Big(n_k + \Big[\frac{k}{40}\Big]\Big) = 0,\\
&a(n_k - \sigma) = \cdots = a(n_k - 1) = a(n_k) = \cdots = a\Big(n_k + \Big[\frac{k}{40}\Big]\Big) = 0,\\
&b(n_k - \sigma) = \cdots = b(n_k - 1) = b(n_k) = \cdots = b\Big(n_k + \Big[\frac{k}{40}\Big]\Big) = 0.
\end{aligned}$$

By arguing exactly the same way as the first step, we obtain $C = 0$.

Third step. Suppose that $A \neq 0$, and choose σ such that q^σ does not divide any of the numbers $2Aa(u)a(v)$ for $(u, v) \in \mathbb{N}^2$ with $a(u)a(v) \neq 0$. Choose again $\delta = 40\sigma + 16$, ε as in (30), $\Omega_1 = 1$, $\Omega_2 = 5$, $\theta_1 = 1$, $\theta_2 = 2$ in Theorem 5, and put $n_k = (N_k - 2)/40$. By going on exactly as in the first and second steps, one can prove that $A = 0$.

Fourth step. Thus we have $D\alpha_q + E\beta_q + F = 0$. It can be proved, by elementary means, this is impossible. Hence Theorem 3 is proved.

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References

[1] D. Duverney: *Propriétés arithmétiques d'une série liée aux fontions thêta*, Acta Arith. **64** (1993), 175–187.

[2] D. Duverney: *Sommes de deux carrés et irrationalité de valeurs de fonctions thêta*, C.R. Acad. Sci. Paris, Ser. I, **320** (1995), 1041–1044.

[3] D. Duverney: *Propriétés arithmétiques d'un produit infini lié aux fonctions thêta*, J. Reine Angew. Math. **477** (1996), 1–12.

[4] D. Duverney: *Some arithmetical consequences of Jacobi's triple product identity*, Math. Proc. Camb. Phil. Soc. **122** (1997), 393–399.

[5] D. Duverney, Keiji Nishioka, Kumiko Nishioka and Iekata Shiokawa: *Transcendence of Rogers-Ramanujan continued fraction*, Proc. Japan Acad. **73A7** (1997), 140–142.
[6] G. Gasper and M. Rahman: *Basic hypergeometric series*, Cambridge University Press, 1990.
[7] S. Graham: *On Linnik's constant*, Acta Arith. **39** (1981), 163–179.
[8] G.H. Hardy and E.M. Wright: *An introduction to the Theory of Numbers, Fifth Edition*, Oxford Science Publications, 1989.
[9] W.B. Jones and W.J. Thron: *Continued fractions: Analytic theory and applications*, Addison-Wesley, 1980.
[10] H. Lebesgue: *Leçons sur les constructions géométriques*, Gauthiers-Villars, 1949.
[11] T. Matala-Aho: *On diophantine approximations of the Rogers-Ramanujan continued fraction*, J. Number Theory, **45** (1993), 215–227.
[12] Yu V. Nesterenko: *Modular functions and transcendence problems*, Sbornik Math. **187** (1996), 1319–1348.
[13] C.F. Osgood: *The diophantine approximation of certain continued fractions*, Proc. Amer. Math. Soc. **3** (1977), 1–7.
[14] I. Shiokawa: *Rational approximations to the values of certain hypergeometric functions*, in Number Theory and Combinatorics, Japan 1984, Word Scientific, Singapore, 1985.
[15] I. Shiokawa: *Rational approximations to the Rogers-Ramanujan continued fraction*, Acta Arith. **50** (1988), 23–30.

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