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# SHORT PROOFS OF HIRAMINE' RESULTS ON CHARACTER VALUES

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## 1. Introduction

Let $G$ and $H$ be finite groups of order $n$. A mapping $f$ from $G$ into $H$ is called a planar function of degree $n$ if, for each element $\tau \in H$ and $u \in G^* = G - \{1\}$, there exists exactly one $x \in G$ such that $f(ux)f(x)^{-1} = \tau$. In [2] Hiramine has shown that if both $G$ and $H$ are abelian groups of order $3p$ with $p(\geq 5)$ a prime, then there exists no planar function from $G$ into $H$. To prove this he has established two results on character values. Their proofs are slightly complicated. In this note we shall give short proofs.

In section 2 we shall present Proposition 2 which is useful for the proof of Result 2. In section 3 we shall state Hiramine' results and give short proofs.

We follow the notation and terminology of [2].

## 2. Planar Functions and Equations in Group Algebras

Let $G$ and $H$ be finite groups of order $n$. Throughout this article elements of $G$ will be denoted by small Roman letters and elements of $H$ by small Greek letters.
Let $f$ be a mapping from $G$ into $H$ and $S_\alpha = \{x \in G | f(x) = \alpha\}$, $\alpha \in H$. If $S_\alpha \neq \emptyset$, we set $\hat{S_\alpha} = \sum_{x \in S_\alpha} x \in C[G]$ and $\hat{S}_\alpha^{-1} = \sum_{x \in S_\alpha} x^{-1} \in C[G]$, otherwise $\hat{S_\alpha} = \hat{S}_\alpha^{-1} = 0$, where $C[G]$ is the group algebra of $G$ over the complex number field $C$. Let $G_0 = G \times H$ be the direct product of groups $G, H$.

To prove the results we need two propositions.

The following is Proposition 2.1 [2].

**Proposition 1.** *The following are equivalent.*

(i) *The function $f$ is planar.*

(ii) *In the group algebra $C[G]$ of $G$,*

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha} \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_{\alpha\tau}^{-1} \hat{S}_\alpha = \begin{cases} \hat{G} + n - 1 & \textit{if } \tau = 1, \\ \hat{G} - 1 & \textit{otherwise.} \end{cases}$$

REMARK 1. If $\tau \neq 1$, then it follows from the equation in (ii) of the proposition above that in the group algebra $C[G_0]$ of $G_0$,

$$\sum_{\alpha \in H} \hat{S}_{\tau\alpha}\tau\alpha\hat{S}_{\alpha}^{-1}\alpha^{-1} = (\hat{G}-1)\tau.$$

We prove the following

**Proposition 2.** *We have in* $C[G_0]$,

$$(\sum_{\alpha \in H} \hat{S}_{\alpha}\alpha)(\sum_{\beta \in H} \hat{S}_{\beta}^{-1}\beta^{-1}) = \hat{G} + n - 1 + \sum_{\tau \in H,\, \tau \neq 1} (\hat{G}-1)\tau.$$

Proof of Proposition 2.

$$\begin{aligned}(\sum_{\alpha \in H} \hat{S}_{\alpha}\alpha)(\sum_{\beta \in H} \hat{S}_{\beta}^{-1}\beta^{-1}) &= \sum_{\tau \in H}(\sum_{\beta \in H} \hat{S}_{\tau\beta}\tau\beta\hat{S}_{\beta}^{-1}\beta^{-1}) \\ &= \hat{G} + n - 1 + \sum_{\tau \in H,\, \tau \neq 1} (\hat{G}-1)\tau, \quad \text{by Remark 1.}\end{aligned}$$

We complete the proof of Proposition 2. □

## 3. Proofs of Hiramine' Results

We start with the following well-known facts about character theory. These facts play important parts in the proofs of his results.

FACT 1. *Let $G$ be an abelian group and $\chi$ an arbitrary (linear) character of $G$. Then $\chi$ is a homomorphism from $G$ into $C^* = C - \{0\}$. So we can extend this homomorphism $\chi$ to an algebra homomorphism $\overline{\chi}$ from $C[G]$ into $C$.*

FACT 2. *Let $H_1, H_2$ be finite groups and $G_1$ the direct product of $H_1, H_2$. Then all irreducible characters of $G_1$ are obtained as follows. Let $\chi_0, \ldots, \chi_{s-1}$ be the irreducible characters of $H_1$, $\rho_0, \ldots, \rho_{t-1}$ the irreducible characters of $H_2$. Then $G_1$ has exactly st irreducible characters $\Psi_{ij} (0 \leq i \leq s-1,\ 0 \leq j \leq t-1)$, satisfying $\Psi_{ij}(h_1 h_2) = \chi_i(h_1)\rho_j(h_2)$, where $h_1 \in H_1,\ h_2 \in H_2$.*

Proof. See[1, p.54]. □

REMARK 2. In Fact 2 if both $\chi_i$ and $\rho_j$ are linear characters, then $\Psi_{ij}$ is a homomorphism from $G_1$ to $C^*$. As in Fact 1, we have an algebra homomorphism $\overline{\Psi}_{ij}$ from $C[G_1]$ into $C$ which is an extension of $\Psi_{ij}$.

Now we shall start to state Hiramine' results and prove them. In the remainder of this section we assume that $f$ is a planar function and that $G$ is an abelian group of order $n$. Let $\chi_0(= 1_G), \ldots, \chi_{n-1}$ be the irreducible (linear) characters of $G$, where $1_G$ denote the principal character of $G$. We set

$$d_i^{(\alpha)} = \begin{cases} \sum_{x \in S_\alpha} \chi_i(x) & \text{if } S_\alpha \neq \emptyset, \\ 0 & \text{if } S_\alpha = \emptyset \end{cases}$$

for each $0 \leq i \leq n-1$ and for each $\alpha \in H$. Now we state one of Hiramine' results[2].

RESULT 1. *The following hold*

(i) $d_0^{(\alpha)} = |S_\alpha|$ *and*

$$\sum_{\alpha \in H} d_0^{(\tau\alpha)} d_0^{(\alpha)} = \sum_{\alpha \in H} d_0^{(\alpha\tau)} d_0^{(\alpha)} = \begin{cases} 2n-1 & \text{if } \tau = 1, \\ n-1 & \text{otherwise.} \end{cases}$$

(ii) *For* $i \neq 0$,

$$\sum_{\alpha \in H} d_i^{(\tau\alpha)} \overline{d_i^{(\alpha)}} = \sum_{\alpha \in H} \overline{d_i^{(\alpha\tau)}} d_i^{(\alpha)} = \begin{cases} n-1 & \text{if } \tau = 1, \\ -1 & \text{otherwise.} \end{cases}$$

(*Here* $\bar{d}$ *denotes the complex conjugate of* $d \in C$.)

Proof. Since $G$ is abelian, from Fact 1 we note that for each $0 \leq i \leq n-1$, $\overline{\chi_i}$ is an algebra homomorphism from $C[G]$ into $C$. We shall prove (i). It is immediate that $d_0^{(\alpha)} = |S_\alpha|$. If $\tau = 1$, then the equation in (ii) of Proposition 1 becomes

$$\sum_{\alpha \in H} \hat{S}_\alpha \hat{S}_\alpha^{-1} = \sum_{\alpha \in H} \hat{S}_\alpha^{-1} \hat{S}_\alpha = \hat{G} + n - 1.$$

We apply $\overline{\chi_0}$ to this equation. Then

$$\sum_{\alpha \in H} \overline{\chi_0}(\hat{S}_\alpha) \overline{\chi_0}(\hat{S}_\alpha^{-1}) = \sum_{\alpha \in H} \overline{\chi_0}(\hat{S}_\alpha^{-1}) \overline{\chi_0}(\hat{S}_\alpha) = \overline{\chi_0}(\hat{G} + n - 1),$$

which implies the equation for $\tau = 1$ in (i). Similarly we can prove the equation for $\tau \neq 1$ in (i). We have proved (i). Next we shall prove (ii). By applying the algebra homomorphism $\overline{\chi_i}$ $(i \neq 0)$ to the equation in (ii) of Proposition 1 we can prove (ii). This completes the proof of Result 1. □

We state another result on character values in [2].

RESULT 2. *With the same notation and assumption as in Result 1, suppose that $H$ is abelian and let $\rho_0(=1_H), \ldots, \rho_{n-1}$ be the irreducible characters of $H$. Set $z_{ij} = \sum_{\alpha \in H} d_i^{(\alpha)} \rho_j(\alpha)$. Then,*

(i) $z_{0,0} = n$, *and* $z_{i,0} = 0 (i \neq 0)$

(ii) *For* $j \neq 0$, $z_{ij}\overline{z_{ij}} = n$.

Proof. Since $\chi_i$ and $\rho_j$ are linear, from Remark 2 we see that $\overline{\Psi}_{ij}$ is an algebra homomorphism from $C[G_0]$ into $C$. First we shall prove (ii). We apply $\overline{\Psi}_{ij}$ $(j \neq 0)$ to the equation in Proposition 2. Then we get (ii). We have proved (ii). Next we shall prove (i). Similarly by using $\overline{\Psi}_{i0}$ we can prove (i). This completes the proof of Result 2. $\square$

REMARK 3. Nakagawa[3] has proved Result 2 by using Gaussian sums.

## References


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